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Robert W Grubbström , The time-averaged L4L solution - a condition for long-run stability applying MRP theory, 2012, International Journal of Production Research, (50), 21, 6099-6110.
International Journal of Production Research is available online at informaworld™:
http://dx.doi.org/10.1080/00207543.2011.653457

The Time-Averaged Lot-for-Lot Solution - A Condition for Long-Run Stability Applying MRP Theory

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Abstract
MRP Theory provides a theoretical background for multi-level, multi-stage production-inventory systems (Material Requirements Planning in a general sense) together with their economic evaluation, in particular applying the *Net Present Value Principle*. The theory combines the use of *Input-Output Analysis* and *Laplace transforms*, the former for capturing product structures, and the latter for incorporating timing, including time lags, lead times, and output delays.

In this paper, the we consider any production policy, when given any external demand as a vector-valued function of time. It is shown that in order for available inventory to be kept at finite levels at any time, the Lot-for-Lot (L4L) solution must be valid for the time averages of production and deliveries, irrespective of the policy followed. This analysis is carried out using properties the Laurent expansions of the transforms involved.

Keywords: *MRP theory, Lot-For-Lot, Laplace transform, input-output analysis, stability*.

1. Introduction

MRP Theory provides a theoretical background for multi-level, multi-stage production-inventory systems (Material Requirements Planning [Orlicky, 1975] in a general sense) together with their economic evaluation, in particular applying the Net Present Value Principle. The theory combines the use of *Input-Output Analysis* [Leontief, 1928] and *Laplace transforms* [Aseltine, 1958], the former for capturing product structures, and the latter for incorporating time lags, lead times, and output delays, etc. For an early overview of MRP Theory, please consult [Grubbström and Tang, 2000].

A basic entity concerning product structures is the *input matrix* **H**, and concerning lead times the *lead time matrix* $\tilde{\boldsymbol{\tau}}(s)$, where s is the Laplace frequency. The diagonal matrix $\tilde{\boldsymbol{\tau}}(s)$ contains elements $e^{-s\tau_i}$, $i = 1, 2, \ldots$, which function as operators moving the function multiplied by in time by τ_i backwards in time.

The Laplace transform is defined by the integral equation [Aseltine, 1958]

$$\tilde{f}(s) = \pounds\left\{ f(t) \right\} = \int_{t=0}^{\infty} f(t) e^{-st} dt , \tag{1}$$

which translates a time function $f(t)$, $t \geq 0$, from the time domain of t into the complex frequency domain of s. In all practical cases, there is a one-to-one-correspondence between the time function and its transform. The tilde and £ are two alternative notations, the first indicating that the transform is a function of a new variable *s*, the second indicating the origin of the transform. The inverse transform, translating a transform back into the time domain, is normally written:

$$£^{-1}\left\{ \tilde{f}(s) \right\} = f(t). \tag{2}$$

The product $\mathbf{H}\tilde{\boldsymbol{\tau}}(s)$ has been named the *generalised input matrix,* and it includes information on all requirements and their requested advanced timing in relation to completion dates. The Lot-for-Lot (L4L) policy (also named "As Required") prescribes production in such a way that there is never any addition to available inventory. If total production is $\tilde{\mathbf{P}}(s)$, this requires internal (dependent) demand of items in the amounts of $\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\tilde{\mathbf{P}}(s)$, so *net production* (for an assembly system) is $\left(\mathbf{I}-\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)\tilde{\mathbf{P}}(s)$, where $\mathbf{I}$ is the identity matrix. The matrix $\left(\mathbf{I}-\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)$ is named the *technology matrix*.

If given the planned external demand (Master Production Schedule) to be exported from the system as a transformed vector $\tilde{\mathbf{D}}(s)$, the L4L solution determines production $\tilde{\mathbf{P}}(s)$ as to amounts of all items together with their completion times by:

$$\tilde{\mathbf{P}}(s) = \left(\mathbf{I}-\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\tilde{\mathbf{D}}(s). \tag{3}$$

If available inventory is empty at time zero, this formula is universally valid for any MRP system, with any number of items, any product structures, and with any given lead times.

Other common policies in practice, but not possible to express in any equally simple equation, are the Fixed Order Quantity (FOQ), by which production batches always have the same size, or the Fixed Period Requirements policy (FPR), by which production always covers demand (internal and external) during a fixed number of periods, cf. [Grubbström and Huynh, 2006].

In this paper, we consider any production policy, when given any external demand $\tilde{\mathbf{D}}(s)$. It is shown that in order for available inventory to be kept at finite levels at any time, the L4L solution must be valid for the time averages of production and demand, irrespective of the policy followed. Recently, [Grubbström et al., 2009, Grubbström and Tang, 2012] it has been shown that the L4L solution determines all possible times when internal demand events can occur, so we may regard the currently considered property of the L4L solution as its third rôle.

The analysis that follows uses properties the Laurent expansions of the transforms involved. In the next section, we briefly show generalisations of the L4L solution to other than assembly systems, followed by a section dealing with time averages as properties of Laplace transforms. In Section 4, we extend our results to the case of stochastic demand, which is assumed to follow a compound renewal process. Our fifth section offers an example of a simple assembly system applying a Fixed Order Quantity policy illustrating our general results, and a conclusions section summarises our findings.

2. Generalisations of L4L solution

In cases when also divergent processes, such as extraction, recycling or transportation, the output matrix **I** needs to be exchanged for a more general matrix **G**. In the case that we are dealing with a pure arborescent system, the input matrix will be the identity matrix **I**, and the output matrix **G** will be triangular with non-zero elements below its main diagonal. Net production in such a case will be $\left(\tilde{\boldsymbol{\Delta}}(s)\mathbf{G}-\mathbf{I}\right)\tilde{\mathbf{P}}(s)$, where $\tilde{\boldsymbol{\Delta}}(s)$ is a diagonal matrix, the *output delay matrix,* with operators $e^{-s\Delta_i}$ along its diagonal providing the delay in delivery for each respective product. A further generalisation is obtained when allowing for the possibility of adding output delays related to processes and represented by the new diagonal matrix $\tilde{\boldsymbol{\Delta}}'(s)$, cf. [Bogataj and Grubbström, 2012], and we then obtain net production as

$$\left(\tilde{\boldsymbol{\Delta}}(s)\mathbf{G}\tilde{\boldsymbol{\Delta}}'(s)-\mathbf{I}\right)\tilde{\mathbf{P}}(s). \tag{4}$$

We use the notation $\tilde{\mathbf{D}}(s)$ for external deliveries (from and to the system), so for the pure arborescent system we have the equation:

$$\left(\tilde{\boldsymbol{\Delta}}(s)\mathbf{G}\tilde{\boldsymbol{\Delta}}'(s)-\mathbf{I}\right)\tilde{\mathbf{P}}(s)=\tilde{\mathbf{D}}(s), \tag{5}$$

when no inventory is accumulated.

