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Optimal Power Allocation for Hybrid ARQ with Chase Combining in i.i.d. Rayleigh Fading Channels

Tumula V. K. Chaitanya and Erik G. Larsson

Abstract—We consider the optimization of Chase combining (CC)-based hybrid-automatic repeat request (HARQ) schemes with a limit on the maximum number of retransmissions. We formulate two optimization problems: (i) minimizing the packet drop probability (PDP) under a total average transmit power constraint, and (ii) minimizing the average transmit power under a fixed PDP constraint. Towards solving these equivalent optimization problems, we provide a closed-form expression for the outage probability of a CC-HARQ scheme. We then show that solving the optimization problems using an exact expression of the outage probability becomes complex with an increase in the maximum number of retransmissions. We propose an alternative approach in which we approximate the optimization problems by using an approximate outage probability expression and formulate the two optimization problems as two equivalent geometric programming problems (GPPs), which can be solved efficiently even for a large limit on the maximum number of retransmissions.

The results show that the optimal power allocation solution provides significant gains over the equal power allocation solution. For PDP values below 10^{-3} , the optimal solution provided by the GPP approach has a performance close to that of the solution provided by solving the optimization problem exactly using nonlinear optimization techniques.

Index Terms—Hybrid-ARQ, diversity combining, optimal power allocation, outage probability, geometric programming.

I. INTRODUCTION

Fading in wireless channels causes the signal strength to vary at the receiver, and this results in a loss of transmitted data packets. Diversity is an approach used in wireless systems to combat the effects of fading and thereby to provide reliable data transfer [1]. In many practical systems like WiMAX and 3GPP-LTE, retransmission schemes are employed to mitigate the effects of fading and perform link adaptation in the absence of perfect channel knowledge at the transmitter. Early versions of retransmission schemes were called automatic repeat request (ARQ) schemes. These schemes are used when a feedback channel from the receiver to the transmitter is available. Many types of ARQ schemes have been proposed for efficient usage of resources [2]. However, ARQ schemes suffer from a reduction in throughput. In hybrid-ARQ (HARQ) schemes, the throughput performance is improved by combining conventional ARQ mechanisms with forward error correction (FEC) schemes. There are many different HARQ schemes

proposed in the literature, and they are mainly classified into two types [3]. In *type-I HARQ schemes*, the receiver does not combine the information across different (re)transmissions. By way of contrast, in *type-II HARQ schemes*, the receiver combines the information from different (re)transmissions to decode the packet. These *type-II HARQ schemes* are further classified into:

- *Chase Combining (CC) Schemes*: These are *type-II HARQ schemes* in which all the retransmissions carry the same coded bits [4]. The receiver uses maximum-ratio-combining (MRC) to decode the data packet.
- *Incremental Redundancy (IR) Schemes*: In these *type-II HARQ schemes*, additional parity bits are sent to the destination during the retransmissions, thereby effectively forming a longer codeword with a smaller rate [5], [6]. The receiver uses code combining to decode the data packet.

One disadvantage of retransmission schemes in practice is that additional delays are introduced in the system. However in many practical systems, the number of retransmissions allowed is limited to avoid an unacceptable time delay before the successful transmission of a packet. In this work, we assume that the maximum number of transmissions for a packet is limited to L , i.e., if the destination is not able to decode a packet after L transmissions, the packet is dropped.

A. Related Work

Recently, adaptive as well as optimal resource allocation for HARQ schemes has attained much research interest [7] – [17]. Performance metrics such as throughput, energy efficiency in circuitry, outage probability, delay etc. were considered as the objectives for the optimization. Related to the adaptive resource allocation schemes, the transmission rate adaptation was shown to improve the energy efficiency in the circuitry [7]. An adaptive power ramping scheme was proposed in [8] to improve the throughput. In case of optimal resource allocation, a general approach for joint power and rate allocation under delay and packet-overflow constraints for type-I HARQ systems was presented in [9]. An optimal bit and power allocation for an adaptive modulated OFDM system with ARQ retransmissions was presented in [10].

In wireless channels, providing the transmitter with instantaneous channel state information (CSI) is difficult. Optimization of resources for HARQ schemes very much depends on the form of CSI available at the transmitter. Recent research related to HARQ schemes also focused on improving the

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performance of HARQ schemes along with methods to provide efficient CSI feedback. Optimal power allocation for improving the average rate performance of HARQ schemes was presented in [11], [12] for quasi-static fading channels with different forms of CSI feedback. Similarly, throughput performance of IR-HARQ schemes over block fading channels was studied in [13], where the feedback bits were split between the request for retransmission and providing the CSI to the transmitter.

Conventionally, performance of HARQ schemes is evaluated under the assumption of the same transmission power in different ARQ rounds, for example as in [7]. In [14], [15], the optimization problem of minimizing the PDP under an average transmit power constraint was considered for IR-HARQ schemes, and the results were presented for the special case of $L = 2$. An optimal power allocation scheme for a space-time coded HARQ scheme was presented in [16]. Our work is mainly inspired by [17], in which the optimal power assignment for minimizing the average total transmission power was considered for CC-based HARQ schemes. However in the work of [17], a quasi-static Rayleigh fading channel assumption was used in the analysis.¹ This assumption may not be practical in the sense that many practical systems have a significant time delay between the original transmission and the retransmissions of a packet, and hence the channel gains often become independent in different ARQ rounds.

B. Contributions

In this work, we consider the problem of optimal power allocation for CC-HARQ schemes under the assumption of statistical channel state information at the transmitter (CSIT) and also assuming that different ARQ rounds experience i.i.d. Rayleigh fading channel gains. The statistical CSIT assumption is practical as it requires only a modest amount of feedback from the receiver. The i.i.d. Rayleigh fading channel assumption also fits well with HARQ schemes in practical mobile wireless systems. The specific optimization problems we are interested in are:

- *Problem 1:* Minimizing the PDP under an average transmit power constraint.
- *Problem 2:* Minimizing the average transmit power under a specified PDP.

Towards solving these equivalent optimization problems, using a result from [22], we provide a closed-form expression for the outage probability of a CC-HARQ scheme with a fixed value of L . We then show that the optimization problems can be solved using standard nonlinear optimization techniques. However finding the optimal solution becomes complex as the value of L increases. We then approximate the original optimization problems by using an approximation of the outage probability expression. This approximation allows us to formulate the optimization problems as geometric programming problems (GPPs) which can be solved efficiently even for large values of L . We also prove that, to obtain a fixed target PDP value with two different spectral efficiency values, it is

¹ All the ARQ transmission rounds experience the same channel gain.

Figure 1. System model for the CC-HARQ scheme considered in this work.

sufficient to solve the optimum power allocation problem for one of the spectral efficiency values and scale the so-obtained optimum power allocation values to obtain the solution for the other case. We also show that, when the average channel gain value is fixed, and to achieve a fixed PDP value, the change required in the average power for a given change in the spectral efficiency value is independent of the power allocation method and also independent of the limit on the maximum number of retransmissions.

The numerical results show that the optimal power allocation solution provides significant gains over the equal power allocation solution. The suggested approximation using the GPP approach works well for the case of $L = 2$. For the values of $L > 2$, the GPP approach provides gains over the equal power allocation for the values of PDP below 10^{-3} . In the region where the values of PDP are below 10^{-3} , the optimal power allocation provided by the GPP approach has performance close to that of the solution provided by solving the optimization problem exactly using nonlinear optimization techniques. The PDP values of 10^{-3} and below are still relevant in practical systems using HARQ schemes [30], and hence the suggested approximation can still be useful for finding a low complexity solution in this region. To the best of our knowledge, the problem of optimal power allocation in i.i.d. Rayleigh fading channels has not been solved before. Our work differs from the works [11] – [13] in that, we consider the minimization of outage probability or the minimization of average transmit power as the objectives with only statistical knowledge of the channel at the transmitter.

