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Robust finite-frequency $\mathcal{H}_2$ analysis of uncertain systems with application to flight comfort analysis

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Abstract

In many applications, design or analysis is performed over a finite-frequency range of interest. The importance of the $\mathcal{H}_2$ norm highlights the necessity of computing this norm accordingly. This paper provides different methods for computing upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm for systems with structured uncertainties. An application of the robust finite-frequency $\mathcal{H}_2$ norm for a comfort analysis problem of an aero-elastic model of an aircraft is also presented.

Keywords: robust $\mathcal{H}_2$ norm, uncertain systems, robust control, flight comfort analysis.

1. Introduction

The $\mathcal{H}_2$ norm has been one of the pivotal design and analysis criteria in many applications, such as structural dynamics, acoustics, colored noise disturbance rejection, etc, Caracciolo et al. (2005); Marro and Zattoni (2005); Zattoni (2006); Alazard (2002); Banjerdpongchai and How (1998). This norm also plays an important role in the field of robust control, where there has been a substantial amount of research on computation, analysis and design based on this measure in the presence of uncertainty, many of which consider the use of Linear Matrix Inequalities (LMIs) and Riccati equations for this purpose, e.g., Doyle et al. (1989); Stoovogel (1993); Boyd et al. (1994); Iwasaki (1996); Paganini (1997, 1999a); Sznajder et al. (2002). A survey of methods in robust $\mathcal{H}_2$ analysis is provided in Paganini and Feron (2000).

Most of the methods presented in the literature consider the whole frequency range for calculating the $\mathcal{H}_2$ and robust $\mathcal{H}_2$ norm. However, in some applications it is beneficial to concentrate only on a finite-frequency range and calculate the design/analysis measures accordingly. This can be due to different reasons, e.g., the model is only valid for a specific frequency range or the design is targeted for a specific frequency interval. This motivates the importance of computing the (robust) finite-frequency $\mathcal{H}_2$ norm.

Frequency limitations in several analysis and design problems relevant to control systems have been addressed

in Iwasaki and Hara (2005), by introducing a generalization of the celebrated KYP lemma. However, it is not known that this result can be used to compute the robust finite-frequency $\mathcal{H}_2$ norm. In Gawronski (2000), a method for calculating the finite-frequency $\mathcal{H}_2$ norm for systems without uncertainty is presented, where the key step is to compute the finite-frequency observability Gramian. This is accomplished by first computing the regular observability Gramian and then scaling it by a system dependent matrix.

In this paper, we introduce two methods for calculating an upper bound of the robust finite-frequency $\mathcal{H}_2$ norm for systems with structured uncertainties. We also assume that a LFT (Linear Fractional Transformation) representation of these systems are available, which is a common assumption in many fields concerning uncertain systems, e.g., in aeronautics, Cockburn and Morton (1997); Poussot-Vassal and Roos (2012); Ferreres (2011); Zhou et al. (1996). The first method combines the notion of finite-frequency Gramians, introduced in Gawronski (2000), with convex optimization tools, Boyd and Vandenberghe (2004), commonly used in robust control, and it calculates an upper bound by solving an underlying optimization problem, Masi et al. (2010). The second method, provides a computationally cheaper algorithmic method for calculating such upper bounds. In contrast to the first approach, the second method performs frequency gridding and breaks the original problem into smaller problems, which are possibly easier to solve. Then it uses the ideas presented in Roos and Biannic (2010), on computing upper bounds of structured singular values, for solving the smaller problems. The results of the smaller problems are then combined to compute an upper bound over the whole desired frequency range, Pakazad et al. (2011).

This paper is structured as follows. First some of the notations used throughout the paper are presented. Sec-

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tion 2 introduces the problem formulation. Mathematical preliminaries are presented in Section 3, which covers the notion of finite-frequency Gramians and reviews the calculation of upper bounds of the robust $\mathcal{H}_2$ norm. Sections 4 and 5 provide the details of the two methods for calculating upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm. In Section 6 a numerical example is presented to illustrate the main features of the proposed techniques. The application of the two methods to a robust comfort analysis problem for an aero-elastic model of a civil aircraft is presented in Section 7. Section 8 provides more insight by discussing advantages and disadvantages of the proposed methods, and finally Section 9 concludes the paper with some final remarks.

