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Linköping University Post Print

N.B.: When citing this work, cite the original article.

Original Publication:
http://dx.doi.org/10.1137/120868074
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http://www.siam.org/

Postprint available at: Linköping University Electronic Press
http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-87969
ON THE LINEAR WATER WAVE PROBLEM IN THE PRESENCE OF A CRITICALLY SUBMERGED BODY

I. V. KAMOTSKI† AND V. G. MAZ'YA‡

Abstract. We study the two-dimensional problem of propagation of linear water waves in deep water in the presence of a critically submerged body (i.e., the body touching the water surface). Assuming uniqueness of the solution in the energy space, we prove the existence of a solution which satisfies the radiation conditions at infinity as well as at the cusp point where the body touches the water surface. This solution is obtained by the limiting absorption procedure. Next we introduce a relevant scattering matrix and analyze its properties. Under a geometric condition introduced by V. Maz’ya in 1978, we prove an important property of the scattering matrix, which may be interpreted as the absence of total internal reflection. This property also allows us to obtain uniqueness and existence of a solution in some function spaces (e.g., $H^2_{loc} \cap L^\infty$) without use of the radiation conditions and the limiting absorption principle, provided a spectral parameter in the boundary conditions on the surface of the water is large enough. The fact that the existence and uniqueness result does not rely on either the radiation conditions or the limiting absorption principle is the first result of this type known to us in the theory of linear wave problems in unbounded domains.

Key words. water waves, limiting absorption principle, radiation conditions, uniqueness, domains with cusps

AMS subject classifications. 35J05, 35C20, 35J25, 35P25, 76B15

DOI. 10.1137/120868074

1. Introduction. We study the problem of propagation of linear water waves in a domain $\Omega$, which represents water of infinite depth in the presence of a critically submerged body $\tilde{\Omega}$. Let us describe the domain $\Omega$. We fix a Cartesian system $x = (x_1, x_2)$ with the origin $O$ and consider a bounded domain $\tilde{\Omega} \subset \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ (notice that the axis $x_2$ points downward). We assume that $S := \partial \tilde{\Omega}$ is smooth and touches the water surface $\Gamma := \{x_2 = 0\}$ only at the origin $O$. Further we define $\Omega := \mathbb{R}^2_+ \setminus \overline{\tilde{\Omega}}$ and set $\Omega_\tau := \Omega \cap \{|x_1| < \tau, \ x_2 < \tau\}$, where $\tau$ is a small positive number. We assume that $\Omega_\tau$ coincides with the set

\begin{equation}
\{x : |x_1| < \tau, \ 0 < x_2 < \phi(x_1)\},
\end{equation}

where $\phi$ is a function from $C^2[-\tau, \tau]$, such that

\begin{equation}
\phi(0) = \phi'(0) = 0
\end{equation}

and

\begin{equation}
\kappa := \phi''(0) > 0.
\end{equation}

Moreover, let $\phi$ be strongly decreasing on $(-\tau, 0)$ and strongly increasing on $(0, \tau)$. The governing equations are the following:

\begin{equation}
\Delta u = f \ 	ext{in} \ \Omega,
\end{equation}
\[ (1.5) \quad \partial_n u = g_1 \text{ on } S \setminus O, \]
\[ (1.6) \quad \partial_n u - \nu u = g_2 \text{ on } \Gamma \setminus O, \]
where \( n \) is the external normal to \( \Omega \), \( \nu > 0 \) is a fixed spectral parameter, and \( f, g_1, g_2 \) are given functions.

The linear water wave problems for fully submerged and semisubmerged bodies in deep water has been studied extensively; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The presence of a critically submerged body implies that the domain \( \Omega \) contains two external cusps. The problems in domains with cusps were studied from various points of view in [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44] (see also [45], where more references can be found).

Our main condition on \( \Omega \) is as follows.

**Condition 1.** The homogeneous problem (1.4)-(1.6) does not have nontrivial solutions in the energy space \( V(\Omega) = \{ u : \int_\Omega |\nabla u|^2 \, dx + \int_{\partial \Omega} |u|^2 \, ds < +\infty \} \).

This condition may hold for many fully submerged bodies. For example, it is well known (see [3, 10]) that the following geometric condition implies the uniqueness for fully submerged bodies.

**Condition 2.** Let \( n(x) = (n_1(x), n_2(x)) \) be the unit normal to \( S \), external to \( \Omega \). Then we have
\[ (1.7) \quad x_1(x_1^2 - x_2^2)n_1(x) + 2x_1^2x_2n_2(x) \geq 0, \quad x \in S. \]

One of the results of this paper is that Condition 2 still implies uniqueness for the case of critically submerged bodies (see also [46], where similar problem without cusps had been considered); in fact we can say more—see Theorems 4.4, 4.6, and 4.7.

We are interested in the existence of solutions which satisfy an outgoing radiation condition at infinity (see (2.5) below for the precise definition):
\[ (1.8) \quad u \sim d^+ e^{-i\nu x_1 - \nu x_2} \quad \text{as} \quad x_1 \to +\infty \quad \text{and} \quad u \sim d^- e^{i\nu x_1 - \nu x_2} \quad \text{as} \quad x_1 \to -\infty, \]
where \( d^+ \) and \( d^- \) are some constants.

If \( \Omega \) is completely submerged (i.e., there is no cusp, \( \overline{\Omega} \in \mathbb{R}^2 \)), the existence of a solution to (1.4)-(1.6) satisfying radiation conditions at infinity follows immediately under Condition 1; see [10] and the references therein. Our situation is more subtle, for two reasons: the first reason is purely technical, namely, that we cannot directly apply the method of [10], which was based on integral equations, due to the presence of the cusps.

The other reason is that, depending on the parameter \( \nu \), the solutions might not be in \( H^1_{\text{loc}}(\Omega) \). The situation is in fact even more complicated: there may be many “reasonable” solutions, and so we need to select only one. The latter implies that we need to additionally employ new conditions at the cusp. We will refer to them as radiation conditions at the cusp since they play similar role to radiation conditions (1.8). To be more precise, we prove, under suitable conditions on \( f, g_1, g_2 \) and assuming Condition 1, that there is a unique solution to (1.4)-(1.6) satisfying radiation conditions at infinity and such that, provided \( \nu > \kappa/8 \),
\[ (1.9) \quad u \sim c_1 x_1^{-1/2+i\sqrt{\frac{\nu}{8} - \frac{\kappa}{8}}}, \quad x_1 \to +0, \quad u \sim c_2 |x_1|^{-1/2+i\sqrt{\frac{\nu}{8} - \frac{\kappa}{8}}}, \quad x_1 \to -0. \]
In the case \( \nu < \kappa/8 \) we have
\[
(1.10) \quad u \sim c_1 x_1^{-1/2+\sqrt{\frac{\nu}{\kappa}}}, \quad x_1 \rightarrow +0, \quad u \sim c_2 |x_1|^{-1/2+\sqrt{\frac{\nu}{\kappa}}}, \quad x_1 \rightarrow -0.
\]
In the above formulae \( c_1 \) and \( c_2 \) are some constants. For the case \( \nu = \kappa/8 \) we have the same expressions as in (1.9), but Condition 1 needs to be modified; see Condition 1' in section 3.

Let us mention that radiation conditions for the water wave problems in finite geometry have been studied in [18] and [42].

The presence of radiation conditions both at infinity and at the origin presents new challenges. In particular, we need to employ to this end a nonstandard version of the **limiting absorption principle**; cf., e.g., [47].

Asymptotic representations (1.9) and (1.10) show, in particular, that if \( \nu < \kappa/8 \), then the solution is in the space \( H^1_{loc}(\Omega) \). In the case \( \nu \geq \kappa/8 \) the solution does not belong to \( H^1_{loc}(\Omega) \), in general. Moreover, in the latter case there are many solutions with a similar type of behavior; we show, however, that the condition (1.9) fixes the unique one.

Expressions (1.8) and (1.9) can be interpreted as “outgoing waves” and their complex conjugates as “incoming waves.” This introduces, for \( \nu > \kappa/8 \), a \( 4 \times 4 \) scattering matrix describing the relation between the incoming and outgoing solutions at both infinities but also at two cusps.

We study properties of this scattering matrix and show that, apart from the standard ones of unitarity and symmetry, it has more subtle “block properties”; see Theorem 4.3. The latter ensures in particular that any combination of waves coming in from the infinities will at least partially “scatter in the cusps” (and vice versa). This may be interpreted as the absence of an analogue of total internal reflection (i.e., of “infinity to infinity” or of “cusp to cusp” scattering).

The crucial ingredient for establishing the above properties of the scattering matrix is the uniqueness theorem, Theorem 4.4, roughly in the class of arbitrary combinations of the cusp incoming and outgoing waves (in fact in the class of functions which can grow as \( |x_1|^{-M} \) for any finite \( M \), as \( |x| \rightarrow 0 \)). We prove this by showing that, under Condition 2, the method of multipliers (see, e.g., [3, 10]) surprisingly works also in the presence of functions singular at the cusp.

Moreover, the properties of the scattering matrix allow us to establish the uniqueness and existence results for problem (1.4)-(1.6) in various functional spaces. We prove that if \( \nu > \kappa/8 \), and \( f, g \) are regular enough and have a compact support, then there exists a unique solution of problem (1.4)-(1.6) in the space \( H^2_{loc}(\Omega) \cap L^\infty(\Omega) \). Under the same conditions we also establish the existence and uniqueness results in the space \( H^p_{loc}(\Omega \setminus O) \cap L^p(\Omega), \quad p \in (2, 6) \) (the result is not true for other values of \( p \); see (1.8) and (1.9)). The former may be interpreted as a solution with no waves either incoming or outgoing to the cusps (hence bounded), and the latter with no similar waves either from or to infinity (hence a localized solution in some sense). In particular these spaces of functions do not differentiate between the incoming and the outgoing waves and the radiation conditions are no longer employed.

The paper is organized as follows: In section 2 we consider the problem without a submerged body and derive some useful estimates which are employed in section 3. There we prove the existence of the solution of (1.4)-(1.6) in the space of functions with the radiation conditions, using the limiting absorption principle. In the last section we introduce the scattering matrix for problem (1.4)-(1.6), prove some of its properties, and establish the uniqueness and existence results for (1.4)-(1.6) in various spaces of functions without radiation conditions.
2. Problem in $\mathbb{R}^2_+$. Here we consider an auxiliary problem in the entire half-space:

\begin{align}
\Delta u &= f \text{ in } \mathbb{R}^2_+, \\
\partial_n u - \nu u &= 0 \text{ on } \Gamma.
\end{align}

(Here $f$ is not necessarily the same as before; the precise assumption on $f$ are given below.) We are interested in solutions which satisfy the following *radiation condition at infinity*: $u$ can be represented as a sum of two outgoing waves and of a function decaying at infinity. To make this more precise we define the *outgoing waves at infinity*, e.g.,

\begin{align}
(2.3)\quad u^-(x) &= \chi(x_1)e^{-\nu x_1 - \nu x_2}, \quad u^-(x) = \chi(-x_1)e^{i\nu x_1 - \nu x_2},
\end{align}

where $\chi$ is the cut-off function, such that

\begin{align}
(2.4)\quad \chi \in C^\infty(\mathbb{R}), \quad \chi(t) = 0 \text{ for } t < N, \quad \chi(t) = 1 \text{ for } t > 2N,
\end{align}

and $N$ is a fixed positive number. (Physically, function $u^-$ represents an outgoing wave moving to the right, respectively, the outgoing wave $u^+$ moving to the left.) Then we say that $u$ satisfies *radiation condition at infinity* (see, e.g., [10]) if

\begin{align}
(2.5)\quad u = c_1u^+ + c_2u^- + \tilde{u}, \quad \text{in } \Omega \setminus B_N, \quad |\tilde{u}| + |x||\nabla \tilde{u}| = O(|x|^{-1}) \text{ as } |x| \to \infty,
\end{align}

where $c_1$ and $c_2$ are some constants and $B_N = \{x : ||x|| < N\}$.

The existence of a solution which satisfies radiation conditions (under certain assumptions on $f$) is well known; see, e.g., [10]. Below we discuss the relation of this solution to the limiting absorption principle and derive some useful estimates which we will apply in the next section.