The rest of the paper is organized as follows. In Section II, we describe the system model considered and formulate the two optimization problems. In Section III, we present a closed-form expression for the outage probability of a CC-HARQ scheme, and discuss about the limitations of solving the optimization problems. In Section IV, we derive an approximation to the outage probability expression, and using this, we formulate the two optimization problems as GPPs in Section V. Finally, we present some numerical results in Section VI and conclusions in Section VII.

II. SYSTEM MODEL

The system model considered for a CC-HARQ scheme is shown in Fig. 1. N_I information bits $b_1, b_2, \dots, b_{N_I}$ are en-

coded to obtain a code-block with coded bits $c_1, c_2, \dots, c_{N_C}$, where N_C denotes the number of coded bits. These N_C coded bits are then mapped onto a constellation $\mathcal{S}$ to obtain N modulation symbols $s_1, s_2, \dots, s_N$. These modulation symbols are then transmitted to the destination using a CC-HARQ scheme. The number of transmission rounds for the CC-HARQ scheme is limited to L . The destination uses MRC to combine the packets it received in different ARQ rounds. We also assume that one modulation symbol is transmitted per channel use in each ARQ round, i.e., we have N channel uses in each ARQ round.² In [15], for the case of IR-HARQ (as opposed to CC-HARQ considered here), we showed that using different channel uses in each ARQ round did not result in any appreciable gain in terms of minimizing the PDP.³ We assume that the modulation symbols have unit average energy, i.e., $\mathbb{E}|s_n|^2 = 1$. We consider a block-fading Rayleigh channel in which the channel gain remains constant in each ARQ round and varies independently between different ARQ rounds. Let h_l denote the channel gain for the l th ARQ round, and let $\lambda = \mathbb{E}|h_l|^2, \forall l = 1, 2, \dots, L$.⁴ We assume that the transmitter has statistical knowledge of the channel gains, i.e., it has the knowledge about their distribution. This assumption is practical as it does not require instantaneous channel gain feedback from the receiver. We also assume that the same transmission power is used in all the channel uses of the l th ARQ round and it is equal to P_l .

With the above described system model, we can write the received signal at the destination during the i th channel use of the l th ARQ round as:

$$y_{l,i} = \sqrt{P_l} h_l s_{l,i} + e_{l,i}, \quad l = 1, 2, \dots, L, i = 1, 2, \dots, N \quad (1)$$

where $s_{l,i} \in \mathcal{S}$ and $e_{l,i}$ denote the modulated symbol and the additive white Gaussian noise sample in the i th channel use of the l th ARQ round, respectively. We assume that $e_{l,i}, \forall l = 1, 2, \dots, L$ and $i = 1, 2, \dots, N$ are i.i.d. with distribution $\mathcal{CN}(0, 1)$. Assuming a Gaussian codebook with a sufficiently large value of N , the probability of a decoding error at each ARQ round can be well approximated by the information outage probability [18]. The information outage probability is defined as the probability that the instantaneous spectral efficiency is smaller than the target spectral efficiency (or equivalently the total accumulated mutual information is smaller than the total number of information bits). Under the assumptions made, we can write the total accumulated mutual

information at the destination till the l th ARQ round as:⁵

$$I_l = N \log_2 \left(1 + \sum_{m=1}^l P_m |h_m|^2 \right) = N \log_2 \left(1 + \sum_{m=1}^l \alpha_m \right) \quad (2)$$

where $\alpha_m = P_m |h_m|^2$ with $\alpha_m \sim \exp(P_m \lambda), \forall m = 1, 2, \dots, l$. We assume that the feedback signaling from the receiver is based on the total accumulated mutual information at the destination. The receiver sends an acknowledgement (ACK) signal after the l th ARQ round if $I_l \geq N_I$ and a negative ACK (NACK) signal otherwise. We also assume that the ACK/NACK feedback signaling from the receiver to the transmitter is error-free. The probability that the packet is in outage after l ARQ rounds can be written as [19]:

$$p_{\text{out},l} = \Pr(I_l < N_I) \quad (3)$$

and the PDP, which is the probability that the packet is in outage after L ARQ rounds is given by:

$$p_{\text{drop}} \triangleq p_{\text{out},L} = \Pr(I_L < N_I) \quad (4)$$

Let $p_{\text{out},0} = 1$, by default. The average transmission power is defined as:

$$P_{\text{avg}} \triangleq \sum_{l=1}^L P_l p_{\text{out},l-1} \quad (5)$$

Now we formulate the two optimization problems of interest.

Problem 1: Here, the objective is to minimize the packet drop probability under an average transmit power constraint. This problem can be formulated as follows:⁶

$$\begin{aligned} & \min_{(P_1, P_2, \dots, P_L)} p_{\text{drop}} \\ \text{Subject to} \quad & 0 \leq P_l, \quad \text{for } 1 \leq l \leq L \\ & P_{\text{avg}} = \sum_{l=1}^L P_l p_{\text{out},l-1} \leq \bar{P}_{\text{given}} \end{aligned} \quad (6)$$

Problem 2: Here, the objective is to minimize the average transmit power under a fixed packet drop probability constraint. This can be formulated as:

$$\begin{aligned} & \min_{(P_1, P_2, \dots, P_L)} P_{\text{avg}} \\ \text{Subject to} \quad & 0 \leq P_l, \quad \text{for } 1 \leq l \leq L \\ & p_{\text{drop}} \leq p_{\text{drop}}^{\text{target}} \end{aligned} \quad (7)$$

The two optimization problems in (6) and (7) are equivalent as shown in Appendix A.

²For CC-HARQ, N is equal for all transmissions by construction.

³Using optimal allocation of channel uses might bring in gains if one considers other performance metrics such as the maximization of throughput or the minimization of delay.

⁴Extending the analysis to the case with different average channel gains in different ARQ rounds, i.e., $\lambda_l = \mathbb{E}|h_l|^2, \forall l = 1, 2, \dots, L$ is straightforward but requires some additional notation.

⁵This assumes a Gaussian codebook. However, for many practical schemes with adaptive modulation and coding, we can write $I_l \approx N \log_2 \left(1 + \sum_{m=1}^l \zeta_m P_m |h_m|^2 \right)$ for $l = 1, 2, \dots, L$, where ζ_m are penalty factors that represent the distance from the Shannon capacity [20]. For the treatment of information outage probability of finite block-length codes in the context of ARQ, see [21].

⁶We can write the expression for the average energy spent per data packet as $E_{\text{avg}} \triangleq N \sum_{l=1}^L P_l p_{\text{out},l-1}$. Using this, we can equivalently consider the optimization problem with a constraint on the energy spent.