Therefore the solution in $\tilde{\mathbf{P}}(s)$ corresponding to L4L in such a case obeys:

$$\tilde{\mathbf{P}}(s)=\left(\tilde{\boldsymbol{\Delta}}(s)\mathbf{G}\tilde{\boldsymbol{\Delta}}'(s)-\mathbf{I}\right)^{-1}\tilde{\mathbf{D}}(s). \tag{6}$$

In a similar way we may generalise the input matrix by allowing for additional lead times related to products and represented by the diagonal matrix $\boldsymbol{\tau}'(s)$, giving the generalised input matrix $\tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)$. Still further generalisations have been introduced in [Bogataj and Grubbström, 2012].

Whereas the Leontief inverse $\left(\mathbf{I}-\tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1}$ contains only elements with non-negative coefficients, as shown using the Neumann expansion

$$\left(\mathbf{I}-\tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1}=\sum_{i=0}^{\infty}\left(\tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{i}=\mathbf{I}+\sum_{i=1}^{\infty}\left(\tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{i}, \tag{7}$$

the technology matrix inverse $\left(\tilde{\boldsymbol{\Delta}}(s)\mathbf{G}\tilde{\boldsymbol{\Delta}}'(s)-\mathbf{I}\right)^{-1}$ will contain only non-positive elements, also shown from a Neumann expansion

$$\left(\tilde{\boldsymbol{\Delta}}'\mathbf{G}\tilde{\boldsymbol{\Delta}}(s)-\mathbf{I}\right)^{-1}=-\left(\mathbf{I}-\tilde{\boldsymbol{\Delta}}'\mathbf{G}\tilde{\boldsymbol{\Delta}}(s)\right)^{-1}=-\mathbf{I}-\sum_{i=1}^{\infty}\left(\tilde{\boldsymbol{\Delta}}'\mathbf{G}\tilde{\boldsymbol{\Delta}}(s)\right)^{i}, \tag{8}$$

in which no element can have a positive value.

The vector $\tilde{\mathbf{P}}(s)$ represents the activity levels of the processes, and contains by definition non-zero components, so the external flow $\tilde{\mathbf{D}}(s)$ for the pure arborescent system will have only non-positive components, reflecting that these represent input flows of materials into the processes belonging to the system.

Available inventory is defined as the inventory of items which are not ear-marked for production according to a given production plan $\tilde{\mathbf{P}}(s)$ (the activity vector). Net production, when a component is positive, adds to available inventory, and when negative, reduces available inventory. Available inventory is therefore the cumulative value of net production, and may be written

$$\tilde{\mathbf{R}}(s) = \frac{\mathbf{R}(0) + \left(\tilde{\mathbf{G}}(s) - \tilde{\mathbf{H}}(s)\right)\tilde{\mathbf{P}}(s) - \tilde{\mathbf{D}}(s)}{s}, \tag{9}$$

where $\mathbf{R}(0)$ is the vector of initial available inventory, and $\tilde{\mathbf{D}}(s)$ is the net outflow from the system for a positive component, and a net inflow into the system for a negative component. The division by s is a time integration in terms of the Laplace transform. The matrix $\tilde{\mathbf{G}}(s) = \tilde{\mathbf{\Delta}}(s)\mathbf{G}\tilde{\mathbf{\Delta}}'(s)$ is the generalised output matrix, and $\tilde{\mathbf{H}}(s) = \tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)$ is the generalised input matrix.

In cases when cumulative external demand might exceed cumulative production, we have a situation when backlogs are built up, or sales are lost. The case of all excessive demand being backlogged, when external demand is a stochastic compound renewal process, is treated in Section 4.

The Lot-for-Lot solution requires production $\tilde{\mathbf{P}}(s)$ exactly to satisfy the externally given flows $\tilde{\mathbf{D}}(s)$, i. e. the *Master Production Schedule* in the case of an assembly system, so assuming no initial available inventory $\mathbf{R}(0) = \mathbf{0}$, the activity vector $\tilde{\mathbf{P}}(s)$ in the L4L case will be given by Eq. (3) above. Available inventory may never be negative, since this would make the production plan $\tilde{\mathbf{P}}(s)$ infeasible, so we have the available inventory constraint:

$$£^{-1}\left\{\tilde{\mathbf{R}}(s)\right\} \geq \mathbf{0}. \tag{10}$$

3. Time average considerations

According to Laplace transform methodology the time average (written $\overline{f}$) of a time function $f(t)$, and defined as $\lim_{t\to\infty}\frac{1}{t}\int_{u=0}^{t} f(u)du$, obeys, cf. [Aseltine, 1958]:

$$\overline{f} = \lim_{t\to\infty}\frac{1}{t}\int_{u=0}^{t} f(u)du = \lim_{s\to 0} s\tilde{f}(s), \tag{11}$$

provided that either limit exists. This will also be the limit value

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\tilde{f}(s), \tag{12}$$

in the case that fluctuations in $f(t)$ cease to exist as $t \to \infty$.

A *Laurent expansion* of a function $\tilde{f}(s)$ around a point s_0 in the complex plane is written:

$$\tilde{f}(s) =$$

$$= \dots \frac{a_{-2}}{\left(s-s_0\right)^2} + \frac{a_{-1}}{s-s_0} + a_0 + a_1\left(s-s_0\right) + a_2\left(s-s_0\right)^2 + \dots = \sum_{k=-\infty}^{\infty} a_k\left(s-s_0\right)^k . \tag{13}$$

We will refer to the coefficients a_k as *Laurent coefficients*. The highest negative index *k* for which the coefficient a_k is non-zero, determines the *order* of the singularity at s_0, and *f*(*s*) is then said to have a *pole* "of order *k*" at s_0. If this value of *k* is -1, the pole is *simple*.