Consider now the problem with a small absorption described by $\varepsilon > 0$:

\begin{align}
(2.6)\quad \Delta u_\varepsilon - i\varepsilon u_\varepsilon &= f \text{ in } \mathbb{R}^2_+, \\
(2.7)\quad \partial_n u_\varepsilon - \nu u_\varepsilon &= g_2 \text{ on } \Gamma.
\end{align}

In order to describe precisely a solution of (2.6), (2.7) we introduce the following spaces. Denote $\langle x_j \rangle = (1 + x_j^2)^{1/2}$, $j = 1, 2$, and for real $\beta, \gamma$ and $l = 0, 1, \ldots$, for relevant domain $\Theta$, let

\begin{align}
(2.8)\quad W^l_{\beta, \gamma}(\Theta) &= \left\{ u : \sum_{|\delta| \leq l} \int_\Theta e^{2\beta \langle x_1 \rangle} \langle x_1 \rangle^{2\gamma} \langle x_2 \rangle^2 |\nabla^\delta u|^2 dx < \infty \right\}, \\
(2.9)\quad \dot{H}^l(\Theta) &= \left\{ u : \sum_{1 < |\delta| \leq l} \int_\Theta |\nabla^\delta u|^2 dx + \int_\Theta \langle x_2 \rangle^{-2} |u|^2 dx < \infty \right\}, \quad l \geq 1,
\end{align}

with the corresponding definitions of the norm and of the trace spaces. For the case $\Theta = \mathbb{R}^2_+$ we omit the dependence on the domain in the notation.

Application of the Fourier transform with respect to $x_1$ and shift of the contour of integration (see, e.g., [20, 48]) yields the following result.
LEMMA 2.1. Let \( \beta > 0 \) and let \( \varepsilon \) be such that \( \beta > -\operatorname{Im}(\nu^2 - i\varepsilon)^{1/2} \) and \( \varepsilon^{1/2} > -\operatorname{Im}(\nu^2 - i\varepsilon)^{1/2} \). Suppose further that \( f \in W_{2,\gamma}^0 \) and \( g_2 \in W_{1/2}(\Gamma) \), and \( \gamma \in \mathbb{R} \). Then there exists a unique solution \( u_\varepsilon \in W_{2,\gamma}^2 \) of problem (2.6)–(2.7), and the following representation holds:

\[
\tag{2.10} u_\varepsilon = b_1^\varepsilon U_1^\varepsilon + b_2^\varepsilon U_2^\varepsilon + \hat{u}_\varepsilon, \quad \text{where} \quad \hat{u}_\varepsilon \in W_{2,\gamma}^2, \quad \beta^* < \min \left\{ \beta, \frac{\varepsilon^{1/2}}{\sqrt{2}} \right\}.
\]

Here

\[
\tag{2.11} U_1^\varepsilon(x) = \chi(x_1) e^{i\nu^2 x_1 - \nu x_2},
\]

\[
\tag{2.12} U_2^\varepsilon(x) = \chi(-x_1) e^{i\nu^2 x_1 + \nu x_2},
\]

\( b_1^\varepsilon, b_2^\varepsilon \) are constants, and

\[
\tag{2.13} |b_1^\varepsilon| + |b_2^\varepsilon| + \|\hat{u}_\varepsilon\|_{W_{2,\gamma}^2} \leq c(\|f\|_{W_{2,\gamma}^0} + \|g_2\|_{W_{1/2}^0}).
\]

Remark 2.1. Clearly we have \( U_1^0 = u_j^- \), \( j = 1, 2 \).

The constant \( c \) appearing in (2.13) depends on \( \varepsilon \). The estimate which appears in the next lemma overcomes this disadvantage.

THEOREM 2.2. Suppose that the conditions of Lemma 2.1 hold, and additionally let us assume that \( \gamma = 1 \) and \( g_2 = 0 \). Then the following estimate holds:

\[
\tag{2.14} |b_1^\varepsilon| + |b_2^\varepsilon| + \|\hat{u}_\varepsilon\|_{H^2} \leq c\|f\|_{W_{2,1}^0},
\]

where \( c \) does not depend on \( \varepsilon \).

(Henceforth \( c \) is a constant whose value may change from line to line.)

Theorem 2.2 is proved in the appendix.

The above statement allows us to pass to the limit in (2.6), (2.7), and we have \( b_1^\varepsilon \to b_1 \), \( b_2^\varepsilon \to b_2 \), and \( \hat{u}_\varepsilon \) converges to \( \hat{u} \) weakly in the space \( H^2 \) as \( \varepsilon \to 0 \). As a result we obtain a solution \( u \) to problem (2.1), (2.2), which can be represented in the form \( u = b_1 u_1 + b_2 u_2 + \hat{u} \).

3. Critically submerged body. Consider now the original problem (1.4)–(1.6) with critically submerged body \( \tilde{\Omega} \). Let us associate with this problem an “energy space” \( V = \{ u : \int_\Omega |\nabla u|^2 \, dx + \int_{\partial \Omega} |u|^2 \, dS < \infty \} \). Let us notice that

\[
\tag{3.1} \int_\Omega (x_2^2 + 1)^{-1}|u|^2 \lesssim \int_\Omega |\nabla u|^2 + \int_{\partial \Omega} |u|^2 \, dS.
\]

(From now on, \( \lesssim \) denotes \( \leq \) with a constant \( c \).) This inequality follows from two obvious inequalities,

\[
\tag{3.2} \int_{\Omega \setminus \{|x_1| > N\}} \frac{|u|^2}{x_2^2 + 1} \, dx \lesssim \int_{\Omega \setminus \{|x_1| > N\}} \frac{\partial u}{\partial x_2}^2 \, dx + \int_{\{x_2 = 0, |x_1| > N\}} |u|^2 \, dx_1,
\]

\[
\tag{3.3} \int_{\Omega \setminus B_N \cap \{|x_1| < N\}} \frac{|u|^2}{x_2^2 + x_1^2} \, dx \lesssim \int_{\Omega \setminus B_N} \frac{|u|^2}{x_2^2 + x_1^2} \, dx \lesssim \int_{\Omega \setminus B_N} \frac{1}{r} \frac{|\partial u}{\partial \theta}|^2 \, d\theta \, dr + \int_{\{x_2 = 0, |x_1| > N\}} |u|^2 \, dx_1,
\]
and the Friedrichs inequality
\begin{equation}
\int_{\Omega \cap B_N} |u|^2 \, dx \leq \int_{\Omega \cap B_N} |\nabla u|^2 \, dx + \int_{\partial (\Omega \cap B_N)} |u|^2 \, dS,
\end{equation}
which is valid for any bounded domain; see [49, section 6.11.1]. Here we assume that constant \( N \) from the previous section is such that \( \Omega \subset B_N := \{ x : \|x\| < N \} \).

We are planning to find a solution to problem (1.4)–(1.6) by employing the principle of limiting absorption. In fact we need an absorption in (1.4) and in the boundary conditions (1.5), (1.6), locally in the neighborhood of the origin.

Let us fix a cut-off function \( \mu \in C_c^\infty (\Gamma) \), such that \( \mu(x_1) = 1 \), \( |x_1| < N \), and \( \mu(x_1) = 0 \), \( |x_1| > 2N \). Consider now the following problem with a small absorption \( \varepsilon \geq 0 \),
\begin{align}
\Delta v_\varepsilon - i \varepsilon v_\varepsilon &= f \quad \text{in } \Omega, \\
\partial_n v_\varepsilon + i \varepsilon v_\varepsilon &= g_1 \quad \text{on } S, \\
\partial_n v_\varepsilon - (\nu - i \varepsilon \mu) v_\varepsilon &= g_2 \quad \text{on } \Gamma,
\end{align}
and the corresponding energy space:
\[ V := \left\{ u : \int_{\Omega} |\nabla u|^2 \, dx + \int_{\partial \Omega} |u|^2 \, dS + \int_{\Omega} |u|^2 \, dx < \infty \right\}. \]

**Lemma 3.1.** Let \( g_1 \in L_2(S) \), \( g_2 \in L_2(\Gamma) \), and \( f \in L_2(\Omega) \). Then for any \( \varepsilon > 0 \) there exists a unique solution \( v_\varepsilon \in V \) of problem (3.5)–(3.7).

**Proof.** Let us associate with (3.5)–(3.7) a variational problem: Find \( v_\varepsilon \) such that
\begin{equation}
a_\varepsilon(v_\varepsilon, \varphi) := F(\varphi) \quad \forall \varphi \in V.
\end{equation}
Here
\begin{align}
a_\varepsilon(v, \varphi) &:= \int_{\Omega} \nabla v \cdot \nabla \varphi \, dx - \int_{\Omega} (\nu - i \varepsilon \mu(x_1)) \varphi v \, dx_1 \\
&\quad + i \varepsilon \int_{\Gamma} \varphi dS + i \varepsilon \int_{\Omega} \varphi v_\varepsilon \, dx
\end{align}
and
\begin{equation}
F(\varphi) := -\int_{\Omega} \varphi \, dx + \int_{\Gamma} g_2 \varphi \, dx_1 + \int_{\Gamma} g_1 \varphi \, dS.
\end{equation}
The sesquilinear form \( a_\varepsilon(\cdot, \cdot) \) is clearly continuous and coercive on \( V \), and \( F \) is an antilinear continuous functional on \( V \), and the application of the Lax–Milgram lemma (see, e.g., [50]) gives us a unique solution \( v_\varepsilon \) from energy space \( V \). Due to ellipticity, the local estimates give us \( v_\varepsilon \in H^2(\Omega \setminus B_\sigma) \) for any positive \( \sigma \). \( \square \)

We aim to pass to the limit in (3.8) as \( \varepsilon \to 0 \). The main difficulty is the absence of compactness of embedding of \( H^1(\Omega) \) into \( L_2(\partial \Omega) \) and \( L_2(\Omega) \) due to the fact that \( \Omega \) is unbounded, and lack of compactness of embedding of \( H^1(\Omega) \) into \( L_2(\partial \Omega) \) in the neighborhood of the origin due to the presence of the external quadratic cusps.

To overcome this problem, we need to employ more detailed information about properties of the solutions.
The following theorem was proved in [42].

We start from the description of \( v_\varepsilon \) in right and left neighborhood of the origin:

\[
\Omega^+_\varepsilon := \Omega \cap \{0 < x_1\}, \quad \Omega^-_\varepsilon := \Omega \cap \{x_1 < 0\}, \quad \Gamma^\pm := \partial \Omega^\pm \cap \partial \Omega.
\]

In order to describe precisely a solution of (3.8), we need to introduce the following weighted Sobolev spaces: Let \( \Xi \) be a domain and let \( \gamma \) be real, \( l = 0, 1, \ldots \); then we define \( W^0_\gamma(\Xi) \) and \( V^1_\gamma(\Xi) \) as the closures of the set \( C^\infty(\Xi \setminus O) \) with respect to the norms

\[
\|u\|_{W^0_\gamma(\Xi)} := \sum_{|\delta| \leq l} \int_{\Xi} |x_1|^{4(\gamma - l + |\delta|)} |\partial_\delta^\varepsilon u|^2 dx,
\]

\[
\|u\|_{V^1_\gamma(\Xi)} := \sum_{|\delta| \leq l} \int_{\Xi} |x_1|^{2(\gamma - l + |\delta|)} |\partial_\delta^\varepsilon u|^2 dx,
\]

respectively, where \( \delta \in \mathbb{Z}^2_+ \) is the usual multi-index. Furthermore, for \( l \geq 1 \) we define \( W^{l-1/2}_\gamma(\partial \Xi) \) and \( V^{l-1/2}_\gamma(\partial \Xi) \) as the trace space for \( W^l_\gamma(\Xi) \) and \( V^l_\gamma(\Xi) \) on the boundary \( \partial \Xi \).

Finally we define the space

\[
V^2_\gamma(\Omega^+_\varepsilon) = \{ u \in V^2_\gamma(\Omega^+_\varepsilon) : P_2 u \in W^2_\gamma(\Omega^+_\varepsilon) \},
\]

with the norm

\[
\|u\|_{V^2_\gamma(\Omega^+_\varepsilon)} = \|u\|_{V^2_\gamma(\Omega^+_\varepsilon)} + \|P_2 u\|_{W^2_\gamma(\Omega^+_\varepsilon)}.
\]

Here the projection operator \( P_2 \) is defined as follows. We represent \( u \in V^2_\gamma(\Omega^+_\varepsilon) \) as

\[
u(x_1, x_2) = u_1(x_1) + u_2(x_1, x_2), \quad 0 < x_1 < \tau, \quad 0 < x_2 < \phi(x_1),
\]

where

\[
u_1(x_1) = \phi(x_1)^{-1} \int_0^{\phi(x_1)} u(x_1, x_2) dx_2,
\]

and we define

\[
P_1 u := u_1, \quad P_2 u := u - u_1 := u_2.
\]

We also define the fully analogous space \( V^2_\gamma(\Omega^-_\varepsilon) \).

