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On automorphisms groups of cyclic p-gonal Riemann surfaces

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Abstract

In this work we obtain the group of conformal and anticonformal automorphisms of real cyclic p-gonal Riemann surfaces, where $p \geq 3$ is a prime integer and the genus of the surfaces is at least $(p-1)^2+1$. We use Fuchsian and NEC groups, and cohomology of finite groups.

Keywords:

1. Introduction

A closed Riemann surface X_g, $g \geq 2$, is a *cyclic p-gonal* Riemann surface, where p is a prime integer, if it is a regular p-sheeted covering f from X_g to the Riemann sphere, f is called a *cyclic p-gonal morphism*. The morphism f is a cyclic covering and the cyclic group C_p of deck-transformations of the p-gonal morphism is called the *p-gonality group*. When $p = 2$ these surfaces are the hyperelliptic surfaces, if $p = 3$ the surfaces are called cyclic trigonal surfaces.

A cyclic p-gonal Riemann surface X_g is called *real cyclic p-gonal* if there is an anticonformal involution (symmetry) σ of X_g which is a lift of the complex conjugation by the covering f. A real cyclic p-gonal Riemann surface can be represented by a complex algebraic curve admitting a real polynomial

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equation of the form:

$$y^p = \prod (x - a_i) \prod (x - b_j)^2 \cdots \prod (x - m_k)^{p-1}.$$

Cyclic p-gonal and real cyclic p-gonal Riemann surfaces have been extensively studied, see (1), (3), (4), (5), (6), (8), (9), (10), (11), (12), (13), (14), (16), (15), (21), (22).

Bujalance et al (5) have calculated the groups of automorphisms of hyperelliptic Riemann surfaces and more recently Bujalance, Cirre and Gromadzki list the automorphisms groups of cyclic trigonal Riemann surfaces (see (6)). Recently Sanjeewa (18) has obtained the automorphisms groups of cyclic n-gonal algebraic curves over fields of any characteristics. Sanjeewa and Shaska determined equations of families of cyclic n-gonal algebraic curves (19).

We list and classify the groups of conformal and anticonformal automorphisms of real cyclic p-gonal Riemann surfaces for $p \geq 3$ a prime integer and when the p-gonalily group is normal in the full group of automorphisms of the surfaces (in particular when the genus of the surfaces is at least $(p-1)^2+1$). It interesting to remark that some exceptional groups occur for $p = 3$ and $p \equiv 1 \bmod 6$ (see Theorem 3). As a tool we calculate first the group of conformal automorphisms of cyclic p-gonal Riemann surfaces.

2. Riemann surfaces and Fuchsian groups

Let $\mathcal{H}$ be the upper half-plane, i.e. the set of complex numbers z with imaginary part $\operatorname{Im} z > 0$. A cocompact, discrete subgroup $\overline{\Delta}$ of $\mathcal{G} = Aut(\mathcal{H})$ of conformal and anticonformal automorphisms of $\mathcal{H}$ is called an *(NEC) non-euclidean crystallographic group*. An NEC group Δ consisting only of orientation-preserving elements is a Fuchsian group. The subgroup of an NEC group $\overline{\Delta}$ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* $\overline{\Delta}$.

If a Fuchsian group Δ has a canonical presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1 \ldots x_k | x_1^{m_1} = \cdots = x_k^{m_k} = \prod x_i \prod [a_i, b_i] = 1 \right\rangle \qquad (1)$$

we say that Δ has *signature*

$$s(\Gamma) = (g; m_1, \ldots, m_k). \qquad (2)$$

In the signatures we shall use the notation p^r meaning $p, \overset{r}{\ldots}, p$. The generators in the presentation (1) will be called the *canonical generators*. The hyperbolic area of the orbifold $\mathcal{H}/\Delta$ coincides with the hyperbolic area of an arbitrary fundamental region of Δ it is

$$\mu(\Delta) = 2\pi \left(2g - 2 + \sum_{i_1}^{r} (1 - \frac{1}{m_i}) \right) \tag{3}$$

Let X be a Riemann surface uniformized by a surface Fuchsian group Γ_g, i.e. a group with signature $(g; -)$ (g must be > 1). A finite group G is a group of automorphisms of X if and only if there is a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$ such that $\ker \theta = \Gamma_g$. The epimorphism θ is the monodromy of the covering $f : X \to X/G = \mathcal{H}/\Delta$. The Fuchsian group Δ is the lifting of G to the universal covering $\pi : \mathcal{H} \to X$: the *universal covering transformations group of* (X, G).

