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On isolated strata of pentagonal Riemann surfaces in the branch locus of moduli spaces

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Abstract

The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g has orbifold structure, and the set of singular points of such orbifold is the *branch locus* $\mathcal{B}_g$. For $g \not\equiv 3 \; mod \; 4$, $g \geq 26$, $g \neq 37$, there exists isolated strata corresponding to families of pentagonal Riemann surfaces.

1 Introduction

In this article we study the topology of moduli spaces of Riemann surfaces. The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g being the quotient of the Teichmüller space by the discontinuous action of the mapping class group, has the structure of a complex orbifold, whose set of singular points is called the *branch locus* $\mathcal{B}_g$. The branch locus $\mathcal{B}_g$, $g \geq 3$ consists of the Riemann surfaces with symmetry, i. e. Riemann surfaces with non-trivial automorphism group. Our goal is to study of the topology of $\mathcal{B}_g$ through its connectedness. The connectedness of moduli spaces of hyper-elliptic, $p-$gonal and real Riemann surfaces has been widely studied, for instance by [?], [?], [?], [?], [?].

It is known that $\mathcal{B}_2$ is not connected, since R. Kulkarni (see [?] and [?]) showed that the curve $w^2 = z^5 - 1$ is isolated in $\mathcal{B}_2$, i. e. this single surface is a isolated component of $\mathcal{B}_2$, furthermore $\mathcal{B}_2$ has exactly two connected components (see [?] and [?]). It is also known that the branch loci $\mathcal{B}_3$, $\mathcal{B}_4$ and $\mathcal{B}_7$ are connected and $\mathcal{B}_5, \mathcal{B}_6$, $\mathcal{B}_8$ are connected with the exception of isolated points (see [?] and [?]).

In this article we prove that $\mathcal{B}_g$ is disconnected for $g \not\equiv 3 \; mod \; 4$, $g \geq 26$; more concretely we find equisymmetric isolated strata induced by order 5 automorphisms of Riemann surfaces of genera $g \not\equiv 3 \; mod \; 4$. In [?] it is proved that $\mathcal{B}_g$ is disconnected for $g \geq 65$.

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2 Riemann surfaces and Fuchsian groups

Let X be a Riemann surface and assume that $Aut(X) \neq \{1\}$. Hence $X/Aut(X)$ is an orbifold and there is a Fuchsian group $\Gamma \leq Aut(\mathcal{D})$, such that $\pi_1(X) \lhd \Gamma$:

$$\mathcal{D} \to X = \mathcal{D}/\pi_1(X) \to X/Aut(X) = \mathcal{D}/\Gamma$$

where $\mathcal{D} = \{z \in \mathbb{C} : \|z\| < 1\}$.

If the Fuchsian group Γ is isomorphic to an abstract group with canonical presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1 \ldots x_k | x_1^{m_1} = \cdots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle, \quad (1)$$

we say that Γ has *signature*

$$s(\Gamma) = (g; m_1, \ldots, m_k). \quad (2)$$

The generators in presentation (**??**) will be called *canonical generators*.

Let X be a Riemann surface uniformized by a surface Fuchsian group Γ_g, i.e. a group with signature $(g; -)$. A finite group G is a group of automorphisms of X, i.e. there is a holomorphic action a of G on X, if and only if there is a Fuchsian group Δ and an epimorphism $\theta_a : \Delta \to G$ such that $\ker \theta_a = \Gamma_g$. The epimorphism θ_a is the monodromy of the covering $f_a : X \to X/G = \mathcal{D}/\Delta$.

The relationship between the signatures of a Fuchsian group and subgroups is given in the following theorem of Singerman:

Theorem 1 *([**?**]) Let Γ be a Fuchsian group with signature (**??**) and canonical presentation (**??**). Then Γ contains a subgroup Γ' of index N with signature*

$$s(\Gamma') = (h; m'_{11}, m'_{12}, ..., m'_{1s_1}, ..., m'_{k1}, ..., m'_{ks_k}).$$

if and only if there exists a transitive permutation representation $\theta : \Gamma \to \Sigma_N$ satisfying the following conditions:

1. The permutation $\theta(x_i)$ has precisely s_i cycles of lengths less than m_i, the lengths of these cycles being $m_i/m'_{i1}, ..., m_i/m'_{is_i}$.