1.1. Notation

The notation in this paper is standard. The set of $m \times n$ real and complex matrices are denoted by $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$, respectively. Given a matrix A , A^T is its transpose and A^* is its conjugate transpose. By I_n , we denote the $n \times n$ identity matrix. The symbols $\preceq$ and $\prec$ denote the inequality relation between matrices and by $\ln(A)$ we denote the standard matrix logarithm. Given matrices A_i for $i = 1, \dots, n$, $\text{diag}(A_1, \dots, A_n)$ denotes a block-diagonal matrix with A_i s as its diagonal blocks. The min and max represent the minimum and maximum of a function or a set, and similarly sup represents the supremum of a function. A transfer function matrix in terms of state-space data is denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := C(j\omega I - A)^{-1}B + D. \quad (1)$$

By $\|\cdot\|_2$, we denote the Euclidian or 2-norm of a vector or the norm of a matrix induced by the 2-norm. For the sake of brevity of notation, unless needed for clarity, we drop the dependence of functions on frequency.

2. Problem formulation

Consider the following stable system in state space form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (2)$$

and define $G(s)$ as its corresponding transfer function. Then the $\mathcal{H}_2$ norm of the system in (2) is defined as

$$\|G\|_2^2 = \int_{-\infty}^{\infty} \text{Tr} \{G(j\omega)^* G(j\omega)\} \frac{d\omega}{2\pi}, \quad (3)$$

which can also be expressed as

$$\begin{aligned} \|G\|_2^2 &= \int_0^{\infty} \text{Tr} \{B^T e^{A^T t} C^T C e^{At} B\} dt \\ &= \text{Tr} \left\{ B^T \left(\int_0^{\infty} e^{A^T t} C^T C e^{At} dt \right) B \right\} \\ &= \text{Tr} \{B^T W_o B\} = \text{Tr} \{C W_c C^T\}, \end{aligned} \quad (4)$$

where W_o and W_c are the observability and controllability Gramians of the system, respectively.

As can be seen from the equation in (3), computing the $\mathcal{H}_2$ norm requires the integration over the whole frequency range. We define the finite-frequency $\mathcal{H}_2$ norm of the system by limiting the integration bounds to finite values as

$$\|G\|_{2,\bar{\omega}}^2 = \int_{-\bar{\omega}}^{\bar{\omega}} \text{Tr} \{G(j\omega)^* G(j\omega)\} \frac{d\omega}{2\pi}, \quad (5)$$

where the integration interval $\mathcal{I}_d = [-\bar{\omega}, \bar{\omega}]$ represents the frequency range of interest. Similarly, we can extend this notion to uncertain systems. Consider the following uncertain system in state space form

$$\begin{cases} \dot{x} = Ax + B_q q + B_w w \\ p = C_p x + D_{pq} q \\ z = C_z x + D_{zq} q \\ q = \Delta p \end{cases} \quad (6)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $z \in \mathbb{R}^l$ and $p, q \in \mathbb{R}^d$, see Figure 1. The perturbation block Δ , which represents the uncertainty in (6), is a causal linear time invariant operator, bounded in the $\mathcal{L}_2$ induced norm, and has the following structure

$$\Delta = \text{diag} [\delta_1 I_{r_1} \quad \dots \quad \delta_L I_{r_L} \quad \Delta_{L+1} \quad \dots \quad \Delta_{L+F}], \quad (7)$$

where $\delta_i \in \mathbb{R}$, for $i = 1, \dots, L$, and $\Delta_{L+i} \in \mathbb{C}^{m_i \times m_i}$, for $i = 1, \dots, F$, such that $\sum_{i=1}^L r_i + \sum_{i=1}^F m_i = d$. It is assumed that $\Delta \in B_\Delta$ where B_Δ is the unit ball for the induced $\mathcal{L}_2$ norm, i.e., $B_\Delta = \{\Delta : \|\Delta\|_2 \leq 1\}$. This structure of Δ is standard in robust control and allows one to represent both real parametric uncertainties and unmodeled system dynamics using real and complex blocks, respectively.