$$p_{\text{out},1} = 1 - e^{-\left(\frac{Z}{P_1\lambda}\right)} \quad (15)$$

$$p_{\text{out},2} = \begin{cases} p_{\text{out},2}^{\text{equal}} \triangleq 1 - e^{-\left(\frac{Z}{P_1\lambda}\right)} \left[1 + \frac{Z}{P_1\lambda}\right], & \text{if } P_1 = P_2 \\ p_{\text{out},2}^{\text{diff}} \triangleq 1 - \left[\frac{P_1 e^{-\left(\frac{Z}{P_1\lambda}\right)} - P_2 e^{-\left(\frac{Z}{P_2\lambda}\right)}}{P_1 - P_2} \right], & \text{if } P_1 \neq P_2 \end{cases} \quad (16)$$

$$p_{\text{out},3} = \begin{cases} p_{\text{out},3}^{\text{equal}} \triangleq 1 - e^{-\left(\frac{Z}{P_1\lambda}\right)} \left[1 + \frac{Z}{P_1\lambda} + \frac{Z^2}{2P_1^2\lambda^2}\right], & \text{if } P_1 = P_2 = P_3 \\ p_{\text{out},3}^1 \triangleq 1 - \left[e^{-\left(\frac{Z}{P_1\lambda}\right)} \left(-\frac{P_1 P_3}{(P_1 - P_3)^2} + \frac{P_1}{P_1 - P_3} \left(1 + \frac{Z}{P_1\lambda}\right) \right) + \frac{P_3^2}{(P_3 - P_1)^2} e^{-\left(\frac{Z}{P_3\lambda}\right)} \right], & \text{if } P_1 = P_2 \neq P_3 \\ p_{\text{out},3}^2 \triangleq 1 - \left[\frac{P_1^2}{(P_1 - P_2)^2} e^{-\left(\frac{Z}{P_1\lambda}\right)} + e^{-\left(\frac{Z}{P_2\lambda}\right)} \left(-\frac{P_1 P_2}{(P_2 - P_1)^2} + \frac{P_2}{P_2 - P_1} \left(1 + \frac{Z}{P_2\lambda}\right) \right) \right], & \text{if } P_1 \neq P_2 = P_3 \\ p_{\text{out},3}^3 \triangleq 1 - \left[e^{-\left(\frac{Z}{P_1\lambda}\right)} \left(-\frac{P_1 P_2}{(P_1 - P_2)^2} + \frac{P_1}{P_1 - P_2} \left(1 + \frac{Z}{P_1\lambda}\right) \right) + \frac{P_2^2}{(P_2 - P_1)^2} e^{-\left(\frac{Z}{P_2\lambda}\right)} \right], & \text{if } P_1 = P_3 \neq P_2 \\ p_{\text{out},3}^{\text{diff}} \triangleq 1 - \left[\frac{P_1^2 e^{-\left(\frac{Z}{P_1\lambda}\right)}}{(P_1 - P_2)(P_1 - P_3)} + \frac{P_2^2 e^{-\left(\frac{Z}{P_2\lambda}\right)}}{(P_2 - P_1)(P_2 - P_3)} + \frac{P_3^2 e^{-\left(\frac{Z}{P_3\lambda}\right)}}{(P_3 - P_1)(P_3 - P_2)} \right], & \text{if } P_1 \neq P_2 \neq P_3 \end{cases} \quad (17)$$

III. OUTAGE ANALYSIS AND SOLUTION TO THE OPTIMIZATION PROBLEMS

In this section, we first provide a closed-form expression for $p_{\text{out},l}$ for $1 \leq l \leq L$. From (2) and (3), we have:

$$\begin{aligned} p_{\text{out},l} &= \Pr \left(N \log_2 \left(1 + \sum_{m=1}^l \alpha_m \right) < N_I \right) \\ &= \Pr \left(\sum_{m=1}^l \alpha_m < \underbrace{2^{\delta} - 1}_{\triangleq Z} \right) \end{aligned} \quad (8)$$

where $\delta \triangleq N_I/N$ is spectral efficiency per ARQ round expressed in bits per channel use (bpcu). In (8), $\Theta \triangleq \sum_{m=1}^l \alpha_m$ is a sum of l independent and non-identically distributed (i.n.i.d) exponential random variables. Using a result from [22], we can express $p_{\text{out},l}$ as:

$$p_{\text{out},l} = 1 - \sum_{i=1}^{M_l} \sum_{j=1}^{\tau_{i,l}} \sum_{k=0}^{j-1} \frac{\chi_{i,j}(\mathbf{P}_l)}{k!} \left(\frac{Z}{P_{\langle i \rangle}} \right)^k e^{-\left(\frac{Z}{P_{\langle i \rangle}}\right)} \quad (9)$$

where $\mathbf{P}_l = \text{diag}(P_1\lambda, P_2\lambda, \dots, P_l\lambda)$, and M_l , $1 \leq M_l \leq l$ is the number of distinct diagonal elements of $\mathbf{P}_l$. $P_{\langle 1 \rangle}, P_{\langle 2 \rangle}, \dots, P_{\langle M_l \rangle}$ denote these distinct diagonal elements of $\mathbf{P}_l$. $\tau_{i,l}$ denotes the multiplicity of $P_{\langle i \rangle}$. $\chi_{i,j}(\mathbf{P}_l)$ is the (i,j) th characteristic coefficient of $\mathbf{P}_l$ such that [23]:

$$\begin{aligned} \det(\mathbf{I} + \nu \mathbf{P}_l)^{-1} &= \frac{1}{(1 + \nu P_{\langle 1 \rangle})^{\tau_{1,l}} \dots (1 + \nu P_{\langle M_l \rangle})^{\tau_{M_l,l}}} \\ &= \sum_{i=1}^{M_l} \sum_{j=1}^{\tau_{i,l}} \frac{\chi_{i,j}(\mathbf{P}_l)}{(1 + \nu P_{\langle i \rangle})^j}. \end{aligned} \quad (10)$$

where ν is a scalar constant such that $\mathbf{I} + \nu \mathbf{P}_l$ is nonsingular. We can write

$$\chi_{i,j}(\mathbf{P}_l) = \left(\frac{-1}{P_{\langle i \rangle}} \right)^{\omega_{i,j}} \sum_{\substack{k_1 + \dots + k_{M_l} = \omega_{i,j} \\ k_i = 0 \\ k_n \in \{0, \omega_{i,j}\} \text{ for } n \neq i}} \prod_{\substack{n=1 \\ n \neq i}}^{M_l} \binom{\tau_{n,l} + k_n - 1}{k_n}$$

$$\left(\frac{P_{\langle n \rangle}^{k_n}}{\left(1 - \frac{P_{\langle n \rangle}}{P_{\langle i \rangle}}\right)^{\tau_{n,l} + k_n}} \right) \quad (11)$$

where $\omega_{i,j} = \tau_{i,l} - j$. For the special case of equal power allocation $P_i = P$ for $1 \leq i \leq l$, we have $M_l = 1$, $\tau_{1,l} = l$, and we can simplify (9) as:

$$p_{\text{out},l}^{\text{equal}} = 1 - \sum_{k=0}^{l-1} \frac{1}{k!} \left(\frac{Z}{P\lambda} \right)^k e^{-\frac{Z}{P\lambda}}. \quad (12)$$

Similarly, when all the transmission powers P_i in different ARQ rounds are distinct, we have $M_l = l$, and $\tau_{i,l} = 1$, $1 \leq i \leq l$. For this case, the characteristic coefficients can be written as [23]:

$$\chi_{i,1}(\mathbf{P}_l) = \prod_{\substack{j=1 \\ j \neq i}}^l \left(1 - \frac{P_j}{P_i} \right)^{-1}, \quad 1 \leq i \leq l \quad (13)$$

Substituting (13) in (9), we have:

$$p_{\text{out},l}^{\text{diff}} = 1 - \sum_{i=1}^l \left\{ \prod_{\substack{j=1 \\ j \neq i}}^l \left(1 - \frac{P_j}{P_i} \right)^{-1} \right\} e^{-\frac{Z}{P_i\lambda}}. \quad (14)$$

In general, some of the powers P_i , $1 \leq l \leq l$ may be equal and some may be distinct. For example, in the general case, the expressions for $p_{\text{out},l}$ for $l = 1, 2$, and 3 are shown at the top of this page. As exemplified in (15)–(17), as the value of l is increasing, $p_{\text{out},l}$ will have $2^l - l$ different cases depending on the combinations of P_i , for $1 \leq i \leq l$. Clearly, this makes it complex to solve *Problem 1* (or equivalently *Problem 2*) using (9). More specifically, in order to solve *Problem 1* for a fixed value of L , we need to consider $2^L - L$ different subproblems each with corresponding constraints and choose the solution of the subproblem which gives the minimum among them as the solution to *Problem 1*. It is important to note that even though $p_{\text{out},3}^1, p_{\text{out},3}^2$, and $p_{\text{out},3}^3$ in (17) have similar expressions, when solving the subproblems, each of them have different feasible regions.