The Riemann-Hurwitz give us $\mu(Ker(\theta)) = |G| \, \mu(\Delta)$. Singerman (20) determined the relation between the signatures of a Fuchsian group Δ and a finite index subgroup of Δ.

Given an odd prime p, a cyclic p-gonal Riemann surface is a pair (X_g, f), where f is a cyclic p-gonal morphism. By Lemma 2.1 in (1) the p-gonality group C_p is normal in $Aut(X_g)$ and $Aut^{\pm}(X_g)$ if the genus $g \geq (p-1)^2 + 1$, since the p-gonal morphism is unique. There are families of p-gonal Riemann surfaces of genus $(p-1)^2$ admitting two such p-gonal morphisms (see (12), (13) and (21)). From now on, we shall assume either the genera will satisfy the condition above or the p-gonality group C_p is normal in $Aut^{\pm}(X_g)$.

Costa and Izquierdo ((9) and (10)) gave the following characterization of p-gonal Riemann surfaces X of genus $g > 1 :$ X admits a p-gonal morphism f if and only if there is a Fuchsian group Λ with signature $(0; p^r)$, $r = \frac{2g}{p-1} + 2$, and an epimorphism $\theta : \Lambda \to C_p$, such that X is conformally equivalent to $\mathcal{H}/Ker(\theta)$ with $Ker(\theta)$ a surface Fuchsian group.

3. Automorphism Groups of Cyclic *p*-gonal Riemann Surfaces

We want to find the groups of automorphisms of cyclic p-gonal Riemann surfaces where the p-gonality group is a normal subgroup of $Aut(X)$.

Lemma 1. *Let (X, f) be a cyclic p-gonal Riemann surface where the p-gonality group is a normal subgroup of $Aut(X)$. Then $Aut(X)$ is an extension of C_p by a finite group of automorphisms of the Riemann sphere.*

Proof. The quotient group $Aut(X)/C_p$ acts on X/C_p that is the Riemann sphere. $\square$

A finite group $\overline{G}$ of conformal automorphisms of the Riemann sphere is a subgroup of the following groups: C_q, D_q, A_4, Σ_4, A_5, for any integer $q > 0$. Lemma 1 says that any group G of automorphisms of a cyclic p-gonal Riemann surface is an extension

$$1 \to C_p \to G \to \overline{G} \to 1$$

of C_p by a group $\overline{G}$ of automorphisms of the Riemann sphere listed above.

Consider an extension $1 \to N \to G \to Q \to 1$ with inclusion $\mu : N \to G$ and quotient $\epsilon : G \to Q$. It defines a *transversal function* (in general no homomorphism) $\tau : Q \to G$ satisfying $\tau\epsilon = 1$. This yields a function (in general no homomorphism) $\lambda : Q \to Aut(N)$, two such functions $\lambda, \lambda' : Q \to Aut(N)$ differ by an inner automorphism of N. So an extension $1 \to N \to G \to Q \to 1$ of a normal subgroup N of a group G by a quotient group Q induces a homomorphism $\eta : Q \to Out(N)$, *the coupling of Q to N*. Two equivalent extensions (in the natural sense) induce the same coupling. A coupling $\eta : Q \to Out(N)$ induces a structure as Q-module on $Z(N)$, where $Z(N)$ the center of N, and we have:

Theorem 2. *((17), (2)) Let N and Q be groups and let $\eta : Q \to Out(N)$ be a coupling of Q to N. Assume that η is realized by at least one extension of N by Q. Then there is a bijection between the equivalence classes of extensions of N by Q with coupling η and the elements of $H^2_\eta(Q, Z(N))$, with $Z(N)$ the center of N with structure of Q-module given by η.*

We say that an extension $1 \to N \to G \to Q \to 1$ *splits* if the transversal function $\tau : Q \to G$ is an (injective) homomorphism, in this case the function $\lambda : Q \to Aut(N)$ is a homomorphism and Q acts as a group of automorphisms of N. An extension $1 \to N \to G \to Q \to 1$ splits if and only if Q is a complement to N in G, i.e. G is a semidirect product $N \rtimes Q$. In case of N being Abelian the set of classes of extensions of N by Q is in bijection with $H^2_\eta(Q, N)$ and the set of classes of complements of N in $G = N \rtimes Q$ is in bijection with $H^1_\eta(Q, N)$. See (2) and (17).

