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The Space of Solution Alternatives in the Optimal Lotsizing Problem for General Assembly Systems Applying MRP Theory

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Abstract

MRP Theory combines the use of Input-Output Analysis and Laplace transforms, enabling the development of a theoretical background for multi-level, multi-stage production-inventory systems together with their economic evaluation, in particular applying the Net Present Value principle (NPV).

In a recent paper [Grubbström et al. 2010], a general method for solving the dynamic lotsizing problem for a general assembly system was presented. It was shown there that the optimal production (completion) times had to be chosen from the set of times generated by the Lot-For-Lot (L4L) solution. Thereby, the problem could be stated in binary form by which the values of the binary decision variables represented either to make a production batch, or not, at each such time. Based on these potential times for production, the problem of maximising the Net Present Value or minimising the average cost could be solved, applying a single-item optimal dynamic lotsizing method, such as the Wagner-Whitin algorithm or the Triple Algorithm, combined with dynamic programming.

This current paper follows up the former paper by investigating the complexity defined as the number of possible feasible solutions (production plans) to compare. We therefore investigate how properties of external demand timing and properties of requirements (Bill-of-Materials) have consequences on the size of this solution space. Explicit expressions are developed for how the total number of feasible production plans depends on numbers of external demand events on different levels for, in particular, the two extreme cases of a serial system and a full system (the latter, in which items have requirements of all existing types of subordinate items). A formula is also suggested for general systems falling in between these two extremes. For the most complex full system, it is shown that the number of feasible plans will be the product of elements taken from *Sylvester's sequence* (an instance of doubly exponential sequences) raised to a powers depending on numbers of external demand events.

Keywords: MRP Theory, optimal lotsizing, assembly system, Laplace transform, complexity.

1. Introduction

Materials requirements planning (MRP) is a management information system commonly used in industry for determining production schedules in multi-level, multi-item systems. By taking into account the internal demand relationship and process lead times, MRP reduces a master schedule of finished products into a time-phased schedule of requirements for subordinate items. This parts explosion approach needs a clear description of product structure and effect of lead times. *MRP Theory* , which has been developed in the past decades, combines the use of Input-Output Analysis and Laplace transforms, enabling the development of a compact theoretical background for such production-inventory systems [cf. Grubbström and Tang 2000].

One import issue in MRP is defining the lotsizing policy. In practice, this is often done in a sequential manner, i.e. lot sizing is determined for the single item, starting from the top level and then exploding to the rest of the items. An integrated lot sizing policy is seldom used in practice, due to its computational complexity. In a recent paper [Grubbström et al. 2010], a general method for solving the dynamic lotsizing problem for a general assembly system was presented. It was shown there that the optimal production (completion) times had to be chosen from the set of times generated by the Lot-For-Lot (L4L) solution. Thereby, the problem could be stated in binary form by which the values of the binary decision variables represented either to make a production batch, or not, at each such time. Based on these potential times for production, the problem of maximising the Net Present Value or minimising the average cost could be solved.

Nevertheless, one issue has not been discussed: namely what is the computational complexity in applying such an approach? It is only after we understand this issue that the previously proposed method can receive attention for its potential implementation. This current paper follows up the former paper by investigating the complexity defined as the number of possible feasible solutions to compare. We therefore investigate how properties of external demand timing and properties of requirements (Bill-of-Materials) have consequences on the size of this solution space. Explicit expressions are developed for how the total number of feasible production plans depends on numbers of external demand events on different levels for, in particular, the two extreme cases of a serial system and a full system (the latter, in which items have requirements of all existing types of subordinate items). A formula is also suggested for general systems falling in between these two extremes.

In order to help readers to understand the problem and modelling background, we briefly review two streams of literature relevant to current study: MRP Theory and lotsizing policy.

One breakthrough in developing the theoretical background of MRP is the combination of Laplace transform (or z-transform) and Input-Output Analysis methodology [Grubbström and Ovrin 1992, Grubbström and Molinder 1994]. In a linear production system, the Input-Output model can be used to describe the internal demand relationships. This has been further enhanced by using the Laplace transform. The transform has been applied for describing time developments and lags in the relevant production, demand and inventory variables in a compact way including effects of order flows and lead times. A generalised input matrix is developed to describe the product structure as well as the timing relationships between items. Fundamental equations can be established to describe the dynamics of the system, and to evaluate economic consequences.

Apart from the above advantages, the transform has been applied as a moment-generating function to capture stochastic properties, which is also important in production/inventory systems (but of no relevance in the current treatment). Furthermore, the transform has been used for assessing cash flows adopting the Net Present Value (NPV) principle, or the annuity stream, which is a variation of the NPV. This has made the analysis compact and distinct [Grubbström 1967, 1980, Grubbström and Lundquist 1977, Buser 1986, Thorstenson 1988, Grubbström and Jiang 1990, Molinder, 1995, and Tang 2000].

The background mentioned above has built up the foundation of MRP theory. With the support of this theory, applications have been extended in different directions to interconnect with other fields such as game theory [Horvat and Bogataj 1999], rescheduling and replanning [Grubbström and Tang, 2000, Tang and Grubbström 2002], supply chain risk management [Bogataj and Bogataj 2007], global supply chains [Bogataj et al. 2010]. An overview of such developments can be found in [Grubbström and Tang 2000].

The second stream of literature is lotsizing, a classic operations management problem with an aim to balance the sum of inventory holding and ordering costs. A wide range of policies have been proposed in the single-item case, for instance economic order quantity, fixed order quantity, periodic order quantity, Silver-Meal heuristic, part period balance, McLaren's order moment, among others. However, only few approaches for optimal solutions have been proposed, the classic being the Wagner-Whitin algorithm [Wagner and Whitin 1958], based on dynamic programming, cf also the Triple Algorithm [Grubbström 2005]. Detailed descriptions of the above policies can be found in [Silver et al. 1998]. The computational complexity is discussed in different studies for instance [Federgruen and Tzur 1991]

Many authors have tested to sequentially apply single-item lotsizing methods to multi-item systems, while ignoring the interdependencies between items. This yields a substantial cost increase, with or without modifying the cost parameters [Blackburn and Millen 1982, and Gupta and Brennan 1992]. A dynamic programming formulation has been used for the multi-item system when the end item has a periodic demand [Crowston et al. 1973]. Other authors adopt echelon, commonality concepts into lotsizing problems with general product structures [Steinberg and Napier 1980, Afentakis et al. 1984, Afentakis and Gavish 1986, McKnew et al. 1991]. Due to the computational complexity in such multi-item systems, in particular when capacity constraints are considered, heuristic approaches have been developed such as simulated annealing and genetic algorithms [Kuik and Salomon 1990, Kim and Kim 1996, Kimms 1999, Dellaert and Jeunet 2000, Delleart et al. 2000, Tang 2004, Kaku et al. 2009], among others. Often simplifications are introduced such as setting all lead times to zero or to unity such as in [Kaku et al. 2009, p.551] reducing complexity substantially. Nevertheless, the determination of computational complexity is still much an open question.

Below, standard notation will be listed and a few new concepts will be introduced and defined. As mentioned, MRP Theory treats problems concerning multi-level, multi-stage production-inventory systems, by using input matrices (from Input-Output Analysis) for defining product structures and the Laplace transform (or z-transform) for capturing effects, in particular, from lead times on the time development of relevant variables, as well as for the evaluation of the economic consequences when applying the Net Present Value (NPV) principle. The average cost consequences may be obtained as an approximation from NPV expressions derived.

The dynamic lotsizing problem behind our treatment concerns the question to choose a production plan that maximises the NPV when given a set of external demand events (the Master Production Schedule), given product structure(s) and corresponding lead times, and given economic parameters. All external and internal demand needs to be satisfied, so no backlogs or other shortages are allowed. External demand is given and deterministic.

In this paper we concentrate on the question of how complex the problem is depending on product structure properties and number of external demand events for items on different levels in the product structure. As a measure of complexity, we use the number of feasible production plans to choose between. This would correspond to the standard complexity definition of a complete enumeration algorithm. Complexity of a problem is of interest as such, even when the problem is not solved. But a high complexity might suggest that further simplifying assumptions should be introduced, such as often has been the case in the literature mentioned [cf. also Grubbström 2001]. Complexity might also suggest which method to use, the needs of processing time and memory, how much time of new solution development that would be expected, or indicate that a solution currently is unachievable.

Section 2 below is devoted to a problem background and Section 3 to our main theoretical developments concerning serial and full systems, the latter with items requiring all types of subordinate items as inputs. In Section 4 an in-between example from [Grubbström et al. 2010] is analysed in which a type of degeneracy occurs because of specific timing assumptions.

2. Notation and Background

The following notation is used:

s	Laplace (complex) frequency
$\tilde{f}(s)$	Laplace transform of time function $f(t)$, $t \geq 0$
$\mathbf{I}$	identity matrix
$\mathbf{H}$	input matrix (Bill-of-Materials)
$\mathbf{H}'$	*modified input matrix*, with unit elements replacing non-zero elements in $\mathbf{H}$
τ_i	lead time for the assembly/production of item i
$\tilde{\boldsymbol{\tau}}(s)$	diagonal lead time matrix with element $e^{s\tau_i}$ in its ith diagonal position, where τ_i is the lead time for sub-items entering item i
$\tilde{\mathbf{H}}(s) = \mathbf{H}\tilde{\boldsymbol{\tau}}(s)$	generalised input matrix
$\left(\mathbf{I} - \mathbf{H}\right)^{-1}$	Leontief inverse
$\left(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1}$	generalised Leontief inverse
$\left(\mathbf{I} - \mathbf{H}'\right)^{-1}$	modified Leontief inverse
$\tilde{\mathbf{D}}(s)$	transform of vector of externally demanded items
$\tilde{\mathbf{D}}'(s)$	transform of modified vector of externally demanded items
$\tilde{\mathbf{P}}(s)$	transform of vector of the production of items (at completion times)
M_i	number of external demand events for item i
n	number of levels (items) in the assembly system, which equals the dimension of $\mathbf{H}$ and other matrices and the dimension of $\tilde{\mathbf{P}}(s)$ and other vectors
$\binom{m}{n} = \frac{m!}{(m-n)!n!}$	Binomial coefficient, where $m! = \prod_{i=1}^{m} i = 1 \cdot 2 \cdot \ldots \cdot m$, etc.

