Analysis of Dynamics of the Tippe Top

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They have proven quite effectively that bumblebees indeed can fly against the field’s authority.

-Einstürzende Neubauten
Abstract

The Tippe Top is a toy that has the form of a truncated sphere with a small peg. When spun on its spherical part on a flat supporting surface it will start to turn upside down to spin on its peg. This counterintuitive phenomenon, called inversion, has been studied for some time, but obtaining a complete description of the dynamics of inversion has proven to be a difficult problem. This is because even the most simplified model for the rolling and gliding Tippe Top is a non-integrable, nonlinear dynamical system with at least 6 degrees of freedom. The existing results are based on numerical simulations of the equations of motion or an asymptotic analysis showing that the inverted position is the only asymptotically attractive and stable position for the Tippe Top under certain conditions. The question of describing dynamics of inverting solutions remained rather intact.

In this thesis we develop methods for analysing equations of motion of the Tippe Top and present conditions for oscillatory behaviour of inverting solutions.

Our approach is based on an integrated form of Tippe Top equations that leads to the Main Equation for the Tippe Top (METT) describing the time evolution of the inclination angle \( \theta(t) \) for the symmetry axis of the Tippe Top.

In particular we show that we can take values for physical parameters such that the potential function \( V(\cos \theta, D, \lambda) \) in METT becomes a rational function of \( \cos \theta \), which is easier to analyse. We estimate quantities characterizing an inverting Tippe Top, such as the period of oscillation for \( \theta(t) \) as it moves from a neighborhood of \( \theta = 0 \) to a neighborhood of \( \theta = \pi \) during inversion. Results of numerical simulations for realistic values of physical parameters confirm the conclusions of the mathematical analysis performed in this thesis.
**Populärvetenskaplig sammanfattning**


I denna avhandling presenteras ekvationerna för Tippe Top i flera former; vi studerar dem dels i vektor-form, dels i koordinat-form och dels i formen av en separationsekvation, bestånd utav energin för systemet. Dessa former belyser olika egenskaper av vändningsrörelsen som en Tippe Top utför och hjälper oss att förklara både varför den vänder sig och hur man ska beskriva vändningen asymptotiskt. Det bevisades redan på 1950-talet att glidfrictionen mellan Tippe Top och underlaget är den huvudsakliga mekanismen som driver vändningsrörelsen. Man kan vidare visa att om Tippe Top:en är konstruerad på ett korrekt sätt och den startar med tillräckligt hög vertikal rotationshastighet så kommer vändning alltid ske. Även om detta svarar på frågorna *varför* och *när* vändning av Tippe Top sker har vi ingen ordentlig beskrivning på dynamiken i vändningsrörelsen, det vill säga *hur* den vänder sig på sig.

Vi utvecklar matematiska metoder att studera differentialekvationer som beskriver Tippe Tops rörelse och visar att den centrala frågan om rörelse av vinkeln $\theta(t)$ mellan Tippe Tops symmetriaxel och vertikalaxeln kan reduceras till en studie av en enklare ekvation för $\theta(t)$.

Denna huvudekvation för Tippe Top liknar en separabel energiekvation som uppstår i specialfallet av en rullande Tippe Top.

För en rullande och glidande Tippe Top är dock rörelsekonstanter i denna ekvation tidsberoende, vilket försvårar analys. Vi kan ändå visa hur potentialfunktionen i ekvationen deformeras för en Tippe Top som vänder på sig. Speciellt så kan vi hitta parametervärden så potentialen blir en rationell funktion som är mycket enklare att analysera. Vi uppskattar relevanta storheter som karaktäriserar en vändande Tippe Top, bland annat svängningstiden för $\theta(t)$ då den oscillatorar under vändningen.
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Part I

Introduction
Introduction

1 Overview of Papers

This thesis consists of an introduction and three papers.

Paper 1

Tippe Top equations and equations for the related mechanical systems

We model the Tippe Top (TT) as an axially symmetric sphere rolling and gliding on a flat surface according to the Newton equations for motion of a rigid body. The equations of motion are nonintegrable and are difficult to analyse. The only existing arguments about TT inversion are based on analysis of stability of asymptotic solutions and on a LaSalle type theorem. These arguments do not explain the dynamics of inversion. To approach this problem we review and analyse here the equations of motion for the rolling and gliding TT in three equivalent forms; a vector form, an Euler angle form and an integrated form, each one providing different bits of information about motion of TT. They lead to the Main Equation for the tippe top, which describes well the oscillatory character of motion of the symmetry axis \( \hat{3} \) during the inversion. We show also that the equations of motion of TT in a suitable limit give rise to equations of motion for two other simpler mechanical systems: the gliding heavy symmetric top and the gliding eccentric cylinder. Analysis of equations for these systems is useful for understanding the dynamics of the inverting Tippe Top.
4

Paper 2

High frequency behaviour of a rolling ball and simplification of the Main Equation for the Tippe Top

The Chaplygin separation equation for a rolling axisymmetric ball has an algebraic expression for the effective potential $V(z = \cos \theta, D, \lambda)$ which is difficult to analyse. We simplify this expression for the potential and find a 2-parameter family for when the potential becomes a rational function of $z = \cos \theta$. Then this separation equation becomes similar to the separation equation for the heavy symmetric top. For nutational solutions of a rolling sphere, we study a high frequency $\omega_3$-dependence of the width of the nutational band, the depth of motion above $V(z_{\text{min}}, D, \lambda)$ and the $\omega_3$-dependence of nutational frequency $\frac{2\pi}{T}$. These results have bearing for understanding the inverting motion of the Tippe Top, modeled by a rolling and gliding axisymmetric sphere.

Paper 3

Dynamics of an inverting Tippe Top

In this paper we tackle the difficult question of describing dynamics of inverting solution of the Tippe Top (TT) when the physical parameters of the TT satisfy $1 - \alpha^2 < \gamma < 1$ and the initial conditions are such that Jellett’s integral satisfies $\lambda > \lambda_{\text{thres}} = \frac{\sqrt{mgR^3 I_3 \alpha (1 + \alpha)^2}}{\sqrt{1 + \alpha - \gamma}}$. Our approach is based on studying an equivalent integrated form of TT equations that leads to the Main Equation for the Tippe Top (METT), an equation describing time evolution of the inclination angle $\theta(t)$ of inverting TT. We study how the effective potential $V(\cos \theta, D, \lambda)$ in METT deforms as TT is inverting and show that its minimum moves from a neighborhood of $\theta = 0$ to a neighborhood of $\theta = \pi$. We formulate conditions for oscillatory behaviour of $\theta(t)$ when it moves toward $\theta = \pi$. Estimates for the maximal value of the oscillation period of $\theta(t)$ are given.