In the following theorem and in the rest of the work, the operator $\rtimes$ means a semidirect product including the direct product. In case the group is not a direct product we will denote the product by $\rtimes_n$, with n the order of the action of the non-normal factor as a group of automorphisms of the normal factor.

Theorem 3. *(Automorphisms of cyclic p-gonal Riemann surfaces) Let (X_g, f) be a p-gonal Riemann surface of genus $g \geq (p-1)^2+1$ with p an odd prime integer. Then the possible (conformal) automorphisms groups of X_g are*

1. C_{pq}
2. D_{pq}
3. $C_p \rtimes C_q$, *where $\rtimes$ means any semidirect product (including the direct product).*
4. $C_p \rtimes D_q$, *where $\rtimes$ means any semidirect product (including the direct product).*
5. $C_p \times A_4$, $(C_p \times A_4) \rtimes_2 C_2 = C_p \rtimes_2 \Sigma_4$, $C_p \times \Sigma_4$, $C_p \times A_5$
6. *Exceptional Case 1.* $((C_2 \times C_2) \rtimes_3 C_9)$ *for* $p = 3$ *and* $Aut(X_g)/C_p = \overline{G} = A_4$
7. *Exceptional Case 2.* $(C_p \times C_2 \times C_2) \rtimes_3 C_3$ *for* $p \equiv 1$ *mod* 6, $\overline{G} = A_4$
8. *Exceptional Case 3.* $((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2$ *for* $p = 3$, $\overline{G} = \Sigma_4$

Proof. Let (X_g, f) be a cyclic p-gonal Riemann surface with p-gonal morphism f induced by the automorphism φ of X_g of odd prime order p such that the cyclic group $C_p = \langle\varphi\rangle$ is normal in $G = Aut(X_g)$ with quotient group $\overline{G} = C_q$, D_q, A_4, Σ_4 or A_5. By Lemma 2.1 in (1) this condition is satisfy for genera $g \geq (p-1)^2 + 1$, since the p-gonal morphism is unique.

By Lemma 1 we have to find all the equivalence classes of extensions

$$1 \to C_p \to G \to \overline{G}.$$

First of all (Zassenhaus Lemma), if $(|\overline{G}|, p) = 1$, then the extension splits and all the complements of C_p in G are conjugated, since C_p is solvable. See (17).

By Shur-Zassenhaus Lemma (17) an extension $1 \to C_p \to G \to \overline{G}$ splits if and only if all the extensions of C_p by any t-Sylow subgroup of $\overline{G}$ splits, with $t \mid |\overline{G}|$.

Since C_p is an Abelian group, by Theorem 2, the coupling $\eta : Q \to Aut(N)$ will be realized by an extension given by an element of $H^2(\overline{G}, C_p)$ with the $\overline{G}$-module structure of C_p given by η. The split extension $G = C_p \rtimes \overline{G}$ corresponds to $1 \in H^2(\overline{G}, C_p)$. See (2) and (17).

1. $H^2(A_5, C_p) = \{1\}$ for $p \geq 3$ and since the only homomorphism $\lambda : A_5 \to C_{p-1}$ is trivial, $G = C_p \times A_5$.