For applying the time average theorem or limit value theorem (11)-(12) above, we are interested in expanding functions around the origin $s_0 = 0$, and the Laurent expansion then has the form:

$$\tilde{f}\left(s\right) = \dots \frac{a_{-2}}{s^2} + \frac{a_{-1}}{s} + a_0 + a_1 s + a_2 s^2 + \dots = \sum_{k=-\infty}^{\infty} a_k s^k . \tag{14}$$

Clearly, according to (14), the time average of the function *f*(*t*) will only exist if its transform $\tilde{f}\left(s\right)$ at most has a simple pole at $s = 0$.

We now apply the time average theorem to available inventory. In order for the vector-valued time function **R**(*t*) to stay limited (i. e. have a limited time average), its transform $\tilde{\mathbf{R}}(s)$ must have the property

$$\bar{\mathbf{R}} = \lim_{s\to 0} s\tilde{\mathbf{R}}(s) = \lim_{s\to 0} \mathbf{R}(0) + \left(\tilde{\mathbf{G}}(s) - \tilde{\mathbf{H}}(s)\right)\tilde{\mathbf{P}}(s) - \tilde{\mathbf{D}}(s) \neq \pm\infty . \tag{15}$$

This implies that all Laurent coefficients of net production $\left(\tilde{\mathbf{G}}(s) - \tilde{\mathbf{H}}(s)\right)\tilde{\mathbf{P}}(s)$ and of deliveries $\tilde{\mathbf{D}}(s)$ must equal each other for all negative values of *k*, including especially *k* = -1. Since $\left(\tilde{\mathbf{G}}(s) - \tilde{\mathbf{H}}(s)\right)$ has no poles, we concentrate our attention on $\tilde{\mathbf{P}}(s)$ and $\tilde{\mathbf{D}}(s)$.

Taking a Laurent expansion of $\tilde{\mathbf{P}}(s)$, we have its time average written as the vector $\bar{\mathbf{P}} = \lim_{s\to 0} s\tilde{\mathbf{P}}(s) = \mathbf{P}_{-1}$, with $\mathbf{P}_{-1}$ collecting the Laurent coefficients of the first negative order. With a similar notation for external deliveries, let its vector-valued time average be written $\bar{\mathbf{D}} = \lim_{s\to 0} s\tilde{\mathbf{D}}(s) = \mathbf{D}_{-1}$.

So, since $\left(\tilde{\mathbf{G}}(s) - \tilde{\mathbf{H}}(s)\right) \to \left(\mathbf{G} - \mathbf{H}\right)$, when $s \to 0$, we have

$$\begin{aligned} \lim_{s\to 0} s^2\tilde{\mathbf{R}}(s) &= \lim_{s\to 0}\left(s\mathbf{R}(0) + s\left(\tilde{\mathbf{G}}(s) - \tilde{\mathbf{H}}(s)\right)\tilde{\mathbf{P}}(s) - s\tilde{\mathbf{D}}(s)\right) = \\ &= \lim_{s\to 0}\left(\left(\mathbf{G} - \mathbf{H}\right)s\tilde{\mathbf{P}}(s) - s\tilde{\mathbf{D}}(s)\right) = \left(\mathbf{G} - \mathbf{H}\right)\bar{\mathbf{P}} - \bar{\mathbf{D}} = \mathbf{0} , \end{aligned} \tag{16}$$

if average available inventory $\lim_{s\to 0} s\tilde{\mathbf{R}}(s)$ is to be kept finite. Therefore we state our main theorem:

Theorem 1

For available inventory to be kept finite over time, average production $\bar{\mathbf{P}}$ *(the collection of activity levels) must be the following Lot-for-Lot (L4L) solution for the system when considering average external deliveries*

$$\bar{\mathbf{P}} = \left(\mathbf{G}-\mathbf{H}\right)^{-1}\bar{\mathbf{D}}, \tag{17}$$

where $\bar{\mathbf{D}}$ *collects the time averages of external demand.*

This is the main result of this paper, showing that the Lot-for-Lot Solution plays a third important rôle, apart from its original detailed rôle as determining a production policy and its second rôle as generating all possible internal (dependent) demand events. An extension to stochastic demand is provided in the next section.

Looking at the long-run average of inventories, we search for the limit value $\bar{\mathbf{R}} = \lim_{s\to 0} s\tilde{\mathbf{R}}(s)$ in (15). From (9), we have

$$s\tilde{\mathbf{R}}(s) = \mathbf{R}(0) + \left(\tilde{\boldsymbol{\Delta}}'(s)\mathbf{G}\tilde{\boldsymbol{\Delta}}(s) - \tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)\tilde{\mathbf{P}}(s) - \tilde{\mathbf{D}}(s). \tag{18}$$

Expanding the diagonal matrices $\tilde{\boldsymbol{\Delta}}'(s)$, $\tilde{\boldsymbol{\Delta}}(s)$, $\tilde{\boldsymbol{\tau}}'(s)$, and $\tilde{\boldsymbol{\tau}}(s)$ into Maclaurin series, for $\tilde{\boldsymbol{\Delta}}'(s)$ we have

$$\tilde{\boldsymbol{\Delta}}'(s) = \begin{bmatrix} e^{-s\Delta_1'} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-s\Delta_n'} \end{bmatrix} = \mathbf{I} - s\begin{bmatrix} \Delta_1' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta_n' \end{bmatrix} + s^2\begin{bmatrix} \Delta_1' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta_n' \end{bmatrix}^2 / 2 - \mathbf{O}(s^3), \tag{19}$$

where $\mathbf{O}(s^3)$ is a matrix vanishing at least as s^3, when $s \to 0$. Similar expressions are obtained for the three other diagonal matrices with exponential operators. Laurent expansions of production and demand give

$$\tilde{\mathbf{P}}(s) = \frac{\mathbf{P}_{-1}}{s} + \mathbf{P}_0 + \mathbf{P}_1 s + \ldots, \tag{20}$$

$$\tilde{\mathbf{D}}(s) = \frac{\mathbf{D}_{-1}}{s} + \mathbf{D}_0 + \mathbf{D}_1 s + \ldots, \tag{21}$$

where $\mathbf{P}_{-1} = \bar{\mathbf{P}}$ and $\mathbf{D}_{-1} = \bar{\mathbf{D}}$, $\mathbf{P}_0$ and $\mathbf{D}_0$ being the zeroth-order and $\mathbf{P}_1$ and $\mathbf{D}_1$ the first-order Laurent coefficients, respectively. So the multiplications in (15) provide the following low-indexed Laurent coefficients

$$s\tilde{\mathbf{R}}(s) = \frac{\left(\mathbf{G}-\mathbf{H}\right)\mathbf{P}_{-1} - \mathbf{D}_{-1}}{s} + \mathbf{R}(0) + (\mathbf{G}-\mathbf{H})\mathbf{P}_0 - \mathbf{D}_0$$