2 Background

A Tippe Top is a small toy, built as a truncated ball with a peg as a handle. When it is spun fast enough on a flat surface with the handle pointing upward, the top will start to turn upside down in a nutating manner until it ends up spinning on its handle. This counterintuitive and fascinating phenomenon is called inversion.

In this thesis we study a model of the Tippe Top with the aim of understanding how the dynamics of inversion follows from the properties of the modeling equations.

It turns out that even the most simple comprehensive model of a Tippe Top which exhibits the inversion phenomenon constitute a non-integrable, non-linear
dynamical system of (at least) six degrees of freedom. This makes dynamics of
the Tippe Top a challenging problem to analyse.

Due to the fascinating nature of the problem, analysis of Tippe Top inversion
has a rich history. In the 1950s several papers [3, 10, 13, 21, 22] established a
working physical model for the Tippe Top, essentially reducing it to a rolling
and gliding axisymmetric sphere. There was also some debate over what is the
main cause of the inversion phenomenon, if it is the shift of the center of mass
along symmetry axis in the spherical part, or the gliding friction between the
Tippe Top and the flat supporting surface. Del Campo [8] has shown definitely
that the gliding friction is the only mechanism, within the model, giving rise to
inversion.

This result was later affirmed by Cohen [7] in a paper where he presented
the first numerical simulations of Tippe Top inversion.

Since the 1990s the focus has shifted to analysis of the mathematical nature
of the problem; analysis of the integrable subcase of a rolling Tippe Top [11, 15]
and analysis of asymptotic behaviour of the Tippe Top [2, 6, 9]. These works
provided among other things criteria for how the Tippe Top has to be built
and how fast it has to be spun in order to enable inversion. The equations
for purely rolling Tippe Top are known to be integrable since Chaplygin and
Routh [4, 25]. When however friction between Tippe Top and the supporting
plane is introduced the dynamical equations acquire two additional variables
and become a non-integrable dynamical system of 6 degrees of freedom. A
rigorous analysis of these equations and of the dynamics of inversion remains
an unexplored field.

Most of the work in that direction focused on numerical simulations for var-
nious initial conditions and for various assumptions about the friction [20, 26].

This thesis reviews the existing results regarding the asymptotics and dy-
namics of the model of the Tippe Top and provides analysis of the dynamics of
inversion. We study in particular the separation equation for the rolling Tippe
Top as a starting tool for understanding inverting solutions of the rolling and
gliding Tippe Top. We show that for a certain range of parameters this equation
can be simplified so that precise estimates of behaviour of nutational solutions
can be found.

In this introduction we review the basics of rigid body motion, illustrate it
for the well known example of a Heavy Symmetric Top and summarize the main
points in our study of dynamics of the Tippe Top. Finally, to illustrate the inher-
ent complexity of Tippe Top inversion, we numerically integrate the equations of
motion for the rolling and gliding TT for a set of physical parameters and initial
conditions corresponding to an inverting TT. The resulting graphs of evolution
of relevant functions and variables are discussed at length.
3 Preliminaries

The basic notation we use in this thesis to describe the motion for rigid bodies is similar to conventions used in Landau [16] and Goldstein et al. [12]. We shall use the Euler angles \((\theta, \varphi, \psi)\) to describe the orientation of the body.

We consider a rigid body of mass \(m\) with instantaneous contact with the supporting plane at the point \(A\). In this thesis we will exclusively consider axisymmetric bodies. Let \(CM\) be the center of mass of the body and \(O\) the geometric center (not necessarily the same).

We choose a fixed inertial reference frame \(K_0 = (\hat{X}, \hat{Y}, \hat{Z})\) with \(\hat{X}\) and \(\hat{Y}\) parallel to the supporting plane and with vertical \(\hat{Z}\). We place the origin of this system in the supporting plane. Let \(K = (\hat{x}, \hat{y}, \hat{z})\) be a frame defined through rotation around \(\hat{Z}\) by an angle \(\varphi\), where \(\varphi\) is the angle between the plane spanned by \(\hat{X}\) and \(\hat{Z}\) and the plane spanned by the points \(CM, O\) and \(A\) (see figure 1).

The third reference frame is \(\tilde{K} = (\hat{1}, \hat{2}, \hat{3})\), with origin at \(CM\), defined through a rotation around \(\hat{y}\) by an angle \(\theta\), where \(\theta\) is the angle between \(\hat{Z}\) and the symmetry axis. Note that this frame is not fully fixed in the body. We let \(\hat{3}\) be parallel to the symmetry axis. The angle \(\psi\) is the rotation angle of the body about the symmetry axis \(\hat{3}\).

The Euler angles for this body relative to the vertical \(\hat{z}\) will be denoted \((\theta, \varphi, \psi)\) and the angular velocity of the frame \(\tilde{K}\) w.r.t. \(K_0\) is \(\omega_{ref} = \theta \hat{2} + \varphi \hat{2} = -\varphi \sin \theta \hat{1} + \theta \hat{2} + \varphi \cos \theta \hat{3}\). The total angular velocity of the body is found by adding the rotation around the symmetry axis \(\hat{3}\): \(\omega = \omega_{ref} + \psi \hat{3} = -\varphi \sin \theta \hat{1} + \theta \hat{2} + (\psi + \varphi \cos \theta) \hat{3}\), and we shall refer to the third component of this vector as \(\omega_3 = \psi + \varphi \cos \theta\). In this thesis we will always let the "dot-notation" mean derivative w.r.t. time.

Let \(s\) denote the position vector of \(CM\) w.r.t. \(K_0\). Let \(\rho\) be the vector from \(CM\) to an arbitrary point \(P\) in the body. The velocity of the point \(P\) w.r.t. \(K_0\) can
then be written as
\[ v = \dot{s} + \omega \times \rho. \] (1)

Note that \( \dot{\rho} = \omega \times \rho. \)

If we think of a rigid body as composed of fixed masses \( m_i \) (with the total mass \( m = \sum_i m_i \)) at positions \( \rho_i \) w.r.t. CM then the angular momentum of the body about the CM is defined as
\[ L = \sum_i m_i \rho_i \times v_i, \] (2)

where \( v_i \) denotes the velocity w.r.t. \( K_0 \) of the \( i \)th mass. From (1) we see that this expression can be written as
\[ L = \sum_i m_i \rho_i \times (\dot{s} + \omega \times \rho_i). \]

But by the definition of the center of mass, \( \sum_i m_i \rho_i = 0 \), so only the second sum remains. We can then define a linear operator acting on a vector \( u \) in the body:
\[ I u = \sum_i m_i \rho_i \times (u \times \rho_i). \]

We call \( I \) the inertia operator (or inertia tensor). We then write the angular momentum as \( L = I \omega \). The inertia operator is symmetric and positive, which means we can find an ON-basis where \( I \) is diagonal. The components of \( I \) in this basis are called the principal moments of inertia and the corresponding vectors the principal axes. Since the body is axisymmetric, the principal axes are not unique; to the principal moments of inertia \( I_1 = I_2 \) there corresponds a whole plane of principal vectors. Therefore, the vectors \( \hat{1}, \hat{2} \) in the moving frame \( \tilde{K} \) will be principal vectors, along with \( \hat{3} \).

