A problem in number theory

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Abstract

This thesis focuses on a function which moves the last digit of an integer to the first position, e.g. $A(123) = 312$. The objective of this thesis is to show how one can find all solutions $x$ to the equation $A(x) = kx$, where $k$ is a rational number. It also explains the connection between the solutions and certain periodic decimal numbers, and in which way these decimal numbers can be used to solve the equation. Finally, the problem is generalized to other bases than 10.

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Nomenclature

Most of the reoccurring abbreviations and symbols are described here.

Symbols

\( \phi(n) \)  Euler’s \( \phi \)-function
\( \lambda(n) \)  Carmichael’s function
\( A(x) \)  function which moves the last digit of \( x \) in base 10 to the first position
\( A_B(x) \)  function which moves the last digit of \( x \) in base \( B \) to the first position

Abbreviations

gcd  greatest common divisor
lcm  least common multiple
ord\( _m(n) \)  multiplicative order of \( n \) modulo \( m \)
max\( (\alpha, \beta) \)  maximum of \( \alpha \) and \( \beta \)
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Chapter 1

Introduction

This text is written as a Bachelor of Science thesis at Linköpings universitet by Hannah Schäfer Sjöberg with Anders Björn as supervisor and examiner in 2013.

Let, for an integer $x$, $A(x)$ be the integer which is obtained by moving the last digit of $x$ to the first position, e.g. $A(1024) = 4102$. Consider the equation $A(x) = kx$. Jimmie Enhäll wrote 2004/2005 a Bachelor’s thesis about this problem, in which he solved $A(x) = kx$ for $k \in \mathbb{Z}$. He also generalized the problem to other bases than 10. His solutions indicate that there might be a connection between the solutions of the above equations and certain periodic decimal numbers.

The objective of this Bachelor’s thesis is to examine this connection and by that contribute to a better understanding of the problem. It also deals with examining the corresponding equation in which $k$ is not an integer (but of course a rational number). In Chapter 2 the problem is restricted to base 10, whereas in Chapter 3 the problem is generalized to all bases. There is also an appendix containing tables with solutions $x$ to $A(x) = kx$ for some rational numbers $k$. 

Chapter 2

\[ A(x) = kx \]

In this chapter we study the equation \( A(x) = kx \) in base 10. First, we will focus on the case where \( k \) is an integer, and observe the relation between \( x \) and certain periodic decimal numbers, e.g. \( A(142857) = 5 \cdot 142857 \) and \( 1/7 = 0.\overline{142857} \). Next, we consider the equation \( A(x) = (1/l)x \), where \( l \) is an integer. Then, the problem is generalized to \( A(x) = (p/q)x \), where \( p \) and \( q \) are integers. At the end of this chapter, we consider the differences between minimal and repeating solutions.

2.1 \( k \) is an integer

Consider the equation \( A(x) = kx \) for \( k \in \mathbb{Z}_+ = \{1, 2, \ldots\} \). \( A(x) \) is defined as the integer which is obtained by moving the last digit of \( x \) to the first position. While examining the minimal\(^1\) solutions, we can see that every solution corresponds to a periodic decimal number, as shown in the following table [1]:

\(^1\)To every minimal solution there exist infinitely many other solutions by repeating the same digits. For example, for the minimal solution \( x = 102564 \), repeating solutions are \( x = 102564102564102564 \), etc.

\[ A(x) = kx \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>105263157894736842</td>
<td>( b = k )</td>
</tr>
<tr>
<td>3</td>
<td>103448257862068965517243793</td>
<td>( b = k )</td>
</tr>
<tr>
<td>4</td>
<td>1016949152542372881355932203</td>
<td>( b = 1, k + 1, \ldots, 9 )</td>
</tr>
<tr>
<td>5</td>
<td>101492275362318840579</td>
<td>( b = k )</td>
</tr>
</tbody>
</table>

\( k = 1, 2, \ldots, 9 \)
2.1. $k$ is an integer

<table>
<thead>
<tr>
<th>8</th>
<th>1012658227848</th>
<th>8/79 = 0.1012658227848</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1139240506329</td>
<td>9/79 = 0.1139240506329</td>
</tr>
<tr>
<td>9</td>
<td>1011235955056179775...</td>
<td>9/89 = 0.1011235955056179775...</td>
</tr>
<tr>
<td></td>
<td>...2808988764044943820224719</td>
<td>...2808988764044943820224719</td>
</tr>
</tbody>
</table>

How can we explain this connection?

$A(x) = kx$

can be written as

$$10^n b + a = k(10a + b),$$

where $x = 10a + b$ and $n$ is the number of digits of $a$, and $n + 1$ is the number of digits of $x$. We observe that for every solution $x$ to this equation in the table above, we obtain a corresponding decimal number,

$$0.x = \frac{b}{10k - 1}.$$  

(For instance, for $x = 142857$ we have $A(x) = 714285 = 5 \cdot 142857$ and $0.142857 = 7/49$.) Why do all decimal numbers of the form

$$0.x = \frac{b}{10k - 1}, \quad b = k, k+1, \ldots, 9,$$

have the characteristic that moving the last digit of $x$ first gives a number which is a multiple of $x$? We can write $0.x$ as

$$\frac{x}{10^{n+1} - 1}.$$  

This gives us

$$\frac{x}{10^{n+1} - 1} = \frac{b}{10k - 1}$$

or equivalently

$$10a + b = \frac{b(10^{n+1} - 1)}{10k - 1} \iff 100ka + 10kb - 10a - b = 10^{n+1}b - b \iff k(10a + b) = 10^n b + a,$$

which is equivalent to

$$kx = A(x).$$

This means that all the numbers $x$ of the form

$$x = \frac{b(10^{n+1} - 1)}{10k - 1}$$

are solutions to

$$A(x) = kx,$$

provided that $x$ has $n + 1$ digits. So for every fraction of the form

$$\frac{b}{10k - 1}, \quad k = 2, \ldots, 9, \quad b = k, k+1, \ldots, 9,$$
we can find a corresponding \( x \) which is a solution to
\[
A(x) = kx.
\]

On the other hand, every \( x \) which is a solution to
\[
A(x) = kx,
\]
corresponds to a fraction
\[
\frac{b}{10k - 1}.
\]
Multiplying a purely periodic fraction with
\[
10^m - 1,
\]
where \( m \) is the length of the period, gives us the repeating decimals as a number. Here, \( m = n + 1 \), where \( n \) is the number of digits of \( a \).