(22)

$$-\left(\begin{bmatrix}\Delta_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Delta_n\end{bmatrix}\mathbf{G}+\mathbf{G}\begin{bmatrix}\Delta_1' & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Delta_n'\end{bmatrix}\right)\mathbf{P}_{-1}$$

$$-\left(\begin{bmatrix}\tau_1' & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \tau_n'\end{bmatrix}\mathbf{H}+\mathbf{H}\begin{bmatrix}\tau_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \tau_n\end{bmatrix}\right)\mathbf{P}_{-1}+\mathbf{O}(s),$$

where $\mathbf{O}(s)$ is a vector vanishing with *s*.

So, for a stable production policy obeying our Theorem 1 above, the first term in (22) vanishes, and introducing (17), we obtain:

$$\bar{\mathbf{R}} = \lim_{s\to 0} s\tilde{\mathbf{R}}(s) =$$

$$= \mathbf{R}(0)+(\mathbf{G}-\mathbf{H})\mathbf{P}_0 - \left(\begin{matrix}\begin{bmatrix}\Delta_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Delta_n\end{bmatrix}\mathbf{G}+\mathbf{G}\begin{bmatrix}\Delta_1' & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Delta_n'\end{bmatrix}\\ +\begin{bmatrix}\tau_1' & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \tau_n'\end{bmatrix}\mathbf{H}+\mathbf{H}\begin{bmatrix}\tau_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \tau_n\end{bmatrix}\end{matrix}\right)\left(\mathbf{G}-\mathbf{H}\right)^{-1}\mathbf{D}_{-1}-\mathbf{D}_0. \qquad (23)$$

This equation clearly indicates the various influences that time delays and lead times have on the average inventory level, and therefore also on holding costs.

It should be pointed out that these developments are valid for any stable production policy and any demand situation, whether or not the policy is feasible, so it holds for all feasible production policies.

4. Extension to stochastic demand

Turning our attention to the case of demand being a sequence of demand events separated by stochastic time intervals, we derive the following

Theorem 2

For available inventory and backlogs to be kept finite over time, average production $\bar{\mathbf{P}}$ *(the collection of activity levels) must be the following Lot-for-Lot (L4L) solution for the system when considering expected average external deliveries*

$$\bar{\mathbf{P}} = \left(\mathbf{G}-\mathbf{H}\right)^{-1}\mathrm{E}\left[\bar{\mathbf{D}}\right], \qquad (24)$$

where $\mathrm{E}\left[\bar{\mathbf{D}}\right]$ *collects the time averages of expected external demand.*

The proof is limited to demand being a compound renewal process, i. e. that (i) the time intervals between demand events are independent stochastic variables with the same individual probability distribution, (ii) that the volumes of demand at these events are independent stochastic variables with the same distribution, and (iii) that the sequences of intervals and of demanded volumes are independent.

We introduce the following additional notation:

T_{ij}	Stochastic time interval between (*j* - 1)st and *j*th demand events for item *i*. Intervals are independent with same probability distribution for the same item.
$f_i(t)$	Probability density function (pdf) of T_{ij}, $f_i(t) = (d/dt)\Pr\left(T_{ij} \le t\right)$.
$\tilde{f}_i(s) = \pounds\{f_i(t)\}$	Laplace transform of pdf $f_i(t)$.
D_{ij}	Size of stochastic external demand for item *i* at *j*th demand event.
$g_i(x)$	Probability density function of D_{ij}, $g_i(x) = (d/dx)\Pr\left(D_{ij} \le x\right)$.
$\tilde{g}_i(u) = \hat{\pounds}\{g_i(x)\}$	Laplace transform of pdf $g_i(x)$, where $\hat{\pounds}\{g_i(x)\} = \int_{x=0}^{\infty} e^{-ux} g_i(x)dx$ and u is the Laplace frequency for a transform of a function with x as argument.
$\dot{\mathbf{v}}_i(t)$	Stochastic sequence of external demand events for item *i*. $\dot{\mathbf{v}}_i(t)$ is a Dirac impulse if an event occurs at *t*, and otherwise zero.
$\mathbf{v}_i(t) = \int_{\alpha=0}^{t} \dot{\mathbf{v}}_i(\alpha)d\alpha$	Cumulative number of demand events up to and including time *t*, staircase function.
μ_i^T	Mean value of stochastic variable T_{ij}.
μ_i^D	Mean value of stochastic variable D_{ij}.

We should note that external demand might not exist for some items, in which case the probability density function (pdf) for these $g_i(x)$ would be concentrated as an impulse at the point zero $x = 0$.

Let us reconsider Eq. (9) explaining the development of available inventory:

$$\tilde{\mathbf{R}}(s) = \frac{\mathbf{R}(0) + \left(\tilde{\mathbf{G}}(s) - \tilde{\mathbf{H}}(s)\right)\tilde{\mathbf{P}}(s) - \tilde{\mathbf{D}}(s)}{s}. \tag{25}$$

In our currently considered extension, $\tilde{\mathbf{D}}(s)$ is stochastic, and we also allow for $\mathbf{R}(t)$ having negative components, a negative value for one of the components meaning the negative of the level of cumulative backlogs. Hence we are reinterpreting $\mathbf{R}(t)$ as net inventory. Our question now concerns the conditions for $\mathbf{R}(t)$ keeping a limited value through time. In Eq. (25), $\tilde{\mathbf{D}}(s)$ and the resulting $\tilde{\mathbf{R}}(s)$ are stochastic.