One of characteristic properties of the above scale of spaces is the following. If \( u \in V^2_\gamma(\Omega^+_\varepsilon ) \), then for

\[
L u := (\Delta - i\varepsilon) u
\]

and

\[
B u := \left\{ (\partial_n + i\varepsilon) u |_{\Gamma^+_\varepsilon}, (\partial_n - i\varepsilon) u |_{\Gamma^-_\varepsilon} \right\}
\]

we have \( \{L u, Bu\} \in W^{0}_{\gamma}(\Omega^+_\varepsilon) \times W^{1/2}_{\gamma}(\Gamma^+_\varepsilon) \). Denote

\[
\lambda_\varepsilon := 2(\nu - 2i\varepsilon)/\kappa - 1/4.
\]

The following theorem was proved in [42].
Theorem 3.2 (see [42, Theorem 4.3]). Let \( \gamma \neq \frac{1}{2} \pm \frac{1}{2}|\text{Im}\sqrt{\lambda_e}| \). Suppose that \( u_\varepsilon \in V^2_\gamma(\Omega^+_{\delta}) \) is a solution of the problem

\begin{align}
(3.19) & \quad \Delta u_\varepsilon - i \varepsilon u_\varepsilon = f \text{ in } \Omega^+_\delta, \\
(3.20) & \quad \partial_n u_\varepsilon + i \varepsilon u_\varepsilon = g_1 \text{ on } S \cap \gamma^+_\delta, \\
(3.21) & \quad \partial_n u_\varepsilon - (\nu - i \varepsilon) u_\varepsilon = g_2 \text{ on } \Gamma \cap \gamma^+_\delta,
\end{align}

where \( g := (g_1, g_2) \) and \( (f, g) \in W^0_\gamma(\Omega^+_\delta) \times W^{1/2}_\gamma(\gamma^+_\delta) \). Then for any \( \varepsilon \), there exists \( \delta_0 > 0 \) such that, for any \( 0 < \delta < \delta_0 \), solution \( u_\varepsilon \) satisfies the estimate

\begin{equation}
(3.22) \quad \| u \|^2_{V^2_\gamma(\Omega^+_\delta)} \leq c \left( \| f \|^2_{W^0_\gamma(\Omega^+_\delta)} + \| g \|^2_{W^{1/2}_\gamma(\gamma^+_\delta)} + \| u \|^2_{L^2(\Omega^+_\delta \setminus \Omega^+_\delta/2)} \right).
\end{equation}

Here constant \( c \) is independent of \( f, g, \) and \( u_\varepsilon \).

The next theorem from [42] describes the structure of the solution.

Theorem 3.3. Let \( \gamma, \gamma_1 \) be real numbers, and let \( \gamma, \gamma_1 \neq \frac{1}{2} \pm \frac{1}{2}|\text{Im}\sqrt{\lambda_e}| \) and \( (f, g) \in W^0_{\gamma_1}(\Omega^+_\delta) \times W^{1/2}_{\gamma_1}(\gamma^+_\delta) \). Suppose that \( v_\varepsilon \in V^2_{\gamma_1}(\Omega) \) is a solution of the problem (3.19)–(3.21). Then the solution \( v_\varepsilon \) admits the representation

\begin{equation}
(3.23) \quad v_\varepsilon = c^+ Y^+_e + c^- Y^-_e + \tilde{Y}_e \text{ in } \Omega^+_\delta
\end{equation}

for sufficiently small positive \( \delta \). Here \( \tilde{Y}_e \in V^2_{\gamma_1}(\Omega) \), \( Y^\pm_e \) are solutions of the homogeneous problem (3.19)–(3.21), and \( c^\pm \) are constants.

Properties of the special solutions \( Y^\pm_e \) had been described in [42] (see Theorem 4.4), which can be reformulated in our context as follows.

Theorem 3.4. Suppose that in (3.18) \( \lambda_e \neq 0 \). Then there exist solutions \( Y^+_e \) and \( Y^-_e \) of the homogeneous problem (3.19)–(3.21) in \( \Omega^+_\delta \) for small enough positive \( \delta \), such that

\begin{equation}
(3.24) \quad Y^\pm_e(x) = y^+_1(x_1, \varepsilon) + y^+_2(x_1, \varepsilon), \quad \int_0^{\phi(x_1)} y^+_2(x) dx_2 = 0, \quad x_1 < \delta,
\end{equation}

where

\begin{equation}
(3.25) \quad y^+_1(x_1, \varepsilon) = x_1^{-\Lambda^+_e} + \tilde{y}^+_1(x_1, \varepsilon), \quad y^+_i \in V^2_{\gamma^+_e}(\Omega_\delta) \forall \gamma^+_e > \mp \text{Im}\sqrt{\lambda_e}/2,
\end{equation}

\begin{equation}
(3.26) \quad y^+_2(x_1, x_2, \varepsilon) = x_1^{2-\Lambda^+_e} Q^+_e(\varepsilon/\phi(x_1)) + \tilde{y}^+_2(x_1, \varepsilon), \quad \tilde{y}^+_2 \in W^2_{\gamma^+_e}(\Omega_\delta) \forall \gamma^+_e > \mp \text{Im}\sqrt{\lambda_e}/2.
\end{equation}

Here

\begin{equation}
(3.27) \quad Q^+_e(\varepsilon) = \kappa \left( \nu - i \varepsilon \right) \left( \frac{(z-1)^2}{2} - \frac{1}{6} \right) - \frac{\kappa}{2} \left( \kappa \Lambda^+_e + i \varepsilon \right) \left( \frac{z^2}{2} - \frac{1}{6} \right)
\end{equation}

and

\begin{equation}
\Lambda^+_e = 1/2 \pm i \sqrt{\lambda_e}.
\end{equation}

Remark 3.1. Let us note that \( Y^\pm_e \in V^2_{\sigma^\pm_e}(\Omega^+_\delta) \forall \sigma^\pm_e > \mp \text{Im}\sqrt{\lambda_e}/2 + 1/2 \). It will be useful in what follows to use another representation for \( Y^\pm_e \) instead of (3.24), namely,

\begin{equation}
(3.27) \quad Y^\pm_e = v^\pm_e + \tilde{v}^\pm_e,
\end{equation}
where
\[
(3.28) \quad \mathbf{v}_\varepsilon^\pm(x) = |x_1|^{-\Delta_\varepsilon^\pm} + |x_1|^2 - \frac{\Lambda_\varepsilon^\pm}{2} \mathcal{Q}_\varepsilon^\pm(x_2/\phi(x_1))
\]
and
\[
\mathbf{v}_\varepsilon^\pm \in \mathcal{V}_\gamma^2(\Omega_\delta^+ \sigma) \quad \forall \gamma > \mp \text{Im} \sqrt{\lambda_\varepsilon}/2.
\]

Remark 3.2. In the case $\lambda_\varepsilon = 0$, i.e., $\varepsilon = 0$ and $\nu = \kappa/8$ (see (3.18)), functions $Y_\varepsilon^\pm$ do still exist and belong to $\mathcal{V}_\sigma^2(\Omega_\delta^+ \sigma) \forall \sigma > 1/2$. We have the following representation for them:

\[
(3.30) \quad \mathbf{v}_\varepsilon^\pm + \mathbf{w}_\varepsilon^\pm,
\]

where
\[
(3.31) \quad \mathbf{v}_\varepsilon^\pm(x) = |x_1|^{-1/2} + |x_1|^{3/2} \mathcal{Q}_0^\pm(x_2/\phi(x_1))
\]
and
\[
\mathbf{w}_\varepsilon^\pm \in \mathcal{V}_\gamma^2(\Omega_\delta) \forall \gamma > 0.
\]

Here $\mathcal{Q}_0^\pm$ is defined according to (3.26) ($\mathcal{Q}_0^- = \mathcal{Q}_0^+$ in this case) and

\[
(3.32) \quad \mathcal{Q}(z) = -\frac{\kappa^2}{2} \left( \frac{z^2}{2} - \frac{1}{6} \right).
\]

As we have seen, the structure of the solution crucially depends on the relation between $\nu$ and $\kappa/8$ (which determines the real part of $\lambda_\varepsilon$ according to (3.18)). Let us start from the most singular case, $\nu > \kappa/8$.

Let us check the implications of the above theorems for the solution $\mathbf{v}_\varepsilon$ of the boundary value problem (3.5)–(3.7), under assumption that the pair

\[
(f, g) \in \mathcal{W}_0^0(\Omega^+_\tau) \times \mathcal{W}_0^{1/2}(\Omega^+_\tau)
\]
and has compact support. It follows from Theorems 3.3 and 3.4 and Remark 3.1 that

\[
\mathbf{v}_\varepsilon = d_\varepsilon \mathbf{v}_\varepsilon^+ + c_\varepsilon \mathbf{v}_\varepsilon^- + w_\varepsilon \quad \text{in } \Omega^+_\tau,
\]
where $w_\varepsilon \in \mathcal{V}_0^2(\Omega^+_\tau)$ and $c_\varepsilon$, $d_\varepsilon$ are some constants. Moreover, Theorem 3.4 provides information about functions $\mathbf{v}_\varepsilon^+$ and $\mathbf{v}_\varepsilon^-$, i.e.,

\[
\mathbf{v}_\varepsilon^+(x) = O \left( x_1^{-1/2-i(\lambda_0-4i\varepsilon\kappa)^{-1/2}} \right), \quad \mathbf{v}_\varepsilon^-(x) = O \left( x_1^{-1/2+i(\lambda_0-4i\varepsilon\kappa)^{-1/2}} \right) \quad \text{as } x_1 \to 0,
\]
and consequently $d_\varepsilon = 0$ since $\mathbf{v}_\varepsilon \in H^1(\Omega)$.

It is easy to see that $\mathbf{w}_\varepsilon \in \mathcal{V}_0^2(\Omega^+_\tau)$ solves the problem

\[
(3.33) \quad \Delta \mathbf{w}_\varepsilon = f - (\Delta - i\varepsilon)c_\varepsilon \mathbf{v}_\varepsilon^- + i\varepsilon \mathbf{w}_\varepsilon \quad \text{in } \Omega^+_\tau,
\]
\begin{equation}
\partial_n w^\varepsilon = g_1 - (\partial_n + i\varepsilon) c_\varepsilon v^\varepsilon_\varepsilon - i\varepsilon w^\varepsilon \text{ on } S \cap \Gamma^+,
\end{equation}

and for sufficiently small \( \varepsilon > 0 \), we obtain

\begin{equation}
\begin{aligned}
\| v^\varepsilon \|_{L^2(\Omega^\varepsilon_{\delta/2})} &\leq c \left( \| f \|_{W^1_0(\Omega^\varepsilon_{\delta/2})} + \| g \|_{W^{1/2}(\Gamma^+_{\delta/2})} + \| w^\varepsilon \|_{L^2(\Omega^\varepsilon_{\delta/2} \setminus \Omega^\varepsilon_{\delta/4})} + |c_\varepsilon| + \varepsilon \| v^\varepsilon \|_{W^1_0(\Omega^\varepsilon_{\delta/2})} \right) \\
\| w^\varepsilon \|_{L^2(\Omega^\varepsilon_{\delta/2})} &\leq c \left( \| f \|_{W^1_0(\Omega^\varepsilon_{\delta/2})} + \| g \|_{W^{1/2}(\Gamma^+_{\delta/2})} + \| w^\varepsilon \|_{L^2(\Omega^\varepsilon_{\delta/2} \setminus \Omega^\varepsilon_{\delta/4})} + |c_\varepsilon| \right).
\end{aligned}
\end{equation}

Clearly, for any \( \varepsilon > 0 \) such that \( \gamma \neq 1/2 - 1/2 \text{Im} \sqrt{\lambda} \), we have

\begin{equation}
\| v^\varepsilon \|_{L^2(\Omega^\varepsilon_{\delta/2})} \leq c \left( \| f \|_{W^1_0(\Omega^\varepsilon_{\delta/2})} + \| g \|_{W^{1/2}(\Gamma^+_{\delta/2})} + \| w^\varepsilon \|_{L^2(\Omega^\varepsilon_{\delta/2} \setminus \Omega^\varepsilon_{\delta/4})} + |c_\varepsilon| \right).
\end{equation}

Let us emphasize that the constant \( c \) in the above formula is independent of \( \varepsilon, f, \gamma, g, \) and \( v^\varepsilon \).