Remark 1. The assumption on Δ can be relaxed to cope with more general uncertainty models, such as time-varying or nonlinear operators. In such cases, the notion of the $\mathcal{H}_2$ norm of a system must be carefully reconsidered (see Paganini (1999b) and Sznaier et al. (2002) for a thorough discussion on this issue).

The transfer function matrix for the uncertain system in (6) is defined as below, see Figure 1,

$$M(j\omega) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_q & B_w \\ \hline C_p & D_{pq} & 0 \\ C_z & D_{zq} & 0 \end{array} \right], \quad (8)$$

where $M \in \mathbb{C}^{(d+l) \times (d+m)}$, $M_{11} \in \mathbb{C}^{d \times d}$, $M_{12} \in \mathbb{C}^{d \times m}$, $M_{21} \in \mathbb{C}^{l \times d}$ and $M_{22} \in \mathbb{C}^{l \times m}$. In the upcoming sections, we also utilize the following partitioning of this transfer function matrix

$$M(j\omega) = \begin{bmatrix} M_1 & M_2 \end{bmatrix} = \left[\begin{array}{c|cc} A & B_q & B_w \\ \hline C & D & 0 \end{array} \right], \quad (9)$$

Figure 1: Uncertain system with structured uncertainty

where $M_1 \in \mathbb{C}^{(d+l) \times (d)}$, $M_2 \in \mathbb{C}^{(d+l) \times (m)}$ and

$$C = \begin{bmatrix} C_p \\ C_z \end{bmatrix}, D = \begin{bmatrix} D_{pq} \\ D_{zq} \end{bmatrix}. \quad (10)$$

In analysis of uncertain systems, the transfer function between the signals $w(t)$ and $z(t)$ is of interest. This transfer function is given by the upper LFT representation

$$(\Delta * M) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}. \quad (11)$$

which is a special case of the so-called Redheffer product, Zhou et al. (1996). Having (11), the robust $\mathcal{H}_2$ norm of the system in (6) is defined as

$$\begin{aligned} & \sup_{\Delta \in B_\Delta} \|\Delta * M\|_2^2 \\ &= \sup_{\Delta \in B_\Delta} \int_{-\infty}^{\infty} \text{Tr} \{(\Delta * M)^*(\Delta * M)\} \frac{d\omega}{2\pi}, \end{aligned} \quad (12)$$

and similarly the robust finite-frequency $\mathcal{H}_2$ norm of the system in (6) is defined as

$$\begin{aligned} & \sup_{\Delta \in B_\Delta} \|\Delta * M\|_{2,\bar{\omega}}^2 \\ &= \sup_{\Delta \in B_\Delta} \int_{-\bar{\omega}}^{\bar{\omega}} \text{Tr} \{(\Delta * M)^*(\Delta * M)\} \frac{d\omega}{2\pi}. \end{aligned} \quad (13)$$

This paper proposes methods for calculating upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm of such systems. Next section provides the mathematical background for these methods.

3. Mathematical preliminaries

3.1. Finite-frequency observability Gramian

As was mentioned in Section 1, one of the ways of computing the $\mathcal{H}_2$ norm of the system in (2) is by using its observability Gramian, see (4). We can compute the observability Gramian by solving the following Lyapunov equation

$$A^T W_o + W_o A + C^T C = 0. \quad (14)$$

Using Parseval's identity and (4), the observability Gramian can also be expressed as

$$W_o = \int_{-\infty}^{\infty} H(j\omega)^* C^T C H(j\omega) \frac{d\omega}{2\pi}, \quad (15)$$

where $H(j\omega) = (j\omega I - A)^{-1}$. This allows us to define the finite-frequency observability Gramian, as proposed in Gawronski (2000), as

$$W_o(\bar{\omega}) = \int_{-\bar{\omega}}^{\bar{\omega}} H(j\omega)^* C^T C H(j\omega) \frac{d\omega}{2\pi}. \quad (16)$$

The next lemma provides a way to express $W_o(\bar{\omega})$ in terms of the observability Gramian, W_o .