As the limit on the number of retransmissions L increases, writing all the $2^L - L$ subproblems each with a different objective function and a different feasible set and solving each of these subproblems (some of which needs to be further divided if they involve inequality constraints) also becomes difficult.⁷ Hence in the next section, we provide an approximation to $p_{\text{out},l}$, for $1 \leq l \leq L$ and this allows us to approximate the problems in (6), (7) and formulate them as GPPs with a single objective function for the entire feasibility region.

IV. APPROXIMATION TO OUTAGE PROBABILITY EXPRESSION

In this section we provide an approximation to the outage probability expression in (9). We then use this approximated expression in (6) and (7). Before deriving the approximation for the outage probability, we state the following Lemma.

Lemma 1. For given distinct real numbers $a_1, \dots, a_M$ and non-negative integers $\tau_1, \dots, \tau_M$ with $\tau_k \geq 1$ for $1 \leq k \leq M$ such that $\sum_{k=1}^M \tau_k = L$, the partial fraction expansion coefficients $b_{k,j}$, for $1 \leq k \leq M$ and $1 \leq j \leq \tau_k$ in the following expansion

$$\frac{1}{(1 + xa_1)^{\tau_1} \dots (1 + xa_M)^{\tau_M}} = \sum_{k=1}^M \sum_{j=1}^{\tau_k} \frac{b_{k,j}}{(1 + xa_k)^j}, \quad (18)$$

satisfy the following $L + 1$ linear relations

$$\sum_{k=1}^M \sum_{j=1}^{\tau_k} b_{k,j} = 1 \quad (19)$$

and

$$A_i \triangleq \sum_{k=1}^M \sum_{j=1}^{\min(i, \tau_k)} \frac{(-1)^{j-1} \binom{i-1}{j-1} b_{k,j}}{a_k^i} = \begin{cases} 0 & \text{for } i = 1, 2, \dots, L-1 \\ \frac{(-1)^{L-1}}{a_1^{\tau_1} a_2^{\tau_2} \dots a_M^{\tau_M}} & \text{for } i = L \end{cases} \quad (20)$$

Proof: Please see Appendix B. ■

Remark: For given values of $P_1, \dots, P_l$ and λ , we can use any l linear equations from the system of $l+1$ relations in (19) and (20) to solve for the characteristic coefficients $\chi_{i,j}(\mathbf{P}_l)$.

Theorem 1. $p_{\text{out},l}$, for $1 \leq l \leq L$ in (9) can be written as:⁸

$$p_{\text{out},l} = \frac{Z^l}{l! (\lambda^l P_1 P_2 \dots P_l)} - \underbrace{\sum_{i=1}^{M_l} \sum_{m=l+1}^{\infty} \frac{1}{m!} \left(\frac{-Z}{P_{(i)}} \right)^m \left[\sum_{j=1}^{\tau_{i,l}} \sum_{k=0}^{j-1} \frac{\chi_{i,j}(\mathbf{P}_l)}{k!} \left(\frac{Z}{P_{(i)}} \right)^k \right]}_{\triangleq \Xi} = \frac{Z^l}{l! (\lambda^l P_1 P_2 \dots P_l)} + O\left(\frac{1}{P_{\min}^{l+1}}\right) \quad (21)$$

⁷The $2^L - L$ different objective functions correspond to $2^L - L$ different cases for $p_{\text{out},L}$ depending on whether some of the $P_l, 1 \leq l \leq L$ values are the same or not.

⁸ $f(x) = O(g(x))$ means that there exist $\Omega \in \mathcal{R}$ and $M \in \mathcal{R}$ such that $\frac{|f(x)|}{|g(x)|} \leq M$ whenever $x > \Omega$.

Figure 2. Plot showing the exact outage probability using (9) as well as approximate outage probability using the expression in (22) after different ARQ rounds as a function of channel power-to-noise ratio (λ). Parameters used are $\delta = 5$ bpcu, $L = 4$. Normalized power values are $P_1 = P_2 = 0$ dB and $P_3 = P_4 = 10$ dB.

where $Z = 2^\delta - 1$, and $P_{\min} \triangleq \min(P_1, P_2, \dots, P_l)$.

Proof: Please see Appendix C. ■

It can be proved directly from (12) that if the same power is allocated in all ARQ rounds, i.e., if $P_1 = P_2 = \dots = P_l = P$, then $p_{\text{out},l}^{\text{equal}} = \frac{1}{l!} \left(\frac{Z}{P\lambda} \right)^l + O\left(\frac{1}{P^{l+1}}\right)$. One can also use an alternate method using Vandermonde matrix determinants and prove from (14) that $p_{\text{out},l}^{\text{diff}} = \frac{Z^l}{l! (\lambda^l P_1 P_2 \dots P_l)} + O\left(\frac{1}{P_{\min}^{l+1}}\right)$.⁹ We approximate $p_{\text{out},l}$ by neglecting the $O(\cdot)$ term from (21) in the further analysis, i.e., we write

$$p_{\text{out},l} \approx \frac{Z^l}{l! (\lambda^l P_1 P_2 \dots P_l)}. \quad (22)$$

The motivation for doing the approximation in (22) is that i) after l ARQ rounds, the maximum achievable diversity order in an i.i.d. fading environment is l and the approximation in (22) also achieves this diversity order ii) the term Ξ in (21) approaches zero faster than the term $\frac{Z^l}{l! (\lambda^l P_1 P_2 \dots P_l)}$ as $P_{\min} \rightarrow \infty$ iii) The approximation also gives us a single objective function for the entire feasibility region as it is independent of $\chi_{i,j}(\mathbf{P}_l)$, this is not the case for the exact outage probability expression in (9). To show the validity of the approximation in (22), we compare the exact and approximate outage probability as a function of channel power-to-noise ratio (CNR) values (λ) in Fig. 2. In the figure, we have plotted outage probability after each ARQ round for the case of $L = 4$, (i) computed using the expression in (9) and (ii) by using the approximate expression in (22). As we can see from the figure, the approximation in (22) is close to the exact value at high CNR values for all the ARQ rounds. Even though a formal mathematical proof is still elusive, we conjecture that Ξ in (21) is non-negative and its magnitude is smaller than $\frac{Z^l}{l! (\lambda^l P_1 P_2 \dots P_l)}$, and hence the approximation in (22) is an upper bound on $p_{\text{out},l}$, this can also be seen from Fig. 2.

⁹The alternate proofs for these special cases are not provided here because of the space constraints.

V. GEOMETRIC PROGRAMMING FORMULATION

In this section, we use the approximation in (22) to formulate the optimization problems in (6) and (7) as GPPs. Using the approximation in (22), we can rewrite the Problems 1 and 2 equivalently as follows.

Problem 1':

$$\begin{aligned} & \min_{(P_1, P_2, \dots, P_L)} \frac{Z^L}{L! \lambda^L P_1 P_2 \dots P_L} \\ \text{Subject to } & 0 \leq P_l, \quad \text{for } 1 \leq l \leq L \\ & \sum_{l=1}^L P_l \frac{Z^{l-1}}{(l-1)! (\lambda^{l-1} P_1 P_2 \dots P_{l-1})} \leq \bar{P}_{\text{given}} \end{aligned} \quad (23)$$

and

Problem 2':

$$\begin{aligned} & \min_{(P_1, P_2, \dots, P_L)} \sum_{l=1}^L P_l \frac{Z^{l-1}}{(l-1)! (\lambda^{l-1} P_1 P_2 \dots P_{l-1})} \\ \text{Subject to } & 0 \leq P_l, \quad \text{for } 1 \leq l \leq L \\ & \frac{Z^L}{L! \lambda^L P_1 P_2 \dots P_L} \leq p_{\text{drop}}^{\text{target}} \end{aligned} \quad (24)$$

where $P_0 = 1$. As we can see, (23) and (24) are GPPs. It can be easily proved that the average power constraint and the PDP constraint in (23) and (24) are active at their respective optimal solutions, from which we can deduce that (23) and (24) are also equivalent.