We limit our attention to a compound renewal process for each item. The probability that at time *t* the cumulative number of demand events is *j* is given by:

$$\Pr\left(\nu_i(t) = j\right) = \Pr\left(\nu_i(t) < j+1)\right) - \Pr\left(\nu_i(t) < j\right) = \Pr\left(\sum_{k=1}^{j+1} T_{ik} > t\right) - \Pr\left(\sum_{k=1}^{j} T_{ik} > t\right). \tag{26}$$

In Laplace terms, $£\left\{\Pr\left(\sum_{k=1}^{j+1} T_{ik} > t\right)\right\} = (1-\tilde{f}_i(s)^{j+1})/s$ and $£\left\{\Pr\left(\sum_{k=1}^{j} T_{ik} > t\right)\right\} = (1-\tilde{f}_i(s)^{j})/s$, so

$$£\left\{\Pr\left(\nu_i(t)=j\right)\right\} = \tilde{f}(s)^j\left(1-\tilde{f}(s)\right)/s. \tag{27}$$

For the cumulative demand for item *i*, we have the probability of it being less than or equal to *x* conditional on $\nu_i(t)=k$:

$$\Pr\left(\bar{D}_i(t) \le x \middle| \nu_i(t)=k\right) = \Pr\left(\sum_{j=1}^{k} D_{ij} \le x\right). \tag{28}$$

In transform terms the right-hand member is

$$\hat{£}\left\{\Pr\left(\sum_{j=1}^{k} D_{ij} \le x\right)\right\} = \tilde{g}_i(u)^k/u. \tag{29}$$

So

$$\Pr\left(\nu_i(t)=k \wedge \sum_{j=1}^{k} d_{ij}(t) \le x\right) = £^{-1}\left\{\tilde{f}_i^k(s)(1-\tilde{f}_i(s))/s\right\}\hat{£}^{-1}\left\{\tilde{g}_i(u)^k/u\right\}, \tag{30}$$

where $\wedge$ is the logical operator “and”. The expectation of cumulative demand for item *i* at time *t* is therefore

$$\begin{aligned} E\left[\bar{D}_i(t)\right] &= \int_{x=0}^{\infty}\sum_{k=0}^{\infty} £^{-1}\left\{\tilde{f}_i(s)^k\left(1-\tilde{f}_i(s)\right)/s\right\}\hat{£}^{-1}\left\{\tilde{g}_i(u)^k\right\} x\,dx = \\ &= £^{-1}\left\{\frac{1-\tilde{f}_i(s)}{s}\int_{x=0}^{\infty}\hat{£}^{-1}\left\{\frac{1}{1-\tilde{f}_i(s)\tilde{g}_i(u)}\right\} x\,dx\right\}. \end{aligned} \tag{31}$$

We make use of the moment theorem of the Laplace transform, by which $\int_{x=0}^{\infty} h(x)xdx = -\lim_{u\to 0} d\tilde{h}(u)/du$ for any function $h(x)$,

$$\begin{aligned} \int_{x=0}^{\infty}\hat{£}^{-1}\left\{\frac{1}{1-\tilde{f}_i(s)\tilde{g}_i(u)}\right\} xdx &= -\lim_{u\to 0}\frac{\partial}{\partial u}\frac{1}{1-\tilde{f}_i(s)\tilde{g}_i(u)} \\ &= -\lim_{u\to 0}\frac{\tilde{f}_i(s)\left(d\tilde{g}_i(u)/du\right)}{\left(1-\tilde{f}_i(s)\tilde{g}_i(u)\right)^2} = \mu_i^D\frac{\tilde{f}_i(s)}{\left(1-\tilde{f}_i(s)\right)^2}. \end{aligned} \tag{32}$$

where $\mu_i^D = -\lim_{u\to 0} d\tilde{g}_i(u)/du$ is the mean of the demand for item *i* at each event, and where we have used $\tilde{g}_i(0)=1$, $g_i(x)$ being a pdf. Inserting (32) into (31), we obtain

$$\mathrm{E}\left\{\bar{D}_i(t)\right\} = \mu_i^D £^{-1}\left\{\frac{\tilde{f}_i(s)}{s\left(1-\tilde{f}_i(s)\right)}\right\}, \tag{33}$$

so the transform of expected cumulative demand may be written

$$£\left\{\mathrm{E}\left\{\bar{D}_i(t)\right\}\right\} = \mathrm{E}\left\{\tilde{\bar{D}}_i(s)\right\} = \frac{\mu_i^D}{s}\cdot\frac{\tilde{f}_i(s)}{1-\tilde{f}_i(s)}, \tag{34}$$

and the expected value of demand

$$£\left\{\mathrm{E}\left\{D_i(t)\right\}\right\} = \mathrm{E}\left\{\tilde{D}_i(s)\right\} = \mu_i^D\cdot\frac{\tilde{f}_i(s)}{1-\tilde{f}_i(s)}, \tag{35}$$

where $\tilde{f}_i(s)/\left(s\left(1-\tilde{f}_i(s)\right)\right)$ is recognised as the *renewal function*.

The long-run time average of expected demand becomes

$$\mathrm{E}\left\{\bar{D}_i\right\} = \lim_{s\to 0} s£\left\{\mathrm{E}\left\{D_i(t)\right\}\right\} = \mu_i^D\cdot\lim_{s\to 0}\frac{s\tilde{f}_i(s)}{1-\tilde{f}_i(s)} = \mu_i^D/\mu_i^T, \tag{36}$$

where l'Hôpital's rule has been used on the expression $s\tilde{f}_i(s)\left(1-\tilde{f}_i(s)\right)$, which is of the type 0/0, and where $\tilde{f}_i(0)=1$ and $\mu_i^T = -\lim_{s\to 0} d\tilde{f}_i(s)/ds$ have been applied.

In the case that the demand events for item i are a Poisson process with parameter λ, we have $\tilde{f}_i(s)=\lambda/(\lambda+s)$ and therefore $\tilde{f}_i(s)/\left(s\left(1-\tilde{f}_i(s)\right)\right)=\lambda/s^2$, which is a ramp in time with a slope of λ. Instead with a constant interval, we have $\tilde{f}_i(s)=e^{-s\hat{T}}$, and $\tilde{f}_i(s)/\left(s\left(1-\tilde{f}_i(s)\right)\right)$ becomes a staircase function with unit steps separated in time by $\hat{T}$.