Additional notation will be introduced as the need arises.

In the article [Grubbström et al. 2010], a dynamic programming procedure was proposed for solving the optimal dynamic lotsizing problem for a general assembly system within the framework of MRP Theory. Although there are infinitely many production plans that satisfy a given external demand, it was shown that the inner-corner principle previously applied to the single-item dynamic lotsizing problem also is valid for a multi-level system. By the inner-corner principle, the optimal plan in a dynamic lotsizing problem must be chosen from the finite set of plans that involve production to take place (be completed) among the times at which there is demand (internal or external). This principle is valid both when using the NPV or the total/average cost as a criterion function.

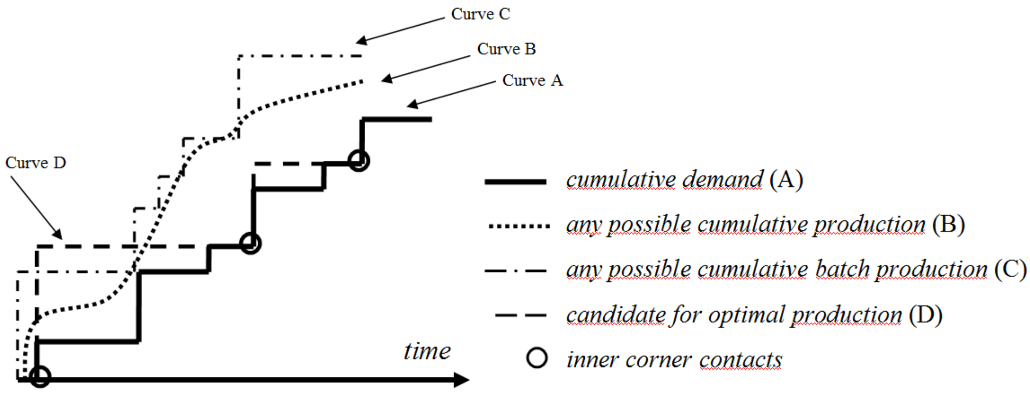

Figure 1. Illustration to the inner-corner condition.

This inner-corner condition is most easily explained from a graph illustrating cumulative demand versus cumulative production as illustrated in Figure 1. A continuous time scale is assumed (but this is no limitation). Demand events (a finite number thereof), each representing a quantity of finite size, occur at discrete points in time, so cumulative demand will be a given staircase function (Curve A). Production completions need to satisfy demand, if no stockouts are allowed. So any function representing cumulative production above or on the cumulative demand staircase will be possible (such as Curve B). If production takes place in batches, cumulative production will also be a staircase function (such as Curve C). The difference between cumulative production and cumulative demand is the current (available) inventory level. Because of economic consequences in the form of holding costs for inventories, the set of feasible production plans candidating as optimal will be limited to encompass staircase functions for which (i) each production event coincides with a demand event and (ii) there is no inventory immediately prior to this production event. If this were not so and a batch was produced strictly before the nearest future demand event, then the time holding inventory could be lowered by postponing production which results in a cost reduction. And if inventory were not zero at the time of producing a batch, then the level of cumulative production could be lowered by decreasing current inventories from reducing the size of the previous batch, thereby also reducing costs. The consequence of this will be that an optimal production plan never includes production at times with no demand, and never should there be any inventory on the occasion of a production event (Curve D). The number of possible plans is reduced from infinity to a finite number of feasible plans. This observation is similar to the binary principle reported in [Veinott 1969].

From the inner-corner principle, the dynamic lotsizing problem can be turned into a binary variable problem of choosing whether to or not produce at each of these times. For a one-level (single-item) system the number of such plans will be 2^{M-1}, where M is the number of demand events, since there must always be production at the first demand event. We call a plan meeting the inner-corner condition a *feasible plan* (although, strictly speaking, there are infinitely many plans meeting the requirement of not generating shortages). In this sense, a feasible plan is a plan that qualifies as a candidate for optimality by the inner-corner condition.

In the dynamic lot-sizing problem, there are two extreme solutions, on the one hand the Lot for Lot solution (L4L) by which production takes place at each point in time a demand event occurs, on the other the All at Once ($\forall@1$) solution, when all production is made at only one point in time, i. e. when the first demand event occurs. For a single-level system, the L4L solution is optimal when the setup cost is very low in comparison with holding costs, and it maximises the number of setups (times at which production occurs), whereas the $\forall@1$ solution is optimal in the converse case when the setup cost is very high compared to holding costs, and the number of setups are then reduced to only one.

In [Grubbström et al. 2010] was concluded that the only times that need to be considered in a general assembly system, are those generated by the L4L solution, since demand (neither external nor internal) can never appear at any other time.

When studying the complexity of the multi-level dynamic lotsizing problem, then neither the size of demand at a certain point in time, nor the values of the elements entering the input matrix (the technical coefficients of the assembly) are of importance. Only the fact that demand occurs at a given point, and that there is a non-zero element in the input matrix $\mathbf{H}$ at a certain position, is essential. If external demand of a particular item would happen to be five units, then internal demand will be generated by non-zero amounts *at the same times* as if external demand had had any other non-zero size. Hence the size of external demand is of no significance apart from being strictly positive. The same holds for internal demand generated by other internal demand. Similarly, if an element in $\mathbf{H}$ happens to be seven, the matrix will generate internal demand for the sub-item in question at certain earlier times from higher-level production, but this *timing* is entirely unaffected by the size of the element, as long as it is positive.

We can therefore "normalise" the description of demand by assuming only events of unit size to occur, collecting external demand in a vector $\tilde{\mathbf{D}}'(s)$ (the non-zero components of which are sums of Dirac impulses). In a similar way, we can "normalise" the input matrix by exchanging all non zero-valued elements for unit-valued elements. We call the resulting input matrix the *modified input matrix* and denote it $\mathbf{H}'$. The corresponding Leontief inverse $(\mathbf{I}-\mathbf{H}')^{-1}$, is called the *modified Leontief inverse*.

The number of non-zero elements belonging to an input matrix obviously affects the complexity of the system. We can then distinguish two extremes, the simplest case which is the *serial system*, and the most complex case, cf. [Tang 2004], with a *full* matrix denoted $\hat{\mathbf{H}}$, when each item in the structure has an input from all kinds of items on lower levels. We will study both of these cases below. In the serial system, each item only requires inputs from one kind of sub-item, so the number of unit elements in the modified input matrix will be $(n-1)$, whereas the full modified input matrix has unit elements everywhere below its main diagonal, and the number of such elements is then therefore $(n-1)n/2$.

It might be noted that all types of assembly systems, involving one or more end-item products, however complicated, are covered by only one input matrix. But we will ignore cases when end products are completely independent of each other, not sharing any common component whatsoever. Such systems would be better treated as independent sub-systems with several unconnected input matrices, thereby reducing the maximum dimension of the matrices involved. Therefore, the simplest type of system under investigation below will be the serial system, by which each item requires only one type of other items as an input.

A second property that influences complexity is the number of external demand events for items on different levels. For lower-levels, these items can be interpreted as spare parts, etc. By coincidence, times at which external demand occurs for a lower-level item can be the same as when an internal demand event might occur. This would happen, for instance, when the distance in time between an external event for a higher-level item and a similar event for a lower-level item exactly coincides with the cumulative lead time between the corresponding levels, so that requirements for the lower-level item were needed just at the time an external demand event occurred. Such a coincidence lowers the number of possible production times in the L4L solution and thereby reduces complexity. In a sense, one can consider such a case as degenerate. In the following we shall assume that a certain point in time in the L4L solution, cannot be generated from two different sources of demand.

From standard expressions in MRP Theory, assuming no initial inventories, if external demand is given as a vector of transformed demand events $\tilde{\mathbf{D}}(s)$, then the L4L solution implies production $\tilde{\mathbf{P}}(s)$ to take place according to:

$$\tilde{\mathbf{P}}(s) = \left(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1} \tilde{\mathbf{D}}(s), \tag{1}$$

where $\tilde{\boldsymbol{\tau}}(s)$ is the *lead time matrix* (diagonal) and $\left(\mathbf{I} - \mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}$ the *generalised Leontief inverse*. The product $\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\tilde{\mathbf{P}}(s)$ is internal (dependent) demand (including amounts and advanced timing), Production in the L4L case must exactly meet the sum of external and internal demand, i.e. $\tilde{\mathbf{P}}(s) = \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\tilde{\mathbf{P}}(s) + \tilde{\mathbf{D}}(s)$ resulting in (1). When $\mathbf{H}$, showing amounts required, multiplies the diagonal vector $\tilde{\boldsymbol{\tau}}(s)$ with operators $e^{\tau_i s}$, $i = 1,\ 2,\ ,,,$, along its diagonal, so that the generalised input matrix $\mathbf{H}\tilde{\boldsymbol{\tau}}(s)$ is obtained, this new matrix will have elements giving amounts as well as their advanced timing compared to completion times embedded in $\tilde{\mathbf{P}}(s)$.