We can use GPP toolboxes like GGPLAB [26] to solve (23) and (24) efficiently even for large values of L . Note that for the case of $L = 2$, the optimization problems (23) and (24) resemble the optimization problems for IR-HARQ presented in [14]. This is because, for both CC-HARQ and IR-HARQ schemes with $L = 2$, $p_{\text{drop}} \propto \frac{1}{P_1 P_2}$.

VI. NUMERICAL RESULTS

In this section, we present numerical results comparing the performance of optimal power allocation compared to equal power allocation. For optimal power allocation (OPA) case, we solve the optimization problem (labeled as “Exact method” in the plots) in (6) by splitting the main problem into subproblems with corresponding feasible sets and solving the subproblems using an interior-point algorithm. Note that these nonlinear optimization techniques do not guarantee that we find the global optimum, however we have verified that the solution returned by the interior-point algorithms matches very closely with the optimal solution found by an exhaustive grid search over the possible $P_l, 1 \leq l \leq L$ values.¹⁰ For equal power allocation (EPA) problem, the optimal solution can be obtained by solving for P using the relation $P \sum_{l=1}^L p_{\text{out}, l-1}^{\text{equal}} = \bar{P}_{\text{given}}$.

A. Optimal Power Allocation vs Equal Power Allocation

Fig. 3 shows the performance comparison of the optimal power allocation with the equal power allocation for $L = 2$

(a) $L = 2$.

(b) $L = 3$.

Figure 3. Comparison of the performance of the optimal power allocation using the exact method with the equal power allocation for $L = 2$ and 3. For the equal power allocation case, we have solved for P using $P \sum_{l=1}^L p_{\text{out}, l-1}^{\text{equal}} = \bar{P}_{\text{given}}$. We have plotted p_{drop} for different values of δ .

and 3. In the figure we have plotted p_{drop} vs. the average signal-to-noise ratio (SNR) per channel use for different values of δ . We define the average SNR per channel use in dB as $10 \log_{10} (\bar{P}_{\text{given}} \lambda)$, and it is varied by varying $\bar{P}_{\text{given}}$ and fixing $\lambda = 1$. From Figs. 3(a) and 3(b), we can see that the performance difference between the optimal power allocation and the equal power allocation solution for a given L is independent of the spectral efficiency per ARQ round (δ). Because of this, at a PDP of 10^{-4} and $L = 2$, the optimal power allocation solution has a performance gain of 3.5 dB over the equal power allocation solution for $\delta = 0.1, 1, 2$, and 3 bpcu. Similarly, for $L = 3$, the gain for the optimal power allocation over the equal power allocation at a PDP of 10^{-4} is 3 dB for $\delta = 0.1, 1, 2$, and 3 bpcu. The reason for this can be explained using the following proposition.

Proposition 1. For a given spectral efficiency $\delta_1 > 0$ bpcu, maximum number of transmissions L and a target PDP

¹⁰Within the resolution used for the exhaustive grid search.

Figure 4. Plot showing the optimal power values in different ARQ rounds as a function of $\bar{P}_{\text{given}}$ for $L = 2$ and 3. Parameters used are $\delta = 1$ bpcu and $\lambda = 1$.

value of ρ , let the optimal power allocation solution be $(P_{1,\delta_1}, P_{2,\delta_1}, \dots, P_{L,\delta_1})$ and the average SNR per channel use be SNR_1 . The optimal power values and the average SNR per channel use required to achieve the same PDP value of ρ with a spectral efficiency of $\delta_2 \neq \delta_1$ are given by

$$P_{l,\delta_2} = P_{l,\delta_1} \left(\frac{2^{\delta_2} - 1}{2^{\delta_1} - 1} \right), 1 \leq l \leq L \quad (25)$$

and

$$\text{SNR}_2 = \text{SNR}_1 + 10 \log_{10} \left(\frac{2^{\delta_2} - 1}{2^{\delta_1} - 1} \right) \text{ dB} \quad (26)$$

Proof: Please see Appendix D. ■

This result leads to the conclusion that one does not need to solve the optimization problems separately for different values of δ . It is sufficient to solve for the optimum power values for a single δ value and scale the optimum power values for other values of δ . Note that the result in Proposition 1 also holds for the equal power allocation method, which can be proven using similar techniques as in the proof of Proposition 1. Hence, for a given change in the spectral efficiency value, both the optimal power allocation and the equal power allocation methods require the same amount of increase in average SNR to achieve a fixed target PDP value. This increase in the average SNR value is also independent of the value of L as can be seen from (26) and also from Figures 3(a), 3(b).

B. Optimal Power Values

Fig. 4 illustrates the optimal power values $P_l, 1 \leq l \leq L$ for $L = 2$ and 3. As we can see, at low average power values and for a fixed L value, the optimal P_l values used in different ARQ rounds are quite similar. As the value of $\bar{P}_{\text{given}}$ increases, the values of $p_{\text{out},l-1}, 1 \leq l \leq L$ decrease significantly with increasing l . In this case, the “later ARQ rounds”, which are towards the end of an ARQ process are rarely used and their effect on the average power constraint is also minimal. An ARQ process uses later ARQ rounds only if the packet is still in outage. Since the objective of the optimization is to minimize the PDP after L ARQ rounds, we

(a) $L = 2, \delta = 1$ bpcu.

(b) $L = 3, \delta = 1$ bpcu.

(c) $L = 4, \delta = 5$ bpcu.

Figure 5. Figure comparing the performance of the optimal power allocation with the equal power allocation. We have solved the optimal power allocation problem using exact method by solving $2^L - L$ subproblems as described in Section III and the GPP approach presented in Section V.

need to improve the probability of successful decoding during these later ARQ rounds. In other words, the cost associated with an unsuccessful decoding during these later ARQ rounds towards PDP is high. Hence for large $\bar{P}_{\text{given}}$ values, optimal power allocation assigns high power values towards these later ARQ rounds as can be seen from Fig. 4. However this high power is rarely spent.

C. Comparison of GPP and Exact Solutions

In this section, we compare the performance of the optimal power allocation solution obtained using the GPP approach (labeled as “GPP approach”) described in Section V with the exact solution as well as the equal power allocation solution.¹¹

As we can see from Figs. 5(a) and 5(b), the equal power allocation is strictly sub-optimal compared to optimal power allocation solution obtained using the exact method. For $L = 2$ and $\delta = 1$ bpcu, the optimal power allocation solution obtained using the GPP approach has similar performance as that of the solution obtained using exact method. For the cases of $L = 3$ and 4, the optimal power allocation solution obtained using GPP has better performance than the equal power allocation solution for PDP values smaller than 10^{-2} and 10^{-3} respectively. The “loss” for the solution obtained using the GPP approach at higher PDP values is because the approximation in (22) is an upper bound on $p_{\text{out},l}$ and hence the L.H.S. of the average power constraint in (23) is always larger than or equal to the L.H.S. of the average power constraint in (6). Note that even though the gains obtained with the optimum power allocation scheme using the exact method and the GPP approach are significant only for small PDP values, these gains are still relevant in practical wireless systems. For example, in the current LTE systems, the HARQ retransmissions are implemented as a part of the medium access control (MAC) layer. The target block error rate (BLER) values after the retransmissions at this layer are typically in the order of 10^{-3} – 10^{-4} [30]. Another example of a practical application which requires very small target PDP values is the communication in smart grids [31].