The motion of each mass \( m_i \) is described by the Newton equation
\[ m_i \ddot{v}_i = F_i, \]
where \( F_i \) is a force acting on the mass \( m_i \). Then by summing these equations for the motion of the center of mass \( s(t) \) we have obtained the Newton equation
\[ m \ddot{s} = F, \]

where \( F = \sum_i F_i \) is the net external force acting on the body. Since \( \omega = I^{-1} L \), the time-evolution of the angular velocity \( \omega(t) \) can be found by solving the equation obtained when differentiating (2):
\[ \dot{L} = \sum_i (m_i \dot{\rho}_i \times v_i + m_i \rho_i \times \dot{v}_i) \]
\[ = \sum_i m_i (\omega \times \rho_i) \times (\dot{s} + \omega \times \rho_i) + \sum_i \rho_i \times (m_i \dot{v}_i) \]
\[ = (\dot{s} \times \omega) \times \sum_i m_i \rho_i + \sum_i \rho_i \times (m_i \dot{v}_i) = \sum_i \rho_i \times F_i, \]

where \( F_i \) is the force acting on the \( i \)th particle. The right hand side in this equation is the sum of all external torques acting on the body w.r.t. CM (if the
angular momentum is written w.r.t. another point in the body, the sum should be the sum of external torques acting on the body w.r.t. that point).

In the following example we will look at a simple application of deriving the equations of motion for a specific rigid body, the Heavy Symmetric Top (HST). We also find the separation equation for the motion of this body. This is instructive since the technique of deriving the separation equation for the Tippe Top involves the same steps. The equations for the HST are also of interest since in paper 1 we derive equations for a non-integrable generalization of this rigid body (the gliding HST). In paper 2 the method used in [1, 12] to analyse nutational motion and to estimate relevant quantities for the HST is applied in study of certain nutational solutions of the model for the rolling Tippe Top.

--- Example 3.1 ---

The Heavy Symmetric Top (also called the Lagrange Top) is an axially symmetric rigid body with one point fixed, moving under the action of vertical gravitational force. We let the fixed point $A$ coincide with the origin of the inertial system $K_0$. As we see in figure 2 the center of mass $CM$ is at distance $l$ from the origin and on the $\hat{3}$-axis.

![Diagram of the heavy symmetric top.](image)

We note that the forces acting on the HST are the reaction force $F$ applied at the point $A$ and the force of gravity acting at $CM$ (we let $g = 9.82 \text{ m} \cdot \text{s}^{-2}$ be the gravitational acceleration, as usual). Here we have $s = l\hat{3} = -a$, so the equations of motion in vector notation are

$$m\ddot{s} = F - mg\hat{z}, \quad \dot{L}_A = s \times (-mg\hat{z}).$$

The inertia tensor w.r.t. the principal axes is $I = I_1\hat{1}\hat{1} + I_2\hat{2}\hat{2} + I_3\hat{3}\hat{3}$, but $I_1 = I_2$ since the HST is axisymmetric (here the superscript $t$ means the transpose of a vector). Using Steiner’s theorem [1], we also have the “shifted” inertia tensor w.r.t. the fixed point $A$: $I^* = I_1^* (\hat{1}\hat{1}t + \hat{2}\hat{2}t + \hat{3}\hat{3}t)$, where $I_1^* = I_1 + ml^2$. The
angular momentum w.r.t. the point $A$ is written as $\mathbf{L}_A = I^* \omega$. To make the system of equations (3) closed we need to add the kinematic equation for the motion of the symmetry axis:

$$\dot{\mathbf{3}} = \omega \times \mathbf{3} = \left((I^*)^{-1}\mathbf{L}_A\right) \times \mathbf{3} = \frac{1}{I_1} \mathbf{L}_A \times \mathbf{3}.$$

The dynamics of the HST is found by solving the closed system of equations

$$\dot{\mathbf{L}}_A = -mg\|\mathbf{3} \times \mathbf{z}\|, \quad \dot{\mathbf{3}} = \frac{1}{I_1} \mathbf{L}_A \times \mathbf{3}.$$ 

The solution to the system gives the value of the total reaction force $\mathbf{F} = m\ddot{s} + mg\|\mathbf{z}\|$ needed to keep the tip of the HST fixed at the point $A$.

It is well known that the HST admits three integrals of motion, the energy $E = \frac{1}{2} \omega \cdot \mathbf{L}_A + mg\|\mathbf{z}\| \cdot \mathbf{z}$ and the angular momentum projected on the $\mathbf{3}$ and $\mathbf{z}$-axis $L_3 = \mathbf{L}_A \cdot \mathbf{3}$, $L_z = \mathbf{L}_A \cdot \mathbf{z}$. These integrals of motion make it possible to find a separation equation for system (3). Since the angular velocity $\omega = -\dot{\phi} \sin \theta \mathbf{1} + \dot{\theta} \mathbf{2} + (\dot{\psi} + \dot{\phi} \cos \theta) \mathbf{3}$, the integrals are:

$$E = \frac{1}{2} \mathbf{L}_A \cdot \omega + mg\|s\| \cdot \mathbf{z} = \frac{I^*}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{L_3}{2}(\psi + \phi \cos \theta)^2 + mgl \cos \theta, \quad (4)$$
$$L_z = \mathbf{L}_A \cdot \mathbf{z} = (I_1^* \sin^2 \theta + I_3 \cos^2 \theta) \dot{\phi} + I_3 \dot{\psi} \cos \theta, \quad (5)$$
$$L_3 = \mathbf{L}_A \cdot \mathbf{3} = I_3 (\dot{\psi} + \phi \cos \theta). \quad (6)$$

By solving equations (5), (6) for $\dot{\phi}$, $\dot{\psi}$ and by eliminating $\phi$, $\psi$ from the energy integral (4) we obtain a separable differential equation for $\theta$:

$$E = \frac{I^*}{2} \theta^2 + \frac{L_3^2}{2I_3} + \frac{(L_z - L_3 \cos \theta)^2}{2I_1^* \sin^2 \theta} + mgl \cos \theta.$$

By renaming constants, this equation becomes

$$\frac{2E}{I_1^*} = \theta^2 + \frac{I_1^* a^2}{I_3} + \frac{(b - a \cos \theta)^2}{\sin^2 \theta} + \beta \cos \theta, \quad (7)$$

where

$$\beta = \frac{2mgl}{I_1^*}, \quad a = \frac{L_z}{I_1^*}, \quad b = \frac{L_3}{I_1^*}.$$ 

The solution to (7) gives the function $\theta(t)$, which makes it possible to find $\phi(t)$ and $\psi(t)$ expressed by quadratures in terms of elliptic functions. The trajectory of the solutions is then represented by a curve on the unit sphere $S^2$ that is drawn by the symmetry axis $\mathbf{3}$. 
4 The Tippe Top Model

The model for the Tippe Top (TT) that we shall use is a rigid, eccentric ball as an approximation of the truncated ball with handle. The truncation shifts the center of mass of the sphere from its geometric center along its symmetry axis.