Following the same reasoning, for \( v^\varepsilon \) in \( \Omega^- \) we obtain

\begin{equation}
v^\varepsilon(x) = b_\varepsilon v^\varepsilon_\varepsilon(x) + w^\varepsilon(x), \quad x \in \Omega^-,
\end{equation}

where \( b_\varepsilon \) is some constant, \( w^\varepsilon \in V^2_0(\Omega^-) \) for any \( \gamma > 0 \), and

\begin{equation}
\| v^\varepsilon \|_{V^2_0(\Omega^-_{\delta/2})} \leq c \left( \| f \|_{V^1_0(\Omega^-_{\delta/2})} + \| g \|_{V^{1/2}(\Gamma^-_{\delta/2})} + \| w^\varepsilon \|_{L^2(\Omega^-_{\delta/2} \setminus \Omega^-_{\delta/4})} + |b_\varepsilon| \right),
\end{equation}

where the constant \( c \) is independent of \( \varepsilon, f, g, b_\varepsilon, \) and \( w^\varepsilon \).

Let us now consider \( v^\varepsilon \) in \( \Omega \setminus B_{2N} \). Using Theorem 2.2 (we choose \( N \) such that \( \text{supp } g \subset (-N, N) \)), we obtain

\begin{equation}
v^\varepsilon = b^\varepsilon U^\varepsilon_1 + b^\varepsilon U^\varepsilon_2 + \bar{w}^\varepsilon
\end{equation}

and

\begin{equation}
|b^\varepsilon_1| + |b^\varepsilon_2| + \| \bar{w}^\varepsilon \|_{H^2(\Omega \setminus B_{2N})} \leq c \left( \| f \|_{W^1_0(\Omega \setminus B_{2N})} + \| w^\varepsilon \|_{L^2(\Omega \setminus B_{2N})} \right),
\end{equation}

where the constant \( c \) in the above formula is independent of \( \varepsilon, f, g, \) and \( w^\varepsilon \).

In the intermediate region, say \( \Omega \setminus B_{2N} \setminus \Omega_{\delta/2} \), we apply the usual elliptic estimates, yielding

\begin{equation}
\| v^\varepsilon \|_{H^2(\Omega \setminus B_{2N} \setminus \Omega_{\delta/2})} \leq c \left( \| f \|_{L^2(\Omega \setminus B_{2N} \setminus \Omega_{\delta/4})} + \| g \|_{H^{1/2}(\partial \Omega)} + \| v^\varepsilon \|_{L^2(\Omega \setminus B_{2N} \setminus \Omega_{\delta/4})} \right),
\end{equation}

where \( c \) obviously does not depend on \( \varepsilon \).

Now we are going to combine estimates (3.37), (3.39), (3.41), and (3.42). With this purpose we introduce the weighted space

\begin{equation}
\mathcal{H}^2_\gamma(\Omega) := \{ \bar{v} \in H^2_\text{loc}(\Omega \setminus O) : \bar{v} \in V^2(\Omega^-), \bar{v} \in H^2(\Omega \setminus B_{\varepsilon/2}) \},
\end{equation}

\begin{equation}
\| \bar{v} \|^2_{\mathcal{H}^2_\gamma(\Omega)} := \| \bar{v} \|^2_{V^2(\Omega^-)} + \| \bar{v} \|^2_{H^2(\Omega \setminus B_{\varepsilon/2})}.
\end{equation}
and a space with “detached asymptotics”:

\[ (3.47) \quad v \in H^2_{\gamma, \epsilon}(\Omega) \Leftrightarrow v = \sum_{j=1}^{4} c_j U_j^\epsilon + \bar{v}, \quad \bar{v} \in H^2_{\gamma}(\Omega), \quad c_j \in \mathbb{C}, \quad j = 1, \ldots, 4. \]

Here \( U_1^\epsilon \) and \( U_2^\epsilon \) are as introduced in (2.11), and

\[ (3.45) \quad U_1^\epsilon(x) := \zeta_1^+(x)v_1^\epsilon(x), \quad U_2^\epsilon(x) := \zeta_1^-(x)v_1^\epsilon(x), \]

\( \zeta_1^\pm \in C^\infty(\overline{\Omega}\setminus O) \) are cut-off functions, such that \( \zeta_1^\pm = 1 \) in \( \Omega_{\epsilon/2}^\pm \) and \( \zeta_1^\pm = 0 \) in \( \Omega \setminus \Omega_{\epsilon}^- \).

We will refer to \( H^2_{\gamma, \epsilon}(\Omega) \) as space with radiation conditions at the infinity and at the origin.

Finally, we obtain the following lemma.

**Lemma 3.5.** Let \( \{f, g\} \in W^{0}_2(\Omega) \times W^{1/2}_0(\partial\Omega) \) and have compact support. Then for any \( \epsilon > 0 \) the unique solution \( v_\epsilon \in V \) of problem (3.5)–(3.7) ensured by Lemma 3.1 belongs to the space \( H^2_{\gamma, \epsilon}(\Omega) \) for any \( \gamma > 0 \), in particular

\[ (3.46) \quad v_\epsilon = \sum_{j=1}^{4} c_j^\epsilon U_j^\epsilon + \bar{v}_\epsilon. \]

Moreover, for any \( \gamma \in (0, 1/2) \) and sufficiently small \( \epsilon \), the following estimate is valid:

\[ (3.47) \quad \|\bar{v}_\epsilon\|_{W^2_0(\Omega)} + \|\bar{v}_\epsilon\|_{H^2(\Omega \setminus B_{\epsilon/2})} \leq c \left( \|f\|_{W^2_0(\Omega)} + \|g\|_{W^{1/2}_0(\partial\Omega)} + \|v_\epsilon\|_{L^2(\Omega \cap B_{4\epsilon} \setminus \Omega_\epsilon)} + \sum_{j=1}^{4} |c_j^\epsilon| \right), \]

where \( c \) does not depend on \( f, g, c_j, \) and \( \epsilon \).

In order to pass to the limit in (3.5)–(3.7), we need to demonstrate that the “extra” quantity which appears on the right-hand side of (3.47), namely,

\[ b_\epsilon := \|v_\epsilon\|_{L^2(\Omega \cap B_{4\epsilon} \setminus \Omega_\epsilon)} + \sum_{j=1}^{4} |c_j^\epsilon|, \]

is bounded.

**Lemma 3.6.** Under Condition 1, we have

\[ (3.48) \quad b_\epsilon \leq c \left( \|f\|_{W^2_0(\Omega)} + \|g\|_{W^{1/2}_0(\partial\Omega)} \right), \]

where \( c \) does not depend on \( \epsilon, f, \) and \( g \).

**Proof.** Let us assume that \( b_\epsilon \) is not bounded; then there exists a subsequence \( \epsilon_k \), such that \( b_{\epsilon_k} \) is not bounded. Consider a “normalized” subsequence of \( \bar{v}_{\epsilon_k} \), \( \bar{u}_{\epsilon_k} := \frac{v_{\epsilon_k}}{b_{\epsilon_k}} \) (which we still denote \( \bar{u}_{\epsilon} \)). The following representation is now valid for this subsequence (cf. (3.46)):

\[ u_\epsilon = \bar{u}_\epsilon + \sum_{j=1}^{4} \alpha_j^\epsilon U_j^\epsilon \]
and

\begin{equation}
\|u_\varepsilon\|_{L^2(\Omega \cap B_{\varepsilon/4} \setminus \Omega_\varepsilon)} + \sum_{j=1}^{\nu} |\alpha_j^\varepsilon| = 1.
\end{equation}

Then it follows from (3.47) that we can choose a subsequence (which we still denote \(u_\varepsilon\)) such that \(u_\varepsilon\) converges weakly to \(\tilde{u}\) as \(\varepsilon \to 0\). Here the “weak convergence with radiation conditions,” denoted \(\text{rad}\), is understood in the following sense:

1. \(u\) can be represented as

\[ u = \tilde{u} + \sum_{j=1}^{\nu} \alpha_j U_j^0. \]

2. \(\alpha_j^\varepsilon \to \alpha_j\), \(\tilde{u}_\varepsilon\) converges weakly to \(\tilde{u}\) in \(H^2_{\gamma}(\Omega)\) for any \(\gamma > 0\), and \(U_j^\varepsilon \to U_j^0\) in \(H^2(K)\), where \(K\) is any compact set not containing the singularity point \(O\).

Let us note that convergence of \(U_j^\varepsilon\) to \(U_j^0\) follows from explicit expressions for \(U_1^\varepsilon\), \(U_2^\varepsilon\) (see (2.11), (2.12)) and explicit formulae for \(U_j^\varepsilon\) (see formula (3.45) and Remark 3.1 to Theorem 3.4). This allows us to pass to the limit in (3.5)–(3.7) (with \(u_\varepsilon\) instead of \(v_\varepsilon\) and \(\frac{1}{\varepsilon^2}\{f,g\}\) instead of \(\{f,g\}\)) and conclude that \(u \in \mathcal{H}^2_{\gamma,0}(\Omega)\) for any \(\gamma > 0\) and is a solution of the homogeneous problem. A standard trick with integration by parts (see Remark 4.1 below) shows that \(\alpha_j = 0\), \(j = 1, \ldots, 4\), and consequently \(u \in \mathcal{H}^2_{\gamma}(\Omega)\) for any \(\gamma > 0\) and in particular \(u \in V\), which, due to Condition 1, implies \(u = 0\). This contradicts (3.49), since, clearly, weak convergence of \(\tilde{u}_\varepsilon\) in \(\mathcal{H}^2_{\gamma}(\Omega)\) for any \(\gamma > 0\) and convergence of \(\alpha_j^\varepsilon\), \(j = 1, \ldots, 4\), imply the strong convergence of \(u_\varepsilon\) in \(L^2_{\gamma}\) on compact sets.

Finally, combining Lemmas 3.1 and 3.5, and then treating \(v_\varepsilon\) in the same way as \(u_\varepsilon\) in Lemma 3.6, we arrive at the following results.

**Theorem 3.7.** Suppose Condition 1 is satisfied and \(\nu > \kappa/8\). Assume further that \(\{f,g\} \in W^0_\nu(\Omega) \times W^{1/2}_0(\partial \Omega)\) and have compact support. Then there exists the unique solution of (1.4)–(1.6), \(v \in \mathcal{H}^2_{\gamma,0}(\Omega)\) for any \(\gamma > 0\), in particular

\begin{equation}
\begin{align*}
\alpha_j \sum_{j=1}^\nu c_j U_j^0 + \tilde{v},
\end{align*}
\end{equation}

and for any \(\gamma \in (0,1/2)\), the following estimate is valid:

\begin{equation}
\begin{align*}
|\alpha_j| \|\tilde{v}\|_{W^2_{\gamma}(\Omega_\varepsilon)} + \|\tilde{v}\|_{H^2(\Omega \setminus B_{\varepsilon/2})} & \\
& \leq c \left( \|f\|_{W^\nu(\Omega)} + \|g\|_{W^{1/2}_{\gamma}(\partial \Omega)} \right),
\end{align*}
\end{equation}

where \(c\) does not depend on \(f\) and \(g\).

Moreover, this solution can be obtained as the result of the limiting absorption procedure, namely, \(v^{\text{rad}}\) \(v\), where \(v^{\text{rad}}\) is a solution of (3.5)–(3.7).

If \(\nu = \kappa/8\), then we apply arguments as above, employing Remark 3.2 instead of Theorem 3.4. However, in order to obtain the result of Theorem 3.7, we need to employ a stronger condition.
Condition 1'. The homogeneous problem (1.4)-(1.6) does not have nontrivial solutions in the space $V'(\Omega) = \{ u : u = e^+ x_1^{-\frac{1}{2}} \zeta_+^+ + e^- x_1^{-\frac{1}{2}} \zeta_-^- + \bar{u}, \; e^\pm \in \mathbb{C}, \; \int_\Omega |\nabla u|^2 dx + \int_{\partial \Omega} |\overrightarrow{u}|^2 ds < +\infty \}$. The difference comes from the fact that we are able only to prove, in the analogue of Lemma 3.6, that the solution of the homogeneous problem from the space with radiation conditions is actually in energy space $V'(\Omega)$. (For the case $\nu > \kappa/8$ we were able to deduce that the solution is in $V(\Omega)$.) As a result, we conclude with the following corollary.

Corollary 3.8. If Condition 1' is satisfied, then condition $\nu > \kappa/8$ in Theorem 3.7 can be relaxed to $\nu \geq \kappa/8$.