In [Grubbström et al. 2010] was shown that in the case of $\mathbf{H}$ being a full modified input matrix $\hat{\mathbf{H}}'$, its corresponding Leontief inverse would be given by:

$$\left[\left(\mathbf{I}-\hat{\mathbf{H}}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\right]_{ij} = \begin{cases} 0, \ \ for \ \ i<j, \\ 1, \ \ for \ \ i=j, \\ e^{\tau_j s}, \ \ for \ \ i=j+1, \\ e^{\tau_j s}\prod_{k=j+1}^{i-1}\left(1+e^{\tau_k s}\right), \ \ for \ \ i>j+1. \end{cases}, \tag{2}$$

We now supplement this theorem by a similar theorem applicable to the simpler serial system:

Theorem 1

Let $\mathbf{H}'\boldsymbol{\tau}(s) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ e^{s\tau_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & e^{s\tau_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & e^{s\tau_{n-1}} & 0 \end{bmatrix}$ *be a modified generalised input matrix for a serial system. Then its generalised Leontief inverse* $\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}$ *will be:*

$$\left[\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\right]_{ij} = \begin{cases} 0, \ \ for \ \ i<j, \\ 1, \ \ for \ \ i=j, \\ \prod_{k=j}^{i-1} e^{s\tau_k}, \ \ for \ \ i>j. \end{cases}, \tag{3}$$

Proof

Since the matrix $\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)$ is triagonal with unit elements along its main diagonal, its determinant is unity. It suffices to compute $\sum_{k=1}^{n}\left[\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)\right]_{ik}\left[\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\right]_{kj} =$

$$-e^{s\tau_{i-1}}\left[\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\right]_{i-1,j} + 1\cdot\left[\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\right]_{ij} = \begin{cases} 0, \ i<j, \\ 1, \ i=j, \\ -e^{s\tau_{i-1}}\prod_{k=j}^{i-2} e^{s\tau_k} + \prod_{k=j}^{i-1} e^{s\tau_k} = 0, \ i>j. \ \blacksquare \end{cases}$$

If we let $s \to 0$ in (2) and (3), the modified Leontief inverses become

$$\left[\left(\mathbf{I}-\hat{\mathbf{H}}'\right)^{-1}\right]_{ij} = \begin{cases} 0, \ \ for \ \ i>j, \\ 1, \ \ for \ \ i=j, \\ 2^{i-j-1}, \ \ for \ \ i<j, \end{cases} \tag{4}$$

for the full system, and for the serial system

$$\left[\left(\mathbf{I}-\mathbf{H}'\right)^{-1}\right]_{ij}=\begin{cases}0, & \text{for } i<j,\\ 1, & \text{for } i\geq j,\end{cases} \tag{5}$$

the latter matrix being the full input matrix to which unit elements are added also along its main diagonal. This is of interest for showing the numbers of internal demand events that can be generated by external events in the L4L solution.

3. Complexity from Internal and External Demand Events

We now turn to complexity considerations. Complexity is here defined as the number of feasible production plans that a vector of given demand developments over time may generate. The numbers of external demand events for different levels will be considered as input events together with internally generated events. The numbers of external events are denoted M_i for levels $i = 1, 2, \ldots, n$. They are conveniently collected in a vector $\mathbf{M}$. The internal demand events generated from level i will be called output events. External demand events concern both finished items (top level) and items on lower levels. The latter are manufactured, for instance, also for the sake of satisfying external demand for spare parts.

We define *degeneracy* as a situation when demand (internal or external) happens to occur from different sources at the same time. Degeneracy obviously reduces the number of demand events occurring at different points in time, and thereby reduces complexity. Throughout this section we assume that there is no degeneracy from lead times adding up to differences between times of external events. But in Section 4 we will show an example when one instance of degeneracy has a substantial effect on reducing complexity.

Hence, with our definition of degeneracy, this situation occurs when a demand event (internal and/or external) for an item at some point in time is generated at least twice. A degenerative situation would thus have the consequence that the L4L solution $\tilde{\mathbf{P}}(s)$ with modified external demand and a modified input matrix has a value of at least two at some specific point in time, where $\tilde{\mathbf{P}}(s)=\left(\mathbf{I}-\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\tilde{\mathbf{D}}'(s)=\sum_{i=0}^{\infty}\left(\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{i}\tilde{\mathbf{D}}'(s)$. If external demand event times are randomly distributed, this effect could only happen due to properties of the generalised input matrix (Bill-of-Materials with advanced timing), otherwise the probability of such a coincidence would be zero. If, in addition, lead times were randomly distributed, the event of any degeneracy at all would have a zero probability. Investigating which properties of $\mathbf{H}'$, $\tilde{\boldsymbol{\tau}}(s)$ and $\tilde{\mathbf{D}}'(s)$ that would have the consequence of degeneracy is a clearly involved issue and left for future research. In practice, when timing is approximated as a discrete sequence of natural numbers some appearing both in $\tilde{\mathbf{D}}'(s)$ and in $\tilde{\boldsymbol{\tau}}(s)$, then the opportunity for degeneracy is possible as is also illustrated in Section 4 below. Degeneracy therefore occurs only by chance and it is highly improbable to occur, if the times of external demand and lead times are arbitrary positive real numbers.

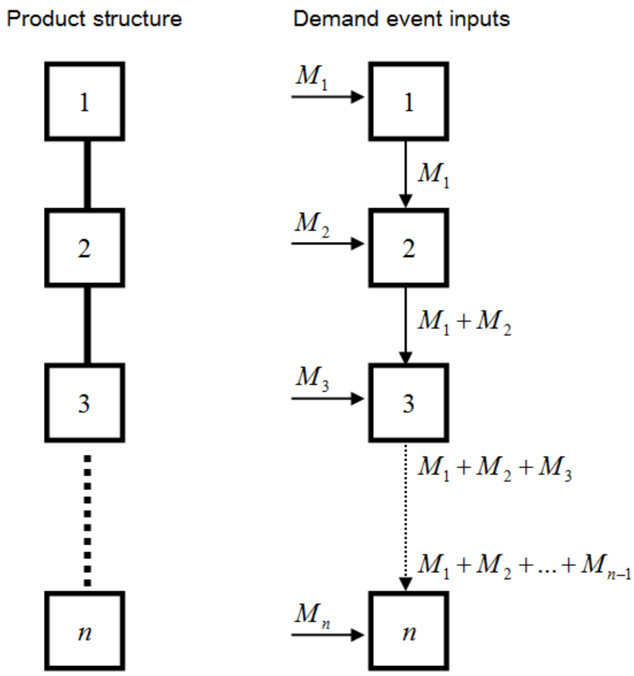

Figure 2. The serial structure with internal and external inputs of demand events.

The Serial System

We start out by investigating consequences from the simplest serial structure with only two items, please see two top level boxes in Figure 2. The modified input matrix is $\mathbf{H}' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, the modified Leontief Inverse is $(\mathbf{I}-\mathbf{H}')^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and the number of external demand events is $\mathbf{M} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$. Assume to begin with that M_2 is zero. Considering the opportunities to choose different lotsizing plans for the top item, there will be $\begin{pmatrix} M_1-1 \\ 0 \end{pmatrix} = 1$ cases with only one production setup ($\forall$@1), $\begin{pmatrix} M_1-1 \\ 1 \end{pmatrix} = M_1-1$ cases with two production setups, … , and $\begin{pmatrix} M_1-1 \\ M_1-1 \end{pmatrix} = 1$ when all setup opportunities are used (L4L).

The number of feasible plans for producing the second-level item is therefore $\begin{pmatrix} M_1-1 \\ 0 \end{pmatrix} \cdot 2^0$ from the first set of top-level cases, $\begin{pmatrix} M_1-1 \\ 1 \end{pmatrix} \cdot 2^1$ for the next set of cases, … , and $\begin{pmatrix} M_1-1 \\ M_1-1 \end{pmatrix} \cdot 2^{M_1-1}$ for the final L4L set of cases. Together this yields the number of different feasible plans for the first- and second-level items together:

$$\sum_{k=0}^{M_1-1} \begin{pmatrix} M_1-1 \\ k \end{pmatrix} \cdot 2^{M_1-1-k} = 3^{M_1-1}. \tag{6}$$

An example with $M_1 = 4$ is shown in Figure 3.

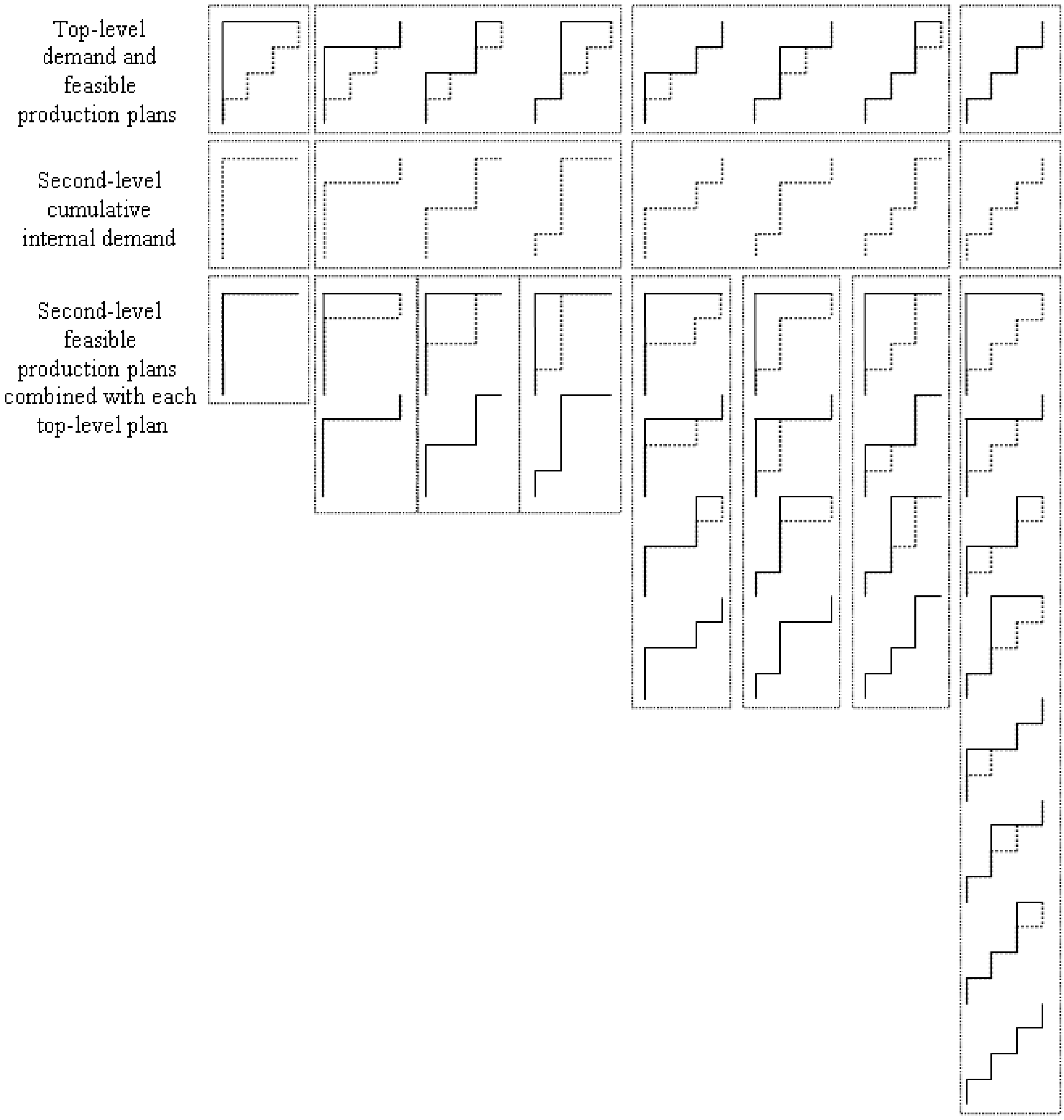

Figure 3. Example with two-level system, four external demand events on top level $M_1 = 4$ and no external demand for second item $M_2 = 0$. Cumulative demand is dotted and cumulative production solid. Each column shows combinations of top-level plan and feasible second-level plans. Total number of feasible production plans becomes $3^{M_1 - 1} = 27$.