Fig. 6 shows the optimal power values obtained using the GPP approach for $L = 5, 7$, and 8. We can observe from the figure that the GPP approach also assigns high power to later ARQ rounds as discussed in Section VI-B.

VII. CONCLUSIONS AND DISCUSSION

We considered the optimization of power for a CC-HARQ scheme with a fixed number of retransmissions (L). Solving the optimization problem exactly involves solving at least $2^L - L$ nonlinear optimization subproblems for a given value of L . Using a GPP approach, one can solve this optimization problem efficiently even for large values of L . However because of the PDP approximation, the GPP solution is close to the exact solution only at low PDP values. From the results, we can conclude that for relatively high average SNR values, it is important to ramp up power towards the end of the

Figure 6. Plot showing the optimal power values in different ARQ rounds obtained using the GPP approach for normalized $\bar{P}_{\text{given}} = 6$ dB and for $L = 5, 7$, and 8. Parameters used are $\delta = 1$ bpcu and $\lambda = 1$.

ARQ process in order to maximize the chance of successful decoding.

Possible extensions to this work include alternate approaches to come up with tighter approximations of the PDP expression, which will minimize the “loss” seen in the case of optimal power allocation with the GPP approach. However, these tighter approximations may include the terms of both positive and negative signs in them, and may restrict the use of the GPP approach in this case. Other possible extensions include the optimization of power in the case where the retransmissions contain only a part of the actual codeword (partial repetition). It may also be interesting to study the power allocation problem when the channel gains in different ARQ rounds have a non-zero correlation.

APPENDIX A

PROOF THAT (6) AND (7) ARE EQUIVALENT

We prove this statement by proving that the average power constraint and the PDP constraints in (6) and (7) are active at the optimal solutions of (6) and (7), respectively.

Since $\log_2(1+x)$ is strictly concave for $x > 0$ and that $\beta_m = |h_m|^2$, $1 \leq m \leq L$ are exponentially distributed, for some $\epsilon > 0$, we have

$$\begin{aligned} & \Pr \left(\log_2 \left(1 + \sum_{m=1}^L P_m \beta_m + \epsilon \beta_L \right) < \delta \right) \\ & < \Pr \left(\log_2 \left(1 + \sum_{m=1}^L P_m \beta_m \right) < \delta \right) \\ & < \Pr \left(\log_2 \left(1 + \sum_{m=1}^L P_m \beta_m - \epsilon \beta_L \right) < \delta \right). \end{aligned} \quad (27)$$

Let $(P_1^*, P_2^*, \dots, P_L^*)$ denote the optimal solution to (6) and assume that $\sum_{l=1}^L P_l p_{\text{out},l-1} \big|_{(P_1^*, P_2^*, \dots, P_L^*)} < \bar{P}_{\text{given}}$. We could increase P_L^* by ϵ such that

$$P_L^* = P_L^* + \epsilon \text{ and } \sum_{l=1}^L P_l p_{\text{out},l-1} \big|_{(P_1^*, P_2^*, \dots, P_L^*)} = \bar{P}_{\text{given}}.$$

¹¹We solve the optimization problem in (23) and once optimal power values are obtained, we compute p_{drop} using (9).

Using (27), we have

$$p_{\text{drop}} \left| (P_1^*, P_2^*, \dots, P_L^*) \right| < p_{\text{drop}} \left| (P_1^*, P_2^*, \dots, P_L^*) \right|$$

which contradicts the assumption that $(P_1^*, P_2^*, \dots, P_L^*)$ is the optimal solution to (6).

Now let $(\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L)$ be the optimal solution to (7) and assume that

$$p_{\text{drop}} \left| (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L) \right| < p_{\text{drop}}^{\text{target}}$$

We could now reduce $\tilde{P}_L$ by $\epsilon > 0$ such that

$$\tilde{P}'_L = \tilde{P}_L - \epsilon \text{ and } p_{\text{drop}} \left| (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}'_L) \right| = p_{\text{drop}}^{\text{target}}.$$

Now we have

$$\sum_{l=1}^{L-1} \tilde{P}_{l, \text{out}, l-1} + \tilde{P}'_{L, \text{out}, L-1} < \sum_{l=1}^{L-1} \tilde{P}_{l, \text{out}, l-1} + \tilde{P}_{L, \text{out}, L-1}$$

which contradicts the assumption that $(\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_L)$ is the optimal solution to (7). Since the optimization problems in (6) and (7) satisfy the average power constraint and the PDP constraint with equalities, the optimal solutions should be the same.

APPENDIX B PROOF OF LEMMA 1

Before proving Lemma 1, we state some results from combinatorics which will be useful for the proof. For fixed integers n, p , and m with $n > p > m$, the Vandermonde convolution formula is given by [29]:

$$\binom{n+p}{m} = \sum_{k=0}^m \binom{n}{m-k} \binom{p}{k} \quad (28)$$

Using (28) together with the relation that $\binom{n+m-1}{m} = (-1)^m \binom{-n}{m}$, we can prove that

$$\binom{n-p}{m} = \sum_{k=0}^m (-1)^k \binom{n}{m-k} \binom{p+k-1}{k}, \quad (29)$$

and

$$\sum_{k=1}^p (-1)^{k-1} \binom{n-k}{m} \binom{p}{k} = \binom{n}{m}, \text{ for } 0 \leq m \leq p-1. \quad (30)$$

Now to prove the relations in (19) and (20), we first multiply both sides of (18) with $(1 + xa_1)^{\tau_1} \dots (1 + xa_M)^{\tau_M}$ to obtain

$$1 = \sum_{k=1}^M \sum_{j=1}^{\tau_k} b_{k,j} (1 + xa_k)^{\tau_k-j} \prod_{\substack{l=1 \\ l \neq k}}^M (1 + xa_l)^{\tau_l} \quad (31)$$

Let $c_i, 1 \leq i \leq L$ denote the coefficient of x^{L-i} in the expansion on the R.H.S. of (31), then we have

$$d_i \triangleq \frac{c_i}{a_1^{\tau_1} a_2^{\tau_2} \dots a_M^{\tau_M}} \quad (32)$$

$$= \sum_{k=1}^M \sum_{j=1}^{\min(i, \tau_k)} \frac{b_{k,j}}{a_k^j} \left(\sum_{\substack{k_1 + \dots + k_M = i-j \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\binom{\tau_l - j \delta_{lk}}{k_l}}{a_l^{k_l}} \right), \quad 1 \leq i \leq L$$

where

$$\delta_{lk} = \begin{cases} 1 & \text{if } l = k \\ 0 & \text{otherwise} \end{cases} \quad (33)$$

We can note from (31) that $c_i = d_i = 0$ for $1 \leq i \leq L-1$. Considering (32) for $i = L$, we have

$$d_L = \frac{1}{a_1^{\tau_1} a_2^{\tau_2} \dots a_M^{\tau_M}} = \sum_{k=1}^M \sum_{j=1}^{\tau_k} \frac{b_{k,j}}{a_1^{\tau_1} a_2^{\tau_2} \dots a_M^{\tau_M}}, \quad (34)$$

from (34), we have the result in (19).¹² Now using $i = 1$ in (32), we have

$$d_1 = \frac{c_1}{a_1^{\tau_1} a_2^{\tau_2} \dots a_M^{\tau_M}} = \sum_{k=1}^M \frac{b_{k,1}}{a_k} = A_1 = 0. \quad (35)$$

We prove the remaining identities corresponding to $A_i, 2 \leq i \leq L$ by considering the following recursive relation:

$$\sum_{m=1}^{i-1} (-1)^{m-1} A_{i-m} \left(\sum_{\substack{k_1 + \dots + k_M = m \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\binom{\tau_l}{k_l}}{a_l^{k_l}} \right) + (-1)^{i-1} d_i = A_i \quad (36)$$