If $\nu < \kappa/8$, then there is no need to isolate waves in the cusp. Let us introduce the space with “detached asymptotics” at infinity only:

\begin{equation}
(3.52) \quad v \in F^2_{\gamma, \varepsilon}(\Omega) \Leftrightarrow v = \sum_{j=1}^{2} c_j U_j + \tilde{v}, \quad \tilde{v} \in H^2(\Omega),
\end{equation}

where $c_j \in \mathbb{C}$.

We have the following analogue of Lemma 3.5.

Lemma 3.9. Let $\{f, g\} \in W^{0}_{1/2}(\Omega) \times W^{1/2}_{1/2}(\partial \Omega)$ and have compact support. Then for any $\varepsilon > 0$ the unique solution $v_\varepsilon$ of (3.5)-(3.7), delivered by Lemma 3.1, belongs to the space $F^{2}_{1/2, \varepsilon}(\Omega)$, in particular

\begin{equation}
(3.53) \quad v_\varepsilon = \sum_{j=1}^{2} c_j^\varepsilon U_j + \tilde{v}_\varepsilon.
\end{equation}

Moreover, for sufficiently small $\varepsilon$, the following estimate is valid:

\begin{equation}
(3.54) \quad \|\tilde{v}_\varepsilon\|_{V^{0}_{1/2}(\Omega)} + \|\tilde{v}_\varepsilon\|_{H^2(\Omega \setminus B_{r/2})} \leq c \left( \|f\|_{W^{0}_{1/2}(\Omega)} + \|g\|_{W^{1/2}_{1/2}(\partial \Omega)} + \|v_\varepsilon\|_{L^2(\Omega \setminus B_{4r/3} \setminus \Omega)} + \sum_{j=1}^{2} |c_j^\varepsilon| \right),
\end{equation}

where $c$ does not depend on $\varepsilon$, $f$, and $g$.

Employing similar arguments to the above, we arrive at the following theorem.

Theorem 3.10. Suppose Condition 1 is satisfied and $\nu < \kappa/8$. Assume further that $\{f, g\} \in W^{0}_{1/2}(\Omega) \times W^{1/2}_{1/2}(\partial \Omega)$ and have compact support. Then there is the unique solution of (1.4)-(1.6), $v \in F^{2}_{1/2, 0}(\Omega)$, in particular

\begin{equation}
(3.55) \quad v = \sum_{j=1}^{2} c_j U_j + \tilde{v},
\end{equation}

and the following estimate is valid:

\begin{equation}
(3.56) \quad \sum_{j=1}^{2} |c_j| + \|\tilde{v}\|_{V^{0}_{1/2}(\Omega)} + \|\tilde{v}\|_{H^2(\Omega \setminus B_{r/2})} \leq c \left( \|f\|_{W^{0}_{1/2}(\Omega)} + \|g\|_{W^{1/2}_{1/2}(\partial \Omega)} \right),
\end{equation}

where $c$ does not depend on $f$ and $g$. 

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Moreover, this solution can be obtained as the result of the limiting absorption procedure, namely, $v^\text{rad} \to v$ in the space $\mathbb{H}^2_{1/2,0}$, where $v_z$ is a solution of (3.5)–(3.7).

Remark 3.3. Now formula (1.10) follows from Theorems 3.3 and 3.4.

Finally let us comment on the applicability of Condition 1. It has been proved in [3] that in the case of a fully submerged body Condition 2 implies uniqueness. Various examples of bodies satisfying Condition 2 can be found in [10]. In particular, Condition 2 is satisfied by ellipses whose major axis is parallel to the $x_2$ axis; see [51].

One can apply the method of [3] to the case of a critically submerged body. This method is based on multipliers techniques and integration by parts. So we only need to verify that integration by parts in the neighborhood of the origin is justified. In other words, for the case $\nu > \kappa/8$ Condition 2 implies Condition 1. The same remains true if $\nu < \kappa/8$. Then the solution of the homogeneous problem (1.4)–(1.6), which is in $V$, belongs to the space $\mathcal{H}^2_{\gamma}(\Omega)$ for some $\gamma < 1/2$ (see Theorem 3.3), and analysis of possible singularities shows that we still can integrate by parts (see Theorem 4.4 below for details).

The critical case $\nu = \kappa/8$, where we need to check Condition 1’, is more subtle, but still one can prove that Condition 2 implies Condition 1’; see Corollary 4.5.

4. Scattering matrix and its properties. Let us define the usual scattering matrix for $\nu < \kappa/8$. In this case we need to employ only waves at the infinity. First we need to introduce “incoming waves”:

$$\begin{align*}
u_1(x) &= \chi(x_1)e^{\nu x_1 - \nu x_2}, \\ \nu_2(x) &= \chi(-x_1)e^{-\nu x_1 + \nu x_2};
\end{align*}$$

compare the above with (2.3).

Theorem 4.1. Suppose that $\nu < \kappa/8$ and Condition 1 is satisfied. Then there exist two linearly independent solutions of the homogeneous problem (1.4)–(1.6), $\eta_j, j = 1, 2$, such that

$$\eta_j = \nu_j + \sum_{n=1}^{2} s_{jn} \nu_n + \tilde{\eta}_j,$$

where $\tilde{\eta}_j \in \mathcal{H}^2_{1/2}$. Condition (4.2) determines $\eta_j$ uniquely. The scattering matrix $s = (s_{jn})_{j,n=1}^2$ is unitary and $s_{jn} = s_{jn}, j, n = 1, 2$.

Proof. The arguments are standard; see, e.g., [48]. Consider, for example, case $j = 1$. We are looking for $\eta_1$ in the form

$$\eta_1(x) = e^{\nu x_1 - \nu x_2} + \xi_1(x),$$

where $\xi_1(x)$ is a solution of (1.4)–(1.6) with $f = 0, g_2 = 0$, and $g_1 = -\partial_n e^{\nu x_1 - \nu x_2}|s$.

The solution to this problem exists in the space $\mathbb{H}^2_{1/2,0}(\Omega)$, due to Theorem 3.10. In particular

$$\xi_1 = \sum_{j=1}^{2} c_j U_j^0 + \bar{v},$$

where $\bar{v} \in \mathcal{H}^2_{1/2}(\Omega)$. Since $U_j^0 = \nu_j$ (compare (2.11), (2.12) with (2.3)), we see that

$$\eta_1(x) = e^{\nu x_1 - \nu x_2} + \xi_1(x)$$

satisfies (4.2) with $s_{11} = c_1$ and $s_{12} = c_2 + 1$. Clearly this solution is unique. The same argument applies to $\eta_2$. 

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Next we verify the properties of scattering matrix $s$. We know that $\eta_j$ solves the problem

\begin{align}
\Delta \eta_j &= 0 \quad \text{in } \Omega, \\
\partial_n \eta_j &= 0 \quad \text{on } S, \\
\partial_n \eta_j &= \nu \eta_j \quad \text{on } \Gamma.
\end{align}

Let us multiply (4.5) by $\overline{\eta_m}$, $n = 1, 2$, integrate over $\Omega \cap \{|x_1| < M\}$, and integrate by parts twice (which is justified since $\eta_j \in \mathcal{H}_1^{1/2}(\Omega_\tau)$ and due to conditions (2.5)). We have

\begin{align}
0 &= \int_0^{+\infty} \eta_n \partial_{x_1} \eta_j |x_1| = M dx_2 - \int_0^{+\infty} \eta_n \partial_{x_1} \eta_j |x_1| = -M dx_2 \\
&- \int_0^{+\infty} \eta_j \partial_{x_1} \eta_j |x_1| = M dx_2 + \int_0^{+\infty} \eta_j \partial_{x_1} \eta_j |x_1| = -M dx_2.
\end{align}

Next, using (4.2) and (4.1), we pass to the limit as $M \to +\infty$ and obtain

\begin{align}
0 &= \delta_{jn} - \sum_{p=1}^2 \overline{s_{np}} s_{jp}.
\end{align}

So $s$ is indeed unitary.

Now the symmetry property $s_{jn} = s_{nj}$, $j, n = 1, 2$, follows easily, since (4.5)–(4.7) is a problem with real coefficients, $s$ is unitary, and

\begin{align}
u_j^- = \overline{u_j^+}, \quad j = 1, 2. \quad \Box
\end{align}

In the case \( \nu > \kappa/8 \), there are four linearly independent solutions to the homogeneous problem (1.4)–(1.6), viewed as solutions of a scattering problem. First we renormalize functions $U_j^0$, $j = 3, 4$ (see (3.28) and (3.45)):

\begin{align}
u_3^-(x) := (\omega \kappa)^{-\frac{1}{4}} U_3^0(x) = (\omega \kappa)^{-\frac{1}{4}} v_0^- (x) \zeta_\tau^+ \\
&= (\omega \kappa)^{-\frac{1}{4}} |x_1|^{-\frac{1}{4} + i \omega} (1 + x_1^2 Q_0^- (x_2/\phi(x_1))) \zeta_\tau^+
\end{align}

and

\begin{align}
u_4^-(x) := (\omega \kappa)^{-\frac{1}{4}} U_4^0(x) = (\omega \kappa)^{-\frac{1}{4}} v_0^- (x) \zeta_\tau^- \\
&= (\omega \kappa)^{-\frac{1}{4}} |x_1|^{-\frac{1}{4} + i \omega} (1 + x_1^2 Q_0^- (x_2/\phi(x_1))) \zeta_\tau^-,
\end{align}

\begin{align}
\omega := \sqrt{\frac{2\nu}{\kappa} - \frac{1}{4}}.
\end{align}

Similarly to (2.3), we will refer to these functions as outgoing waves in the cusps. Namely, $u_3^+$ is the outgoing wave in the right cusp $\Omega_\tau^+$ and $u_4^-$ is the outgoing wave in the left cusp $\Omega_\tau^-$. In a similar way we introduce incoming waves in the cusps:

\begin{align}
u_3^+(x) := (\omega \kappa)^{-\frac{1}{4}} v_0^+ (x) \zeta_\tau^+ = (\omega \kappa)^{-\frac{1}{4}} x_1^{-\frac{1}{4} - i \omega} (1 + x_1^2 Q_0^+ (x_2/\phi(x_1))) \zeta_\tau^+(x),
\end{align}
exist four linearly independent solutions of the homogeneous problem
\[ \eta \] see (3.28).

**Theorem 4.2.** Suppose that \( \nu > \kappa / 8 \) and Condition 1 is satisfied. Then there exist four linearly independent solutions of the homogeneous problem (1.4)–(1.6), \( \eta_j, j = 1, \ldots, 4 \), such that

\[
(4.16) \quad \eta_j = u_j^+ + \sum_{n=1}^4 S_{jn} u_n^- + \tilde{\eta}_j,
\]

where \( \tilde{\eta}_j \in \mathcal{H}^2_\gamma \forall \gamma > 0 \). Condition (4.16) determines \( \eta_j \) uniquely. The scattering matrix \( S = (S_{jn})_{j,n=1}^4 \) is unitary and \( S_{jn} = S_{jn}, j, n = 1, \ldots, 4 \).