Thus we consider a sphere with radius $R$ and axially symmetric distributed mass $m$ so that the center of mass $CM$ is shifted from the geometric center $O$ along the symmetry axis $\hat{3}$ by $\alpha R$, where $0 < \alpha < 1$. The TT is in instantaneous contact with the supporting plane at $A$, and we consider the TT’s movements as it rolls and glides in the plane. We therefore choose the same reference frames as in the previous section; the vector $s$ is the position of $CM$ with respect to $K_0$ and $a = R(\alpha \hat{3} - \hat{z}) = R(\sin \theta \hat{1} + (\alpha - \cos \theta)\hat{3})$ is the vector from $CM$ to $A$ (figure 3).

![Figure 3: Diagram of the TT. Note that $a = R\alpha \hat{3} - R\hat{z}$.](image)

Newton’s equations for the TT describe the motion of the CM and rotation around the CM:

\[ m\ddot{s} = F - mg\hat{z}, \quad \dot{L} = a \times F, \quad \frac{\dot{3}}{I_3} = \omega \times \hat{3} = \frac{1}{I_1} L \times \hat{3}, \quad (8) \]

where $F$ is the total force acting on the TT at the point $A$. The third equation is the kinematic equation describing the motion of the symmetry axis $\hat{3}$. Here the angular momentum is taken w.r.t. $CM$, $L = I\omega$, where $I = I_1 (\hat{1}\hat{1} + 2\hat{2}\hat{2}) + I_3\hat{3}\hat{3}$ and $\omega = -\dot{\phi} \sin \theta \hat{1} + \dot{\theta} \hat{2} + \omega_3 \hat{3}$ is the angular velocity.

In our model we assume that the point $A$ is always in contact with the supporting plane, which is expressed as the contact constraint $\ddot{z} \cdot (s(t) + a(t)) \equiv 0$. This is an identity with respect to time $t$, so all its time-derivatives have to vanish as well. In particular, $\ddot{z} \cdot (\dot{s} + \omega \times a) = 0$.

Thus the gliding velocity $v_A(t) = \dot{s} + \omega \times a$, which is the velocity of the point that is in instantaneous contact with the plane at time $t$, will have zero $\hat{z}$-component. In the frame $K$, the gliding velocity has components $v_A = v_x \cos \theta \hat{1} + v_y \hat{2} + v_z \sin \theta \hat{3}$, where $v_x, v_y$ are the components in the $\hat{2} \times \hat{z}$ and $\hat{2}$ direction.

Equations (8) also admit Jellett’s integral of motion $\lambda = -L \cdot a [25]$. 

4 The Tippe Top Model

4.1 The frictional force

We assume that the force acting at the point $A$ has the form $F = g_n \hat{z} - \mu g_n v_A$. It consists of a normal reaction force and of a friction force of viscous type, where $g_n$ and $\mu$ are non-negative coefficients. Other frictional forces due to spinning and rolling are ignored. The contact criterion gives that the vertical component $g_n \hat{z}$ is dynamically determined, but the planar component of $F$ has to be specified independently to make equations (8) fully determined.

The original equations of motion (8) can then be written as

$$m \ddot{r} = -\mu g_n v_A, \quad \dot{L} = a \times (g_n \hat{z} - \mu g_n v_A), \quad \frac{\dot{\hat{3}}}{l_1} = \frac{1}{I_1} L \times \hat{3}, \quad (9)$$

where $r = s - s_\ell \hat{z}$. The way of writing the frictional part of the reaction force, $-\mu g_n v_A = -\mu (L, \hat{3}, s, t, g_n(t)) v_A$, indicates that it acts against the gliding velocity $v_A$ and that the friction coefficient can in principle depend on all dynamical variables and on time $t$.

The assumption that the friction force is of viscous type is common in most works about the motion of the TT [6, 9, 18, 19, 24]. Or [20] discusses the possible inclusion of a Coulomb-type friction force, $-\mu C g_n v_A / |v_A|$, along with the viscous friction. Bou-Rabee et al. [2] argues that a Coulomb-type friction force only causes algebraic destabilization of the initially spinning TT, whereas viscous-type friction gives exponential destabilization.

The model for the external force, which we use in equations (9), immediately implies that the energy $E = \frac{1}{2} m s^2 + \frac{1}{2} \omega \cdot L + m g s \cdot \hat{z}$ is decreasing monotonically for the rolling and gliding TT. A short calculation using the equations (9) gives $\dot{E} = F \cdot v_A = -\mu g_n |v_A|^2 < 0$. But since the CM of TT is lifted by $2Ra$ during inversion, the potential energy is increased. This means that kinetic energy decreases during inversion. An argument due to Del Campo [8], which we repeat in paper 1, shows that the planar part of the reaction force is providing the essential mechanism which allows the transfer of energy to occur within the accepted model for the rolling and gliding TT. Thus the gliding friction is the force which gives rise to the inversion phenomenon.

4.2 The dynamical system for the TT model

Since we have specified the friction force for the model of the rolling and gliding TT, equations (9) can be written in Euler angle form and solved for the highest
derivative of each of the variables \((\theta, \varphi, \omega_3, v_x, v_y)\). We get the system

\[
\begin{align*}
\dot{\theta} &= \frac{\sin \theta}{I_1} \left( I_1 \dot{\varphi}^2 \cos \theta - I_3 \omega_3 \dot{\varphi} - R g_n \right) + \frac{R \mu g_n v_y}{I_1} (1 - \alpha \cos \theta), \\
\dot{\varphi} &= \frac{I_3 \dot{\omega}_3 - 2 I_1 \dot{\theta} \dot{\varphi} \cos \theta - \mu g_n v_y R (\alpha - \cos \theta)}{I_1 \sin \theta}, \\
\dot{\omega}_3 &= - \frac{\mu g_n v_y R \sin \theta}{I_3}, \\
v_x &= \frac{R \sin \theta}{I_1} \left( \dot{\varphi} \omega_3 (I_3(1 - \alpha \cos \theta) - I_1) + g_n R \alpha (1 - \alpha \cos \theta) - I_1 \alpha (\dot{\varphi}^2 + \dot{\varphi}^2 \sin^2 \theta) \right) \\
&\quad - \frac{\mu g_n v_x}{m I_1} \left( I_1 + m R^2 (1 - \alpha \cos \theta)^2 \right) + \varphi v_y, \\
v_y &= - \frac{\mu g_n v_y}{m I_1 I_3} \left( I_3 \omega_3 + m R^2 I_3 (\alpha - \cos \theta)^2 + m R^2 I_1 \sin^2 \theta \right) \\
&\quad + \frac{\omega_3 \dot{\theta} R}{I_1} \left( I_3 (\alpha - \cos \theta) + I_1 \cos \theta \right) - \varphi v_x,
\end{align*}
\]