Using $i = 2$ in (36), we have

$$A_1 \left(\sum_{l=1}^M \frac{\tau_l}{a_l} \right) - d_2 = \left(\sum_{k=1}^M \frac{b_{k,1}}{a_k} \right) \left(\sum_{l=1}^M \frac{\tau_l}{a_l} \right) - \sum_{k=1}^M \sum_{j=1}^{\min(2, \tau_k)} \frac{b_{k,j}}{a_k^j} \left(\sum_{\substack{k_1 + \dots + k_M = 2-j \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\binom{\tau_l - j \delta_{lk}}{k_l}}{a_l^{k_l}} \right) = A_2 = 0. \quad (37)$$

Now assuming that (36) holds true for $i = 2, \dots, n \leq L-1$, we prove that (36) holds for $i = n+1$ as well. Using $i = n+1$ in (36), we have

$$\sum_{m=1}^n (-1)^{m-1} A_{n+1-m} \left(\sum_{\substack{k_1 + \dots + k_M = m \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\binom{\tau_l}{k_l}}{a_l^{k_l}} \right) + (-1)^n d_{n+1} = A_n \left(\sum_{l=1}^M \frac{\tau_l}{a_l} \right) - A_{n-1} \left(\sum_{l=1}^M \frac{\tau_l}{a_l^2} + \sum_{l=1}^M \sum_{s>l} \frac{\tau_l \tau_s}{a_l a_s} \right) + \dots + (-1)^{n-1} A_1 \left(\sum_{\substack{k_1 + \dots + k_M = n \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\binom{\tau_l}{k_l}}{a_l^{k_l}} \right) + (-1)^n \sum_{k=1}^M \sum_{j=1}^{\min(n+1, \tau_k)} \frac{b_{k,j}}{a_k^j} \left(\sum_{\substack{k_1 + \dots + k_M = n+1-j \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\binom{\tau_l - j \delta_{lk}}{k_l}}{a_l^{k_l}} \right) = \left(\sum_{k=1}^M \sum_{j=1}^{\min(n, \tau_k)} \frac{(-1)^{j-1} \binom{n-1}{j-1} b_{k,j}}{a_k^j} \right) \left(\sum_{l=1}^M \frac{\tau_l}{a_l} \right)$$

¹²The relation in (19) can also be obtained from (18) or (31) by using $x = 0$ and equating both sides.

$$\begin{aligned}
& - \left(\sum_{k=1}^M \sum_{j=1}^{\min(n-1, \tau_k)} \frac{(-1)^{j-1} \binom{n-2}{j-1} b_{k,j}}{a_k^j} \right) \\
& \left(\sum_{l=1}^M \frac{\tau_l}{a_l^2} + \sum_{l=1}^M \sum_{s>l} \frac{\tau_l \tau_s}{a_l a_s} \right) + \dots \\
& + (-1)^{n-1} \left(\sum_{l=1}^M \frac{\tau_l}{a_l} \right) \left(\sum_{\substack{k_1+\dots+k_M=n \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\tau_l}{a_l^{k_l}} \right) \\
& + (-1)^n \sum_{k=1}^M \sum_{j=1}^{\min(n+1, \tau_k)} \frac{b_{k,j}}{a_k^j} \left(\sum_{\substack{k_1+\dots+k_M=n+1-j \\ k_m \in \{0, \tau_m\}}} \prod_{l=1}^M \frac{\tau_l}{a_l^{k_l}} \right)
\end{aligned} \tag{38}$$

Using (29) and (30) together with some basic algebraic manipulations, we can show that the expression in (38) is equal to A_{n+1} , $2 \leq n \leq L-1$. Since

$$d_n = \begin{cases} 0 & \text{for } 1 \leq n \leq L-1 \\ \frac{1}{a_1^{\tau_1} a_2^{\tau_2} \dots a_M^{\tau_M}} & \text{for } n = L \end{cases}$$

we have the desired result for A_i , $2 \leq i \leq L$ as in (20).

APPENDIX C PROOF OF THEOREM 1

We denote the characteristic coefficients as $\chi_{i,j}$ for notation simplicity. From (9), we can write:

$$\begin{aligned}
P_{\text{out},l} &= 1 - \sum_{i=1}^{M_l} \left\{ e^{-\left(\frac{Z}{P_{\langle i \rangle}}\right)} \sum_{j=1}^{\tau_{i,l}} \sum_{k=0}^{j-1} \frac{\chi_{i,j}}{k!} \left(\frac{Z}{P_{\langle i \rangle}}\right)^k \right\} \\
&= 1 - \sum_{i=1}^{M_l} \left\{ e^{-\left(\frac{Z}{P_{\langle i \rangle}}\right)} \left[\chi_{i,1} + \chi_{i,2} \left(1 + \frac{Z}{P_{\langle i \rangle}}\right) + \dots \right. \right. \\
&\quad \left. \left. + \chi_{i,\tau_{i,l}} \left(1 + \frac{Z}{P_{\langle i \rangle}} + \dots + \frac{Z^{\tau_{i,l}-1}}{(\tau_{i,l}-1)! P_{\langle i \rangle}^{\tau_{i,l}-1}}\right) \right] \right\} \\
&= 1 - \sum_{i=1}^{M_l} \left\{ e^{-\left(\frac{Z}{P_{\langle i \rangle}}\right)} \left[\sum_{j=1}^{\tau_{i,l}} \chi_{i,j} + \frac{Z}{P_{\langle i \rangle}} \left(\sum_{j=2}^{\tau_{i,l}} \chi_{i,j} \right) \right. \right. \\
&\quad \left. \left. + \dots + \frac{Z^{\tau_{i,l}-1}}{(\tau_{i,l}-1)! P_{\langle i \rangle}^{\tau_{i,l}-1}} \right] \right\} \\
&= 1 - \sum_{i=1}^{M_l} \left\{ \left(1 - \frac{Z}{P_{\langle i \rangle}} + \frac{Z^2}{2P_{\langle i \rangle}^2} - \dots + (-1)^l \frac{Z^l}{l! P_{\langle i \rangle}^l} \right) \right. \\
&\quad \left. + \sum_{m=l+1}^{\infty} \frac{(-1)^m}{m!} \left(\frac{Z}{P_{\langle i \rangle}} \right)^m \right) \left[\sum_{j=1}^{\tau_{i,l}} \chi_{i,j} \right. \\
&\quad \left. + \frac{Z}{P_{\langle i \rangle}} \left(\sum_{j=2}^{\tau_{i,l}} \chi_{i,j} \right) + \dots + \frac{Z^{\tau_{i,l}-1}}{(\tau_{i,l}-1)! P_{\langle i \rangle}^{\tau_{i,l}-1}} \right] \right\} \\
&= 1 - \sum_{i=1}^{M_l} \left\{ \left[\left(\sum_{j=1}^{\tau_{i,l}} \chi_{i,j} \right) - \frac{Z}{P_{\langle i \rangle}} \chi_{i,1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{Z^2}{2P_{\langle i \rangle}^2} \left(\sum_{j=1}^{\min(2, \tau_{i,l})} (-1)^{j-1} \binom{1}{j-1} \chi_{i,j} \right) + \dots \\
& + (-1)^l \frac{Z^l}{l! P_{\langle i \rangle}^l} \left(\sum_{j=1}^{\tau_{i,l}} (-1)^{j-1} \binom{l-1}{j-1} \chi_{i,j} \right) \left. \right] \Bigg\} \\
& - \underbrace{\sum_{i=1}^{M_l} \sum_{m=l+1}^{\infty} \frac{1}{m!} \left(\frac{-Z}{P_{\langle i \rangle}} \right)^m \left[\sum_{j=1}^{\tau_{i,l}} \sum_{k=0}^{j-1} \frac{\chi_{i,j}}{k!} \left(\frac{Z}{P_{\langle i \rangle}} \right)^k \right]}_{=\Xi} \\
& = 1 - \sum_{i=1}^{M_l} \sum_{j=1}^{\tau_{i,l}} \chi_{i,j} \\
& + \sum_{k=1}^l (-1)^{k-1} \frac{Z^k}{k!} \sum_{i=1}^{M_l} \sum_{j=1}^{\min(k, \tau_{i,l})} \frac{(-1)^{j-1} \binom{k-1}{j-1} \chi_{i,j}}{P_{\langle i \rangle}^k} - \Xi \\
& \stackrel{(a)}{=} \frac{Z^l}{l! P_{\langle 1 \rangle}^{\tau_{1,l}} \dots P_{\langle M_l \rangle}^{\tau_{M_l,l}}} - \Xi \\
& \stackrel{(b)}{=} \frac{Z^l}{l! P_{\langle 1 \rangle}^{\tau_{1,l}} \dots P_{\langle M_l \rangle}^{\tau_{M_l,l}}} + O\left(\frac{1}{P_{\min}^{l+1}}\right)
\end{aligned}$$