**Proof.** The proof of the existence of \( \eta_1, \eta_2 \) follows the arguments of Theorem 4.1, with reference to Theorem 3.7 instead of Theorem 3.10. As for \( \eta_3 \) and \( \eta_4 \), we need to take some more care. Consider, for example, \( \eta_3 \). Let us look for \( \eta_3 \) in the form

\[
(4.17) \quad \eta_3 = (\omega k)^{-\frac{1}{2}} Y_0^+ \zeta_3^+ + \xi_3,
\]

where \( \nu^+ \) is the function described in Theorem 3.4, \( \delta \) is sufficiently small, and \( \xi_3(x) \) is a solution of (1.4)–(1.6) with \( f = (\omega k)^{-\frac{1}{2}} \Delta (Y_0^+ \zeta_3^+) \) and

\[
(4.18) \quad \xi_3 = \sum_{j=1}^4 c_j U_j^0 + \bar{\nu},
\]

where \( \bar{\nu} \in \mathcal{H}^2_\gamma(\Omega) \) for any \( \gamma > 0 \). Since \( U_j^0 \) is a solution of the homogeneous problem (3.19)–(3.21) (with \( \varepsilon = 0 \)) for small enough \( \delta \), we conclude that \( (f, g) \in W^0_0(\Omega) \times W^{1/2}_{\gamma/2}(\partial \Omega) \) and Theorem 3.7 applies. As a result there is a solution of the problem for \( \xi_3 \) in the space \( \mathcal{H}^2_{\gamma,0}(\Omega) \) for any \( \gamma > 0 \). In particular

\[
(4.19) \quad \Delta \eta_j = 0 \text{ in } \Omega,
\]

\[
(4.20) \quad \partial_n \eta_j = 0 \text{ on } S,
\]

\[
(4.21) \quad \partial_n \eta_j = \nu \eta_j \text{ on } \Gamma.
\]

Let us multiply (4.19) by \( \overline{\eta_m}, n = 1, \ldots, 4 \), integrate over \( \Omega \setminus \Omega_\delta \cap \{|x_1| < M\} \), and integrate by parts twice (which is justified due to conditions (2.5)). As a result
we have

\begin{align}
0 &= \int_{0}^{\infty} (\eta x_n x_{x_1} - M dx_2 - \int_{0}^{\infty} (\eta x_n x_{x_1} - M dx_2 \\
- \int_{0}^{\infty} \eta x_n x_{x_1} - M dx_2 + \int_{0}^{\infty} \eta x_n x_{x_1} - M dx_2 \\
- \int_{0}^{\infty} \eta x_n x_{x_1} - M dx_2 + \int_{0}^{\infty} \eta x_n x_{x_1} - M dx_2 \\
+ \int_{0}^{\infty} \eta x_n x_{x_1} - M dx_2 - \int_{0}^{\infty} \eta x_n x_{x_1} - M dx_2).
\end{align}

Passing to the limits as \( M \to +\infty \) and \( \delta \to 0 \), and using (4.16), (4.11), (4.12), (4.14), (4.15), (2.3), and (4.1), we obtain

\begin{align}
0 = i\delta - i \sum_{p=1}^{4} \delta_{np} S_{np},
\end{align}

so \( S \) is unitary. The property \( S_{jn} = S_{jn}, j, n = 1, \ldots, 4 \), can be verified in the same way as in Theorem 4.1.

**Remark 4.1.** This type of argument with integration by parts implies the fact (which we used in Lemma 3.6) that if \( v \in H^{2}_\gamma(\Omega) \) and is a solution of the homogeneous problem (1.4)–(1.6), then \( v \in H^{2}_\gamma(\Omega) \) and is a solution of the scattering matrix \( S \).

Next we describe an important nonstandard property of the scattering matrix \( S \). First we decompose \( S \) as follows:

\begin{align}
S = \begin{pmatrix} S_{(1,1)} & S_{(1,2)} \\
S_{(2,1)} & S_{(2,2)} \end{pmatrix},
\end{align}

where

\begin{align}
S_{(1,1)} = \begin{pmatrix} S_{1,1} & S_{1,2} \\
S_{2,1} & S_{2,2} \end{pmatrix},
S_{(1,2)} = \begin{pmatrix} S_{1,3} & S_{1,4} \\
S_{2,3} & S_{2,4} \end{pmatrix},
\end{align}

and

\begin{align}
S_{(2,1)} = \begin{pmatrix} S_{3,1} & S_{3,2} \\
S_{4,1} & S_{4,2} \end{pmatrix},
S_{(2,2)} = \begin{pmatrix} S_{3,3} & S_{3,4} \\
S_{4,3} & S_{4,4} \end{pmatrix}.
\end{align}

**Theorem 4.3.** Suppose Condition 2 is satisfied and \( \nu > \kappa/8 \). Then \( \det S_{(1,2)} \neq 0 \), \( \det S_{(2,1)} \neq 0 \), and the absolute values of eigenvalues of the matrices \( S_{(2,2)} \) and \( S_{(1,1)} \) are strictly less than 1.

This theorem will follow from the following uniqueness result.

**Theorem 4.4.** Suppose Condition 2 is satisfied and \( \nu > \kappa/8 \). If \( v \) is a solution of the homogeneous problem (1.4)–(1.6) and \( v \in H^{2}_\gamma(\Omega) \) for some \( \gamma \), then \( v = 0 \).

**Proof.** Due to Theorems 3.3 and 3.4 we have

\begin{align}
v = \sum_{j=3}^{4} c_j^+ u_j^+ + \sum_{j=3}^{4} c_j^- u_j^- + \tilde{v},
\end{align}

where \( c_j^\pm, j = 3, 4 \), are some constants and \( \tilde{v} \in H^{2}_\gamma(\Omega) \) for any \( \gamma > 0 \). Let us consider the real part of \( v \), which we denote by \( u \). It is a solution of the homogeneous problem (1.4)–(1.6) and has the same structure as (4.27).
We start by recalling the method of multipliers of [3, 10], where it was applied for the case of fully submerged bodies. Let \( Z = (Z_1, Z_2) \) be a real vector field in \( \Omega \) with at most linear growth as \( |x| \to \infty \) and \( Z_2(x_1,0) = 0 \) \( \forall x_1 \), and let \( H \) be a constant. The following identity, which can be found in [10, p. 71] can be verified directly:

\[
2\{(Z \cdot \nabla u + Hu)\Delta u\} = 2\nabla \cdot \{(Z \cdot \nabla u + Hu)\nabla u\} + (Q\nabla u) \cdot \nabla u - \nabla \cdot (|\nabla u|^2 Z).
\] (4.28)

Here \( Q \) is a \( 2 \times 2 \) matrix with components \( Q_{ij} = (\nabla \cdot Z - 2H)\delta_{ij} - (\partial_i Z_j + \partial_j Z_i) \), \( i, j = 1, 2 \). Let us choose small positive \( \delta < \tau \), integrate (4.28) over \( \Omega \setminus \Omega_\delta \), and integrate by parts:

\[
0 = 2\int_{\partial \Omega \setminus \partial \Omega_\delta} (Z \cdot \nabla u + Hu)\partial_n u ds + \int_{\Omega \setminus \Omega_\delta} (Q\nabla u) \cdot \nabla u dx - \int_{\partial \Omega \setminus \partial \Omega_\delta} |\nabla u|^2 (Z \cdot n) ds + A_\delta^+ + A_\delta^-,
\] (4.29)

where

\[
A_\delta^\pm = \mp 2\int_0^\phi(\pm \delta) \left( Z \cdot \nabla u + Hu \right) \partial_{x_i} u |_{x_1 = \pm \delta} dx_2 \pm \int_0^{\phi(\pm \delta)} |\nabla u|^2 Z_1 |_{x_1 = \pm \delta} dx_2.
\] (4.30)

Hence,

\[
0 = 2\int_{\Gamma \setminus \partial \Omega_\delta} (Z \cdot \nabla u + Hu)\partial_n u dx_1 + 2\nu\int_{\Gamma \setminus \partial \Omega_\delta} (Z \cdot \nabla u + Hu) u dx_1
\]

\[
+ 2\int_{S \setminus \partial \Omega_\delta} (Z \cdot \nabla u + Hu)\partial_n u ds
\]

\[
+ \int_{\Omega \setminus \Omega_\delta} (Q\nabla u) \cdot \nabla u dx - \int_{S \setminus \partial \Omega_\delta} |\nabla u|^2 (Z \cdot n) ds + A_\delta^+ + A_\delta^-
\]

\[
= 2\nu\int_{\Gamma \setminus \partial \Omega_\delta} (Z_1 \partial_{x_1} u + Hu) u dx_1
\]

\[
+ \int_{\Omega \setminus \Omega_\delta} (Q\nabla u) \cdot \nabla u dx - \int_{S \setminus \partial \Omega_\delta} |\nabla u|^2 (Z \cdot n) ds + A_\delta^+ + A_\delta^-
\]

\[
= \nu\int_{\Gamma \setminus \partial \Omega_\delta} (2H - \partial_{x_2} Z_1)|u|^2 dx_1
\] (4.31)

\[
+ \int_{\Omega \setminus \Omega_\delta} (Q\nabla u) \cdot \nabla u dx - \int_{S \setminus \partial \Omega_\delta} |\nabla u|^2 (Z \cdot n) ds + A_\delta^+ + A_\delta^- + B_\delta^+ + B_\delta^-. 
\]

Here

\[
B_\delta^\pm = \mp \nu Z_1(\pm \delta,0)u^2(\pm \delta,0).
\] (4.32)

Following [10] (see p. 76), we choose

\[
Z(x_1, x_2) = \left( x_1 \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}, \frac{2x_1^2 x_2}{x_1^2 + x_2^2} \right) \quad \text{and} \quad H = 1/2.
\] (4.33)

Then, in particular, the first term on the right-hand side of (4.31) is equal to zero. Moreover it was also verified in [10] (see p. 76) that the quadratic form \((Q\nabla u) \cdot \nabla u\) is
nonpositive. In fact it has been shown in [10], and can be verified by direct inspection, that
\begin{equation}
(Q\nabla u) \cdot \nabla u = - (2x_1 x_2 \partial_{x_1} u + (x_2^2 - x_1^2) \partial_{x_2} u)^2 (x_1^2 + x_2^2)^{-2}.
\end{equation}

Finally, Condition 2 ensures that
\begin{equation}
(Z \cdot n) \geq 0 \text{ on } S.
\end{equation}

Now we need to investigate the behavior of $A_\delta^+ + B_\delta^+$ as $\delta \to 0$. Due to (4.27), we have
\begin{equation}
u(x) = a^+ x_1^{-1/2} \cos(\omega \ln x_1 + b^+) + O(x_1^{1/2}), \ x \in \Omega_\tau^+, \ x_1 \to +0,
\end{equation}
and
\begin{equation}u(x) = a^- (-x_1)^{-1/2} \cos(\omega \ln(-x_1) + b^-) + O(x_1^{1/2}), \ x \in \Omega_\tau^-, \ x_1 \to -0,
\end{equation}
where $a^+$ and $b^+$ are some real constants. Consider, for definiteness, $A_\delta^+ + B_\delta^+$. Then we employ Theorem 3.4 and obtain
\begin{align*}
\partial_{x_1} u(x) &= a^+ x_1^{-3/2} \left( -2^{-1} \cos(\omega \ln x_1 + b) - \omega \sin(\omega \ln x_1 + b) \right) \\
&+ O(x_1^{-1/2}), \ x \in \Omega_\tau^+, \ x_1 \to +0,
\end{align*}
\begin{align*}
\partial_{x_2} u(x) &= O(x_1^{-1/2}), \ x \in \Omega_\tau^+, \ x_1 \to +0.
\end{align*}
Moreover, we have from (4.33)
\begin{align*}
Z_1(x) &= x_1 + O(x_1^{-1}), \ x \in \Omega_\tau^+, \ x_1 \to +0,
\end{align*}
and
\begin{align*}
Z_2(x) &= O(x_1^3), \ x \in \Omega_\tau^+, \ x_1 \to +0,
\end{align*}
and consequently
\begin{align*}
\left( (\partial_{x_1} u(x))^2 + (\partial_{x_2} u(x))^2 \right) Z_1(x) &= 2(Z(x) \cdot \nabla u(x) + H u(x)) \partial_{x_1} u(x) \\
&= -\partial_{x_1} u(x) \left( x_1 \partial_{x_1} u(x) + u(x) \right) + O(x_1^{-1}) \\
&= -(a^+) x_1^{-2} \left( -2^{-1} \cos(\omega \ln x_1 + b) - \omega \sin(\omega \ln x_1 + b) \right) \\
&\times (2^{-1} \cos(\omega \ln x_1 + b) - \omega \sin(\omega \ln x_1 + b)) + O(x_1^{-1}) \\
&= (a^+) x_1^{-1} \left( 4^{-1} \cos^2(\omega \ln x_1 + b^+) - \omega^2 \sin^2(\omega \ln x_1 + b^+) \right) + O(x_1^{-1}) \\
&= (a^+) x_1^{-1} \left( 4^{-1} + \omega^2 \right) \cos^2(\omega \ln x_1 + b^+) + O(x_1^{-1}), \ x \in \Omega_\tau^+, \ x_1 \to +0.
\end{align*}
Next, using
\begin{align*}
\phi(x_1) &= \kappa x_1^2/2 + O(x_1^3),
\end{align*}
we get
\begin{equation}
A_\delta^+ = 2^{-1}(a^+)^2 \kappa \left( 4^{-1} + \omega^2 \right) \cos^2(\omega \ln \delta + b^+) - \omega^2 \right) + O(\delta) \text{ as } \delta \to +0.
\end{equation}
For $B_\delta^+$, we have, using (4.36),
\[ B_\delta^+ = -\nu(a^+)^2 \cos^2(\omega \ln \delta + b^+) + O(\delta) \quad \text{as} \ \delta \to +0. \]

Finally using (4.13), we obtain
\[ (4.38) \quad A_\delta^+ + B_\delta^+ = -2^{-1}(a^+)^2 \kappa \omega^2 + O(\delta) \quad \text{as} \ \delta \to +0. \]

In the same way we derive,
\[ (4.39) \quad A_\delta^- + B_\delta^- = -2^{-1}(a^-)^2 \kappa \omega^2 + O(\delta) \quad \text{as} \ \delta \to +0. \]

Now we can pass to the limit in (4.31) as $\delta \to 0$. We get
\[ (4.40) \quad 0 = \int_{\Omega} (Q \nabla u) \cdot \nabla u \, dx - \int_S |\nabla u|^2 (Z \cdot n) \, ds - 2^{-1} \kappa \omega^2 ((a^-)^2 + (a^+)^2). \]

As result we conclude via (4.34) and (4.35) that $u \equiv 0$.