What do we know about the length of the period or the number of digits of the minimal solutions for \( x \)? We found \( n \) by solving
\[
10^n \equiv k \pmod{10k - 1}
\]
for the smallest possible \( n \). If we find the smallest \( n \) such that
\[
10^n \equiv k \pmod{10k - 1},
\]
then for \( m = n + 1 \) we obtain
\[
10^m = 10^{n+1} \equiv 10k \equiv 1 \pmod{10k - 1}.
\]
Let \( m_1 \) be the smallest positive integer which solves \( 10^{m_1} \equiv 1 \pmod{10k - 1} \). We will show that \( m = m_1 \). Assume that \( m_1 < m \). Then, none of 10, 10\(^2\), ..., 10\(^{m_1-1}\), 10\(^{m_1}\) is congruent to \( k \pmod{10k - 1} \). But then,
\[
10^{m_1+1} \equiv 10, 10^{m_1+2} \equiv 10^2, \ldots, 10^{2m_1} \equiv 10^{m_1} \equiv 1, \ldots \pmod{10k - 1},
\]
which implies that there exists no \( n \) such that \( 10^n \equiv k \pmod{10k - 1} \). But this is a contradiction, since \( n \) is equal to the number of digits of \( a \). Hence, \( m = m_1 \).

So \( m \) is the smallest positive number which solves
\[
10^m \equiv 1 \pmod{10k - 1}.
\]
Observe that 10 and 10\(^k - 1\) are relatively prime, therefore \( m \) is called the multiplicative order of 10 modulo 10\(^k - 1\) ([3], p. 456),
\[
m = \text{ord}_{10k-1}(10).
\]
Hence, we find the length \( m \) of the minimal solutions \( x \) (or, equivalently, the length \( m \) of the period of the fraction \( b/(10k-1) \)) by finding the multiplicative order of 10 modulo 10\(^k - 1\). We do not have to try for every possible \( m = 1, 2, \ldots \) if it solves
\[
10^m \equiv 1 \pmod{10k - 1},
\]
because we know that \( \text{ord}_{10k-1}(10) \) always divides \( \varphi(10k-1) \), Euler’s \( \varphi \)-function ([3], pp. 342-344), which counts the number of positive integers less than or equal to \( 10k-1 \) which are relatively prime to \( 10k-1 \). Euler’s \( \varphi \)-function for a positive integer \( y \) with prime factorization \( y = p_1^{a_1}p_2^{a_2} \ldots p_s^{a_s} \) is

\[
\varphi(y) = (p_1^{a_1} - p_1^{a_1-1})(p_2^{a_2} - p_2^{a_2-1}) \ldots (p_s^{a_s} - p_s^{a_s-1}).
\]

For example for \( k = 2 \), \( \varphi(10k-1) = \varphi(19) = 18 \), so we only have to try \( m = 1, 2, 3, 6, 9, 18 \) to find \( m = \text{ord}_{19}(10) \). An even stronger statement is that the multiplicative order of \( n \) modulo any number coprime to \( n \) also divides \( \lambda(n) \), the value of the Carmichael function ([5], Corollary 9.1.1). Here \( \lambda(10k-1) \) is the smallest integer such that

\[
r^{\lambda(10k-1)} \equiv 1 \pmod{10k-1}
\]

for every integer \( r \) relatively prime to \( 10k-1 \). The Carmichael function ([4], pp. 275-276) for a positive integer \( y \) is defined as

\[
\lambda(y) = \begin{cases} 
\varphi(p^e), & \text{if } y = 2, 4, p^e \text{ or } 2p^e, \\
\frac{\varphi(p^e)}{2}, & \text{if } y = 2^e \text{ and } e \geq 3, \\
\text{lcm}(\lambda(p_1^{e_1}), \lambda(p_2^{e_2}), \ldots, \lambda(p_s^{e_s})), & \text{if } y = \prod_{i=1}^s p_i^{e_i}.
\end{cases}
\]

The following table shows the values of Euler’s \( \varphi \)-function and Carmichael’s function for \( 10k-1 \), and the multiplicative order of 10 modulo \( 10k-1 \) for all possible \( k \geq 2 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 10k-1 )</th>
<th>( \varphi(10k-1) )</th>
<th>( \lambda(10k-1) )</th>
<th>( \text{ord}_{10k-1}(10) )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>19</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>29</td>
<td>28</td>
<td>28</td>
<td>28</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>39</td>
<td>24</td>
<td>12</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>49</td>
<td>42</td>
<td>42</td>
<td>42</td>
<td>41</td>
</tr>
<tr>
<td>6</td>
<td>59</td>
<td>58</td>
<td>58</td>
<td>58</td>
<td>57</td>
</tr>
<tr>
<td>7</td>
<td>69</td>
<td>44</td>
<td>22</td>
<td>22</td>
<td>21</td>
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<tr>
<td>8</td>
<td>79</td>
<td>78</td>
<td>78</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>9</td>
<td>89</td>
<td>88</td>
<td>88</td>
<td>44</td>
<td>43</td>
</tr>
</tbody>
</table>

### 2.2 \( k = \frac{1}{7} \)

Next, we want to find the solutions \( x \) to \( A(x) = kx \) for another special case of \( k \). Consider the equation

\[
A(x) = kx, \quad \text{where } k = \frac{1}{7}, \quad l \in \mathbb{Z},
\]

which is the same as

\[
lA(x) = x.
\]

Letting \( x = 10a + b \), where \( a, b \in \mathbb{Z} \) and \( n \) is the number of digits of \( a \), this is equivalent to

\[
l(10^n b + a) = 10a + b,
\]
where \(10^n - 1 \leq a < 10^n\), \(b = 1 \ldots 9\), \(l = 2 \ldots 9\).

(If \(l = 1\), then \(k = 1\), which is the trivial case \(x = 11b\).)