which, if we add the equation \(\dot{z} = \dot{z} = 0\) and have \(g_n\) be determined from the contact constraint \((\mathbf{a} + \mathbf{s}) \cdot \mathbf{z} = 0\) and we have

\[
g_n = \frac{mg I_1 + m R \alpha (\cos \theta (I_1 \dot{\varphi}^2 \sin^2 \theta + I_3 \dot{\varphi}^2) - I_3 \dot{\varphi} \omega_3 \sin^2 \theta)}{I_1 + m R^2 \alpha^2 \sin^2 \theta - m R^2 \alpha \sin \theta (1 - \alpha \cos \theta) \mu v_z}.
\]

The system has effectively 6 unknowns and is complicated to solve. In section 8 of this introduction we study numerical integration of this system to discern the behaviour of the variables more clearly.

### 4.3 A note on TT models

The rigid body modeling the TT in this thesis is an axisymmetric sphere, which is a rigid body that has been studied extensively in the past \([4, 5, 25]\). This is also the model for the TT used almost exclusively in the articles related to the subject. A valid objection against using this model is that it does not take into account the peg the TT ends up spinning on after inversion. There are articles where an adjusted model for the TT is considered in order to study the “jump” a TT does from rolling on its spherical part to spinning on its peg.

Hugenholtz briefly describes (in the appendix to \([13]\)) the process where the spherical part and the peg is in contact with the plane at the same time at points \(A\) and \(A'\) respectively. The motion of the TT rising up on its peg is characterized by decreasing normal force on \(A\) and increasing normal force on \(A'\). It is shown numerically under simplified circumstances that if \(\dot{\psi}\) is initially large and \(\dot{\varphi}\) small (or \(\dot{\varphi}\) initially large) there exist solutions such that the normal forces changes from a TT resting entirely on its sphere to a TT resting entirely on its peg.
Karapetyan and Zobova [27] studied a model for a TT where the top is constructed from two spherical segments joined by a rod, the smaller of the segments representing the end of the peg. This model requires two sets of equations of motion, similar to (8), each for when the supporting plane is in contact with the corresponding spherical segment. It also requires equations for the boundary case when both segments touch the plane simultaneously, which occurs for some inclination angle during inversion. The dynamics for a TT based on this model has not been explored; the existing works study the stability of steady motion, i.e. when the TT spins in the upright or inverted position or when it rolls on one of its segments, with the center of mass fixed.

These models are mainly an extension of the model based on a rolling and gliding axisymmetric sphere, essentially using two such spheres to model the TT. Thus it seems more interesting to fully understand the dynamics of one rolling and gliding axisymmetric sphere before the model is adjusted to better describe a real TT.

5 Asymptotic results

There is a special subclass of solutions to equations (9) for a rolling and gliding TT with \( F = g_n \hat{z} - \mu g_n v_A \), satisfying the condition \( v_A = 0 \). This gives \( F = g_n \hat{z} \) and equations of motion

\[
\begin{align*}
mr &= 0, \\
L &= R g_n \hat{3} \times \hat{z}, \\
\dot{\hat{3}} &= \frac{1}{I_1} L \times \hat{3},
\end{align*}
\]

where \( r = s - s z \hat{z} \) and the third coordinate is determined by \( m s z = g_n - mg \).

The solutions to this system are important since they describe asymptotic solutions to the equations of motion (9) for the rolling and gliding TT.

In [9] and [24] it is shown that for these solutions the center of mass (CM) remains stationary (\( \dot{s} = 0 \)). These are the situations where the TT is either spinning in the upright position, spinning in the inverted position or rolling around the \( \hat{3} \)-axis in such a way that the CM is fixed (tumbling solutions).

In paper 1, we derive these solutions from the equations of motion in Euler angle form and show that when \( 1 - \alpha < \gamma = I_1 / I_3 < 1 + \alpha \), the stationary tumbling solutions are admissible for all values of \( \cos \theta \in (-1, 1) \).

In [9], the relative stability (in the sense of Lyapunov) of a given spinning and tumbling solution is derived as a relation between the value of the Jellett integral \( \lambda \) and the physical characteristics of the TT. This analysis thus specifies how fast a given TT should be spun to make solutions to (16) stable.

In particular, if \( |\lambda| \) is above the threshold value \( \lambda_{\text{thres}} = \frac{\sqrt{mgR^2 \ln(1 + \alpha)^2}}{\sqrt{1 + \alpha - \gamma}} \), only the inverted spinning position \( \cos \theta = -1 \) is stable [9, 24] (the sign of \( \lambda \) is determined by the direction of the initial spin, so without loss of generality we may assume \( \lambda > 0 \)).

The solutions to (16) characterized here are a subset of the precessional solutions to the system (8) for the rolling TT. But as the analysis in [24] confirms,
these solutions are the asymptotic solutions to (9). A theorem of LaSalle type [24] shows that each solution to these equations, satisfying the contact criterion \( \hat{z} \cdot (a + s) = 0 \) and such that \( g_n(t) \geq 0 \) for \( t \geq 0 \), approaches exactly one solution to (16) as \( t \to \infty \). Thus the solution set to (16) can be seen as an asymptotic set for the inverting solutions of the TT equations.

This application of LaSalle’s theorem [17] says that trajectories of the system (9) approach the asymptotic set, which consists of spinning and tumbling solutions. If a TT satisfy \( 1 - \alpha < \gamma < 1 + \alpha \), the whole interval of \( \theta \in (0, \pi) \) is admissible as tumbling solutions. If moreover \( \lambda \) is above the threshold value \( \lambda_{\text{thres}} \), only the inverted position is stable. Thus we have conditions for when an inverted spinning solution is the only attractive asymptotic state in the asymptotic LaSalle set. Therefore a TT satisfying these conditions has to invert.