Applying the same arguments to imaginary part of $v$, we obtain the same result.

**Corollary 4.5.** It follows from the proof of Theorem 4.4 that Condition 2 implies Condition 1'.

Now Theorem 4.3 easily follows.

**Proof of Theorem 4.3.** It follows from the properties of scattering matrix $S$ (see Theorem 4.2) that it is enough to prove only one of the claims of Theorem 4.3. Let us prove that $\det S(2,1)$ is not zero. If it is not so, then there is a nonzero row $a = (a_1, a_2)$ such that $aS(2,1) = (0, 0)$, and consequently the function $u = a_1 \eta_3 + a_2 \eta_4$ is not identically zero and satisfies the conditions of Theorem 4.4. This delivers a contradiction.

Theorem 4.3 allows us to formulate other existence and uniqueness results.

**Theorem 4.6.** Suppose Condition 2 is satisfied and $\nu > \kappa/8$. Assume further that $\{f, g\} \in L_2(\Omega) \times H^{1/2}(\partial \Omega)$ and have compact supports separated from the origin. Then there is the unique solution of (1.4)--(1.6), $u$, in the space $H^{2}_\text{loc}(\Omega) \cap L^\infty(\Omega)$. Moreover,
\[ (4.41) \quad u = \sum_{j=1}^{2} c_j^+ u_j^+ + \sum_{j=1}^{2} c_j^- u_j^- + \tilde{u}, \]

where $\tilde{u} \in H^2_{\gamma}(\Omega)$ for any $\gamma$ and $c_j^\pm$, $j = 1, 2$, are constants.

**Proof.** Theorem 3.7 delivers us the unique solution of (1.4)--(1.6), $v \in H^2_{\gamma,0}(\Omega)$ for any $\gamma > 0$, i.e.,
\[ (4.42) \quad v = \sum_{j=1}^{4} d_j u_j^- + \tilde{v}, \]

where $\tilde{v} \in H^2_{\gamma}(\Omega) \ \forall \gamma > 0$. Consider the function
\[ (4.43) \quad u := v - a_1 \eta_1 - a_2 \eta_2, \]

where a row $a = (a_1, a_2)$ solves
\[ aS(1,2) = (d_3, d_4). \]
Moreover, \( w(4.47) \) at infinity. But this is the same as proving that any solution of (1.4)–(1.6) and has the structure (4.41) with 
\[
\tilde{u} \in H^2_{\gamma}(\Omega), \quad \text{for } \gamma \leq 0,
\]
follows from Theorems 3.3 and 3.4. Clearly \( u \in H^2_{\text{loc}}(\Omega) \) and uniqueness follows immediately from Theorem 4.4.

Let us prove that representation (4.41) is valid. It follows from Theorems 3.3 and 3.4 that \( u \in H^2_{\gamma}(\Omega) \) for any \( \gamma, \) since \( f \) and \( g \) have support separated from the origin and \( u \in H^2_{\text{loc}}(\Omega) \). Now we need to show that the representation (4.41) is valid at infinity. But this is the same as proving that any solution \( v \in H^2_{\text{loc}}(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+) \) of (2.1), (2.2) with compactly supported \( f \) can be represented as
\[
v = \sum_{j=1}^{2} c_j^+ u_j^+ + \sum_{j=1}^{2} c_j^- u_j^- + \tilde{v}, \tag{4.44}
\]
where \( \tilde{v} \in H^2(\mathbb{R}^2_+) \). Consider the solution \( v_1 \) of (2.1), (2.2) in the space with radiating conditions. Clearly representation (4.44) is valid for \( v_1 \) (in fact coefficients next to outgoing waves are zero) and \( v_1 \in H^2_{\text{loc}}(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+) \). Then \( w := v - v_1 \in H^2_{\text{loc}}(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+) \) is a solution of (2.1), (2.2) with zero right-hand side, and consequently \( w \) is a linear combination of functions \( e^{-i\nu x_1 - \nu x_2} \) and \( e^{i\nu x_1 - \nu x_2} \). We see that representation (4.44) is valid for \( w, v_1 \) and consequently is valid for \( v \). This completes the proof.

In a similar way we prove the next result.

**Theorem 4.7.** Suppose Condition 2 is satisfied and \( \nu > \kappa/8 \). Assume further that \( \{f, g\} \in L^2(\Omega) \times H^{1/2}(\partial\Omega) \) and have compact support separated from the origin. Then there is a unique solution of (1.4)–(1.6), \( w, \) in the space
\[
H^2_{\text{loc}}(\Omega \setminus \Omega) \cap L^p(\Omega), \quad p \in (2, 6).
\]
Moreover, \( w \in H^2_{\gamma}, \) for any \( \gamma > 1/2 \) and can be represented as
\[
w = \sum_{j=3}^{4} c_j^+ u_j^+ + \sum_{j=3}^{4} c_j^- u_j^- + \tilde{w}, \tag{4.45}
\]
where \( \tilde{w} \in H^2_{\gamma}(\Omega) \) for any \( \gamma > 0 \) and \( c_j^\pm, j = 1, 2, \) are some constants.

**Proof.** Theorem 3.7 delivers us the unique solution of (1.4)–(1.6), \( v \in H^2_{\gamma, 0}(\Omega) \) for any \( \gamma > 0, \) i.e.,
\[
v = \sum_{j=1}^{4} d_j u_j^n + \tilde{v}, \tag{4.46}
\]
where \( \tilde{v} \in H^2_{\gamma}(\Omega) \) for any \( \gamma > 0. \) Consider the function
\[
w := v - a_1 \eta_3 - a_2 \eta_4, \tag{4.47}
\]
where the row \( a = (a_1, a_2) \) solves
\[
a S_{(2,1)} = (d_1, d_2).\]
The solution exists due to Theorems 4.3. Then it follows from Theorem 4.2 that $w$ defined by (4.47) is a solution of (1.4)–(1.6) and has the structure (4.45) with $\tilde{w} \in H^2_\text{loc}(\Omega)$ for any $\gamma > 0$. It follows from Theorem 3.4 that $w \in L^p(\Omega)$ for any $p < 6$. On the other hand, (2.5) implies $w \in L^p(\Omega \setminus \Omega_r)$ for any $p > 2$. Consequently $w \in H^2_\text{loc}(\Omega \setminus O) \cap L^p(\Omega), p \in (2, 6)$.

Let us discuss uniqueness. Consider some $w \in H^2_\text{loc}(\Omega \setminus O) \cap L^p(\Omega), p \in (2, 6)$, which is a solution of (1.4)–(1.6). If we additionally know that representation (4.45) is valid for $w$, then uniqueness follows immediately from Theorem 4.4.

Let us prove that representation (4.45) is valid. It is easy to see that since $u \in L^p(\Omega_r)$ for any $p < 6$ and is a solution of (1.4)–(1.6) with $f$ and $g$ having support separated from the origin, $w \in H^2_\text{loc}(\Omega)$ for some large $\gamma$. Then, employing Theorems 3.3 and 3.4, we conclude that $w \in H^2_\text{loc}(\Omega_r)$ for any $\gamma > 1/2$ and representation (4.45) is valid in the neighborhood of the origin.

Let us consider representation (4.45) at infinity. We need to prove that if $w \in H^2_\text{loc}(\Omega \setminus O) \cap L^p(\Omega), p \in (2, 6)$, then $w \in H^2(\Omega \setminus \Omega_r)$. However, this is the same as proving that any solution $u \in H^2_\text{loc}(\mathbb{R}^2_+) \cap L^p(\mathbb{R}^2_+), p \in (2, 6)$, of (2.1), (2.2) with compactly supported $f \in L^2(\mathbb{R}^2_+)$ is in fact in $H^2(\mathbb{R}^2_+)$. Consider solution $u_1$ of problem (2.1), (2.2), which has been obtained in section 2; see the discussion following Theorem 2.2. Clearly $u_1$ can be represented as

$$u_1 = c_1 u_1^+ + c_2 u_2^- + \tilde{u}_1,$$

where $\tilde{u}_1 \in H^2(\mathbb{R}^2_+)$ and $c_1, c_2$ are some constants. Moreover it follows from (2.5) that $u_1 \in L^p(\mathbb{R}^2_+), p \in (2, 6)$. Then $w := u - u_1 \in H^2_\text{loc}(\mathbb{R}^2_+) \cap L^p(\mathbb{R}^2_+)$ is a solution of (2.1), (2.2) with zero right-hand side, and consequently $w$ is a linear combination of functions $e^{-iv_1x_1 - vx_2}$ and $e^{iv_1x_1 - vx_2}$. As a result, we see that

$$u = c_1 u_1^+ + c_2 u_2^- + b_1 e^{-iv_1x_1 - vx_2} + b_2 e^{iv_1x_1 - vx_2} + \tilde{u}_1,$$

where $b_1$ and $b_2$ are some constants. We know that $u \in H^2_\text{loc}(\mathbb{R}^2_+) \cap L^p(\mathbb{R}^2_+), p \in (2, 6)$; on the other hand, functions $u_1^+(x), u_2^-(x), e^{-iv_1x_1 - vx_2}$, and $e^{iv_1x_1 - vx_2}$ do not belong to $L^p(\mathbb{R}^2_+), p \in (2, 6)$, and are linearly independent. Consequently $u$ coincides with $\tilde{u}_1$, which is in $H^2(\mathbb{R}^2_+)$. This ends the proof.

Solutions $u$ and $w$, delivered by Theorems 4.6 and 4.7, do not satisfy radiation conditions in general, and cannot be obtained as a result of the limiting absorption procedure. However, their description is simple: $u$ is bounded and $w$ decays at the infinity. It is worth emphasizing that these results rely on Condition 2.

We conclude with brief remarks on how one can define a scattering matrix for the case $\nu = \kappa/8$. We follow constructions which appeared in [48] for domains with conical points and [52, 53, 54] for periodic media. First we introduce incoming and outgoing waves in the cusps for the threshold case $\nu = \kappa/8$:

$$u_1^+ := (v^+ \pm iv^-) \zeta^+,$$

$$u_2^+ := (v^+ \mp iv^-) \zeta^-;$$

see (3.30) and (3.31). Waves $u_1^+$ and $u_2^+$ are defined according to (2.3) and (4.1). Arguing in the same way as for the case $\nu > \kappa/8$, we obtain the following theorem.

**Theorem 4.8.** Suppose that $\nu = \kappa/8$ and Condition 1′ is satisfied. Then there exist four linearly independent solutions of the homogeneous problem (1.4)–(1.6),
\( \eta_j, j = 1, \ldots, 4, \) such that
\[
\eta_j = u_j^+ + \sum_{n=1}^{4} a_{jn} u_n^- + \tilde{\eta}_j,
\]
where \( \tilde{\eta}_j \in H_0^2 \) for any \( \gamma > 0 \). Condition (4.50) determines \( \eta_j \) uniquely.

**Appendix A.** In this appendix we prove Theorem 2.2 and obtain some results which may be of interest in and of themselves.