This equation expressed in \(a\) is

\[
a = \frac{b(10^n l - 1)}{10 - l},
\]

where

\[
10^n > a = \frac{b(10^n l - 1)}{10 - l}.
\]

(2.1)

Since \((10^n l - 1) > 10^n\) for \(l \geq 2\),

\[
b < 10 - l.
\]

Moreover,

\[
(10 - l) \mid b(10^n l - 1)
\]

and as

\[
(10 - l) \nmid b
\]

it follows that

\[\gcd(10 - l, 10^n l - 1) \neq 1.\] (2.2)

Observe that

\[
2 \nmid (10^n l - 1) \quad \text{and} \quad 5 \nmid (10^n l - 1),
\]

and

\[
3 \nmid (10^n l - 1) \quad \text{can be written as} \quad 3 \nmid ((10^n - 1)l + (l - 1)),
\]

where

\[
3 \nmid (10^n - 1),
\]

so

\[
3 \nmid (10^n l - 1) \quad \text{happens if and only if} \quad 3 \nmid (l - 1).\] (2.3)

In the following table we can see which values for \(l\) could be possible:

<table>
<thead>
<tr>
<th>(l)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 - (l)</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Prime factorization of 10 - (l)</td>
<td>2(^4)</td>
<td>7</td>
<td>2 \cdot 3</td>
<td>5</td>
<td>2(^2)</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(\gcd(10 - l, 10^n l - 1))</td>
<td>1</td>
<td>*</td>
<td>*</td>
<td>1</td>
<td>1</td>
<td>*</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The remaining possible values for \(l\) are 3, 4 and 7. (They are marked by *.) We will examine each case to find all solutions \(x\).

**Case 1.** \(l = 7\). Consider \(\gcd(10 - l, 10^n l - 1) = \gcd(3, 7 \cdot 10^n - 1)\). From (2.3) follows that \(3 \nmid (7 \cdot 10^n - 1)\) because \(3 \nmid 6\), so \(\gcd(3, 7 \cdot 10^n - 1) = 3\), i.e. condition (2.2) is satisfied. But through restriction (2.1),

\[
a = \frac{b(7 \cdot 10^n - 1)}{3} < 10^n
\]

we obtain

\[
b(7 \cdot 10^n - 1) < 3 \cdot 10^n, \quad \text{where} \quad b \geq 1.
\]
2.2. \( k = \frac{1}{7} \)

Hence, there are no solutions for \( l = 7 \).

**Case 2.** \( l = 4 \). Through restriction (2.1),

\[
\frac{b(4 \cdot 10^n - 1)}{6} < 10^n
\]

we obtain

\[
b(4 \cdot 10^n - 1) < 6 \cdot 10^n.
\]

Observe that

\[
gcd(10 - l, 10^n l - 1) = gcd(6, 4 \cdot 10^n - 1) = 3.
\]

Therefore, 2 must divide \( b \). So,

\[
b(4 \cdot 10^n - 1) < 6 \cdot 10^n, \quad b \geq 2
\]

which means that there are no solutions for \( l = 4 \).

**Case 3.** \( l = 3 \). Condition (2.2) says that

\[
gcd(10 - l, 10^n l - 1) = gcd(7, 3 \cdot 10^n - 1) \neq 1,
\]

and because 7 is a prime, this implies that 7 divides \( 3 \cdot 10^n - 1 \). We obtain

\[
3 \cdot 10^n \equiv 1 \pmod{7}
\]

and hence

\[
3 \cdot 10^n \equiv 3 \cdot 3^n \equiv 3^{n+1} \equiv 1 \pmod{7}.
\]

Let \( m = n + 1 \),

\[
3^m \equiv 1 \pmod{7}, \quad \text{where } gcd(3, 7) = 1.
\]

The smallest such \( m \) is called the multiplicative order of 3 modulo 7, \( \text{ord}_7(3) \).

We find that \( \text{ord}_7(3) = 6 \), which implies that

\[
3^m \equiv 1 \pmod{7} \quad \text{for all } m = 6j, \ j \in \mathbb{Z}.
\]

So all \( n \) must be of the form \( n = 6j - 1 \). The smallest possible \( n \) is 5. Through (2.1) we obtain a restriction on \( b \),

\[
\frac{b(3 \cdot 10^n - 1)}{7} < 10^n,
\]

that is,

\[
b(3 \cdot 10^n - 1) < 7 \cdot 10^n,
\]

which restricts \( b \) to 1 and 2. Now we can calculate the minimal solutions \( x \) for \( b = 1, 2 \). For \( n = 5 \) and \( b = 1 \), we have

\[
a = \frac{3 \cdot 10^5 - 1}{7} = 42857,
\]

\[
x = 428571.
\]
For \( n = 5 \) and \( b = 2 \), we have
\[
    a = \frac{2(3 \cdot 10^5 - 1)}{7} = 85714,
\]
\[
    x = 857142.
\]

**Solution.** \( 3A(x) = x \) has two minimal solutions, \( x_1 = 428571 \) and \( x_2 = 857142 \). All other solutions are repetitive solutions of \( x_1 \) and \( x_2 \), e.g. \( 428571428571 \) and \( 428571428571428571 \) etc. There are no solutions for \( lA(x) = x, \ l = 2, \ldots, 9 \) where \( l \neq 3 \).

Observe that
\[
    x_1 = 428571 \quad \text{and} \quad \frac{3}{7} = 0.428571,
\]
\[
    x_2 = 857142 \quad \text{and} \quad \frac{5}{7} = 0.714285.
\]

Again, we find a correspondence between the solutions \( x \) and a periodic decimal number,
\[
    0.x = \frac{lb}{10 - l}
\]
or, equivalently,
\[
    x = \frac{lb(10^m - 1)}{10 - l}.
\]
Where does this property come from? We can rewrite the equation as
\[
    10a + b = \frac{lb(10^m - 1)}{10 - l} \iff 100a + 10b - 10la - lb = lb10^m - lb \iff 10a + b = l(a + b10^n),
\]
which is equivalent to
\[
    x = lA(x).
\]
So, any possible solution \( x \) for \( lA(x) = x \) must have the form
\[
    x = \frac{lb(10^m - 1)}{10 - l}.
\]
(Where \( m \) denotes the number of digits in \( x \).)