2. The Riemann-Hurwitz formula

$$\mu(\Gamma')/\mu(\Gamma) = N.$$

where $\mu(\Gamma)$, $\mu(\Gamma')$ are the hyperbolic areas of the surfaces $\mathcal{D}/\Gamma$, $\mathcal{D}/\Gamma'$.

For $\mathcal{G}$, an abstract group isomorphic to all the Fuchsian groups of signature $s = (h; m_1, ..., m_k)$, the Teichmüller space of Fuchsian groups of signature s is:

$$\{\rho : \mathcal{G} \to PSL(2, \mathbb{R}) : s(\rho(\mathcal{G})) = s\}/ \text{ conjugation in } PSL(2, \mathbb{R}) \; = \; T_s.$$

The Teichmüller space T_s is a simply-connected complex manifold of dimension $3g-3+k$. The modular group, $M(\Gamma)$, of Γ, acts on $T(\Gamma)$ as $[\rho] \to [\rho \circ \alpha]$ where $\alpha \in M(\Gamma)$. The moduli space of Γ is the quotient space $\mathcal{M}(\Gamma) = T(\Gamma)/M(\Gamma)$, then $\mathcal{M}(\Gamma)$ is a complex orbifold and its singular locus is $\mathcal{B}(\Gamma)$, called the branch locus of $\mathcal{M}(\Gamma)$. If Γ_g is a surface Fuchsian group, we denote $\mathcal{M}_g = T_g/M_g$ and the branch locus by $\mathcal{B}_g$. The branch locus $\mathcal{B}_g$ consists of surfaces with non-trivial symmetries for $g > 2$.

If $X/Aut(X) = \mathcal{D}/\Gamma$ and genus$(X) = g$, then there is a natural inclusion $i : T_s \to T_g : [\rho] \to [\rho']$, where

$$\rho : \mathcal{G} \to PSL(2,\mathbb{R}),\ \pi_1(X) \subset \mathcal{G},\ \rho' = \rho \mid_{\pi_1(X)}: \pi_1(X) \to PSL(2,\mathbb{R}).$$

If we have $\pi_1(X) \lhd \mathcal{G}$, then there is a topological action of a finite group $G = \mathcal{G}/\pi_1(X)$ on surfaces of genus g given by the inclusion $a : \pi_1(X) \to \mathcal{G}$. This inclusion $a : \pi_1(X) \to \mathcal{G}$ produces $i_a(T_s) \subset T_g$.

The image of $i_a(T_s)$ by $T_g \to \mathcal{M}_g$ is $\overline{\mathcal{M}}^{G,a}$, where $\overline{\mathcal{M}}^{G,a}$ is the set of Riemann surfaces with automorphisms group containing a subgroup acting in a topologically equivalent way to the action of G on X given by the inclusion a, see [?], the subset $\mathcal{M}^{G,a} \subset \overline{\mathcal{M}}^{G,a}$ is formed by the surfaces whose full group of automorphisms acts in the topologically way given by a. The branch locus, $\mathcal{B}_g$, of the covering $T_g \to \mathcal{M}_g$ can be described as the union $\mathcal{B}_g = \bigcup_{G \neq \{1\}} \overline{\mathcal{M}}^{G,a}$, where $\{\mathcal{M}^{G,a}\}$ is the equisymmetric stratification of the branch locus [?]:

Theorem 2 *([?]) Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, G a finite subgroup of the corresponding modular group M_g. Then:*

(1) $\overline{\mathcal{M}}_g^{G,a}$ is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$.

(2) $\mathcal{M}_g^{G,a}$, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in $\overline{\mathcal{M}}_g^{G,a}$.

There are finitely many strata $\mathcal{M}_g^{G,a}$.

An isolated stratum $\mathcal{M}^{G,a}$ in the above stratification is a stratum that satisfies $\overline{\mathcal{M}}^{G,a} \cap \overline{\mathcal{M}}^{H,b} = \varnothing$, for every group H and action b on surfaces of genus g. Thus $\overline{\mathcal{M}}^{G,a} = \mathcal{M}^{G,a}$

Since each non-trivial group G contains subgroups of prime order, we have the following remark:

Remark 3 *([?])*

$$\mathcal{B}_g = \bigcup_{p\ prime} \overline{\mathcal{M}}^{C_p,a}$$

where $\overline{\mathcal{M}}^{C_p,a}$ is the set of Riemann surfaces of genus g with an automorphism group containing C_p, the cyclic group of order p, acting on surfaces of genus g in the topological way given by a.