2. $H^2_1(A_4, C_3) = C_3 = \langle b \rangle$ and similarly to above there are two extensions $G = C_p \times A_4$, corresponding to $1 \in C_3$, and $G = (C_2 \times C_2) \rtimes_3 C_9$ (again, corresponding to b and b^2 in C_3). This last case is the Exceptional Case 1.
   $H^2_i(A_4, C_p) = \{1\}$ for $p \geq 5$, $i = 1, 2$, where the possible homomorphisms $\lambda_i : A_4 \to C_{p-1}$ are $\lambda_1 = 1$ and λ_2 with $Ker(\lambda_2) = C_2 \times C_2$ if $p \equiv 1 \bmod 6$. Then we have two cases $G = C_p \times A_4$ and $G = (C_2 \times C_2 \times C_p) \rtimes_3 C_3$. This last case is the Exceptional Case 2.
3. $H^2_i(\Sigma_4, C_p) = \{1\}$ for $p \geq 5$, $i = 1, 2$, where the possible homomorphisms $\lambda_i : \Sigma_4 \to C_{p-1}$ are $\lambda_1 = 1$ and λ_2 with $Ker(\lambda_2) = A_4$. Then we have two cases $G = C_p \times \Sigma_4$ and $G = (A_4 \times C_p) \rtimes_2 C_2$.
   If $p = 3$, then $H^2_2(\Sigma_4, C_3) = C_3 = \langle b \rangle$, and there are two extensions $G = (A_4 \times C_p) \rtimes C_2$, corresponding to $1 \in C_3$, and $G = ((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2$ (corresponding to b and b^2 in C_3). This last case is the Exceptional Case 3.
4. Consider extensions of $C_p = \langle \varphi \rangle$ by $C_q = \langle b \rangle$. By Zassenhaus Lemma if $(p, q) = 1$ then $G = C_p \times C_q = C_{pq}$ or in general $G = C_p \rtimes C_q$ when $(q, p-1) = d > 1$. In this case the action of C_q on C_p has order a divisor of d.
   Consider now extensions of C_p by C_q with $q = p^k m$, $(p, m) = 1$. Thus, (2),(17):

   - $H^2(C_q, C_p) = \{1\}$ if $\varphi^b \neq \varphi$
   - $H^2(C_q, C_p) = C_p$ if $\varphi^b = \varphi$.

   In the first case we have $G = C_p \rtimes C_q$ where the action of C_q on C_p has order a divisor of $d = (q, p-1)$. In the second case we have $G = C_p \times C_q$ and $G = C_{pq}$ since the extensions given by non-trivial elements of $H^2(C_q, C_p) = C_p$ are isomorphic.
5. Finally consider extensions of $C_p = \langle \varphi \rangle$ by $D_q = \langle s, b \rangle = \langle s, b \mid s^2 = b^q = (sb)^2 = 1 \rangle$. Again by Zassenhaus Lemma if $(p, q) = 1$ then $G = C_p \rtimes D_q$.
   Consider now extensions $C_p = \langle \varphi \rangle$ by D_q with $q = p^k m$, $(p, m) = 1$. First we note that

$$\begin{tikzcd} H^2(D_q, C_p) \arrow[r, "Res_1"] \arrow[rr, bend left, "Res"] & H^2(C_q, C_p) \arrow[r, "Res_2"] & H^2(C_{p^k}, C_p) \end{tikzcd}$$

commutes and the restrictions to the p-Sylow subgroup are injective. Thus if $\varphi^b \neq \varphi$ then $H^2(D_q, C_p) = \{1\}$. Further, the following diagram commutes (see (2)):

$$\begin{tikzcd} H^2(D_q, C_p) \arrow[r, "Res_1"] \arrow[d, "id"] & H^2(C_q, C_p) \arrow[d, "s^*"] \\ H^2(D_q, C_p) \arrow[r, "Res_1"] & H^2(C_q, C_p) \end{tikzcd}$$

Here $s^* : f \mapsto \lambda_s \circ f \circ (\rho_s \times \rho_s)$, where $\lambda_s(a) = a^s$, $a \in C_p$. Now, if $|H^2(D_q, C_p)| > 1$ then s^* has to be trivial thus

- $H^2(D_n, C_p) = \{1\}$ if $\varphi^b \neq \varphi$ or $\varphi^s = \varphi$.
- $H^2(D_n, C_p) = C_p$ if $\varphi^b = \varphi$ and $\varphi^s = \varphi^{-1}$.

In the first case $G = C_p \rtimes C_q$ In the second case $G = C_p \rtimes D_q$ and $G = D_{pq} = \langle s, a \mid s^2 = a^{pq} = (sa)^2 = 1 \rangle$ since the extensions given by non-trivial elements of $H^2(D_q, C_p) = C_p$ are isomorphic.