### 2.3 \( k \) is any rational number

So far, we were able to show that the solutions \( x \) to
\[
    A(x) = kx, \quad k \in \mathbb{Z}
\]
and
\[
    A(x) = \frac{1}{l}x, \quad l \in \mathbb{Z}
\]
can be expressed as
\[
    \frac{x}{10^m - 1} = \frac{b}{10k - 1}
\]
and
\[
\frac{x}{10^m - 1} = \frac{lb}{10 - l},
\]
respectively. (Where \( b \) is the last digit in \( x \) and \( m \) is the number of digits in \( x \).)

We would like to generalize this idea to
\[
A(x) = \frac{p}{q}x, \quad p, q \in \mathbb{Z}.
\]
(with \( 1/10 < k = p/q < 10 \).)

We want to express \( x/(10^m - 1) \) in terms of \( b, p \) and \( q \):
\[
A(x) = \frac{p}{q}x
\]
can be written as
\[
b10^n + a = \frac{p(10a + b)}{q} \iff qb10^n = p(10a + b) - qa
\]
\[
\iff qb10^{n+1} = 10a(10p - q) + 10pb
\]
\[
\iff qb10^{n+1} - qb = 10a(10p - q) + 10pb - qb
\]
\[
\iff \frac{qb}{10p - q} = \frac{10a + b}{10^{n+1} - 1}, \quad (2.4)
\]
which is equivalent to
\[
\frac{x}{10^m - 1} = \frac{qb}{10p - q}.
\]
This means that any solution \( x \) to
\[
A(x) = \frac{p}{q}x
\]
corresponds to the fraction
\[
\frac{qb}{10p - q}.
\]
Solving (2.4) for \( a \) we obtain
\[
a = \frac{b(10^nq - p)}{10p - q}.
\]
So if we want to find the solutions \( x \) to \( A(x) = (p/q)x \), we can as before find \( b \) and the smallest \( n \) such that
\[
10^{n-1} \leq a = \frac{b(10^nq - p)}{10p - q} < 10^n,
\]
where \( a \) is an integer. Alternatively, we can use the fact that
\[
\frac{x}{10^m - 1} = \frac{qb}{10p - q}
\]
to find all possible \( x \). Here,
Chapter 2. \( A(x) = kx \)

gives us a fraction with purely periodic decimal expansion and a period length of \( m \). The decimal expansion of a fraction is said to be purely periodic if the period starts with the first digit in the decimal expansion. This means that

\[
\frac{qb}{10p - q}
\]

also has to be a fraction with purely periodic decimal expansion. We also obtain the bounds\(^2\)

\[
\frac{1}{10} < \frac{x}{10^m - 1} = \frac{qb}{10p - q} < 1.
\]

Solving

\[
\frac{1}{10} < \frac{qb}{10p - q}
\]

and

\[
\frac{qb}{10p - q} < 1
\]

for \( b \), we obtain the following restrictions for \( b \):

\[
\frac{p}{q} - \frac{1}{10} < b < \frac{10p}{q} - 1.
\]

(2.5)

Observe that \( 1 \leq b \leq 9 \), so from

\[
\frac{p}{q} - \frac{1}{10} < b < 9
\]

and

\[
1 \leq b < \frac{10p}{q} - 1
\]

follows that

\[
\frac{1}{5} < \frac{p}{q} < \frac{91}{10}.
\]

Additionally to these restrictions for \( b \), we can obtain further restrictions. We know that

\[
\frac{bq}{10p - q}
\]

has to be a fraction with purely periodic decimal expansion. If

\[
\frac{q}{10p - q}
\]

does not have a purely periodic decimal expansion, then we obtain further restrictions on \( b \). If \( q/(10p - q) \) is a reduced fraction, i.e. if the nominator and the denominator are coprime, then we can make use of a theorem to see if its decimal expansion is purely periodic. First consider the case where \( q/(10p - q) \) is not a reduced fraction, that is, \( q \) and \( 10p-q \) are not coprime. What happens in this case? Assume \( \text{gcd}(q, 10p-q) \neq 1 \), which is equivalent to \( \text{gcd}(q, 10) \neq 1 \). We may assume \( p/q \) to be a reduced fraction, i.e. \( p \) and \( q \) are coprime. Therefore, \( \text{gcd}(q, 10p) \neq 1 \) is equivalent to \( \text{gcd}(q, 10) \neq 1 \). So, \( \text{gcd}(q, 10p-q) \neq 1 \) if

\(^2\)The limit \( \frac{1}{10} < \frac{x}{10^m - 1} \) comes from the fact that \( A(x) \) ignores leading zeros. For example, for \( q = 4 \), \( \frac{1}{10} = 0,025641 \), but \( A(x) = 12564 \neq 102564 = 4 \cdot 25641 \).
and only if 2 \mid q or 5 \mid q. We will show that if 2 divides q, then 2 also has to divide b. Similarly, if 5 divides q, then 5 also has to divide b. Consider
\[ A(x) = \frac{p}{q}x \]
or equivalently
\[ qA(x) = px, \]
so q must divide px, and because p and q are assumed to be coprime, q must divide x, so q \mid (10a + b).
If 2 \mid q, this means that 2 \mid (10a + b), and so it follows that 2 \mid b. Similarly, if 5 \mid q, then 5 \mid (10a + b), so 5 \mid b. From this observation it follows for example that if 10 divides q, then A(x) = (p/q)x has no solutions.
We obtain 4 cases:
Case 1. If \gcd(q, 10) = 1, then
\[ \frac{q}{10p - q} \]
is a reduced fraction.
Case 2. If \gcd(q, 10) = 2, let q = 2q_2 and b = 2b_2. Then,
\[ \frac{b}{10p - q} = 2b_2 \frac{q_2}{2(5p - q_2)} = 2b_2 \frac{q_2}{5p - q_2}, \]
where
\[ \frac{q_2}{5p - q_2} \]
is a reduced fraction, because
\[ \gcd(q_2, p) = 1 \quad \text{and} \quad \gcd(q_2, 5) = 1, \]
and hence
\[ \gcd(q_2, 5p - q_2) = \gcd(q_2, 5p) = 1. \]
Case 3. If \gcd(q, 10) = 5, let q = 5q_5 and b = 5. Then,
\[ \frac{b}{10p - q} = 5 \frac{5q_5}{5(2p - q_5)} = 5 \frac{q_5}{2p - q_5}, \]
where
\[ \frac{q_5}{2p - q_5} \]
is a reduced fraction, because
\[ \gcd(q_5, p) = 1 \quad \text{and} \quad \gcd(q_5, 2) = 1, \]
and hence
\[ \gcd(q_5, 2p - q_5) = \gcd(q_5, 2p) = 1. \]
Case 4. If \gcd(q, 10) = 10, then there are no solutions.
So far, we can express x/(10^m - 1) in terms of b (for which we have some restrictions) and a reduced fraction depending on p and q. Now, if this reduced fraction does not have a purely periodic decimal expansion, then we obtain additional restrictions on b. With the following theorem we can determine whether a reduced fraction has a purely periodic decimal expansion.
Theorem 1 ([2], Theorem 135). If \( \gcd(r, s) = 1, s = 2^a 5^b \), and \( \max(\alpha, \beta) = \mu \), then the decimal expansion of \( r/s \) terminates after \( \mu \) digits. If \( \gcd(r, s) = 1, s = 2^a 5^b Q \), where \( Q > 1, \gcd(Q, 10) = 1, \) and \( v \) is the multiplicative order of 10 \( (\mod Q) \), then the decimal expansion of \( r/s \) has a pre-period of length \( \mu \) and a period of length \( v \).