Leaving the assumption that external demand events for the second-level item are zero, allowing an $M_2 > 0$ opens up for additional feasible plans. With one internal demand event and M_2 external events, there will be 2^{1+M_2-1} plans, with *j* internal events 2^{j+M_2-1}, … , and with M_1 internal events we have $2^{M_1+M_2-1}$ plans. The total number of feasible plans for producing the first- and second-level items is then

$$\sum_{j=0}^{M_1-1} \binom{M_1 - 1}{j} \cdot 2^{M_2+j} = 2^{M_2} 3^{M_1-1}. \tag{7}$$

Of interest in this paper is to classify the number of production plans according to the number of internal demand events that they generate. Writing the summation (7) in a more detailed form and exchanging the order of summation reveals:

$$\sum_{j=0}^{M_1-1}\binom{M_1-1}{j}2^{M_2+j}=\sum_{j=0}^{M_1-1}\binom{M_1-1}{j}\sum_{k=0}^{M_2+j}\binom{M_2+j}{k}=\sum_{k=0}^{M_2+j}\sum_{j=0}^{M_1-1}\binom{M_2+j}{k}\binom{M_1-1}{j}, \tag{8}$$

Here the index k refers to the case that $k+1$ internal demand events are generated. So, the number of plans that generate $k+1$ internal demand events, denoted $N_2(k)$ may be written:

$$N_2(k)=\sum_{j=0}^{M_1-1}\binom{M_2+j}{k}\binom{M_1-1}{j}=\frac{(M_1-1)!}{k!}\sum_{j=0}^{M_1-1}\frac{(M_2+j)!}{j!(M_1-1-j)!(M_2+j-k)!}. \tag{9}$$

As an example we choose the same as in Figure 3 with $M_1=4$, but with one external demand event added for item No 2, $M_2=1$. In this example the added external event has been placed as a last event in time. But this has no influence on the distribution of production plans. Figure 4 displays the cases that these altogether five events generate.

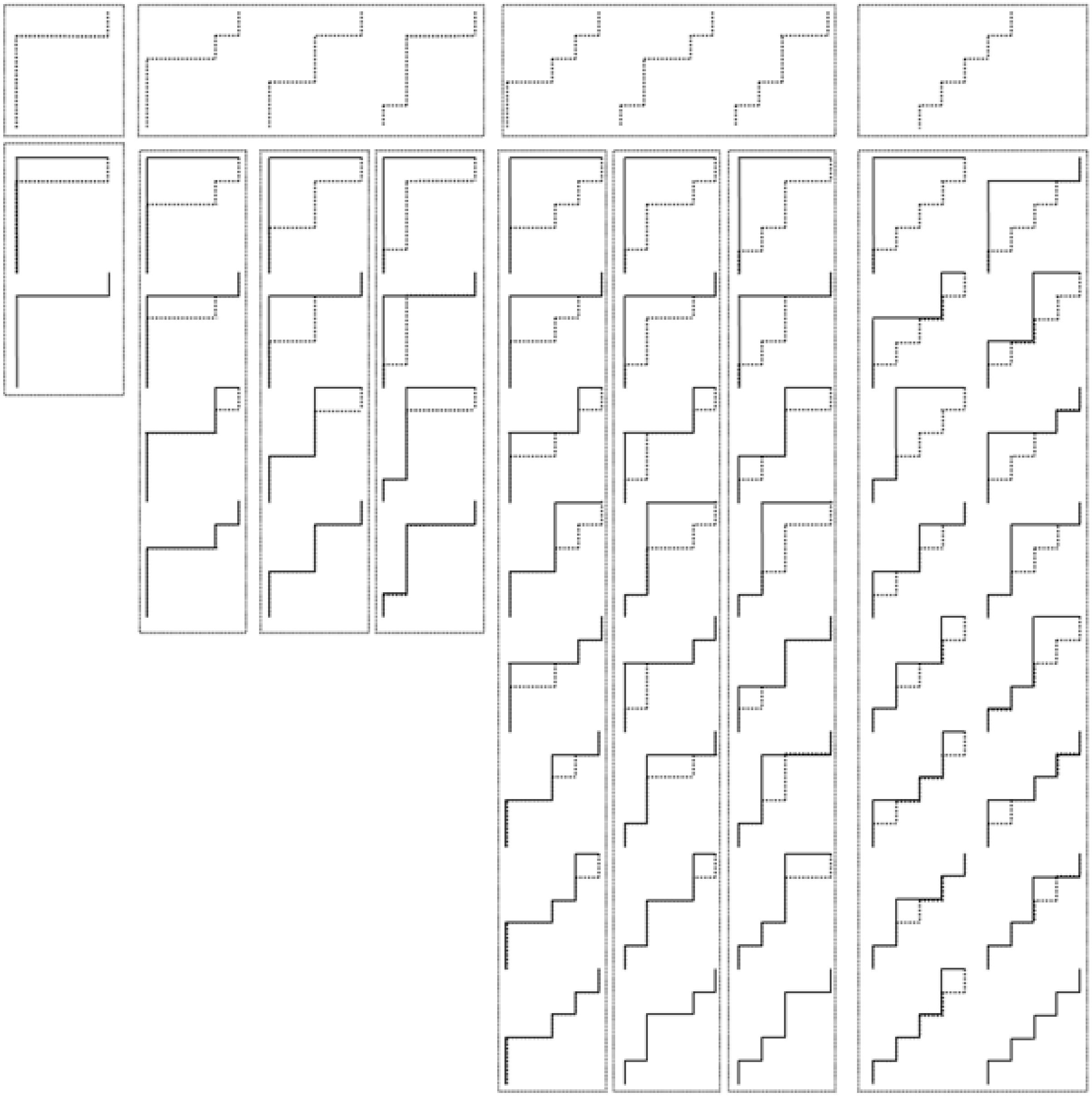

Figure 4. Example with two-level system, four external demand events on top level $M_1 = 4$ *and one external event for second-level item* $M_2 = 1$ *placed as last event. At the top are shown the eight alternative demand patterns for the second-level item, and below the different feasible production plans that these cases provide. Cumulative demand is dotted and cumulative production solid. The total number of production plans will be* $2^{M_2}3^{M_1-1} = 54$.

Table 1 provides an illustration to Equation (9). Using the standard convention that $\binom{M_2 + j}{k} = 0$ for $k > M_2 + j$, the entries in column $j + 1$ are the binomial coefficients $\binom{M_2 + j}{k}$ for $k = 0, 1, \ldots, 4$. The size of each set is $\binom{M_1 - 1}{j}$, $j = 0, 1, \ldots, (M_1 - 1)$.

Cases No	**1**	**2, 3, 4**	**5, 6, 7**	**8**	
Internal plus External input demand events	**1+1**	**2+1**	**3+1**	**4+1**	
Number of generated output events	**Size of set=** $\binom{3}{0}=1$	**Size of set=** $\binom{3}{1}=3$	**Size of set=** $\binom{3}{2}=3$	**Size of set=** $\binom{3}{3}=1$	**Weighted sum of generated output events**
1	1	1	1	1	**8**
2	1	2	3	4	**20**
3	0	1	3	6	**18**
4	0	0	1	4	**7**
5	0	0	0	1	**1**
Weighted sum	**2**	**12**	**24**	**16**	**54**

Table 1. Number of plans categorised by number of internal demand output events generated in example of Figure 4.

We now generalise our investigation to an arbitrarily sized assembly system. To begin with we restrict our attention to a serial system.

Let $N_i(k)$ be the distribution of different plans each generating $k + 1$ output events from producing item i. If the internal demand encompasses $(j + 1)$ events, the number of feasible production plans having $k + 1$ outputs on level i will be $\binom{M_i + j}{k}$. Also let R_i denote the range of input events to level i, which is defined as the maximum number of input events to this level (internal and external).