It is clear that one needs to prove the estimate (2.14) only for small \( \varepsilon \); therefore we will assume that \( \varepsilon \leq 2 \beta \nu \). The desired estimate for the coefficients \( b_1^\varepsilon \) and \( b_2^\varepsilon \) follows from the explicit formulae by multiplying (2.6) by the solutions of the homogeneous problem (2.6), (2.7) and integrating over \( \mathbb{R}_+^2 \). As a result, upon a straightforward integration by parts we get
\[
\begin{align*}
(A.1) \quad b_1^\varepsilon &= i \nu (\nu^2 - i \varepsilon)^{-1/2} \int_{\mathbb{R}_+^2} f(x) e^{i (\nu^2 - i \varepsilon)^{1/2} x_1 - \nu x_2} dx, \\
(A.2) \quad b_2^\varepsilon &= i \nu (\nu^2 - i \varepsilon)^{-1/2} \int_{\mathbb{R}_+^2} f(x) e^{-i (\nu^2 - i \varepsilon)^{1/2} x_1 - \nu x_2} dx.
\end{align*}
\]

The remainder \( \tilde{u}_\varepsilon \in W_{\beta+1}^2 \) (see (2.10)) solves the problem
\[
\begin{align*}
\Delta \tilde{u}_\varepsilon - i \varepsilon \tilde{u}_\varepsilon &= F \text{ in } \mathbb{R}_+^2, \\
\partial_n \tilde{u}_\varepsilon - \nu \tilde{u}_\varepsilon &= 0 \text{ on } \Gamma,
\end{align*}
\]
where
\[
\begin{align*}
(A.5) \quad F(x) &= f(x) - b_1^\varepsilon [\Delta, \chi(x_1)] e^{-i (\nu^2 - i \varepsilon)^{1/2} x_1 - \nu x_2} - b_2^\varepsilon [\Delta, \chi(-x_1)] e^{i (\nu^2 - i \varepsilon)^{1/2} x_1 - \nu x_2},
\end{align*}
\]
and \([A, B] = AB - BA\) is a commutator.

Clearly, via applying the Cauchy–Schwarz inequality to (A.1) and (A.2), we have the estimate
\[
\|F\|_{W_{\beta+1}^2} \lesssim \|f\|_{W_{\beta+1}^2}.
\]

Thus we need to prove the inequality
\[
\|\tilde{u}_\varepsilon\|_{H^2} \lesssim \|F\|_{W_{\beta+1}^2}.
\]

We are going to apply the method of projections to (A.3), (A.4). This method, in the context of linear water waves, probably goes back to [2]. To this end we represent \( \tilde{u}_\varepsilon \) as
\[
\begin{align*}
\tilde{u}(x_1, x_2) &= w_1(x_1) e^{-\nu x_2} + w_2(x_1, x_2),
\end{align*}
\]
where
\[
\begin{align*}
(A.9) \quad \int_0^\infty w_2(x_1, x_2) e^{-\nu x_2} dx_2 &= 0 \forall x_1 \in \mathbb{R}.
\end{align*}
\]
Obviously, by the construction of \( w_1 \) as a projection of \( \tilde{u}_\varepsilon \), we have the estimates
\[
\begin{align*}
(A.10) \quad \|e^{\beta x_1} w_1\|_{H^2(\mathbb{R})} &\leq c\|\tilde{u}_\varepsilon\|_{W_{\beta+1}^2}, \quad \|w_2\|_{W_{\beta+1}^2} \leq c\|\tilde{u}_\varepsilon\|_{W_{\beta+1}^2}.
\end{align*}
\]
Similarly we represent $F$ as

(A.11) $F(x_1, x_2) = f_1(x_1)e^{-\nu x_2} + f_2(x_1, x_2)$, $\int_0^\infty f_2(x_1, x_2)e^{-\nu x_2}dx_2 = 0$, $x_1 \in \mathbb{R}$,

with estimates

(A.12) $\|e^{\beta(x_1)}f_1\|_{L^2(\mathbb{R})} \leq c\|F\|_{W^0_{\beta, 1}}, \|f_2\|_{W^0_{\beta, 1}} \leq c\|F\|_{W^0_{\beta, 1}}$.

Then we get the following by direct inspection of decoupled problems for $w_1$ and $w_2$:

(A.13) $\partial^2_{x_1} w_1(x_1) + (\nu^2 - i\varepsilon) w_1(x_1) = f_1(x_1), \ x_1 \in \mathbb{R}$,

and

(A.14) $\Delta w_2 - i\varepsilon w_2 = f_2$ in $\mathbb{R}^2_+$,

(A.15) $\partial_n w_2 - \nu w_2 = 0$ on $\Gamma$.

Below we demonstrate that both $w_1 e^{-\nu x_2}$ and $w_2$ satisfy the estimate (A.7), but for different reasons.

The estimate for $w_2$ follows from the next lemma, which we prove under less restrictive conditions on $f$.

**Lemma A.1.** Let $(x_2 + 1)f_2 \in L^2(\mathbb{R}^2_+)$, and let the solution $w_2$ of the boundary value problem (A.14), (A.15) satisfy (A.9). Then

(A.16) $\|w_2\|^2_{H^2(\mathbb{R}^2_+)} \leq c\frac{1}{\nu^2}\|(x_2 + 1)f_2\|^2_{L^2(\mathbb{R}^2_+)}$,

where $c$ does not depend on $\varepsilon$.

**Proof.** Let us write down the “energy” identity for problem (A.14), (A.15):

(A.17) $\int_{\mathbb{R}^2_+} |\nabla w_2|^2dx - \nu \int_{-\infty}^{+\infty} |w_2(x_1, 0)|^2dx_1 = -Re\int_{\mathbb{R}^2_+} f_2 w_2 dx$.

Since $w_2$ satisfies (A.9), we have

(A.18) $2\nu|w_2(x_1, 0)|^2 \leq \int_0^{+\infty} |\partial_{x_2} w_2(x_1, x_2)|^2dx_2$.

Now we deduce from (A.18) and (A.17) that

(A.19) $\int_{\mathbb{R}^2_+} |\nabla w_2|^2dx \leq -2Re\int_{\mathbb{R}^2_+} f_2 w_2 dx$.

In order to estimate the right-hand side of (A.19) we employ the Hardy-type inequality (cf., e.g., [55])

(A.20) $\int_0^{+\infty} (x_2 + 1)^{-2}|v|^2dx_2 \leq c\left(\int_0^{+\infty} |\partial_{x_2} v|^2dx_2 + |v(0)|^2\right)$.
It follows from (A.19), (A.20), and (A.18) that, for any $\delta > 0$,

\begin{equation}
\int_{\mathbb{R}^+} |\nabla w_2|^2 dx \leq \delta^{-1} \int_{\mathbb{R}^+} (x_2 + 1)^2 |f|^2 dx + \delta \int_{\mathbb{R}^+} (x_2 + 1)^{-2} |w_2|^2 dx
\end{equation}

\begin{equation}
\leq \delta^{-1} \int_{\mathbb{R}^+} (x_2 + 1)^2 |f|^2 dx + \delta c \int_{\mathbb{R}^+} |\partial_{x_2} w_2|^2 dx_2 + \delta c \int_{-\infty}^{+\infty} |w_2(x_1, 0)|^2 dx_1
\end{equation}

\begin{equation}
\leq \delta^{-1} \int_{\mathbb{R}^+} (x_2 + 1)^2 |f|^2 dx + \delta c \int_{\mathbb{R}^+} |\partial_{x_2} w_2|^2 dx_2 + \frac{\delta c}{2\nu} \int_{\mathbb{R}^+} |\partial_{x_2} w_2|^2 dx.
\end{equation}

This implies

\begin{equation}
\int_{\mathbb{R}^+} |\nabla w_2|^2 dx \leq \int_{\mathbb{R}^+} (x_2 + 1)^2 |f_2|^2 dx,
\end{equation}

\begin{equation}
\int_{-\infty}^{+\infty} |w_2(x_1, 0)|^2 dx_1 \leq \int_{\mathbb{R}^+} (x_2 + 1)^2 |f_2|^2 dx,
\end{equation}

\begin{equation}
\int_{\mathbb{R}^+} (x_2 + 1)^{-2} |w_2|^2 dx \leq \int_{\mathbb{R}^+} (x_2 + 1)^2 |f_2|^2 dx.
\end{equation}

From the boundary value problem (A.14), (A.15) we further have, using standard elliptic estimates and suitable cut-off functions,

\begin{equation}
\int_{\mathbb{R}^+} |\nabla^2 w_2|^2 dx \leq \int_{\mathbb{R}^+} (x_2 + 1)^2 |f_2|^2 dx.
\end{equation}

The estimates (A.22), (A.24), and (A.25) imply (A.16). \qed

The estimate for $w_1 e^{-\nu x^2}$ is ensured by the following lemma, which we formulate in a self-contained form.

**Lemma A.2.** Consider the one-dimensional Schrödinger equation (cf. (A.13))

\begin{equation}
\partial_t^2 u(t) + (\nu^2 - i\varepsilon) u(t) = f(t), \quad t \in \mathbb{R},
\end{equation}

with absorption $\varepsilon \in \mathbb{R}$, $|\varepsilon| < 1$, and rapidly decaying right-hand side $f$. More precisely we assume that $e^{\delta(t)} f \in L^2(\mathbb{R})$ for some $\delta > \frac{|\varepsilon|}{\nu^2}$, $t = (t^2 + 1)^{1/2}$. If, additionally, $e^{\delta(t)} u \in H^2(\mathbb{R})$, then, with constant $c$ independent of $\nu$ and $\varepsilon$,

\begin{equation}
\frac{\nu}{1 + \nu^2} \|u_t\|_{L_2} + \|u_t\|_{L_2} + \nu \|u\|_{L_2} \leq c \left( \int_{-\infty}^{+\infty} |f|^2(t) e^{\frac{|\varepsilon|}{\nu^2}}(t) dt \right)^{1/2}.
\end{equation}

**Proof.** Let us scalar multiply (A.26) by $te^{\alpha|t|}u_t$, where $\alpha = \frac{|\varepsilon|}{\nu^2}$, and integrate by parts. We have

\begin{align*}
&-2\text{Re} \int_{-\infty}^{+\infty} f(t) e^{\alpha|t|} u_{tt} dt \\
&= -2\text{Re} \int_{-\infty}^{+\infty} \partial_t^2 u e^{\alpha|t|} u_{tt} dt - 2\text{Re} \left( \nu^2 - i\varepsilon \right) \int_{-\infty}^{+\infty} u e^{\alpha|t|} u_{tt} dt \\
&= \int_{-\infty}^{+\infty} \left( te^{\alpha|t|} \right)_t |u_t|^2 dt + \nu \int_{-\infty}^{+\infty} \left( te^{\alpha|t|} \right)_t |u|^2 dt + 2\text{Re} \left[ \nu \int_{-\infty}^{+\infty} u e^{\alpha|t|} u_{tt} dt \right] \\
&= \int_{-\infty}^{+\infty} e^{\alpha|t|} (1 + \alpha|t|) |u_t|^2 dt + \nu \int_{-\infty}^{+\infty} e^{\alpha|t|} (1 + \alpha|t|) |u|^2 dt + 2\text{Re} \left[ \varepsilon \int_{-\infty}^{+\infty} e^{\alpha|t|} u_{tt} dt \right] \\
&= \int_{-\infty}^{+\infty} e^{\alpha|t|} |u_t|^2 dt + \int_{-\infty}^{+\infty} \frac{\alpha}{|t|} e^{\alpha|t|} \left| t u_t + \frac{i\varepsilon}{\alpha} |t| u \right|^2 dt + \nu \int_{-\infty}^{+\infty} e^{\alpha|t|} |u|^2 dt.
\end{align*}
From the above we conclude that 
\begin{equation} 
(A.28) \int_{-\infty}^{\infty} e^{\alpha|t|} |u_t|^2 \, dt \leq 4 \int_{-\infty}^{\infty} |f|^2 t^2 e^{\alpha|t|} \, dt 
\end{equation} 
and 
\begin{equation} 
(A.29) \nu^2 \int_{-\infty}^{\infty} e^{\alpha|t|} |u|^2 \, dt \leq \int_{-\infty}^{\infty} |f|^2 t^2 e^{\alpha|t|} \, dt. 
\end{equation} 
Second derivatives can be estimated directly via (A.26), and we obtain 
\begin{equation} 
(A.30) \|u_t e^{\alpha|t|}\|_{L_2(\mathbb{R})} \leq \nu^2 + \|\varepsilon\|_{L_2(\mathbb{R})} + \|f e^{\alpha|t|}\|_{L_2(\mathbb{R})}.
\end{equation} 
Combining (A.28), (A.29), and (A.30) we finally obtain (A.27). 

Finally we obtain the estimate for \(\tilde{u}_z\). From (A.8), (A.27), (A.16), and (A.12) we have 
\begin{equation} 
(A.31) \|\tilde{u}_z\|_{H^2} \leq \|w_1 e^{-\nu x_2}\|_{H^2} + \|w_2\|_{H^2} \leq c \|f(x_1) e^{\alpha x_1}\|_{L_2} + c \|f(x_2)\|_{L_2(\mathbb{R}^+)} \lesssim \|F\|_{W^0_{\beta,1}}.
\end{equation} 
Now estimate (A.6) delivers the desired result (2.14).

Acknowledgments. The authors are grateful to Prof. V. P. Smyshlyaev for helpful discussions and to the referees for valuable remarks.

REFERENCES


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