3 Disconnectedness by pentagonal Riemann surfaces

By the Castelnuovo-Severi inequality [?], the p-gonal morphism of an elliptic- or p- gonal Riemann surface X_g of genus g is unique if $g \geq 2hp + (p-1)^2 + 1$, where $h \in \{0,1\}$ is the genus of the quotient surface.

Let X_g, $g \geq 10h+17$, be an (elliptic-) pentagonal surface, such that $X_g \in \overline{\mathcal{M}}_g^{C_5,a}$ for some action a, let $\langle\alpha\rangle$ be the group of (elliptic-) pentagonal automorphisms of X_g. Consider an automorphism $b \in Aut(X) \setminus \langle\alpha\rangle$, by the Castelnuovo-Severi inequality, b induces an automorphism $\hat{b}$ of order p on the Riemann surface $X_g/\langle a\rangle = Y_h$, of genus h, according to the following diagram:

$$\begin{array}{ccc} X_g = \mathcal{D}/\Gamma_g & \overset{b}{\to} & X_g = \mathcal{D}/\Gamma_g \\ f_a \downarrow & & \downarrow f_a \\ X_g/\langle\alpha\rangle = Y_h(P_1,\ldots,P_k) & \overset{\hat{b}}{\to} & X_g/\langle\alpha\rangle = Y_h(P_1,\ldots,P_k) \end{array}$$

where Γ_g is a surface Fuchsian group and $f_a : X_g = \mathcal{D}/\Gamma_g \to X_g/\langle\alpha\rangle$ is the morphism induced by the group of automorphisms $\langle\alpha\rangle$ with action a. $S = \{P_1,\ldots,P_k\}$ is the branch set in Y_h of the morphism f_a with monodromy $\theta_a : \Delta(h;5,\overset{k}{\ldots},5) \to C_5$ defined by $\theta_a(x_i) = \alpha^{t_i}$, where $t_i \in \{1,2,3,4\}$ for $1 \leq i \leq k$. Let n_j denote the number of times that the exponents j occurs among $t_1,\ldots,t_k$, for $1 \leq j \leq 4$. Then $n_1+n_2+n_3+n_4 = k$ and $1n_1+2n_2+3n_3+4n_4 \equiv 0 \ mod\ 5$.

Now, $\hat{b}$ induces a permutation on S that either takes singular points with monodromy α^j to points with monodromy α^{5-j}, takes points with monodromy α^j to points with monodromy α^{2j}, or it acts on each subset formed by points in S with same monodromy α^{t_j}. Therefore the following conditions force $\hat{b}$ to be the identity on Y_h:

$$\begin{array}{l} 1. |n_1-n_4| + |n_2-n_3| \geq 3+h, \\ 2. |n_1-n_j| \geq 3+h, \text{ for some } n_j \text{ such that } 2 \leq j \leq 4 \text{ and} \\ 3. \text{let } \widehat{n_j} \in \{1,2,3,4\}, \text{ such that } \widehat{n_j} \equiv n_j \,\mathrm{mod}\, p, \text{ then } \sum_{j=1}^{4} \widehat{n_j} \geq 3+h. \end{array} \qquad (3)$$

Theorem 4 *Assume $g \geq 18$ is even, then there exist isolated strata formed by pentagonal surfaces.*

Proof. We will construct monodromies $\theta : \Delta(0;5,\overset{k}{\ldots},5) \to C_5$, where $k = \frac{g}{2}+2$ by the Riemann-Hurwitz formula, such that the conditions (**??**) above are satisfied. Assume $\theta(x_i) = \alpha^{t_i}$, $i = 1,\ldots,k$. Let $n_j = |\{t_i = j; i = 1,\ldots,k\}|$, then we will define the epimorphism θ by the generating vector $(n_1\alpha,\ n_2\alpha^2,\ n_3\alpha^3,\ n_4\alpha^4)$, where $n_j\alpha^j$ means that α^j is the monodromy of n_j different singular points P_i.