Lemma 1. *The finite-frequency observability Gramian can be expressed as*

$$W_o(\bar{\omega}) = L(A, \bar{\omega})^* W_o + W_o L(A, \bar{\omega}), \quad (17)$$

where W_o is defined by (14) or equivalently by (15) and

$$\begin{aligned} L(A, \bar{\omega}) &= \int_{-\bar{\omega}}^{\bar{\omega}} H(j\omega) \frac{d\omega}{2\pi} \\ &= \frac{j}{2\pi} \ln[(A + j\bar{\omega}I)(A - j\bar{\omega}I)^{-1}]. \end{aligned} \quad (18)$$

PROOF. See (Gawronski, 2000, page 100). $\square$

Remark 2. Note that by following the ideas in Gawronski (2000), it is also possible to compute the finite-frequency observability Gramian for general frequency intervals, e.g., $[\underline{\omega}, \bar{\omega}]$.

Remark 3. From (5), (16) and Lemma 1, it is straightforward to observe that the finite-frequency $\mathcal{H}_2$ norm of the system in (2) can be expressed as

$$\|G\|_{2,\bar{\omega}}^2 = \text{Tr} \{B^T W_o(\bar{\omega}) B\}. \quad (19)$$

3.2. An upper bound of the robust $\mathcal{H}_2$ norm

Let $\mathbb{X}$ represent Hermitian, block diagonal positive definite matrices that commute with Δ , i.e., every $\mathcal{X} \in \mathbb{X}$ has the following structure

$$\mathcal{X} = \text{diag} [X_1 \quad \cdots \quad X_L \quad x_{L+1} I_{m_1} \quad \cdots \quad x_{L+F} I_{m_F}]. \quad (20)$$

where the blocks in $\mathbb{X}$ have compatible dimensions with their corresponding blocks in Δ . The following condition, taken from Paganini (1999a), plays a central role throughout this section.

Condition 1. *Consider the system in (6). There exists $\mathcal{X}(\omega) \in \mathbb{X}$, Hermitian $Y(\omega) \in \mathbb{C}^{m \times m}$ and $\epsilon > 0$ such that*

$$M(j\omega)^* \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. \quad (21)$$

The set of matrices $\mathbb{X}$ are the so-called D -scaling matrices. In many cases it is customary to use constant scaling matrices to make the problem easier to handle, Fan et al. (1991), Packard and Doyle (1993). However, it is well

known that the results achieved based on constant scaling matrices can be conservative, Iwasaki and Hara (1998). One of the ways to reduce the conservativeness and keep the computational complexity reasonable is to use special classes of dynamic D -scaling matrices, Scherer and Köse (2007, 2008). This will be investigated in more detail in Section 3.2.2. Also, even less conservative scaling matrices can be considered, like $D - G$ scalings, Fan et al. (1991) or LFT scalings, Iwasaki and Hara (1998).

Next, two methods for computing upper bounds of the robust $\mathcal{H}_2$ norm of systems with structured uncertainties are reviewed. The first method explicitly defines $Y(\omega)$ in Condition 1 and uses $Y(\omega)$ to construct an upper bound of the robust $\mathcal{H}_2$ norm of the system. This method will be referred to as explicit upper bound calculation. The second method calculates an upper bound through computing the observability Gramian via solving a set of LMIs. This method is referred to as Gramian based upper bound calculation.

3.2.1. Explicit upper bound calculation

Consider Condition 1. This condition can be restated as follows.

Lemma 2. *If there exists $\mathcal{X}(\omega) \in \mathbb{X}$ such that*

$$M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega) \prec 0, \quad (22)$$

then Condition 1 is satisfied if and only if there exists $Y(\omega) = Y(\omega)^$ such that,*

$$\begin{aligned} & M_{12}^* \mathcal{X}(\omega) M_{12} + M_{22}^* M_{22} - (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21}) \times \\ & (M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega))^{-1} \times \\ & (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21})^* \preceq Y(\omega). \end{aligned} \quad (23)$$

PROOF. See Appendix A.

Using Condition 1 and Lemma 2, the following theorem provides an upper bound of the gain of the system for all frequencies and will be used to accommodate an upper bound of the robust $\mathcal{H}_2$ norm for systems with structured uncertainty.