□

Remark 4. *Observe that Theorem 3 is valid when the p-gonality group C_p is a normal subgroup of $Aut(X_g)$ and it does not depend on the genus of the p-gonal surface.*

Remark 5. *Theorem 3 has been obtained in (18). The groups $(C_p \times A_4) \rtimes_2 C_2$, $((C_2 \times C_2) \rtimes_3 C_9)$ for $p = 3$, $(C_p \times C_2 \times C_2) \rtimes_3 C_3$ for $p \equiv 1$ mod 6 and $((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2$ for $p = 3$ in Theorem 3 were omitted.*

Theorem 6. *Let G be a finite group isomorphic to one of the groups listed in Theorem 3. Then there exist cyclic p-gonal Riemann surfaces X_g of genus g with automorphisms group isomorphic to G.*

Proof. Consider a group G in the list of theorem 3. The group G contains a normal subgroup $\langle \varphi \rangle$ isomorphic to C_p. To prove the theorem we shall give Fuchsian groups Δ and surface epimorphisms $\theta : \Delta \to G$ with $Ker(\theta) = \Gamma$, where Γ is a surface Fuchsian group uniformizing X_g, such that $s(\theta^{-1}\langle\varphi\rangle) = (0; p^{\frac{2g}{(p-1)}+2})$ (see (9) and (10), this is a consequence of uniformization theorems for two dimensional orbifolds).

We separate the proof in the cases presented in Theorem 3.

1. $G = C_{pq} = \langle \alpha \,|\alpha^{pq} = 1\rangle$. Let us call $(\alpha)^p = \varphi$. Consider a Fuchsian group Δ with signature $(0; p^{\frac{2g}{q(p-1)}}, pq, pq)$ and an epimorphism $\theta : \Delta \to C_{pq}$ defined by:
$$\theta(x_i) = \alpha^q,\ 1 \le i \le \frac{2g}{q(p-1)};$$
$$\theta(x_{\frac{2g}{q(p-1)}+1}) = \alpha^j \text{ such that } (j, pq) = 1 \text{ and } (\frac{2g}{p-1} + j, pq) = 1.$$
By (20), $s(\theta^{-1}\langle\varphi\rangle) = (0; p^r)$, with $r = q \cdot \frac{2g}{q(p-1)} + 1 + 1 = \frac{2g}{(p-1)} + 2$. By (9) (see also (10)), the surface $X_g = \mathcal{H}/Ker(\theta)$ is a cyclic p-gonal Riemann surface with automorphisms group C_{pq}.
2. $G = D_{pq} = \langle s,\, \alpha\,|s^2 = \alpha^{pq} = (s\alpha)^2 = 1\rangle$. Let us call $\alpha^p = \varphi$. Consider a Fuchsian group Δ with signature $(0; p^{\frac{g}{q(p-1)}}, 2, 2, pq)$ and an epimorphism $\theta : \Delta \to D_{pq}$ defined by:
$$\theta(x_i) = \alpha^q,\ 1 \le i \le \frac{g}{q(p-1)};$$
$$\theta(x_{\frac{g}{q(p-1)}+1}) = s\alpha^{\frac{g}{p-1}},\ \theta(x_{\frac{g}{q(p-1)}+2}) = s\alpha,\ \theta(x_{\frac{g}{q(p-1)}+3}) = \alpha^{-1}.$$
Using (20) and (9), we conclude that the surface $X_g = \mathcal{H}/Ker(\theta)$ is a cyclic p-gonal Riemann surface with automorphisms group D_{pq}.
3. $G = C_p \rtimes C_q = \langle\varphi,\, \alpha\rangle$. Consider a Fuchsian group Δ with signature $(0; p^r, q, q)$, with $r \ge 2$, and an epimorphism $\theta : \Delta \to C_p \rtimes C_q$ defined by:
$$\theta(x_i) = \varphi^{j_i},\ 1 \le i \le r \text{ such that } \sum j_i \equiv 0(mod p),$$
$$\text{and } \theta(x_{r+1}) = \alpha,\ \theta(x_{r+2}) = \alpha^{-1}.$$
As before the surface $X_g = \mathcal{H}/Ker(\theta)$ is a cyclic p-gonal Riemann surface with automorphisms group $C_p \rtimes C_q$.
4. $G = C_p \rtimes D_q = \langle\varphi\rangle \rtimes \langle s,\, \alpha\rangle$. Consider a Fuchsian group Δ with signature $(0; p^r, 2, 2, q)$, with $r \ge 2$, and an epimorphism $\theta : \Delta \to C_p \rtimes C_q$ defined by:
$$\theta(x_i) = \varphi^{j_i},\ 1 \le i \le r \text{ such that } \sum j_i \equiv 0(mod p),$$
$$\text{and } \theta(x_{r+1}) = s,\ \theta(x_{r+2}) = s\alpha,\ \theta(x_{r+3}) = \alpha^{-1}.$$
The surface $X_g = \mathcal{H}/Ker(\theta)$ is a cyclic p-gonal Riemann surface with automorphisms group $C_p \rtimes D_q$.