We now turn to our main question of stability. For the time average of expected net inventory **R**(t) in (25) to remain at a final average level over time, we require

$$\lim_{s\to\infty} s\mathrm{E}\left[\tilde{\mathbf{R}}(s)\right] = \mathbf{R}(0) + \lim_{s\to\infty}\left(\left(\tilde{\mathbf{G}}(s)-\tilde{\mathbf{H}}(s)\right)\tilde{\mathbf{P}}(s) - \mathrm{E}\left[\tilde{\mathbf{D}}(s)\right]\right) \neq \pm\infty, \tag{37}$$

To find the balance between the Laurent coefficients for the terms -1, we develop

$$\begin{aligned}\lim_{s\to\infty} s^2\mathrm{E}\left[\tilde{\mathbf{R}}(s)\right] &= \lim_{s\to\infty} s\mathbf{R}(0) + \lim_{s\to\infty} s\left(\left(\tilde{\mathbf{G}}(s)-\tilde{\mathbf{H}}(s)\right)\tilde{\mathbf{P}}(s) - \mathrm{E}\left[\tilde{\mathbf{D}}(s)\right]\right) = \\ &= \left(\mathbf{G}-\mathbf{H}\right)\bar{\mathbf{P}} - \left[\mu_i^D/\mu_i^T\right] = 0,\end{aligned} \tag{38}$$

where $\left[\mu_i^D/\mu_i^T\right]$ is a column vector with the ratios as components.

By inverting (38), we then have

$$\bar{\mathbf{P}} = \left(\mathbf{G}-\mathbf{H}\right)^{-1}\mathrm{E}\left[\bar{\mathbf{D}}\right] = \left(\mathbf{G}-\mathbf{H}\right)^{-1}\left[\mu_i^D / \mu_i^T\right], \tag{39}$$

so once again, the L4L solution applies, this time to long run average production being equal to the product of the Leontief inverse and the expected time average of demand.

Finally, we derive a long-run time average of expected net inventory.

Using $\tilde{f}_i^{(j)}(0)$ for the *j*th derivatives of $\tilde{f}_i(s)$ at $s = 0$, the moment theorem yields $\tilde{f}_i^{(0)}(0) = 1$, $\tilde{f}_i^{(1)}(0) = -\mu_i^T$ and $\tilde{f}_i^{(2)}(0) = \left(\mu_i^T\right)^2 + \left(\sigma_i^T\right)^2$, where σ_i^T is the standard deviation of an interval. Taking a Laurent expansion of (35), we obtain

$$\mu_i^D \cdot \frac{\tilde{f}_i(s)}{1-\tilde{f}_i(s)} = \mu_i^D \cdot \frac{\sum_{j=0}^{\infty} \tilde{f}_i^{(j)}(0) s^j / j!}{1-\sum_{j=0}^{\infty} \tilde{f}_i^{(j)}(0) s^j / j!} = \mu_i^D \cdot \left(\frac{1}{\mu_i^T s} + \frac{1}{2}\left(\left(\frac{\sigma_i^T}{\mu_i^T}\right)^2 - 1\right)\right) + O(s), \tag{40}$$

where $O(s)$ vanishes at least as s. We thus identify the terms corresponding to components of $\mathbf{D}_{-1}$ and $\mathbf{D}_0$ in (23) as $\mathbf{D}_{-1} = \left[\frac{\mu_i^D}{\mu_i^T}\right]$ and $\mathbf{D}_0 = \left[\frac{\mu_i^D}{2}\left(\left(\frac{\sigma_i^T}{\mu_i^T}\right)^2 - 1\right)\right]$.

Therefore, applying the same approach as in (23), we obtain the long-run expected net inventory as:

$$\bar{\mathbf{R}} = \lim_{s\to 0} s\tilde{\mathbf{R}}(s) = \mathbf{R}(0) + (\mathbf{G}-\mathbf{H})\mathbf{P}_0$$

$$-\left(\begin{array}{l}\begin{bmatrix}\Delta_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Delta_n\end{bmatrix}\mathbf{G} + \mathbf{G}\begin{bmatrix}\Delta_1' & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \Delta_n'\end{bmatrix}\\ +\begin{bmatrix}\tau_1' & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \tau_n'\end{bmatrix}\mathbf{H} + \mathbf{H}\begin{bmatrix}\tau_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \tau_n\end{bmatrix}\end{array}\right)\left(\mathbf{G}-\mathbf{H}\right)^{-1}\left[\frac{\mu_i^D}{\mu_i^T}\right] - \left[\frac{\mu_i^D}{2}\left(\left(\frac{\sigma_i^T}{\mu_i^T}\right)^2 - 1\right)\right]. \tag{41}$$

It might be of interest to note, that the coefficient of variation of the time interval between demand events σ_i^T / μ_i^T has a reducing influence on expected long-run net inventory, which would probably be contrary to intuition, and that for a Poisson process, where $\mu_i^T = \sigma_i^T = 1/\lambda$, the $\mathbf{D}_0$ term vanishes entirely. Instead, for the intervals having a Gamma distribution $\tilde{f}_i(s) = (1+as)^{-p} = \left(1+\left(\sigma^2/\mu\right)s\right)^{-\mu^2/\sigma^2}$, with the two parameters a and p, the $\mathbf{D}_0$ term becomes $-\left[\left(\mu_i^D/2\right)\left(1-p\right)/p\right]$, which is independent of the a-parameter.

5. A simple example

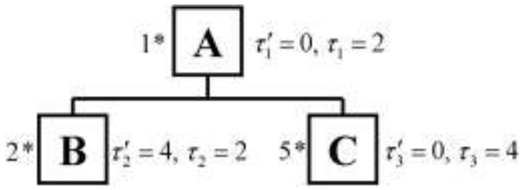

Figure 1. Product structure tree of example. Requirements of inputs, and lead times of products and processes are indicated.