g *mod* 5	k *mod* 5	n_1	n_2	n_3	n_4
$g \equiv 0\ mod\ 5$	$k \equiv 2\ mod\ 5$	$(k-13)$	5	1	7
$g \equiv 1\ mod\ 5$	$k \equiv 0\ mod\ 5$	$(k-7)$	5	1	1
$g \equiv 2\ mod\ 5$	$k \equiv 3\ mod\ 5$	$(k-9)$	1	3	5
$g \equiv 3\ mod\ 5$	$k \equiv 1\ mod\ 5$	$(k-7)$	1	5	1
$g \equiv 4\ mod\ 5$	$k \equiv 4\ mod\ 5$	$(k-9)$	5	1	3

We see that the given epimorphisms satisfy the conditions (**??**) except for $g = 20$, $k = 12$. However, in this case, $g = 20$, $k = 12$, let the epimorphism $\theta : \Delta(0;5,\overset{12}{\ldots},5) \to C_5$ be defined by the generating vector $(\alpha, 7\alpha^2, \alpha^3, 3\alpha^4)$. θ clearly satisfies the conditions **??** above. ■

Remark 5 *The complex dimension of the isolated strata given in the proof of theorem* **??** *is* $0 \times 3 - 3 + k = g/2 + 2 - 3 = g/2 - 1$.

Remark 6 *There are several isolated strata of dimension* $g/2 - 1$ *in* $\mathcal{B}_g$ *for even genera* $g \geq 22$. *For instance consider* $g \equiv 3$ *mod* 5. *The monodromy* θ' *defined by the generating vector* $((k-9)\alpha,\, 5\alpha^2,\, 3\alpha^3,\, 5\alpha^4)$ *induces an isolated stratum different from the one given in the proof of Theorem* **??** *since the actions determined by* θ *and* θ' *are not topologically equivalent, see [***?***].*

Theorem 7 *Assume* $g \geq 29$, $g \equiv 1$ *mod* 4, $g \neq 37$, *then there exists isolated strata formed by elliptic-pentagonal surfaces.*

Proof. Similarly to the proof of Theorem **??**, using the conditions above (**??**), we will construct epimorphisms $\theta : \Delta(1; 5, \overset{k}{\ldots}, 5) \to C_5$, where $k = \frac{g-1}{2}$ by the Riemann-Hurwitz formula. Assume $\theta(x_i) = \alpha^{t_i}$, $i = 1, \ldots, k$. Let $n_j = |\{t_i = j; i = 1, \ldots, k\}|$, the epimorphism θ will be defined by the generating vector $(n_1\alpha,\, n_2\alpha^2,\, n_3\alpha^3,\, n_4\alpha^4)$, where $n_j\alpha^j$ means that α^j appears as the monodromy of n_j different singular points P_i.

g *mod* 5	k *mod* 5	n_1	n_2	n_3	n_4
$g \equiv 0$ *mod* 5	$k \equiv 2$ *mod* 5	$(k-13)$	5	1	7
$g \equiv 1$ *mod* 5	$k \equiv 0$ *mod* 5	$(k-7)$	5	1	1
$g \equiv 2$ *mod* 5	$k \equiv 3$ *mod* 5	$(k-19)$	11	3	5
$g \equiv 3$ *mod* 5	$k \equiv 1$ *mod* 5	$(k-7)$	1	5	1
$g \equiv 4$ *mod* 5	$k \equiv 4$ *mod* 5	$(k-11)$	1	5	5

We see that the given epimorphisms satisfy the conditions set except for $g = 37$, $k = 18$.

■

Remark 8 *The complex dimension of the isolated strata given in the proof of theorem* **??** *is* $1 \times 3 - 3 + k = (g-1)/2$.

Remark 9 *Let* $g \equiv 3$ *mod* 4. *Then there is no isolated stratum in* $\mathcal{B}_g$ *of dimension* $(g-1)/2$. *Such a stratum will consist of elliptic-pentagonal surfaces given by epimorphisms* $\theta : \Delta(1; 5, \overset{k}{\ldots}, 5) \to C_5$, *where* $k = (g-1)/2 \equiv 1$ *mod* 2. *Now such epimorphisms cannot satisfy the third condition in (***??***).*

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