In (a), we have used the results from Lemma 1. In (b), $P_{\min} = \min(P_1, P_2, \dots, P_L)$ and we have represented the Ξ term using the $O(\cdot)$ notation.

APPENDIX D PROOF OF PROPOSITION 1

Let us define $Z_1 \triangleq 2^{\delta_1} - 1$ and $Z_2 \triangleq 2^{\delta_2} - 1$. Let L and ρ denote the maximum number of allowed transmissions and the target PDP value, respectively. We now consider the optimization problem in (7) for two different spectral efficiency values. For a given value of δ_1 , we have the optimization problem

$$\begin{aligned}
& \min_{(P_1, P_2, \dots, P_L)} P_{\text{avg}} = \sum_{m=1}^L P_m P_{\text{out},m-1}(P_1, \dots, P_{m-1}, Z_1) \\
& \text{Subject to } 0 \leq P_m, \quad \text{for } 1 \leq m \leq L \\
& P_{\text{out},L} = \Pr \left(\sum_{m=1}^L P_m \beta_m < Z_1 \right) \leq \rho
\end{aligned} \tag{39}$$

where $\beta_m = |h_m|^2$, $1 \leq m \leq L$. The notation $P_{\text{out},m-1}(P_1, \dots, P_{m-1}, Z_1)$ is used to point out that the outage probability after $m-1$ transmissions is a function of $(P_1, \dots, P_{m-1}, Z_1)$. For given values of Z_1 and ρ , let us define the feasible set for the problem as

$$\begin{aligned}
& F_{\delta_1} \triangleq \left\{ (P_1, P_2, \dots, P_L) \mid P_m \geq 0, \text{ for } 1 \leq m \leq L \right. \\
& \quad \left. \text{and } \Pr \left(\sum_{m=1}^L P_m \beta_m < Z_1 \right) \leq \rho \right\}.
\end{aligned}$$

Let $(P_{1,\delta_1}, P_{2,\delta_1}, \dots, P_{L,\delta_1})$ denote the optimal solution to (39) and let $\text{SNR}_1 = 10 \log_{10} \left(\lambda \sum_{m=1}^L P_{m,\delta_1} P_{\text{out},m-1}(P_{1,\delta_1}, \dots, P_{m-1,\delta_1}, Z_1) \right)$ be the average SNR per channel use. For a spectral efficiency value of $\delta_2 \neq \delta_1$, the optimization problem is

$$\begin{aligned} \min_{(P'_1, P'_2, \dots, P'_L)} P'_{\text{avg}} &= \sum_{m=1}^L P'_m p_{\text{out}, m-1} (P'_1, \dots, P'_{m-1}, Z_2) \\ \text{Subject to } 0 &\leq P'_m, \quad \text{for } 1 \leq m \leq L \\ p_{\text{out}, L} &= \Pr \left(\sum_{m=1}^L P'_m \beta_m < Z_2 \right) \leq \rho \end{aligned} \quad (40)$$

Let

$$\begin{aligned} F_{\delta_2} &\triangleq \left\{ (P'_1, P'_2, \dots, P'_L) \mid P'_m \geq 0, \text{ for } 1 \leq m \leq L \right. \\ &\quad \left. \text{and } \Pr \left(\sum_{m=1}^L P'_m \beta_m < Z_2 \right) \leq \rho \right\} \end{aligned}$$

denote the feasible set for (40). Defining $\Gamma \triangleq Z_2/Z_1$, we can see that if $(P_1, P_2, \dots, P_L) \in F_{\delta_1}$, then $(\Gamma P_1, \Gamma P_2, \dots, \Gamma P_L) \in F_{\delta_2}$ and the average power can be expressed as

$$\begin{aligned} &\sum_{m=1}^L \Gamma P_m p_{\text{out}, m-1} (\Gamma P_1, \dots, \Gamma P_{m-1}, Z_2) \\ &= \Gamma \sum_{m=1}^L P_m p_{\text{out}, m-1} (P_1, \dots, P_{m-1}, Z_1), \end{aligned} \quad (41)$$

since

$$\begin{aligned} &p_{\text{out}, m-1} (\Gamma P_1, \dots, \Gamma P_{m-1}, Z_2) \\ &= p_{\text{out}, m-1} (P_1, \dots, P_{m-1}, Z_1), \text{ for } 1 \leq m \leq L+1. \end{aligned} \quad (42)$$

If $(P'_1, P'_2, \dots, P'_L) \in F_{\delta_2}$, then $\left(\frac{P'_1}{\Gamma}, \frac{P'_2}{\Gamma}, \dots, \frac{P'_L}{\Gamma} \right) \in F_{\delta_1}$.

Assume now that $(P'_{1, \delta_2}, P'_{2, \delta_2}, \dots, P'_{L, \delta_2}) \neq (\Gamma P_{1, \delta_1}, \Gamma P_{2, \delta_1}, \dots, \Gamma P_{L, \delta_1})$ is the optimal solution to (40), which implies that

$$\begin{aligned} &\sum_{m=1}^L P'_{m, \delta_2} p_{\text{out}, m-1} (P'_{1, \delta_2}, \dots, P'_{m-1, \delta_2}, Z_2) \\ &< \Gamma \sum_{m=1}^L P_{m, \delta_1} p_{\text{out}, m-1} (P_{1, \delta_1}, \dots, P_{m-1, \delta_1}, Z_1) \end{aligned} \quad (43)$$

Dividing by Γ on both sides of (43), we have

$$\begin{aligned} &\frac{1}{\Gamma} \sum_{m=1}^L P'_{m, \delta_2} p_{\text{out}, m-1} (P'_{1, \delta_2}, \dots, P'_{m-1, \delta_2}, Z_2) \\ &= \sum_{m=1}^L \frac{P'_{m, \delta_2}}{\Gamma} p_{\text{out}, m-1} \left(\frac{P'_{1, \delta_2}}{\Gamma}, \dots, \frac{P'_{m-1, \delta_2}}{\Gamma}, Z_1 \right) \\ &< \sum_{m=1}^L P_{m, \delta_1} p_{\text{out}, m-1} (P_{1, \delta_1}, \dots, P_{m-1, \delta_1}, Z_1). \end{aligned} \quad (44)$$

(44) contradicts the assumption that $(P_{1, \delta_1}, P_{2, \delta_1}, \dots, P_{L, \delta_1})$ is the optimal solution of (39). Hence $(\Gamma P_{1, \delta_1}, \Gamma P_{2, \delta_1}, \dots, \Gamma P_{L, \delta_1}) \in F_{\delta_2}$ is the optimal solution to (40), from which (26) also follows.

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