So, a fraction has a purely periodic decimal expansion if and only if its denominator is coprime to 10. In case 1, we see that \( q/(10p - q) \) has a purely periodic decimal expansion. Here all restrictions on \( b \) come from (2.5),

\[
\frac{p}{q} - \frac{1}{10} < b < \frac{10p}{q} - 1.
\]

In case 2, the denominator \( 5p - q_2 \) is not divisible by 5, but could be divisible by 2. If this is the case, we need to choose \( b \) such that \( 4 \mid b \). Let \( b = 2b_2 = 4b_4 \).

Then,

\[
b_4\frac{q}{10p - q} = b_2\frac{q}{5p - q_2} = b_4\frac{q}{2(5p - q_2)}.
\]

Now, the denominator \( 1/2(5p - q_2) \) could be coprime to 10, in which case we would have obtained all additional restrictions on \( b \). The denominator could also be divisible by 2. In this case, choose \( b = 8 \):

\[
8\frac{q}{10p - q} = \frac{q}{4(5p - q_2)}.
\]

Now, there are only two possibilities left. Either, \( \gcd(\frac{q}{4(5p - q_2)}, 10) = 1 \), then we know that \( b = 8 \). Or \( 2 \mid \frac{q}{4}(5p - q_2) \), and then there are no solutions.

In case 3, the denominator is not divisible by 2, but could be divisible by 5. If it is not divisible by 5, then \( b = 5 \). If it is divisible by 5, then there are no solutions.

Consider first the case where \( \gcd(q, 10) = 1 \). Here all restrictions on \( b \) come from

\[
\frac{p}{q} - \frac{1}{10} < b < \frac{10p}{q} - 1.
\]

For given \( p \) and \( q \), we can find the restrictions on \( b \). Then we have to find the period length \( m \) of

\[
\frac{q}{10p - q}
\]

or, equivalently, the smallest \( n = m - 1 \) such that

\[
a = \frac{b(q10^n - p)}{10p - q},
\]

where \( a \) is an integer. From Theorem 1 we know that the period length of the decimal expansion of a reduced fraction is equal to the multiplicative order of 10 \( (\mod Q) \), \( \ord_Q 10 \), where \( Q \) is the greatest divisor of the denominator which is coprime to 10. Here, \( Q \) equals \( 10p - q \), so

\[
m = \ord_{10p - q}(10).
\]
As before, to find \( m = \text{ord}_{10p-q}(10) \) it might be useful to calculate the value of the Carmichael function of \( 10p - q \), \( \lambda(10p - q) \). We know that \( \text{ord}_{10p-q}(10) \) must divide \( \lambda(10p - q) \). If we can find \( m \), we find \( x \) by

\[
x = \frac{(10^m - 1)bq}{10p - q}.
\]

In case 2, if \( \gcd(q,10) = 2 \), we need to find the biggest integer \( \alpha \leq 4 \) such that \( 2^\alpha \mid 10p - q \). Note that if \( 2^4 \mid 10p - q \), there are no solutions. If \( \alpha < 4 \), let

\[
Q = \frac{10p - q}{2^\alpha}
\]

and choose \( b \) such that

\[
2^\alpha \mid b
\]

and

\[
\frac{p}{q} - \frac{1}{10} < b < \frac{10p}{q} - 1.
\]

Now we can find \( m \) by

\[
m = \text{ord}_Q(10),
\]

and \( x \) by

\[
x = \frac{(10^m - 1)bq}{10p - q}.
\]

In case 3, where \( \gcd(q,10) = 5 \), we know that \( 5 \mid 10p - q \). We need to check if \( 5^2 \mid 10p - q \). In this case, there are no solutions. If \( 5^2 \nmid 10p - q \), let

\[
Q = \frac{10p - q}{5},
\]

choose \( b = 5 \), and check if

\[
\frac{p}{q} - \frac{1}{10} < b < \frac{10p}{q} - 1.
\]

Now we find \( m \) by

\[
m = \text{ord}_Q(10),
\]

and \( x \) by

\[
x = \frac{(10^m - 1)bq}{10p - q}.
\]