5. The signatures to be considered in these cases are: $(0;p^r,2,3,3)$ for the group $G = C_p \times A_4$, $(0;p^r,2,3,4)$ for the group $G = C_p \times \Sigma_4$, $(0;p^r,2,3,5)$ for the group $G = C_p \times A_5$ and again $(0;p^r,2,3,3)$ for the group $G = (C_p \times A_4) \rtimes_2 C_2$. In all the cases $r \geq 2$. The epimorphisms are defined as in Cases 3 and 4.
6. $G = (C_2 \times C_2) \rtimes_3 C_9 = \langle s,t \rangle \rtimes \langle \alpha \rangle$, where $\varphi = \alpha^3$. Consider the signature $(0;3^r,2,9,9)$, with $r \geq 2$, and an epimorphism $\theta : \Delta \to (C_2 \times C_2) \rtimes_3 C_9$ defined by:
$$\theta(x_i) = \alpha^{3j_i},\ 1 \leq i \leq r \text{ such that } \sum j_i \equiv 0(mod 3),$$
$$\text{and } \theta(x_{r+1}) = s,\ \theta(x_{r+2}) = s\alpha,\ \theta(x_{r+3}) = \alpha^{-1}.$$
7. $G = (C_2 \times C_2 \times C_p) \rtimes_3 C_3 = \langle s,t,\varphi \rangle \rtimes \langle \alpha \rangle$, with $p \equiv 1(mod 6)$. Consider the signature $(0;p^r,2,3,3)$, with $r \geq 2$, and an epimorphism $\theta : \Delta \to (C_2 \times C_2 \times C_p) \rtimes_3 C_3$ defined by:
$$\theta(x_i) = \varphi^{j_i},\ 1 \leq i \leq r \text{ such that } \sum j_i \equiv 0(mod p),$$
$$\text{and } \theta(x_{r+1}) = s,\ \theta(x_{r+2}) = s\alpha,\ \theta(x_{r+3}) = \alpha^{-1}.$$
8. $G = ((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2 = ((\langle s,t \rangle) \rtimes \langle \alpha \rangle) \rtimes \langle v \rangle$, where $\varphi = \alpha^3$. Consider the signature $(0;3^r,2,4,9)$, with $r \geq 2$, and an epimorphism $\theta : \Delta \to ((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2$ defined by:
$$\theta(x_i) = \alpha^{3j_i},\ 1 \leq i \leq r \text{ such that } \sum j_i \equiv 0(mod 3),$$
$$\text{and } \theta(x_{r+1}) = v,\ \theta(x_{r+2}) = v\alpha,\ \theta(x_{r+3}) = \alpha^{-1}.$$

□

Bujalance et al. found in (6) presentations of the groups of automorphsms of trigonal Riemann surfaces X depending on the branching data of the covering $X \to X/Aut(X)$.

Remark 7. *We use GAP package in order to obtain epimorphisms in particular situations of the above theorem, these examples provide us patterns for general cases.*

Example 8. *Let p be an odd prime integer and $q \geq 3$ such that $(p, q) = 1$. Consider surface epimorphisms*

$$\begin{aligned} \theta &: \Delta(0; 2, 2p, qp) \to G \\ &= C_p \times D_q = \langle s,\, \alpha,\, \varphi \,| s^2 = \alpha^q = \varphi^p = (s\alpha)^2 = (s\varphi)^{2p} = (\alpha\varphi)^{pq} = 1\rangle \end{aligned}$$

defined by:

$$\theta(x_1) = s\alpha,\ \theta(x_2) = s\varphi \text{ and } \theta(x_3) = \alpha\varphi^{-1}.$$

By Theorem 6 the epimorphisms θ induce regular dessins d'enfant of type $\{2p, pq\}$ on the cyclic p-gonal surfaces $\mathcal{H}/Ker(\theta)$.