We consider the example of a very simple assembly system shown in Figure 1. The output and input matrices are

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix},$$

and the lead time matrices for products and processes are given by:

$$\boldsymbol{\tau}'(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{4s} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \boldsymbol{\tau}(s) = \begin{bmatrix} e^{2s} & 0 & 0 \\ 0 & e^{2s} & 0 \\ 0 & 0 & e^{4s} \end{bmatrix}.$$

Hence, the generalised technology matrix will be $\left(\mathbf{G} - \tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right) = \begin{bmatrix} 1 & 0 & 0 \\ -2e^{6s} & 1 & 0 \\ -5e^{2s} & 0 & 1 \end{bmatrix}$.

The generalised Leontief inverse is thus $\left(\mathbf{G} - \tilde{\boldsymbol{\tau}}'(s)\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2e^{6s} & 1 & 0 \\ 5e^{2s} & 0 & 1 \end{bmatrix}$.

If external demand (the Master Production Schedule) requires deliveries according to $\tilde{\mathbf{D}}(s) = \begin{bmatrix} \tilde{D}_1(s) \\ \tilde{D}_2(s) \\ \tilde{D}_3(s) \end{bmatrix}$, the L4L solution matching this demand is obtained from:

$$\tilde{\mathbf{P}}_{\mathrm{L4L}}(s) = \begin{bmatrix} \tilde{P}_1(s) \\ \tilde{P}_2(s) \\ \tilde{P}_3(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2e^{6s} & 1 & 0 \\ 5e^{2s} & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{D}_1(s) \\ \tilde{D}_2(s) \\ \tilde{D}_3(s) \end{bmatrix} = \begin{bmatrix} \tilde{D}_1(s) \\ 2e^{6s}\tilde{D}_1(s) + \tilde{D}_2(s) \\ 5e^{2s}\tilde{D}_1(s) + \tilde{D}_3(s) \end{bmatrix}.$$

The components $\tilde{D}_2(s)$ and $\tilde{D}_3(s)$ of external demand may be interpreted as sales of spare parts. Assume for simplicity that the components of demand are constant over time at levels $\hat{D}_1$, $\hat{D}_2$ and $\hat{D}_3$, but that demand does not start until after delays $\hat{t}_1$, $\hat{t}_2$, $\hat{t}_3$ in order to avoid complications with initial shortages (before production has been able to complete its first lots). External demand will then have the transform $\tilde{\mathbf{D}}(s) = \frac{1}{s}\begin{bmatrix} \hat{D}_1 e^{-s\hat{t}_1} \\ \hat{D}_2 e^{-s\hat{t}_2} \\ \hat{D}_3 e^{-s\hat{t}_3} \end{bmatrix}$.

Assume further that production/purchasing is carried out in batches of constant sizes $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_3$ cyclically (a Fixed Order Quantity case), with periods of $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$, respectively, and that there are sufficient amounts of components in initial available inventory for not having to require initial production to be delayed. The production vector will then have the following structure $\tilde{\mathbf{P}}(s) = \begin{bmatrix} \hat{P}_1 / \left(1 - e^{-s\hat{T}_1}\right) \\ \hat{P}_2 / \left(1 - e^{-s\hat{T}_2}\right) \\ \hat{P}_3 / \left(1 - e^{-s\hat{T}_3}\right) \end{bmatrix}$.

Obviously, this production does not satisfy internal and external demand on an L4L basis. Depending on the values of the parameters introduced, there will be inventories building up at times when batches are completed and then reduced linearly in time until a next batch is completed.

Available inventory will behave according to:

$$\tilde{\mathbf{R}}(s) = \mathbf{R}(0)/s + \frac{1}{s}\left(\begin{bmatrix} 1 & 0 & 0 \\ -2e^{6s} & 1 & 0 \\ -5e^{2s} & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{P}_1 / \left(1 - e^{-s\hat{T}_1}\right) \\ \hat{P}_2 / \left(1 - e^{-s\hat{T}_2}\right) \\ \hat{P}_3 / \left(1 - e^{-s\hat{T}_3}\right) \end{bmatrix} - \frac{1}{s}\begin{bmatrix} \hat{D}_1 e^{-s\hat{t}_1} \\ \hat{D}_2 e^{-s\hat{t}_2} \\ \hat{D}_3 e^{-s\hat{t}_3} \end{bmatrix} \right).$$

For available inventory to have a limited time average, the condition $\bar{\mathbf{R}} = \lim_{s\to 0} s\tilde{\mathbf{R}}(s) \neq \infty$ applies. The first division by s immediately cancels with the s multiplying $\tilde{\mathbf{R}}(s)$. However, production has a singularity at $s = 0$ and so does demand. Expanding $\begin{bmatrix} 1 & 0 & 0 \\ -2e^{6s} & 1 & 0 \\ -5e^{2s} & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{P}_1 / \left(1 - e^{-s\hat{T}_1}\right) \\ \hat{P}_2 / \left(1 - e^{-s\hat{T}_2}\right) \\ \hat{P}_3 / \left(1 - e^{-s\hat{T}_3}\right) \end{bmatrix}$ into a Laurent series around $s = 0$, there will be a non-zero coefficient for the first negative power in s amounting to:

$$\lim_{s\to 0} s \begin{bmatrix} 1 & 0 & 0 \\ -2e^{6s} & 1 & 0 \\ -5e^{2s} & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{P}_1 / \left(1 - e^{-s\hat{T}_1}\right) \\ \hat{P}_2 / \left(1 - e^{-s\hat{T}_2}\right) \\ \hat{P}_3 / \left(1 - e^{-s\hat{T}_3}\right) \end{bmatrix} = \lim_{s\to 0} s \begin{bmatrix} 1 & 0 & 0 \\ -2e^{6s} & 1 & 0 \\ -5e^{2s} & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{P}_1 / s\hat{T}_1 \\ \hat{P}_2 / s\hat{T}_2 \\ \hat{P}_3 / s\hat{T}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{P}_1 / \hat{T}_1 \\ \hat{P}_2 / \hat{T}_2 \\ \hat{P}_3 / \hat{T}_3 \end{bmatrix},$$

and the similar Laurent coefficient for demand is $\mathbf{D}_{-1} = \lim_{s\to 0} s\frac{1}{s}\begin{bmatrix}\hat{D}_1 e^{-s\hat{t}_1}\\ \hat{D}_2 e^{-s\hat{t}_2}\\ \hat{D}_3 e^{-s\hat{t}_3}\end{bmatrix} = \begin{bmatrix}\hat{D}_1\\ \hat{D}_2\\ \hat{D}_3\end{bmatrix}$.