Theorem 1. *If there exists $\mathcal{X}(\omega) \in \mathbb{X}$ such that (22) is satisfied for all ω and if we define $Y(\omega)$ as below*

$$\begin{aligned} Y(\omega) = & M_{12}^* \mathcal{X}(\omega) M_{12} + M_{22}^* M_{22} - \\ & (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21}) \times \\ & (M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega))^{-1} \times \\ & (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21})^*, \end{aligned} \quad (24)$$

then $(\Delta * M)(j\omega)^*(\Delta * M)(j\omega) \preceq Y(\omega) \quad \forall \omega$.

PROOF. See Appendix B. $\square$

Corollary 1. *If there exists $\mathcal{X}(\omega) \in \mathbb{X}$ and a frequency interval centered at ω_i , $\mathcal{I}(\omega_i) = [\omega_i + \omega_{\min} \quad \omega_i + \omega_{\max}]$, such that*

$$M_{11}^* \mathcal{X} M_{11} + M_{21}^* M_{21} - \mathcal{X} \prec 0 \quad \forall \omega \in \mathcal{I}(\omega_i), \quad (25)$$

and if we consider $Y(\omega)$ as defined in (24) for the mentioned frequency interval, then

$$\int_{\omega \in \mathcal{I}(\omega_i)} \text{Tr} \{(\Delta * M)^*(\Delta * M)\} \frac{d\omega}{2\pi} \leq \int_{\omega \in \mathcal{I}(\omega_i)} \text{Tr} \{Y(\omega)\} \frac{d\omega}{2\pi}, \quad (26)$$

for all $\Delta \in B_\Delta$, and specifically if $\mathcal{I}(\omega_i)$ covers all frequencies

$$\sup_{\Delta \in B_\Delta} \|\Delta * M\|_2^2 \leq \int_{-\infty}^{\infty} \text{Tr} \{Y(\omega)\} \frac{d\omega}{2\pi}. \quad (27)$$

As a result, using the inequality in (27), it is possible to generate an upper bound of the robust $\mathcal{H}_2$ norm of the system via numerical integration.

3.2.2. Gramian-based upper bound calculation

In this section, we consider a class of dynamic scaling matrices with the following structure

$$\begin{aligned} \mathcal{X}(\omega) &= \psi(j\omega) X \psi(j\omega)^* \\ &= [C_\psi(j\omega I - A_\psi)^{-1} \quad I] X [C_\psi(j\omega I - A_\psi)^{-1} \quad I]^*, \end{aligned} \quad (28)$$

where $A_\psi \in \mathbb{R}^{n_\psi \times n_\psi}$ and $C_\psi \in \mathbb{R}^{d \times n_\psi}$ are fixed matrices with appropriate dimensions such that A_ψ is stable and (C_ψ, A_ψ) is observable. Also note that $X \in \mathbb{R}^{(d+n_\psi) \times (d+n_\psi)}$ is a free basis for the parameters such that $\mathcal{X}(s) \in \mathbb{X}$. In order to derive an upper bound of the robust $\mathcal{H}_2$ norm relying on scaling matrices of the form (28), the following technical result taken from Giusto (1996) will be useful.

Lemma 3. *Consider the partitioning $M = [M_1 \quad M_2]$, defined in (9), for the transfer matrix of system in (6). By replacing $\mathcal{X}(\omega)$ with $\mathcal{X}(\omega)^{-1}$ in (21), the condition in (21) can be restated as*

$$\begin{bmatrix} M_1(j\omega) \mathcal{X}(\omega) M_1(j\omega)^* - \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} & M_2(j\omega) \\ M_2(j\omega)^* & -Y(\omega) \end{bmatrix} \preceq 0. \quad (29)$$

PROOF. See (Giusto, 1996, Lemma 1). $\square$

The upper left block of (29) can be expressed, up to its sign, as

$$\begin{aligned} \mathcal{C}_{11} &:= \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} - M_1(j\omega) \mathcal{X}(\omega) M_1(j\omega)^* \\ &= \begin{bmatrix} \psi & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \psi & 0 \\ 0 & I \end{bmatrix}^* - \begin{bmatrix} M_{11} \psi \\ M_{21} \psi \end{bmatrix} X \begin{bmatrix} M_{11} \psi \\ M_{21} \psi \end{bmatrix}^* \\ &= \begin{bmatrix} M_{11} \psi & \psi & 0 \\ M_{21} \psi & 0 & I \end{bmatrix} \begin{bmatrix} -X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} M_{11} \psi & \psi & 0 \\ M_{21} \psi & 0 & I \end{bmatrix}^*. \end{aligned} \quad (30)$$