For a serial system, we must always have $R_i = R_{i-1} + M_i$, so the range will then be the cumulative number of external demand events on higher levels, this level inclusive (cf Figure 2),

$$R_i = \sum_{j=1}^{i} M_j \text{ , } i = 1, 2, \dots , n, \tag{10}$$

since there is never any internal demand facing the top level. So,

$$N_i(k) = \sum_{j=0}^{R_{i-1}-1} \binom{M_i + k}{j} N_{i-1}(j) . \tag{11}$$

and therefore

$$N_i(k_i) = \sum_{k_{i-1}=0}^{R_{i-1}-1} \binom{M_i + k_{i-1}}{k_i} N_{i-1}(k_i) = \sum_{k_{i-1}=0}^{R_{i-1}-1} \binom{M_i + k_{i-1}}{k_i} \sum_{k_{i-2}=0}^{R_{i-2}-1} \binom{M_{i-1} + k_{i-2}}{k_{i-1}} N_{i-2}(k_{i-2}) =$$

$$= \sum_{k_{i-1}=0}^{R_{i-1}-1} \sum_{k_{i-2}=0}^{R_{i-2}-1} \dots \sum_{k_1=0}^{R_1-1} \binom{M_i + k_{i-1}}{k_i} \binom{M_{i-1} + k_{i-2}}{k_{i-1}} \dots \binom{M_2 + k_1}{k_2} N_1(k_1) . \tag{12}$$

However, we know that the top level generates k_1+1 output events according to $N_1(k_1)=\binom{M_1-1}{k_1}$, so we obtain the following explicit formula for how many plans there are in total with $k+1$ production events on the bottom level of an n-level serial assembly system

$$N_n(k_n)=\sum_{k_{n-1}=0}^{R_{n-1}-1}\sum_{k_{n-2}=0}^{R_{n-2}-1}\cdots\sum_{k_1=0}^{R_1-1}\prod_{l=1}^{n-1}\binom{M_{l+1}+k_l}{k_{l+1}}\binom{M_1-1}{k_1}=\sum_{k_{n-1}=0}^{R_{n-1}-1}\sum_{k_{n-2}=0}^{R_{n-2}-1}\cdots\sum_{k_1=0}^{R_1-1}\prod_{l=0}^{n-1}\binom{M_{l+1}+k_l}{k_{l+1}}, \tag{13}$$

defining k_0 as $k_0=-1$. The grand total of all different plans, which is our measure of complexity, will then be:

$$\sum_{k_n=0}^{R_n-1}N_n(k_n)=\sum_{k_n=0}^{R_n-1}\sum_{k_{n-1}=0}^{R_{n-1}-1}\sum_{k_{n-2}=0}^{R_{n-2}-1}\cdots\sum_{k_1=0}^{R_1-1}\prod_{l=0}^{n-1}\binom{M_{l+1}+k_l}{k_{l+1}}. \tag{14}$$

We first note that the upper summation limits always are at least as large as the upper argument of the corresponding binomial coefficient. Since $k_{i-1}\le R_{i-1}-1$ and $R_i-1=R_{i-1}+M_i-1$, we have $M_i+k_{i-1}\le M_i+R_{i-1}-1=R_i-1$. Therefore the corresponding summations will cover all relevant terms up to this limit.

Reversing the summation order, we first take the summation over k_n, which gives $\sum_{k_n=0}^{R_n-1}\binom{M_n+k_{n-1}}{k_n}=\sum_{k_n=0}^{M_n-1+k_{n-1}}\binom{M_n+k_{n-1}}{k_n}=2^{M_n+k_{n-1}}$, because $M_n+k_{n-1}\le R_n$. For k_{n-1}, the next summation yields $\sum_{k_{n-1}=0}^{R_{n-1}-1}2^{M_n+k_{n-1}}\binom{M_{n-1}+k_{n-2}}{k_{n-1}}=2^{M_n}3^{M_{n-1}+k_{n-2}}$, and so on, until the summation over k_2 which adds the factor n^{M_2}, and finally the summation over k_1 adding the factor $(n+1)^{M_1+k_0}$ (with $k_0=-1$).

Hence, the number of feasible plans in total for the serial system as a whole will be:

$$\sum_{k_n=0}^{R_n-1}N_n(k_n)=\sum_{k_n=0}^{R_n-1}\sum_{k_{n-1}=0}^{R_{n-1}-1}\sum_{k_{n-2}=0}^{R_{n-2}-1}\cdots\sum_{k_1=0}^{R_1-1}\prod_{l=0}^{n-1}\binom{M_{l+1}+k_l}{k_{l+1}}=2^{M_n}3^{M_{n-1}}4^{M_{n-2}}\cdots n^{M_2}(n+1)^{M_1-1}. \tag{15}$$

With positive numbers of external demand events, this number increases progressively with the number of stages n and numbers of external demand events, and with only one external demand input for each level it becomes $n!$. By way of example, if all M_i are 2, 3, or alternatively 4, for different values of n, the resulting number of feasible plans is shown in Table 2.

	All M_i (including M_1) equal to			
n	**1**	**2**	**3**	**4**
1	1	4	8	16
2	2	12	72	432
3	6	144	3456	82944
4	24	2880	345600	41472000
5	120	86400	62208000	44789760000

Table 2. Total number of feasible plans for different sizes of serial assembly system (n) and different numbers of external demand events.

It is often convenient to use the z-transform of distribution functions. Here, we denote the transform of $N_n\left(k_n\right)$ by $\tilde{N}_n(z)$ and use the convention of positive exponents in the definition of the transform (contrary to the standard definition with negative exponents). From (13) we have:

$$\tilde{N}_n(z)=\sum_{k_n=0}^{\infty} N_n\left(k_n\right)z^{k_n}=\sum_{k_n=0}^{\infty} z^{k_n}\sum_{k_{n-1}=0}^{R_{n-1}-1}\sum_{k_{n-2}=0}^{R_{n-2}-1}\cdots\sum_{k_1=0}^{R_1-1}\sum_{k_{n-1}=0}^{R_{n-1}-1}\sum_{k_{n-2}=0}^{R_{n-2}-1}\cdots\sum_{k_1=0}^{R_1-1}\prod_{l=0}^{n-1}\binom{M_{l+1}+k_l}{k_{l+1}}, \tag{16}$$

and reversing the order of summation starting with k_n, then k_{n-1}, and so on, we obtain

$$\sum_{k_n=0}^{\infty} z^{k_n}\binom{M_n+k_{n-1}}{k_n}=\left(1+z\right)^{M_n+k_{n-1}}\quad,\quad\sum_{k_{n-1}=0}^{R_{n-2}-1}\left(1+z\right)^{M_n+k_{n-1}}\binom{M_{n-1}+k_{n-2}}{k_{n-1}}=\left(1+z\right)^{M_n}\left(2+z\right)^{M_n+k_{n-1}}\quad,$$

and so forth, which ends with the transform

$$\tilde{N}_n(z)=\left(1+z\right)^{M_n}\left(2+z\right)^{M_{n-1}}....\left(n+z\right)^{M_1+k_0}=\left(n+z\right)^{-1}\prod_{i=1}^{n}\left(n+1-i+z\right)^{M_i}. \tag{17}$$

Taking the limit $z\to 1$, provides the sum of all plans, as in (15):

$$\lim_{z\to 1}\hat{N}_n(z)=2^{M_n}3^{M_{n-1}}....\left(n+1\right)^{M_1-1}. \tag{18}$$

In the *n*-level problem for the serial system, there are altogether $\sum_{i=1}^{n}R_i=\sum_{i=1}^{n}(n+1-i)M_i$ binary variables, and we could calculate the number of plans on each level assuming there were no dependence between the set of choices on a lower level and what has been decided on an upper level. This would obviously create an overestimate of the correct number of feasible plans. On level 1, we would thus have 2^{M_1-1} plans (knowing that there must be at east one setup), one level 2, we would have $2^{M_1+M_2-1}$ plans, etc. Thus if plans were combined independently of one another, the total number of plans for the system as a whole would be

$$2^{M_1-1}\cdot 2^{M_1+M_2-1}\cdot\ldots\cdot 2^{M_1+M_2+\ldots+M_n-1}=2^{(M_1-1)n+\sum_{i=1}^{n-1}M_{n+1-i}\cdot i},$$

whereas the correct number of plans according to (18) is

$$2^{M_n}3^{M_{n-1}}....\left(n+1\right)^{M_1-1}=2^{(M_1-1)\cdot\log_2(n+1)+\sum_{i=1}^{n-1}M_{n+1-i}\cdot\log_2(i+1)}.$$

Comparing individual terms in the exponents of these two expressions, we find the first to dominate the second, because $i \geq \log_2(i+1)$, since $2^i \geq (i+1)$, with equality only for $i = 1$. This verifies that the first expression is a clear overestimate.

The Full System

Hitherto, we have dealt with a serial system. We now turn our attention to the *full* system in which the modified input matrix is filled below its main diagonal, i. e. the most complex structure, which is the opposite extreme case:

$$\hat{\mathbf{H}}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}. \tag{19}$$

Here the ranges R_i (maximum input event numbers) need to be expanded and redefined for lower levels with i above $i = 2$, as illustrated in Figure 5 for the case that $n = 4$.

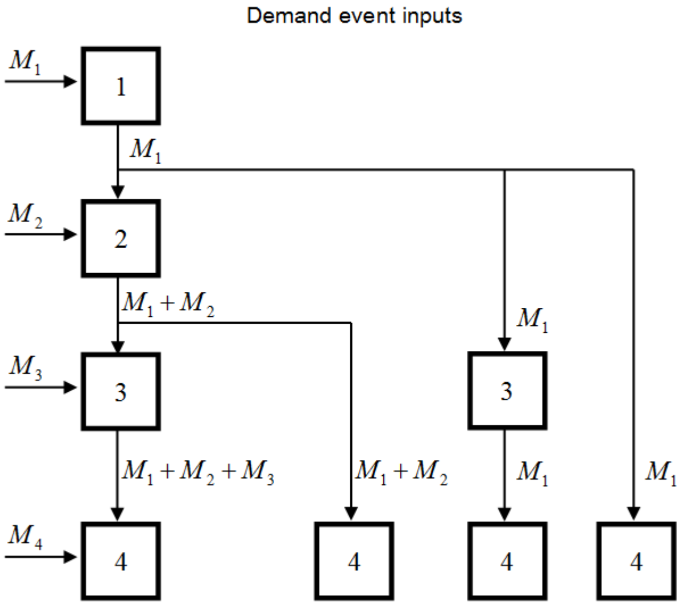

Figure 5. Maximum external and internal demand event inputs to the full system when n = 4.