Example 9. *Cases 5 and 7 in Theorem 6 yield two cyclic heptagonal (7-gonal) Riemann surfaces of genus 66 with a heptagonal morphism ramified in exactly the same 24 points on the Riemann sphere but having non-isomorphic automorphisms groups. Hence the the position of branched points in the Riemann sphere does not provide enought information to know the automorphisms group of a cyclic heptagonal Riemann surface.*

4. Automorphism Groups of Real Cyclic *p*-gonal Riemann Surfaces

In this section we will consider NEC groups. An NEC group $\overline{\Delta}$ with signature

$$(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \qquad (4)$$

corresponds to a quotient orbifold $\mathcal{H}/\overline{\Delta}$ with underlying surface of genus g, having r cone points and k boundary components, each with $s_i \geq 0$ corner points. The signs $+$ and $-$ correspond to orientable and non-orientable orbifolds respectively. For a general reference on NEC groups see (7).

An NEC group Γ without elliptic elements is called a *surface group*; it has signature $s(\Gamma) = (g; \pm; -; \{(-) \ldots (-)\})$. In such a case $\mathcal{H}/\Gamma$ is a *Klein surface*, that is, a surface of topological genus g with a dianalytical structure, orientable or not according to the sign $+$ or $-$ and possibly with boundary. Any Klein surface of genus greater than one can be expressed as $\mathcal{H}/\Gamma$ for Γ a surface NEC group.

Given a Klein (resp. Riemann) surface $X = \mathcal{H}/\Gamma$, with Γ a surface group, a finite group G is a group of automorphisms of X if and only if there exists an NEC group Δ and an epimorphism $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$ (see (7)).

Given an odd prime p, a real cyclic p-gonal Riemann surface is a triple (X, f, σ) where σ is a symmetry of X, f is a cyclic p-gonal morphism and $f \cdot \sigma = c \cdot f$, with c the complex conjugation.

Costa and Izquierdo ((9), (10)) gave the following characterization of real p-gonal Riemann surfaces. Let X be a Riemann surface of genus g. The surface X admits a symmetry σ and a meromorphic function f such that (X, f, σ) is a real cyclic p-gonal surface if and only if there is an NEC group Λ with signature $(0; +; [p^r]; \{(p^s)\})$, $r, s \geq 0$ and an epimorphism $\theta : \Lambda \to D_p$, or $\theta : \Lambda \to C_{2p}$, such that X is conformally equivalent to $\mathcal{H}/Ker(\theta)$ with $Ker(\theta)$ a surface Fuchsian group. In the case of $\theta : \Lambda \to C_{2p}$, then $s(\Lambda) = (0; +; [p^r]; \{(-)\})$.

Given a real cyclic p-gonal surface (X, f, σ), we shall call $\pm$-automorphisms group to the group $Aut^{\pm}(X)$ of conformal and anticonformal automorphisms of X. We want to find the groups of automorphisms of real cyclic p-gonal Riemann surfaces. By Lemma 2.1 in (1) the condition of σ being a lift of an anticonformal involution of the Riemann sphere by the covering f is automatically satisfy for genera $g \geq (p-1)^2 + 1$, since the p-gonal morphism is unique. In this case, this is equivalent to say that the group C_p generated by the p-gonal morphism is normal in $Aut^{\pm}(X)$. As for the case of groups of conformal automorphisms of p-gonal Riemann surfaces we have:

Lemma 10. *Let (X, f, σ) be a real cyclic p-gonal Riemann surface such that the p-gonality group is normal in $Aut^{\pm}(X)$. Then $Aut^{\pm}(X)$ is an extension of C_p by a group of conformal and anticonformal automorphisms of the Riemann sphere.*

A finite group $\overline{G}$ of conformal and anticonformal automorphisms of the Riemann sphere is a subgroup of:

$$D_q,\ C_q \times C_2,\ D_q \rtimes C_2,\ A_4 \times C_2,\ \Sigma_4,\ \Sigma_4 \times C_2,\ A_5 \times C_2.$$

With the same proofs as in Theorems 3 and 6 we obtain the automorphisms groups of real cyclic p-gonal Riemann surfaces in the following theorem:

Theorem 11. *Let (X_g, f, σ) be a real cyclic p-gonal Riemann surface with p an odd prime integer, $g \geq (p-1)^2 + 1$. If the p-gonality group of X_g is $\langle \varphi \rangle$ and $\overline{G} = Aut^{\pm}(X_g)/\langle \varphi \rangle$ then the possible $\pm$-automorphisms groups of X_g are*

1. $C_{pq} \times C_2$ *if* $\langle \varphi, \sigma \rangle = C_{2p}$
   D_{pq} *if* $\langle \varphi, \sigma \rangle = D_p$
2. $D_{pq} \rtimes C_2$, *where* $\rtimes$ *means any possible semidirect product (including the direct product).*
3. $(C_p \rtimes C_q) \rtimes C_2$, *where* $\rtimes$ *means any possible semidirect product (including the direct product).*
4. $(C_p \rtimes D_q) \rtimes C_2$, *where* $\rtimes$ *means any possible semidirect product (including the direct product).*
5. $C_p \rtimes_2 \Sigma_4 = (C_p \times A_4) \rtimes_2 C_2,\ D_p \times A_4,\ D_p \times \Sigma_4,\ D_p \times A_5$ *if* $\langle \varphi, \sigma \rangle = D_p$
   $C_p \times \Sigma_4,\ C_{2p} \times A_4,\ C_{2p} \times \Sigma_4,\ C_{2p} \times A_5$ *if* $\langle \varphi, \sigma \rangle = C_{2p}$
6. *Exceptional Case 1.* $((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2$ *for* $p = 3$ *and* $\overline{G} = \Sigma_4$ *where* $\langle \varphi, \sigma \rangle = D_p$
   $((C_2 \times C_2) \rtimes_3 C_9) \times C_2$ *for* $p = 3$ *and* $\overline{G} = A_4 \times C_2$ *where* $\langle \varphi, \sigma \rangle = C_{2p}$
7. *Exceptional Case 2.* $(C_p \times C_2 \times C_2) \rtimes_3 C_6$ *for* $p \equiv 1$ *mod* 6, $\overline{G} = A_4 \times C_2$ *and* $\langle \varphi, \sigma \rangle = C_{2p}$
8. *Exceptional Case 3.* $(((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2) \times C_2$ *for* $p = 3$ *and* $\overline{G} = \Sigma_4 \times C_2$.

Remark 12.

1. *In order to construct real cyclic p-gonal Riemann surfaces* X *with automorphisms groups* G *isomorphic to groups of the form* $(C_p \rtimes C_q) \rtimes C_2$ *(case 4) it is necessary to consider NEC groups with several essentially different signatures uniformizing the orbifolds* X/G *(this is a point of difference of the proof of the theorems for real Riemann surfaces). More precissely to obtain real p-gonal Riemann surfaces* X *with automorphisms groups* G *isomophic to* $(C_p \rtimes C_q) \times C_2$ *we need to use NEC groups uniformizing the orbifold* X/G *with signatures* $(0; [p^r, q]; \{(-)\})$ *and to obtain automorphisms groups isomorphic to* $C_p \rtimes D_q$ *it is necessary to consider NEC groups with signatures* $(0; [p^r]; \{(q, q)\})$. *In case 2, to obtain automorphisms groups of the form* $D_{pq} \rtimes C_2$ *(not direct product) we need to consider NEC groups with signatures* $(0; [p^r, 2]; \{(qp)\})$ *but to obtain automorphisms groups* $D_{pq} \times C_2$ *we must consider NEC groups with signatures* $(0; [p^r]; \{(2, 2, qp)\})$.
2. *Note that in the exceptional case 2* $\overline{G}$ *cannot be* Σ_4. *Observe that the group* $A_4 = G$ *that appeared in the exceptional case 2 of theorem 3 could be extended a priory to the either* $\overline{G} = A_4 \times C_2$ *or* $\overline{G} = \Sigma_4$. *But this last extension cannot exist.*
3. *In the exceptional case 3, the group*

$$(((C_2 \times C_2) \rtimes_3 C_9) \rtimes_2 C_2) \times C_2$$

contains at least two classes of symmetries: one commuting with the automorphism φ of p-gonality and the order inverting the automorphism of p-gonality.

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