6 The Main Equation for the Tippe Top

If we consider a different model where the supporting surface is such that the TT is not gliding, then the force \( F \) at the contact point \( A \) is dynamically determined and the equations of motion (8) for the TT become equations for a rolling axisymmetric sphere. The problem of determining the motion of this well known [5] integrable system can be reduced to solving a separation equation in \( \theta \). In case of rolling and gliding TT a similar equation has been introduced in [23] where certain parameters are allowed to depend explicitly on time. This equation, which we call the Main Equation for TT, provides a way to study the dynamics of inversion for a rolling and gliding TT.

A purely rolling condition for TT implies that the gliding velocity vanishes, \( v_A = \dot{\theta} + \omega \times a = 0 \). Thus we have \( \dot{\theta} = -\omega \times a \), which allows to eliminate \( F \) from (8), so that we get the equations of motion:

\[
\frac{d}{dt} (I_3 \omega) = m a \times \left( -\frac{d}{dt} (\omega \times a) + g \hat{z} \right), \quad \dot{\hat{z}} = \omega \times \hat{z}. \tag{17}
\]

This system admits three integrals of motion; the Jellett integral

\[
\lambda = -L \cdot a = R I_1 \phi \sin^2 \theta - R I_3 \omega_3 (\alpha - \cos \theta),
\]

the energy

\[
E = \frac{1}{2} m R^2 \left[ (\alpha - \cos \theta)^2 (\dot{\theta}^2 + \phi^2 \sin^2 \theta) + \sin^2 \theta (\dot{\phi}^2 + \omega_3^2 + 2 \omega_3 \phi (\alpha - \cos \theta)) \right] + \frac{1}{2} \left[ I_1 (\phi^2 \sin^2 \theta + \dot{\phi}^2) + I_3 \omega_3^2 \right] + mg R (1 - \alpha \cos \theta), \tag{18}
\]

and the Routh integral

\[
D = \omega_3 \left[ I_1 I_3 + m R^2 I_3 (\alpha - \cos \theta)^2 + m R^2 I_1 \sin^2 \theta \right]^{1/2} := I_3 \omega_3 \sqrt{d(\cos \theta)}, \tag{19}
\]

where \( d(\cos \theta) = \gamma + \sigma (\alpha - \cos \theta)^2 + \sigma \gamma (1 - \cos^2 \theta), \sigma = \frac{m R^2}{I_3} \) and \( \gamma = \frac{I_1}{I_3} \).
The existence of three integrals of motion makes it possible to reduce the system (17) to a separable differential equation for $\theta$. We eliminate $\omega_3 =$ \( \frac{D}{I_3 \sqrt{d(\cos \theta)}} \) and $\dot{\phi} =$ \( \frac{\lambda \sqrt{d(\cos \theta)} + RD(a - \cos \theta)}{R I_1 \sin^2 \theta \sqrt{d(\cos \theta)}} \) from the energy (18), so we get essentially the Chaplygin separation equation [5]:

\[
E = g(\cos \theta) \dot{\theta}^2 + V(\cos \theta, D, \lambda),
\]  

(20)

where $g(\cos \theta) = \frac{1}{2} I_3 \left( \sigma \left( (a - \cos \theta)^2 + 1 - \cos^2 \theta \right) + \gamma \right)$ and

\[
V(z, D, \lambda) = m g R (1 - az) + \frac{(\lambda \sqrt{d(z)} + RD(a - z))^2}{2 R^2 I_1} + \frac{(R^2 D^2 - \sigma \lambda^2)}{2 R^2 I_1}.
\]

(21)

is the effective potential for the TT depending algebraically on $z = \cos \theta$.

The solution $\theta(t)$ of (20) stays between the two turning angles $\theta_0$ and $\theta_1$ determined from the equation $E = V(\cos \theta, D, \lambda)$. The solution curve $(\theta(t), \phi(t))$ will trace out oscillating trajectories between these angles on the unit sphere $S^2$.

For the rolling and gliding TT, $\lambda$ is still an integral of motion, but $D$ and the modified rolling energy $\tilde{E}$ (the part of the energy not involving $v_A$) are no longer integrals of motion. Due to equations (10)–(14), they satisfy

\[
\frac{d}{dt} D(\theta, \omega_3) = \frac{\gamma m}{\sqrt{d(\hat{z} \times a)}} (\hat{z} \times a) \times \hat{v}_A,
\]

\[
\frac{d}{dt} \tilde{E}(\theta, \dot{\theta}, \phi, \omega_3) = m (\omega \times a) \cdot \hat{v}_A.
\]

If we consider the (modified) energy $\tilde{E}$ and $D$ as given functions of time, we can make the same elimination of $\phi$ and $\omega_3$ from the energy expression as in the rolling case and get the Main Equation for the Tippe Top (METT):

\[
\tilde{E}(t) = g(\cos \theta) \dot{\theta}^2 + V(\cos \theta, D(t), \lambda).
\]

(22)

It has the same form as equation (20), but it depends explicitly on time through the functions $D(t)$ and $\tilde{E}(t)$. This is a first order time-dependent ODE which can be analysed given some quantitative knowledge about the functions $D(t)$ and $\tilde{E}(t)$.

Equation (22) describes, for generic values of $\lambda$ and $D$, oscillatory motion of the angle $\theta(t)$ much in the same way as the separation equation for the rolling TT (20). Since $D(t)$ is time-dependent, the potential is deformed during inversion. In this thesis we study the METT for inverting solutions in detail and we deduce behaviour of $\theta(t)$ for this type of solutions.

7 Special case of rational METT

The effective potential $V(z, D, \lambda)$ is an algebraic function in $z$ (due to the presence of $\sqrt{d(z)}$ in (21)), which is complicated to analyse. But in paper 2 we find
that if \(1 - \alpha^2 < \gamma < 1\), which is in the interval \((1 - \alpha, 1 + \alpha)\) needed for inversion, we can choose a positive value for the physical parameter \(\sigma = \frac{1 - \gamma}{\gamma + \alpha^2 - 1}\) such that \(V(z, D, \lambda)\) becomes rational:

\[
V(z, D, \lambda) = mgR(1 - \alpha z) + \left(\frac{\lambda(\alpha - (1 - \gamma)z) + RD\sqrt{\gamma + \alpha^2 - 1}(\alpha - z))^2}{2I_3R^2(\gamma + \alpha^2 - 1)(1 - z^2)} + \frac{R^2D^2(\gamma + \alpha^2 - 1) - (1 - \gamma)\lambda^2}{2R^2I_1(\gamma + \alpha^2 - 1)}\right).
\]

The advantage of a rational potential becomes obvious if we rearrange the separation equation as

\[
z^2 = \frac{(1 - z^2)(E - V(z, D, \lambda))}{g(z)} = f(z).
\]

Real motion of the rolling TT occurs when \(f(z) \geq 0, z \in [-1, 1]\), so analysis of nutational and precessional solutions for a rolling axisymmetric sphere is then reduced to analysis of roots to the rational function \(f(z)\) in this interval. The numerator is a third degree polynomial with negative \(z^3\)-coefficient.