The maximum number of demand event inputs to a certain level (the range) may be found from using the modified input matrix $\hat{\mathbf{H}}'$ in the L4L solution $\left(\mathbf{I} - \hat{\mathbf{H}}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\tilde{\mathbf{D}}'(s)$ from Eq. (1). Looking only at numbers of external input events in $\tilde{\mathbf{D}}'(s)$ given by the vector **M**, the total number of input events to level i (external and internal) will be $\left(\mathbf{I} - \hat{\mathbf{H}}'\right)^{-1}\mathbf{M}$, where $\left(\mathbf{I} - \hat{\mathbf{H}}'\right)^{-1}$ is given by (4) for the full system. So the range for each level, written as a vector **R**, will be:

$$\mathbf{R} = \left(\mathbf{I} - \hat{\mathbf{H}}'\right)^{-1} \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 1 & \cdots & 0 & 0 \\ 4 & 2 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 2^{n-2} & 2^{n-3} & 2^{n-4} & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_n \end{bmatrix}, \tag{20}$$

or, in detailed form:

$$R_i = \sum_{j=1}^{i-1} M_j \cdot 2^{i-1-j} + M_i . \tag{21}$$

For $n = 4$, as in Figure 5, the range of inputs into level 4 becomes $R_4 = = M_1 \cdot 2^2 + M_2 \cdot 2 + M_3 + M_4 = \left(M_1 + M_2 + M_3\right) + \left(M_1 + M_2\right) + M_1 + M_1 + M_4$, which the figure verifies.

As is in the serial case, we have a first-level distribution of plans $N_1\left(k_1\right) = \binom{M_1 - 1}{k_1}$ with $\left(k_1 + 1\right)$ internal output demand events, $k_1 = 0,1,...,M_1 - 1$. This distribution generates an output distribution from the second level as in the serial case, $N_2\left(k_2\right) = \sum_{k_1=0}^{R_1-1} \binom{M_2 + k_1}{k_2}\binom{M_1 - 1}{k_1}$, where k_2 ranges from zero to a maximum of its range less unity, i. e. to $R_2 - 1 = M_1 + M_2 - 1$. However, on the third level the number of input events increases by $k_1 + 1$ generated by the first level, so the distribution of plans with $\left(k_3 + 1\right)$ production events becomes:

$$N_3\left(k_3\right) = \sum_{k_2=0}^{R_2-1} \sum_{k_1=0}^{R_1-1} \binom{M_3 + k_2 + k_1 + 1}{k_3} \binom{M_2 + k_1}{k_2} \binom{M_1 - 1}{k_1}. \tag{22}$$

More generally for the full system, consider internal demand as a vector formed by the product $\hat{\mathbf{H}}'\tilde{\boldsymbol{\tau}}(s)\tilde{\mathbf{P}}(s)$, where $\tilde{\mathbf{P}}(s)$ is the total production vector. If $\tilde{\mathbf{P}}(s)$ contains $\left(k_1 + 1\right)$ production events on level 1 generating the same number of internal demand events for various components, $\left(k_2 + 1\right)$ events on level 2, and so on, then the number of internal demand events as a whole is given by

$$\hat{\mathbf{H}}' \begin{bmatrix} k_1 + 1 \\ k_2 + 1 \\ \vdots \\ k_n + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 + 1 \\ k_2 + 1 \\ \vdots \\ k_n + 1 \end{bmatrix}, \tag{23}$$

since the lead time matrix $\tilde{\boldsymbol{\tau}}(s)$ does not affect the numbers of events, except when degeneracy is present. For the item on level *i* there will thus be $\sum_{j=1}^{i-1}\left(k_j+1\right)$ internal demand events together with M_i external events, and the index k_i will range from zero to $M_i+\sum_{j=1}^{i-1}\left(k_j+1\right)-1$. Generalising (22) we therefore obtain

$$N_n\left(k_n\right)=\sum_{k_{n-1}=0}^{R_{n-1}-1}\cdots\sum_{k_2=0}^{R_2-1}\sum_{k_1=0}^{R_1-1}\binom{M_n+\sum_{i=1}^{n-1}k_i+n-2}{k_n}\cdots\binom{M_3+k_2+k_1+1}{k_3}\binom{M_2+k_1}{k_2}\binom{M_1+k_0}{k_1}. \tag{24}$$

The z-transform of $N_n\left(k_n\right)$ is again given by $\tilde{N}_n(z)=\sum_{k_n=0}^{\infty}N_n\left(k_n\right)z^{k_n}$. Using (24) and reversing the order of summation gives

$$\tilde{N}_n\left(z\right)=\sum_{k_n=0}^{R_n-1}z^{k_n}\sum_{k_{n-1}=0}^{R_{n-1}-1}\cdot\cdot\sum_{k_2=0}^{R_2-1}\sum_{k_1=0}^{R_1-1}\binom{M_n+\sum_{i=1}^{n-1}k_i+n-2}{k_n}\cdots\binom{M_3+k_2+k_1+1}{k_3}\binom{M_2+k_1}{k_2}\binom{M_1+k_0}{k_1}=$$

$$=\sum_{k_1=0}^{R_1-1}\binom{M_1+k_0}{k_1}\sum_{k_2=0}^{R_2-1}\binom{M_2+k_1}{k_2}\cdots\sum_{k_{n-1}=0}^{R_{n-1}-1}\binom{M_{n-1}+K_{n-2}+n-3}{k_{n-1}}\sum_{k_n=0}^{R_n-1}\binom{M_n+K_{n-1}+n-2}{k_n}z^{k_n} \tag{25}$$

where K_j is an abbreviation for the cumulative value $K_j=\sum_{i=1}^{j}k_i$, and as before $k_0=-1$.

We define a sequence of factors $a_i(z)$, by:

$$a_i(z)=\prod_{j=1}^{i-1}a_j(z)+1,\ i=1,2,\ldots,n, \tag{26}$$

where $\prod_{j=1}^{0}a_j(z)$ is defined as *z* by convention (non standard). Also we write $a_i=a_i(1)$, so:

$$a_i=\prod_{j=1}^{i-1}a_j+1=a_{i-1}^2-a_{i-1}+1,\ i=1,2,\ldots,n. \tag{27}$$

These numbers form *Sylvester's sequence* [cf. Sylvester 1880, Aho and Sloane 1973]. The quadratic recursive expression in the right-hand member is obtained from developing $\left(a_i-1\right)/a_{i-1}=\left(\prod_{j=1}^{i-1}a_j\right)/a_{i-1}=\prod_{j=1}^{i-2}a_j=a_{i-1}-1$.

We are now in a position to state

Theorem 2

The z-transform $\tilde{N}_n\left(z\right)$ *of the distribution* $N_n\left(k_n\right)$ *for a full assembly system may be written*

$$\tilde{N}_n(z) = \prod_{i=1}^{n} a_i(z)^{M_{n+1-i}+n-i-1} = \prod_{i=1}^{n} a_{n-i+1}(z)^{M_i+i-2}, \tag{28}$$

and, hence, the total number of feasible plans for a full assembly system as

$$\sum_{k_n=0}^{R_n-1} N_n(k_n) = \lim_{z\to 1} \tilde{N}_n(z) = \prod_{i=1}^{n} a_i^{M_{n+1-i}+n-i-1} = \prod_{i=1}^{n} a_{n-i+1}^{M_i+i-2}, \tag{29}$$

where the $a_i(z)$ *are defined by (26) and* $a_i = a_i(1)$.

Proof

First we show that $M_i + K_{i-1} + n - 2 \le R_i - 1$, for all $i = 1, 2, \dots, n$. Rewriting this inequality and making use of the definition of K_{i-1}, it suffices to show that $K_{i-1} + i - 1 = \sum_{j=1}^{i-1}(k_j + 1) \le R_i - M_i$. Since each k_j ranges from zero to a maximum of its range less unity, $R_j - 1$, we have $\sum_{j=1}^{i-1}(k_j+1) \le \sum_{l=1}^{i-1} R_l$. From (22), we compute $\sum_{l=1}^{i-1} R_l = \sum_{l=1}^{i-1}\left(\sum_{j=1}^{l-1} M_j \cdot 2^{l-j-1} + M_l\right) = \sum_{j=1}^{i-2} M_j \sum_{l=j+1}^{i-1} 2^{l-1-j} + \sum_{l=1}^{i-1} M_l = \sum_{j=1}^{i-2} M_j 2^{i-j-1} + M_{i-1} = R_i - M_i$, proving this statement.

We prove the remainder of this theorem using mathematical induction. First, for $n = 1$ we have $N_1(k_1) = \binom{M_1 - 1}{k_1}$, $R_1 = M_1$, and therefore $\tilde{N}_1(z) = \sum_{k_1=0}^{R_1-1} \binom{M_1 + k_0}{k_1} z^{k_1} = (1+z)^{M_1+k_0}$. So (28) is valid for $n = 1$. Secondly, we note that

$$\sum_{k_{n-j}=0}^{R_{n-j}-1} \binom{M_{n-j} + K_{n-j-1} + n - j - 2}{k_{n-j}} \prod_{i=1}^{j} a_i(z)^{M_{n+1-i}+K_{n-j}+n-i-1} =$$

$$= \prod_{i=1}^{j} a_i(z)^{M_{n+1-i}+K_{n-j-1}+n-i-1} \sum_{k_{n-j}=0}^{R_{n-j}-1} \binom{M_{n-j} + K_{n-j-1} + n - j - 2}{k_{n-j}} \prod_{i=1}^{j} a_i(z)^{k_{n-j}} =$$

$$= \prod_{i=1}^{j} a_i(z)^{M_{n+1-i}+K_{n-j-1}+n-i-1} \left(\prod_{i=1}^{j} a_i(z) + 1\right)^{M_{n-j}+K_{n-j-1}+n-j-2} =$$

$$= \prod_{i=1}^{j} a_i(z)^{M_{n+1-i}+K_{n-j-1}+n-i-1} a_{j+1}(z)^{M_{n-j}+K_{n-j-1}+n-j-2} = \prod_{i=1}^{j+1} a_i(z)^{M_{n+1-i}+K_{n-j-1}+n-i-1}.$$