**Example.** Solve \( A(x) = (17/6)x \). Observe that \( \gcd(q,10) = \gcd(6,10) = 2 \). We need to find \( \alpha \leq 4 \) such that \( 2^\alpha \mid 10 \cdot 17 \cdot 6 \). The biggest such \( \alpha \) is 2, which means that \( 2^2 \) must divide \( b \), so \( b \) can be 4 or 8. From (2.5) it follows that

\[
\frac{17}{6} - \frac{1}{10} < b \leq \frac{170}{6},
\]

which gives us no further restrictions, and hence there are solutions \( x \) with \( b = 4 \) and \( b = 8 \). To find these solutions we need to calculate \( m = \text{ord}_Q(10) \), where

\[
Q = \frac{10 \cdot 17 \cdot 6}{2^2} = 41.
\]
The multiplicate order of 10 modulo 41 must divide \( \lambda(41) \), and since 41 is an odd prime, it follows that \( \lambda(41) = \varphi(41) = 40 \). Then,

\[
x = (10^m - 1) \cdot \frac{b \cdot 6}{10 \cdot 17 - 6}
\]

must be an integer for \( b = 4 \) and \( b = 8 \). Let \( b = 4 \). Then \( m \) is the smallest divisor of 40 such that

\[
x = (10^m - 1) \cdot \frac{4 \cdot 6}{10 \cdot 17 - 6}
\]

becomes an integer. This gives us \( m = 5 \). For \( b = 4 \) we obtain

\[
x = (10^5 - 1) \cdot \frac{4 \cdot 6}{10 \cdot 17 - 6} = 14634,
\]

where

\[
A(x) = \frac{17}{6} x = 41463.
\]

The corresponding periodic fraction is

\[
\frac{6 \cdot 4}{10 \cdot 17 - 6} = \frac{6}{41} = 0.14634.
\]

For \( b = 8 \) we obtain

\[
x = (10^5 - 1) \cdot \frac{8 \cdot 6}{10 \cdot 17 - 6} = 29268,
\]

where

\[
A(x) = \frac{17}{6} x = 82926.
\]

The corresponding periodic fraction is

\[
\frac{6 \cdot 8}{10 \cdot 17 - 6} = \frac{12}{41} = 0.29268.
\]

### 2.4 Minimal solutions

As mentioned before, to every solution \( x \), there are infinitely many repeating solutions. Therefore, we referred to \( x \) as minimal solutions. For example for \( A(x) = (17/6)x \), a minimal solution is \( x = 41634 \), and repeating solutions are \( x = 1463414634 \), \( x = 146341463414634 \) etc. In general, the solutions we find in the way described in the last section will be minimal solutions. But in some cases, for some values of \( b \) there might exist a shorter solution, that is, the solution we found for a certain value of \( b \) is not minimal. This can occur if \( \gcd(b, Q) > 1 \). Then it is sufficient for \( m \) to be equal to \( \text{ord}_Q/\gcd(b, Q)(10) \) instead of \( \text{ord}_Q(10) \). We know that \( \text{ord}_Q/\gcd(b, Q)(10) | \text{ord}_Q(10) \), which implies that we will always find a solution by letting \( m = \text{ord}_Q(10) \), but it might not be the minimal solution. We do not necessarily find a shorter solution if \( \gcd(b, Q) > 1 \).

It can be the case that \( \text{ord}_Q/\gcd(b, Q)(10) = \text{ord}_Q(10) \), there is no general rule for calculating the multiplicative order of an integer. As an example, consider \( A(x) = 5x \). In general, \( x = (10^m - 1)b/49 \), where \( m = \text{ord}_7(10) = 42 \). But for \( b = 7 \), \( x = (10^m - 1) \cdot 7/49 = (10^m - 1) \cdot 1/7 \). Here, we find the minimal solution \( x = 142857 \) with \( m = \text{ord}_7(10) = 6 \).
Chapter 3

Other Bases

In this chapter we will examine what happens to \( A(x) = (p/q)x \) if we define \( A(x) \) for other bases than 10. \( A(x) \) is defined as the integer which is obtained by moving the last digit of \( x \) first. So far, we considered \( x \) represented in the decimal system. Now, we extend the definition of \( A(x) \) to an arbitrary base \( B \geq 2 \). Let \( A_B(x) \) be the integer which is obtained by moving the last digit of \( x \) first, where \( x \) is expressed in base \( B \). This means that

\[
A_B(x) = B^n b + a,
\]

where

\[
x = B a + b, \quad B^{n-1} \leq a < B^n, \quad 1 \leq b < B,
\]

and \( n \) is the number of digits of \( a \) represented in base \( B \). For example,

\[
1234 = 2 \cdot 8^3 + 3 \cdot 8^2 + 2 \cdot 8^1 + 2 \cdot 8^0 = (2322)_8,
\]

\[
A_8(1234) = A_8((2322)_8) = (2232)_8 = 2 \cdot 8^3 + 2 \cdot 8^2 + 3 \cdot 8^1 + 2 \cdot 8^0 = 1178.
\]

Consider now the equation

\[
A_B(x) = \frac{p}{q} x,
\]

with \( 1/B < p/q < B \). Now \( x/(B^m - 1) \) can be expressed in terms of \( b, p \) and \( q \).

(As before, \( m = n + 1 \).)

\[
A_B(x) = \frac{p}{q} x
\]

can be written as

\[
B^n b + a = \frac{p(Ba + b)}{q} \quad \iff \quad B^n qb = a(Bp - q) + pb
\]

\[
\iff \quad B^{n+1} qb - qb = Ba(Bp - q) + Bp b - qb
\]

\[
\iff \quad \frac{qb}{Bp - q} = \frac{Ba + b}{B^{n+1} - 1},
\]

which is equivalent to

\[
\frac{x}{B^m - 1} = \frac{qb}{Bp - q}.
\]
This means that any solution \( x \) to

\[ A_B(x) = \frac{p}{q} x \]

corresponds to the fraction

\[ \frac{qb}{Bp - q}. \]

If \( x \) is a solution, \( \frac{qb}{(Bp - q)} \) will have a purely periodic base \( B \) expansion and the period will be equal to the expression of \( x \) in base \( B \), with

\[ \frac{1}{B} < \frac{x}{B^m - 1} = \frac{q\bar{b}}{Bp - q} < 1. \]

Solving these bounds for \( b \), we obtain

\[ \frac{p}{q} - \frac{1}{B} < b < \frac{Bp}{q} - 1 \]

as restrictions for \( b \). The fraction

\[ \frac{q\bar{b}}{Bp - q} \]

must have a purely periodic base \( B \) expansion, which implies that if

\[ \frac{q}{Bp - q} \]

does not have a purely periodic base \( B \) expansion, we obtain further restrictions on \( b \). A generalization of Theorem 1 to all bases will help analyzing whether the base \( B \) expansion of \( \frac{q}{Bp - q} \) is purely periodic.