When \(f(z)\) has two (unequal) roots \(z_0, z_1\) in \((-1, 1)\), it corresponds to nutational motion of the symmetry axis between two bounding latitudes on the unit sphere \(S^2\). Simple analysis shows that the third root is \(< -1\).

In paper 2 we look at high frequency behaviour of nutational motion for different initial conditions. Thus we assume that the angular velocity along the symmetry axis \(\omega_3\) is large and consider such quantities as the width of the nutational band, the frequency of nutation, the average velocity of precession and the difference between the energy and the minimum of the potential. How these quantities change as functions of \(\omega_3\) can be estimated in a precise but straightforward manner, both in the case of a TT with no initial precessional velocity \(\dot{\phi}(0) = 0\) (similar to the case of a falling HST [1, 12]) and also when this velocity is small.

The rational form of the potential \(V(z, D, \lambda)\) is further used in paper 3, where we look especially at how to show that an inverting solution \(\theta(t)\) to METT is oscillating. We find that we can estimate a maximal period of nutation for a TT with rational potential. When the time of inversion is an order of magnitude larger than this maximal period, the angle \(\theta(t)\) is oscillating during inversion.

\section{Numerical solving of the TT equations}

The aim of solving the dynamical system describing the TT equations numerically is to compare properties of the calculated solutions with the features deduced from the qualitative analysis of TT equations studied in this thesis.

For relating with the published results by Cohen [7] and Ueda et al. [26], we have chosen values of physical parameters close to the values used by these authors. They are summarized in table 1. Other published numerical results by

### Table 1: Physical parameter values used for TT

<table>
<thead>
<tr>
<th>Source</th>
<th>$m$ [kg]</th>
<th>$R$ [m]</th>
<th>$\alpha$</th>
<th>$l_3$ [kg·m²]</th>
<th>$l_1$ [kg·m²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our data</td>
<td>0.02</td>
<td>0.02</td>
<td>0.3</td>
<td>$\frac{3}{2} mR^2$</td>
<td>$\frac{131}{350} mR^2$</td>
</tr>
<tr>
<td>Cohen [7]</td>
<td>0.015</td>
<td>0.025</td>
<td>0.2</td>
<td>$\frac{7}{2} mR^2$</td>
<td>$\frac{7}{2} mR^2$</td>
</tr>
<tr>
<td>Ueda, Sasaki and Watanabe [26]</td>
<td>0.015</td>
<td>0.015</td>
<td>0.1</td>
<td>$\frac{2}{5} mR^2$</td>
<td>$\frac{2}{5} mR^2$</td>
</tr>
</tbody>
</table>

We present here numerical solutions for the special values of physical parameters of TT that give the rational potential $V(z, D, \lambda)$ studied in this thesis ($\sigma = (1 - \gamma) / (\gamma^2 - 1)$ gives $l_1 = \frac{131}{350} mR^2$). We take initial values $\theta(0) = 0.1$ rad, $\omega_3(0) = 155.0$ rad/s (corresponding to a value of $\lambda$ twice the threshold value), $\varphi(0) = 0$, $\dot{\varphi}(0) = 0$ and $\nu_z(0) = 0$. We further let $\mu = 0.3$. But the same calculations performed for close generic values of parameters, giving an algebraic (non-rational) potential, provide similar graphs. In paper 1, figure 4 we have plotted the surface $V(\cos \theta, D, \lambda)$ as function of $(\theta, D)$ for a TT with similar values of physical parameters, giving a non-rational potential, with the difference that $l_1 = \frac{47}{125} mR^2$ there.

The graph for the inclination angle $\theta(t)$ in figure 4.a displays the main feature of an inverting TT. It has a general form of logistic type curve superposed with small amplitude oscillations well visible in the diagram.

In the graphs 4.b and c, we have plotted $\dot{\psi}(t)$ and $\dot{\phi}(t)$ that, together with $\theta(t)$, give $\omega_3 = \dot{\psi} + \dot{\phi} \cos \theta$. These graphs show, in vicinity of time 3 seconds, a non-obvious phenomenon of sudden change of amplitude of $\psi$, $\dot{\phi}$ indicating rapid exchange of angular velocity between these two variables while $\omega_3$ changes very slowly and smoothly (figure 5.a).

An educated guess is that this phenomenon (also visible in corresponding graphs published in [7, 26]) marks initiation of inversion. The exact mechanism of interaction between forces and torques acting on TT remains unexplained yet. This behaviour of $\dot{\psi}(t)$ and $\dot{\phi}(t)$ draws attention to the neighborhood of time at approximately 3 seconds in the $\theta(t)$ graph, where we now notice that initially small oscillations acquire larger amplitude so that the nutational band for $\theta(t)$ almost extends to the north pole of the unit sphere $S^2$.

The graph in figure 5.a shows that $\omega_3(t)$ is an (surprisingly) almost monotonously decreasing function from the initial value of 155 rad/s to about $-85$ rad/s. It is unclear if the derivative $\omega_3(t)$ changes sign at all. The decreasing behaviour of $\omega_3$ reflects inversion of direction of axis $\dot{3}$ and the frictional loss of rotational energy. The curve $\omega_3(t)$ is in the beginning and at the end almost horizontal. During the rapid inversion of TT between 4 and 6.5 seconds the symmetry axis $\dot{3}$ turns upside down, so the projection $\omega \cdot \dot{3}$ of predominantly vertical angular velocity $\omega$ on $\dot{3}$ changes sign. The moment of initiation of inversion of TT at about 3 seconds noticed at graphs for $\dot{\psi}(t)$ and $\dot{\phi}(t)$ is also visible in the graphs 5.b for the
Figure 4: Plots obtained by numerically integrating equations (10)-(14) and equation (15) using the Python 2.7 open source library SciPy [14]. Parameter values are chosen from the first row of table 1 and $\mu = 0.3$. Initial values are $\theta(0) = 0.1 \text{ rad}$, $\omega_3(0) = 155.0 \text{ rad/s}$, $\varphi(0) = 0$, $\phi(0) = \theta(0) = 0$ and $\nu_x(0) = \nu_y(0) = 0$. Plot a shows the evolution of the inclination angle and plots b and c show the evolution of the angular velocities $\dot{\psi}(t)$ and $\dot{\phi}(t)$. 