So

$$\tilde{N}_n(z) = \sum_{k_1=0}^{R_1-1}\binom{M_1+k_0}{k_1}\sum_{k_2=0}^{R_2-1}\binom{M_2+k_1}{k_2}\cdots\sum_{k_{n-1}=0}^{R_{n-1}-1}\binom{M_{n-1}+K_{n-2}+n-3}{k_{n-1}}\sum_{k_n=0}^{R_n-1}\binom{M_n+K_{n-1}+n-2}{k_n} z^{k_n} =$$

$$= \sum_{k_1=0}^{R_1-1}\binom{M_1+k_0}{k_1}\sum_{k_2=0}^{R_2-1}\binom{M_2+k_1}{k_2}\cdots\sum_{k_{n-1}=0}^{R_{n-1}-1}\binom{M_{n-1}+K_{n-2}+n-3}{k_{n-1}} a_1(z)^{M_n+K_{n-1}+n-2} =$$

$$= \sum_{k_1=0}^{R_1-1}\binom{M_1+k_0}{k_1}\sum_{k_2=0}^{R_2-1}\binom{M_2+k_1}{k_2}\cdots\sum_{k_{n-2}=0}^{R_{n-2}-1}\binom{M_{n-2}+K_{n-3}+n-4}{k_{n-2}}\prod_{i=1}^{2} a_i(z)^{M_{n+1-i}+K_{n-2}+n-i-1} =$$

$$= \sum_{k_1=0}^{R_1-1}\binom{M_1+k_0}{k_1}\prod_{i=1}^{n-1} a_i(z)^{M_{n+1-i}+K_1+n-i-1} = \prod_{i=1}^{n} a_i(z)^{M_{n+1-i}+n-i-1},$$

since $K_1 = k_1$ and $k_0 = -1$. The consequence of (29) is then trivial. ■

In particular, for $n = 2$, the serial and full systems coincide (as seen from the input matrix) and (29) gives $\tilde{N}_2(1) = \prod_{i=1}^{2} a_i^{M_{2+1-i}+1-i} = a_1^{M_2} a_2^{M_1-1} = 2^{M_2}3^{M_1-1}$ for the full system, which is the same as from (15) for the serial system.

Numerical values of the first few numbers in Sylvester's sequence a_i are listed in Table 3.

i	a_i	i	a_i
1	2	7	10650056950807
2	3	8	$1.13423713055422 \cdot 10^{26}$
3	7	9	$1.28649386832787 \cdot 10^{52}$
4	43	10	$1.6550664732452 \cdot 10^{104}$
5	1807	11	$2.7392450308603 \cdot 10^{208}$
6	3263443		

Table 3. Values of the factors a_i.

Under all circumstances, one might conclude that the number of feasible plans, although finite, increases astronomically (the number of atoms in the Universe was once estimated to be around 10^{63}). But it is useful to know this.

An example illustrating the number of plans in a simple 3-level case with $M_1 = 2$, $M_2 = M_3 = 0$ is shown in Figure 6. According to (29), the number of feasible plans is $\tilde{N}_3(1) = a_1^{M_3+1} a_2^{M_2} a_3^{M_1-1} = 2^1 \cdot 3^0 \cdot 7^1 = 14$, which the figure verifies.

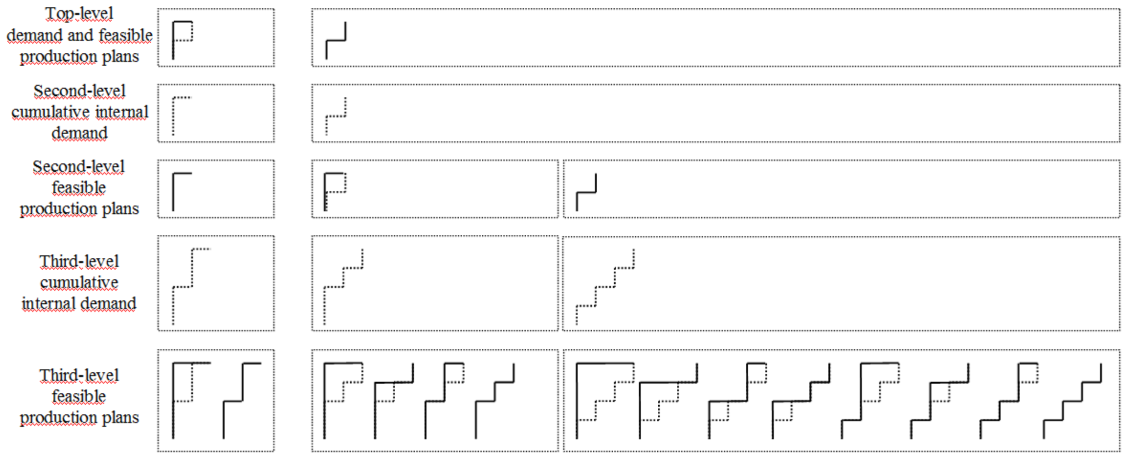

Figure 6. Example showing all feasible plans for a full system with $M_1 = 2$, $M_2 = M_3 = 0$. Dotted lines show cumulative demand and solid lines cumulative production. The third level receives demand also from the first level.

If we instead had a four-level full system with $M_1 = 2$, $M_2 = M_3 = M_4 = 0$, then the number of plans would grow to $a_1^{M_4+2} a_2^{M_3+1} a_3^{M_2} a_4^{M_1-1} = 516$. The case that there is one production event on the top level accounts for 84 possible plans, and with both possible production events on the top level this accounts for the remaining 432 plans. The bottom fourth level is reached by 3 demand events in one case, 4 events in two cases, 5 events in three cases, 6 events in four cases, 7 events in three cases and 8 events in one case. The similar serial system would have had a total of only 4 feasible plans.

Similarly as for the serial system, assuming independence between the different levels and then counting the number of combinations of plans, using (21) gives us

$$2^{R_1-1} \cdot 2^{R_2-1} \cdot \ldots \cdot 2^{R_n-1} = 2^{-n+\sum_{i=1}^{n} R_i} = 2^{\sum_{j=1}^{n}\left(M_j 2^{n-j}-1\right)},$$

whereas from (29) the correct value amounts to

$$\sum_{k_n=0}^{R_n-1} N_n\left(k_n\right) = \tilde{N}_n\left(1\right) = \prod_{i=1}^{n} a_{n-i+1}^{M_i+i-2} = \prod_{i=1}^{n} 2^{\left(M_i+i-2\right)\log_2 a_{n-i+1}} = 2^{\sum_{i=1}^{n}\left(M_i+i-2\right)\log_2 a_{n-i+1}}.$$

To illustrate that the correct value falls short of the former value, we use the proof of Theorem 2, giving us

$$\tilde{N}_n\left(1\right) = \sum_{k_1=0}^{R_1-1}\binom{M_1+k_0}{k_1}\sum_{k_2=0}^{R_2-1}\binom{M_2+k_1}{k_2}\ldots\sum_{k_{n-1}=0}^{R_{n-1}-1}\binom{M_{n-1}+K_{n-2}+n-3}{k_{n-1}}\sum_{k_n=0}^{R_n-1}\binom{M_n+K_{n-1}+n-2}{k_n} \le$$
$$\le \sum_{k_1=0}^{R_1-1}\binom{R_1-1}{k_1}\sum_{k_2=0}^{R_2-1}\binom{R_2-1}{k_2}\ldots\sum_{k_{n-1}=0}^{R_{n-1}-1}\binom{R_{n-1}-1}{k_{n-1}}\sum_{k_n=0}^{R_n-1}\binom{R_n-1}{k_n} = 2^{-n+\sum_{i=1}^{n} R_i},$$

since $M_i + K_{i-1} + i - 1 \le R_i$ and then $\binom{M_i+K_{i-1}+i-2}{k_i} \le \binom{R_i-1}{k_i}$.

4. An intermediate system case with degeneracy and generalisations

As an illustration to a case in between the serial and the full assembly systems we chose the example of a five-level system from [Grubbström et al. 2010], see Figure 7.

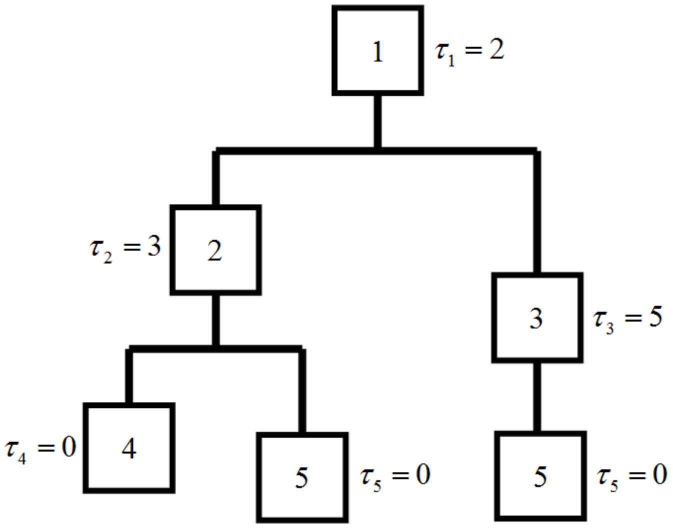

Figure 7. Example of five-level intermediate system with lead times indicated (from [Grubbström et al. 2010]).