**Theorem 2** ([5], Theorem 12.4). *Let \( B \) be a positive integer. Then a periodic base \( B \) expansion represents a rational number. Conversely, the base \( B \) expansion of a rational number either terminates or is periodic. Further, if \( 0 < r/s < 1 \), where \( r \) and \( s \) are coprime, positive integers, and \( s = TQ \), where every prime factor of \( T \) divides \( B \) and \( \gcd(Q, B) = 1 \), then the period length of the base \( B \) expansion of \( r/s \) is \( \text{ord}_{Q}(B) \), the multiplicative order of \( B \) (mod \( Q \)), and the pre-period length is \( \mu \), where \( \mu \) is the smallest positive integer such that \( T \mid B^{\mu} \).*

Now we can find the solutions \( x \) for

\[ A_B(x) = \frac{p}{q} x \]

in the following way: First, find \( d = \gcd(Bp - q, q) \). Observe that \( \gcd(p, q) = 1 \) implies that \( d = \gcd(Bp - q, q) = \gcd(B, q) \). Let \( q_d = q/d \) and \( U = (Bp - q)/d \), so that

\[ \frac{q}{Bp - q} = \frac{dq_d}{dU}. \]
where \(q_d/U\) is a reduced fraction. Consider
\[ A_B(x) = \frac{p}{q} x \]
or equivalently
\[ qA(x) = px, \]
so \(q\) must divide \(px\), and because \(p\) and \(q\) are assumed to be coprime, \(q\) must divide \(x\), so \(q \mid (Ba + b)\). From \(d \mid q\) and \(d \mid (Bp - q)\) it follows that \(d \mid B\), which implies that \(d \mid b\). We have found an additional restriction for \(b\). Now, find \(T\) and \(Q\) such that \(U = TQ\), where every prime factor of \(T\) divides \(B\) and \(\gcd(Q, B) = 1\). \(T\) must divide \(b\), so that the solutions \(x\) will be of the form
\[ b \cdot \frac{q}{Bp - q} = \frac{b \cdot q_d}{T \cdot Q}, \]
where \(q_d/Q\) is a reduced fraction with purely periodic base \(B\) expansion and \(b/T\) is an integer. We know that \(T \mid b\) and \(d \mid b\). Now we want to show that \(dT \mid b\). Consider therefore
\[ \frac{x}{B^m - 1} = b \cdot \frac{q}{Bp - q} \]
or equivalently
\[ \frac{x(Bp - q)}{q} = \frac{bq}{B^m - 1}. \]
With \(x = Ba + b\) and \(Bp - q = dTQ\) we obtain
\[ \frac{(Ba + b)dTQ}{q} = (B^m - 1)b. \]
We know that \(q \mid (Ba + b)\), which leads to \(dT \mid (B^m - 1)b\). All of \(T\)'s prime divisors and \(d\) divide \(B\), which implies that both \(T\) and \(d\) are coprime to \(B^m - 1\). Hence, \(dT\) must divide \(b\). For \(b\) we obtained the additional restrictions \(dT \mid b\). For \(b \nmid Q\) we find \(m\) by
\[ m = \text{ord}_Q(B). \]
As before, if \(\gcd(b, Q) > 1\), the solution might not be minimal. In this case, let \(m = \text{ord}_Q/b, Q)(B)\). For calculating \(m\) it might be helpful to recall that \(\text{ord}_Q(B) \mid \lambda(Q)\) and \(\text{ord}_Q/\gcd(b, Q)(B) \mid \lambda(Q/\gcd(b, Q))\). Now we can find \(x\) by
\[ x = \frac{(B^m - 1)bq}{Bp - q}. \]

**Example.** Solve \(A_8(x) = (1/3)x\). Observe that \(\gcd(q, B) = \gcd(3, 8) = 1\), so all restrictions we obtain for \(b\) come from
\[ 8^{n-1} \leq a = \frac{b(B^np - p)}{Bp - q} = \frac{b(3 \cdot 8^n - 1)}{5} < 8^n. \]
It follows that \(b(3 \cdot 8^n - 1) < 5 \cdot 8^n\), hence \(b = 1\). The equation
\[ \frac{x}{B^m - 1} = \frac{qb}{Bp - q} \]
becomes now
\[ \frac{x}{8^m - 1} = \frac{3}{5}, \]
where \( m \) is the period length of the base 8 expansion of 3/5. One way of solving the equation is by finding the base 8 expansion of 3/5:
\[ \frac{3}{5} = 4 \cdot \frac{1}{8^1} + 6 \cdot \frac{1}{8^2} + 3 \cdot \frac{1}{8^3} + \ldots = (0.4631)_8. \]
This gives us
\[ x = (4631)_8 = 4 \cdot 8^3 + 6 \cdot 8^2 + 3 \cdot 8^1 + 1 \cdot 8^0 = 2457, \]
the only minimal solution to \( A_8(x) = (1/3)x \). For \( x = 2457 \),
\[ A_8(2457) = A_8((4631)_8) = (1463)_8 = 1 \cdot 8^3 + 4 \cdot 8^2 + 6 \cdot 8^1 + 3 \cdot 8^0 = 819 = (1/3) \cdot 2457. \]
Another way of solving the equation is by looking at
\[ a = \frac{3 \cdot 8^n - 1}{5} \]
and finding the smallest \( n \) such that \( a \) becomes an integer. Here, we do not have to convert to another base, because \( a \) is an integer in one base if and only if it is an integer in any other base. The smallest \( n \) such that \( 3 \cdot 8^n \equiv 1 \pmod{5} \) is 3, so \( a = (3 \cdot 8^3 - 1)/5 = 307 = 4 \cdot 8^2 + 6 \cdot 8^1 + 3 \cdot 8^0 = (463)_8 \). Now,
\[ x = 8a + b = (4631)_8 = 2457. \]
Chapter 4