We therefore have average production $\bar{\mathbf{P}} = \mathbf{P}_{-1} = \begin{bmatrix}\hat{P}_1/\hat{T}_1\\ \hat{P}_2/\hat{T}_2\\ \hat{P}_3/\hat{T}_3\end{bmatrix}$, rather expectedly. So for available inventory to stay limited over time, these two coefficients need to cancel each other identically, i. e.

$$\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -5 & 0 & 1\end{bmatrix}\bar{\mathbf{P}} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -5 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{P}_1/\hat{T}_1\\ \hat{P}_2/\hat{T}_2\\ \hat{P}_3/\hat{T}_3\end{bmatrix} = \begin{bmatrix}\hat{D}_1\\ \hat{D}_2\\ \hat{D}_3\end{bmatrix} = \mathbf{D}_{-1} = \bar{\mathbf{D}}.$$

But we may now recognise that this equation is an instance of $(\mathbf{G}-\mathbf{H})\begin{bmatrix}\bar{P}_1\\ \bar{P}_2\\ \bar{P}_3\end{bmatrix} = \begin{bmatrix}\bar{D}_1\\ \bar{D}_2\\ \bar{D}_3\end{bmatrix}$, where again the bars denote averages over time. Hence, this illustrates the time averages necessarily obeying the L4L policy in order for average available inventory to stay limited:

$$\bar{\mathbf{P}} = \begin{bmatrix}\bar{P}_1\\ \bar{P}_2\\ \bar{P}_3\end{bmatrix} = (\mathbf{G}-\mathbf{H})^{-1}\begin{bmatrix}\bar{D}_1\\ \bar{D}_2\\ \bar{D}_3\end{bmatrix} = (\mathbf{G}-\mathbf{H})^{-1}\bar{\mathbf{D}},$$

so the batch sizes must be $\begin{bmatrix}\hat{P}_1\\ \hat{P}_2\\ \hat{P}_3\end{bmatrix} = \begin{bmatrix}\hat{D}_1\hat{T}_1\\ (2\hat{D}_1+\hat{D}_2)\hat{T}_2\\ (5\hat{D}_1+\hat{D}_3)\hat{T}_3\end{bmatrix}$.

Let us finally consider the levels of average available inventory. Expanding the production vector $\tilde{\mathbf{P}}(s)$ to find the zeroth order coefficient of its Laurent series,

$$\tilde{\mathbf{P}}(s) = \begin{bmatrix}\dfrac{\hat{P}_1}{s\hat{T}_1\left(1-s\hat{T}_1/2+\ldots\right)}\\[2ex] \dfrac{\hat{P}_2}{s\hat{T}_2\left(1-s\hat{T}_2/2+\ldots\right)}\\[2ex] \dfrac{\hat{P}_3}{s\hat{T}_3\left(1-s\hat{T}_3/2+\ldots\right)}\end{bmatrix} = \begin{bmatrix}\dfrac{\hat{P}_1\left(1+s\hat{T}_1/2\right)+O(s^2)}{s\hat{T}_1}\\[2ex] \dfrac{\hat{P}_2\left(1+s\hat{T}_2/2\right)+O(s^2)}{s\hat{T}_2}\\[2ex] \dfrac{\hat{P}_3\left(1+s\hat{T}_3/2\right)+O(s^2)}{s\hat{T}_3}\end{bmatrix},$$

we find $\mathbf{P}_0 = \frac{1}{2}\begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \end{bmatrix}$, i. e. half the fixed order quantities. Similarly for external demand, we have

$$\tilde{\mathbf{D}}(s) = \frac{1}{s}\begin{bmatrix} \hat{D}_1 e^{-s\hat{t}_1} \\ \hat{D}_2 e^{-s\hat{t}_2} \\ \hat{D}_3 e^{-s\hat{t}_3} \end{bmatrix} = \frac{1}{s}\begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \\ \hat{D}_3 \end{bmatrix} - \begin{bmatrix} \hat{D}_1\hat{t}_1 \\ \hat{D}_2\hat{t}_2 \\ \hat{D}_3\hat{t}_3 \end{bmatrix} + \mathbf{O}(s),$$

where $\mathbf{D}_{-1}$ is already known from above, and $\mathbf{D}_0$ is identified as $\mathbf{D}_0 = -\begin{bmatrix} \hat{D}_1\hat{t}_1 \\ \hat{D}_2\hat{t}_2 \\ \hat{D}_3\hat{t}_3 \end{bmatrix}$.

From (23) we may now compute average available inventory as:

$$\bar{\mathbf{R}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}\mathbf{P}_0 + \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right)\mathbf{P}_{-1} - \mathbf{D}_0 =$$

$$= \begin{bmatrix} \hat{D}_1\left(\hat{t}_1 + \hat{T}_1/2\right) \\ \hat{D}_2\left(\hat{t}_2 + \hat{T}_2/2\right) + 12\hat{D}_1 \\ \hat{D}_3\left(\hat{t}_3 + \hat{T}_3/2\right) + 10\hat{D}_1 \end{bmatrix}.$$

The example also illustrates that lead times, etc., do not have any effect on the long-term stability behaviour, but the effects on the average available inventory levels are easily found from average demand $\hat{D}_i$, the initial demand delays $\hat{t}_i$, the period of production cycles $\hat{T}_i$, and the more specific lead time influences.

6. Conclusions

In the foregoing we have explored conditions on the relation between production, external demand and system properties in the form of input and output matrices, lead times of assembly processes and output delays of arborescent processes. Using a Laurent expansion combined with the time-average theorem of the Laplace transform, it turned out that the condition for stability, meaning that the time average of available inventory was kept bounded, was to be found in the condition that average production must be the Lot-for-Lot solution with respect to average demand, i. e. Eq. (17) above. Studying the case of stochastic demand generated by compound renewal processes gave similar consequences with respect to expected long-run time averages, Eq. (39).

This means that the Lot-for-Lot solution has taken on a third rôle, the earlier rôles being a production policy, on the one hand, and a solution showing all possible external and internal demand events, on the second.

As side results, we also found expressions for long-run average available inventory Eq. (23), and for long-run expected net inventory in the stochastic case Eq. (41).

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