In this example the effect of degeneracy from equality between lead times and differences between external demand events is also demonstrated. From Figure 7 we find the modified input and lead time matrices to be:

$$\mathbf{H}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \; \tilde{\boldsymbol{\tau}}(s) = \begin{bmatrix} e^{2s} & 0 & 0 & 0 & 0 \\ 0 & e^{3s} & 0 & 0 & 0 \\ 0 & 0 & e^{5s} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This gives us the generalised Leontief inverse:

$$\left(\mathbf{I} - \mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ e^{2s} & 1 & 0 & 0 & 0 \\ e^{2s} & 0 & 1 & 0 & 0 \\ e^{5s} & e^{3s} & 0 & 1 & 0 \\ \left(e^{5s} + e^{7s}\right) & e^{3s} & e^{5s} & 0 & 1 \end{bmatrix}.$$

As in [Grubbström et al. 2010] we assume two external demand events for the top level item and one external demand event for the second-level item. Other levels have no external demand, i. e. $M_1 = 2$, $M_2 = 1$, $M_i = 0$, $i \geq 3$, and the modified demand vector has the timing:

$$\tilde{\mathbf{D}}'(s) = \begin{bmatrix} \tilde{D}_1'(s) \\ \tilde{D}_2'(s) \\ \tilde{D}_3'(s) \\ \tilde{D}_4'(s) \\ \tilde{D}_5'(s) \end{bmatrix} = \begin{bmatrix} e^{-12s} + e^{-14s} \\ e^{-13s} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

Hence one unit of the top-level item is demanded at time 12, and one at time 14, and one unit of the second-level item is demanded at time 13. The L4L solution will then become

$$\tilde{\mathbf{P}}(s) = \left(\mathbf{I} - \mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\right)^{-1} \tilde{\mathbf{D}}'(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ e^{2s} & 1 & 0 & 0 & 0 \\ e^{2s} & 0 & 1 & 0 & 0 \\ e^{5s} & e^{3s} & 0 & 1 & 0 \\ \left(e^{5s} + e^{7s}\right) & e^{3s} & e^{5s} & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-12s} + e^{-14s} \\ e^{-13s} \\ 0 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-12s} + e^{-14s} \\ e^{-10s} + e^{-12s} + e^{-13s} \\ e^{-10s} + e^{-12s} \\ e^{-7s} + e^{-9s} + e^{-10s} \\ e^{-5s} + e^{-7s} + e^{-7s} + e^{-9s} + e^{-10s} \end{bmatrix},$$

which shows all timing opportunities for feasible production plans. We note in particular that demand for the fifth-level item occurs twice at time 7. This is because two different internal demand events happen by coincidence to occur at this point in time, on the one hand from the external demand of the top-level item at time 12, e^{-12s}, creating a requirement for the fifth-level item via internal demand from item 2 the cumulative lead time $\tau_1 + \tau_2 = 5$ before time 12, i. e. at time 7, e^{-7s}, on the second hand from the external demand of the same top-level item at time 14, e^{-14s}, giving item 5 an internal demand from item 3 the cumulative lead time $\tau_1 + \tau_3 = 7$ before time 14, which also happens to be time 7. This coincidence obviously reduces complexity since there is one point in time for production on the fifth level that is duplicated.

We now turn to analysing the complexity of this system. If it had been a five-level serial system with the same external demand and no degeneracy, the number of feasible production plans would have been $(n+1)^{M_1 - 1} \prod_{i=1}^{n-1} (i+1)^{M_{n-i+1}} = 2^0 3^0 4^0 5^1 6^1 = 30$ (using (15)), and had it been a full system this number would have been $\prod_{i=1}^{n} a_i^{M_{n+1-i} + n - i - 1} = a_1^3 a_2^2 a_3^1 a_4^1 a_5^1 = 2^3 \cdot 3^2 \cdot 7 \cdot 43 \cdot 1807 = 39{,}161{,}304$ (from (29) using Table 3).

We first compute the range **R** of the demand events on different levels:

$$\mathbf{R} = \left(\mathbf{I} - \mathbf{H}'\right)^{-1} \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \ \mathbf{R}' = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

However, the degeneracy mentioned above reduces the possible number of demand events from five to four on level 5, leaving us with the vector $\mathbf{R}'$ instead of $\mathbf{R}$.

Using $\mathbf{R}'$ and disregarding the dependencies between different levels, we would have altogether $2^{2-1}\cdot 2^{3-1}\cdot 2^{2-1}\cdot 2^{3-1}\cdot 2^{4-1}=512$ plans. However, several of these are impossible because of contradictions between internal demand and production. For instance if there is one production event for the first-level item, we can never have three internal demand events for the second-level item (which is included as a possibility in the number 512), etc.

As for the full system we consider internal demand as a vector formed by the product $\mathbf{H}'\tilde{\boldsymbol{\tau}}(s)\tilde{\mathbf{P}}(s)$. Assume that $\tilde{\mathbf{P}}(s)$ contains (k_1+1) production events on level 1, (k_2+1) events on level 2, and so on, so the number of internal demand events as a whole is given by:

$$\mathbf{H}'\begin{bmatrix} k_1+1 \\ k_2+1 \\ k_3+1 \\ k_4+1 \\ k_5+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1+1 \\ k_2+1 \\ k_3+1 \\ k_4+1 \\ k_5+1 \end{bmatrix} = \begin{bmatrix} 0 \\ k_1+1 \\ k_1+1 \\ k_2+1 \\ k_2+k_2+2 \end{bmatrix}. \tag{30}$$

Disregarding the degeneracy, we can now compute the transform of the distribution of feasible plans as

$$\tilde{N}_5(z)=\sum_{k_5=0}^{R_5-1} z^{k_5} N_5(k_5)=$$

$$=\sum_{k_5=0}^{R_5-1} z^{k_5}\sum_{k_4=0}^{R_4-1}\sum_{k_3=0}^{R_3-1}\sum_{k_2=0}^{R_2-1}\sum_{k_1=0}^{R_1-1}\binom{M_5+k_3+k_2+2-1}{k_5}\binom{M_4+k_2}{k_4}\binom{M_3+k_1}{k_3}\binom{M_2+k_1}{k_2}\binom{M_1+k_0}{k_1}=$$

$$=2^{M_4}(1+z)^{M_5+1}(2+z)^{M_3}\left(2(1+z)+1\right)^{M_2}\left((2+z)\left(2(1+z)+1\right)+1\right)^{M_1+k_0}, \tag{31}$$

and the total number of plans as

$$\sum_{k_5=0}^{R_5-1} N_5(k_5)=\tilde{N}_5(1)=2^{M_4}2^{M_5+1}3^{M_3}5^{M_2}16^{M_1-1}, \tag{32}$$

by using the same methodology as previously. With $M_1=2$, $M_2=1$ and other $M_i=0$ in this example, the total number of feasible plans is $\tilde{N}_5(1)=2^0 2^1 3^0 5^1 16^1=160$.

These calculations have avoided taking the degeneracy into account. In this example, degeneracy only occurs on the fifth level, due to internal demand simultaneously coming from two different sources at the same time (time 7 for a subset of fifth-level plans). This type of duplication occurs altogether in 50 cases, making the actual total of feasible plans 110. The majority of these cases (32) occur when there is a maximum number of demand events reaching level 5 from levels 2 and 3 (3 and 2, respectively).

We are now in a position to generalise the expression for the number of feasible plans for an arbitrary assembly system with modified input matrix $\mathbf{H}'$, if we still disregard degeneracy.

Defining the ranges by

$$\mathbf{R} = \left(\mathbf{I} - \mathbf{H}'\right)^{-1} \mathbf{M}, \tag{33}$$

and introducing a vector **J** defined by

$$\mathbf{J} = \mathbf{H}' \begin{bmatrix} k_1 + 1 \\ k_2 + 1 \\ \vdots \\ k_n + 1 \end{bmatrix} + \mathbf{M} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{34}$$

the transform of the distribution of the plans becomes

$$\begin{aligned} \tilde{N}_n(z) &= \sum_{k_n=0}^{R_n-1} z^{k_n} N_n(k_n) = \\ &= \sum_{k_n=0}^{R_n-1} z^{k_n} \sum_{k_{n-1}=0}^{R_{n-1}-1} \cdots \sum_{k_2=0}^{R_2-1} \sum_{k_1=0}^{R_1-1} \prod_{i=1}^{n} \binom{J_i}{k_i} = \sum_{k_1=0}^{R_1-1} \sum_{k_2=0}^{R_2-1} \cdots \sum_{k_n=0}^{R_n-1} z^{k_n} \prod_{i=1}^{n} \binom{J_i}{k_i} = \\ &= \sum_{k_1=0}^{R_1-1} \sum_{k_2=0}^{R_2-1} \cdots \sum_{k_{n-1}=0}^{R_n-1} (1+z)^{J_n} \prod_{i=1}^{n-1} \binom{J_i}{k_i}, \end{aligned} \tag{35}$$

and the number of plans will be:

$$\tilde{N}_n(1) = \sum_{k_n=0}^{R_n-1} N_n(k_n) = \sum_{k_1=0}^{R_1-1} \sum_{k_2=0}^{R_2-1} \cdots \sum_{k_{n-1}=0}^{R_{n-1}-1} 2^{J_n} \prod_{i=1}^{n-1} \binom{J_i}{k_i}. \tag{36}$$

The coefficients analogous to $a_i(z)$ in (26) are easily computed, but their expressions will depend heavily on what the structure of the system is, i. e. on $\mathbf{H}'$, that is which of the k_i that are included in each J_i variable.

5. Conclusions

In the foregoing we have examined the complexity of two extreme system structures for assembly systems. On the one hand the simplest system, which is the serial system, on the other, the full system, in which every item needs all kinds of items on lower product structure levels, have been investigated. Throughout this analysis, it has been assumed the condition of no degeneracy due to coincidental equalities between differences in demand event timing and cumulative lead times. Production plans have been enumerated by a binary variable approach according to earlier developments, by which feasible plans are those in which setups (production completions) are located at one or several of the points in time generated by the Lot-for-Lot solution to the problem.

We also analysed a specific five-level example from [Grubbström et al. 2010] falling in between these two extreme types of system. In this example degeneracy appeared, on occasion duplicating the timing of requirements for the fifth-level item.

Expressions have been developed for the total number of feasible plans that exist in these two extreme types of system, Eqs (18), (29), showing explicitly the dependence of this solution space on the number of levels in the system and on the number of external demand events on different levels requiring items to be delivered externally. In all cases, the number of feasible plans increases dramatically in the values of these variables. Also in the intermediate-case example of Section 4, the total number of feasible plans was determined analytically assuming no degeneracy, and it was shown that, in this example, the degeneracy accounted for a reduction in complexity of around 30 per cent (from 160 to 110).

There are, of course, many additional details to be examined. In the future, the expressions might be compared with more simple approximate expressions providing a better overview of the spaces of solutions that are to be dealt with. Also the methodology could be developed to cover more explicit expressions for the in-between types of systems in general, i. e. developments from expressions of the type (36) for assembly systems having an arbitrary structure. It would also be of interest to find more general results concerning when degeneracy occurs, and to quantify the effect that this has on reducing complexity in more general cases.

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