Conclusion

We were able to show how to solve $A(x) = kx$ for arbitrary rational numbers $k$ and to generalize the problem to all bases. The correspondence between the solutions $x$ and certain periodic decimal numbers has been explained. In conclusion, the solutions $x$ to $A_B(x) = (p/q)x$, for given $B, p$ and $q$, are given by

$$x = \frac{(B^m - 1) bq}{Bp - q}.$$  

Then,

$$\frac{x}{B^m - 1} = \frac{b q}{Bp - q}$$

will be a fraction with purely periodic base $B$ expansion, whose reoccurring digits are equal to $x$. The remaining problem is to find $m$ and $b$. We know that there exists a solution $x$ with last digit $b$ if and only if $b$ fulfills the following conditions:

$$1 \leq b < B,$$
$$\frac{p}{q} \frac{1}{B} < b < \frac{Bp}{q} - 1$$

and

$$dT \mid b,$$

where

$$d = \gcd(q, B)$$

and

$$T = \frac{U}{Q}$$

where

$$U = \frac{Bp - q}{d}$$

and

$$Q = \max \{\tilde{Q} | \tilde{Q} \mid U, \gcd(\tilde{Q}, B) = 1\}.$$  

We have shown that if $\gcd(b, Q) = 1$, then

$$m = \text{ord}_Q(B),$$
while if \( \gcd(b, Q) > 1 \), then

\[ m = \text{ord}_{Q/\gcd(b, Q)}(B). \]

Calculating this number \( m \), the number of digits of \( x \), is the biggest challenge, as there is no direct way of finding the multiplicative order of an integer. As a help, we can always calculate \( \lambda(Q) \), the Carmichael number of \( Q \), or if \( \gcd(b, Q) > 1 \), then \( \lambda(Q/\gcd(b, Q)) \). Then we will find \( m \) as the smallest divisor of \( \lambda(Q) \) or \( \lambda(Q/\gcd(b, Q)) \), for which \( x \) becomes an integer.
Bibliography


Appendix A

Solutions to some values of $p$ and $q$ in different bases

The following tables contain the solutions to $A(x) = (p/q)x$ and $A_B(x) = (p/q)x$ for some values of $p$ and $q$. In cases where there are shorter minimal solutions for some values of $b$, these are marked by *. 
Appendix A. Solutions to some values of $p$ and $q$ in different bases

A.1 $A(x) = \frac{q-1}{q}x$

The following table shows the minimal solutions $x$ for $A(x) = ((q - 1)/q)x$, in base 10, where $q$ is an integer, $3 \leq q \leq 10$.

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A.2 \( A(x) = \frac{p}{p-1}x \)

The following table shows the minimal solutions \( x \) for \( A(x) = \frac{p}{(p-1)}x \), in base 10, where \( p \) is an integer, \( 3 \leq p \leq 10 \).

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Appendix A. Solutions to some values of $p$ and $q$ in different bases

A.3 $A(x) = \frac{p}{q}x$

The following table contains the minimal solutions $x$ for $A(x) = (p/q)x$, in base 10, for the remaining values of $p$ and $q$, where $p$ and $q$ are integers, $2 \leq p \leq 9$, $2 \leq q \leq 9$.

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A.3. \( A(x) = \frac{x}{n} \)

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</table>
Appendix A. Solutions to some values of $p$ and $q$ in different bases

A.4 $A_B(x) = kx$

The following tables show the minimal solutions $x$ for $A_B(x) = kx$, where $k$ is an integer, $k \geq 2$, for bases 2 to 9.

**Base 2.** There are no solutions.

**Base 3.**

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**Base 4.**

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**Base 5.**

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### A.4. \( A_B(x) = kx \)

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Appendix A. Solutions to some values of $p$ and $q$ in different bases

Base 9.

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<td>17*</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>204558</td>
<td>121472</td>
</tr>
<tr>
<td>5</td>
<td>10175</td>
<td>6710</td>
</tr>
<tr>
<td></td>
<td>12036</td>
<td>8052</td>
</tr>
<tr>
<td></td>
<td>13787</td>
<td>9394</td>
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<td></td>
<td>15648</td>
<td>10736</td>
</tr>
<tr>
<td>6</td>
<td>1 01467 1 1624 82787 42177 26406 1 16248 27874 21772 64061 01467 1 32030 45233 50756 85843 65538</td>
<td>7314 43232 74264 22602 56480 8533 50438 19974 93036 32560 9752 57643 65685 63470 08640</td>
</tr>
<tr>
<td>7</td>
<td>10126 73558 50647 11405 20254 57228</td>
<td>2324 57720 78428 2656 65976 89632</td>
</tr>
<tr>
<td>8</td>
<td>10112 36067 54045 05630 33720 22473 14618</td>
<td>282 04569 07034 63842 40175 45899 81504</td>
</tr>
</tbody>
</table>
A.5 \( A_B(x) = \frac{1}{l} x \)

The following tables show the minimal solutions \( x \) for \( A_B(x) = (1/l)x \), where \( l \) is an integer, \( k \geq 2 \), for the bases 5, 7, 8 and 9. There are no solutions in the bases 2, 3, 4 and 6.

### Base 5.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( x ) in base 5</th>
<th>( x ) in base 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>31</td>
<td>16</td>
</tr>
</tbody>
</table>

### Base 7.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( x ) in base 7</th>
<th>( x ) in base 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2541</td>
<td>960</td>
</tr>
<tr>
<td></td>
<td>5412</td>
<td>1920</td>
</tr>
<tr>
<td>3</td>
<td>51</td>
<td>36</td>
</tr>
</tbody>
</table>

### Base 8.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( x ) in base 8</th>
<th>( x ) in base 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>52</td>
<td>42</td>
</tr>
<tr>
<td>3</td>
<td>4631</td>
<td>2457</td>
</tr>
</tbody>
</table>

### Base 9.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( x ) in base 9</th>
<th>( x ) in base 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>251</td>
<td>208</td>
</tr>
<tr>
<td></td>
<td>512</td>
<td>416</td>
</tr>
<tr>
<td></td>
<td>763</td>
<td>624</td>
</tr>
<tr>
<td>4</td>
<td>71</td>
<td>64</td>
</tr>
</tbody>
</table>
Appendix A. Solutions to some values of $p$ and $q$ in different bases
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