\begin{align*}
\theta(t) & \quad \text{plot a) inclination angle} \\
\psi(t) & \quad \text{plot b) angular velocity} \\
\phi(t) & \quad \text{plot c) angular velocity}
\end{align*}
Figure 5: Plots obtained as in figure 4. Plot a shows the evolution of the angular velocity $\omega_3(t)$, plot b shows the gliding velocities $v_x(t)$ (blue) and $v_y(t)$ (green) and plot c show the evolution of the energy functions $E(t)$ (green) and $\tilde{E}(t)$ (blue).
Figure 6: Plot $a$ shows the curve $(D(t), \tilde{E}(t))$, calculated by integrating equations (10)-(14) and equation (15) as for figure 4. Plot $b$ shows the evolution of the angular velocity $\theta(t)$.

gliding velocities $v_x(t)$, $v_y(t)$. Both velocities are oscillatory but initially $v_x$ (blue) is larger than $v_y$ (green). In a neighborhood of time 3 seconds $v_y$ becomes larger than $v_x$ and is rapidly, oscillatory increasing by 1–2 orders of magnitude when the inversion takes place. During the inversion the amplitude of $v_x$ also radically increases an order of magnitude but $v_y$ becomes predominantly negative. The loss of total energy during inversion is illustrated in graph $5.c$ where the green curve represents monotonous decrease of the total energy $\dot{E} = -\mu g n |v_A|^2$ and the blue curve represents the modified rotational energy $\tilde{E}(t)$, which is during the inversion consistently larger than $E(t)$. This does not contradict the conservation of energy when we look closer at $\dot{E} = E - \frac{1}{2}m v_A^2 + mv_A \cdot (\omega \times a)$. For an inverting TT the angular velocity $\omega$ remains close to be parallel with the $\hat{z}$ axis and the product $\omega \times a$ points behind the plane of the picture in figure 3. The direction of rotation of TT (with $\omega$ almost parallel to $\hat{z}$) causes $v_y = v_A \cdot \hat{y}$ to be positive and to point also behind the plane of the figure.
Thus the term
\[ \frac{1}{2}mv_A \cdot (2(\omega \times a) - v_A) \approx \frac{1}{2}m(v_x \hat{x} + v_y \hat{y}) \cdot (2(\omega \times a) \hat{y} - (v_x \hat{x} + v_y \hat{y})) \]
\[ \approx -\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y(2|\omega \times a| - v_y) \]
may be positive if \( v_y > v_x \) and \( 2|\omega \times a| - v_y = 2|\omega||a| \sin \theta - v_y > 0 \) is sufficiently large. In figure 5.b we cannot estimate the size of each term, but as we see in figure 3, \(|a|\) is growing during inversion, \(|\omega|\) is decreasing and \(\sin \theta\) is growing until the axis \(\hat{3}\) passes \(\theta = \frac{\pi}{2}\). This may altogether keep the term \(2|\omega \times a| - v_y\) positive and sufficiently large to make (23) positive. This is consistent with the graph 5.c where the difference \(\tilde{E} - E\) becomes largest in the middle of inversion when \(\theta \approx \frac{\pi}{2}\).

The figure 6.a illustrates the curve \((D(t), \tilde{E}(t))\) when it goes from the boundary value at the start of TT inversion \((D_0 \approx 9.381 \cdot 10^{-4}, \tilde{E}_0 \approx 4.912 \cdot 10^{-2})\) to the final boundary value \((D_1 \approx -6.667 \cdot 10^{-4}, \tilde{E}_1 \approx 1.734 \cdot 10^{-2})\) after inverting. The shape of the curve reflects the oscillatory behaviour of \(\tilde{E}(t)\) while \(D(t)\) is decreasing almost monotonously since \(D(t) = I_3\omega_3(t)\sqrt{d(\cos \theta(t))}\).

The last graph in figure 6.b shows how the angular velocity \(\dot{\theta}(t)\) is changing between positive and negative values, thus confirming the oscillatory character of \(\theta(t)\) during inversion.

## 9 Conclusions

In this thesis we have approached the question of qualitative description of dynamics of the TT. The general question of deducing dynamics from equations of motion for a system with 6 degrees of freedom is an impossible task unless equations are integrable.

Since the TT equations are not integrable it is only possible to approach this question for specific types of initial conditions where the main features of the dynamical behaviour are weakly dependent on perturbation of initial conditions.

This is the case of inverting behaviour, which is persistent and easily reproducible for a toy TT. So the point is to explain the features of dynamical equations and of initial conditions that are responsible for the observable effect.

In this thesis we have shown that this is a consequence of the METT equation that is fulfilled by every solution of TT equations with suitable \(D(t), \tilde{E}(t)\) that can be found for any given solution.

In case of inverting TT motion, \(D(t)\) and \(\tilde{E}(t)\) are not exactly known, but we can qualitative describe how they are changing from the initial values \((D_0, \tilde{E}_0)\) to the final values \((D_1, \tilde{E}_1)\).

As usual this work opens more questions than it answers. First, all estimates have been made in the case of rational potential \(V(z, D, \lambda)\), not an algebraic one. The restriction on parameters, which make \(\sqrt{d(z)}\) polynomial in papers 2 and 3 allow us to get estimates of relevant quantities describing dynamics of an
inverting TT. An extension to algebraic \( V(z, D, \lambda) \) seems to be feasible, however technically more complicated.

The connection between the choice of friction law, the value of the friction coefficient \( \mu \) and the time of inversion \( T_{\text{inv}} \) has not been discussed here.

The numerical simulations agree well with the analytically deduced features of inverting solutions and these numerical solutions preserve its character for an open set of values about the values taken here. This is in agreement with the persistent nature of inversion observed in toy experiments.

These numerical solutions indicate also that there may exist a more specific mechanism of how friction works for TT. The graphs show a clear initiation time for an inversion when \( \dot{\psi}, \dot{\phi} \) start to oscillate strongly and \( v_x, v_y \) change signs. The mathematical nature of this mechanism remains unclear, but the mechanism seems to be generic and somehow "has to sit" in the equations.

In this thesis we have clarified the logic of the LaSalle type statements. We have explained what they say about inversion of TT and what they does not say about inversion. We have cleared up the relation of the TT equations to the METT and to equations of two other rigid bodies. We have found that the parameter limit \( R_\alpha = -l, R \to \infty \) of TT equations gives equations for the gliding heavy symmetric top and that for initial conditions with \( D = 0, \lambda = 0 \) we recover equations for a gliding eccentric cylinder from the TT equations. For better understanding of how to work with TT equations we have analysed the rational case.

Finally we have shown that inverting solutions \( \theta(t) \) with \( D(t), \dot{E}(t) \) satisfying the boundary conditions \( (D(0) \approx D_0, \dot{E}(0) \approx \dot{E}_0) \) and \( (D(T_{\text{inv}}) \approx D_1, \dot{E}(T_{\text{inv}}) \approx \dot{E}_1) \) have to oscillate provided that the time of inversion is sufficiently long.

### References