By introducing the following transfer matrix

$$\tilde{C}(j\omega I - \tilde{A})^{-1}\tilde{B}_q + \tilde{D} = \begin{bmatrix} M_{11}\psi & \psi \\ M_{21}\psi & 0 \end{bmatrix}, \quad (31)$$

and setting $\Gamma = [0 \quad I]^T$, (30) can be reformulated as

$$\mathcal{C}_{11} = \left([\tilde{C}(j\omega I - \tilde{A})^{-1} \quad I] \begin{bmatrix} \tilde{B}_q & 0 \\ \tilde{D} & \Gamma \end{bmatrix} \right) \begin{bmatrix} -X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & I \end{bmatrix} \times \\ \left([\tilde{C}(j\omega I - \tilde{A})^{-1} \quad I] \begin{bmatrix} \tilde{B}_q & 0 \\ \tilde{D} & \Gamma \end{bmatrix} \right)^*, \quad (32)$$

with

$$\tilde{A} = \begin{bmatrix} A & B_q C_\psi & 0 \\ 0 & A_\psi & 0 \\ 0 & 0 & A_\psi \end{bmatrix}, \quad \tilde{B}_q = \begin{bmatrix} 0 & B_q & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, \\ \tilde{C} = \begin{bmatrix} C & DC_\psi & C_\psi \\ 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 0 & D & 0 & I \\ 0 & 0 \end{bmatrix}, \quad (33)$$

where $\tilde{A} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, $\tilde{B}_q \in \mathbb{R}^{\tilde{n} \times \tilde{d}}$, $\tilde{C} \in \mathbb{R}^{(l+d) \times \tilde{n}}$ and $\tilde{D} \in \mathbb{R}^{(l+d) \times \tilde{d}}$, with $\tilde{n} = 2n_\psi + n$ and $\tilde{d} = 2n_\psi + 2d$.

Let $\Pi(X, \tilde{B}_q, \tilde{D})$ be an affine function of X , defined as below

$$\Pi(X, \tilde{B}_q, \tilde{D}) = \begin{bmatrix} \tilde{B}_q & 0 \\ \tilde{D} & \Gamma \end{bmatrix} \begin{bmatrix} -X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{B}_q & 0 \\ \tilde{D} & \Gamma \end{bmatrix}^T \\ = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix}, \quad (34)$$

where $\Pi_{11} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, $\Pi_{12} \in \mathbb{R}^{\tilde{n} \times (l+d)}$ and $\Pi_{22} \in \mathbb{R}^{(l+d) \times (l+d)}$. Then the following theorem taken from Paganini (1997) can be used to calculate an upper bound of the robust $\mathcal{H}_2$ norm.

Theorem 2. *If there exist matrix X such that $\mathcal{X}(\omega)$ in (28) satisfies $\mathcal{X}(\omega) \in \mathbb{X}$, and Hermitian matrices $P_-, P_+ \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, $Q \in \mathbb{R}^{n_\psi \times n_\psi}$, $\tilde{W}_o \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, such that*

$$\left\{ \begin{array}{l} P_-, Q \succ 0, \\ \begin{bmatrix} A_\psi Q + Q A_\psi^T & Q C_\psi^T \\ C_\psi Q & 0 \end{bmatrix} - X \prec 0, \\ \begin{bmatrix} \tilde{A}P_- + P_- \tilde{A}^T & P_- \tilde{C}^T \\ \tilde{C}P_- & 0 \end{bmatrix} - \Pi(X, \tilde{B}_q, \tilde{D}) \prec 0, \\ \begin{bmatrix} \tilde{A}P_+ + P_+ \tilde{A}^T & P_+ \tilde{C}^T \\ \tilde{C}P_+ & 0 \end{bmatrix} - \Pi(X, \tilde{B}_q, \tilde{D}) \prec 0, \\ \begin{bmatrix} \tilde{W}_o & I \\ I & P_+ - P_- \end{bmatrix} \succ 0, \\ \text{Tr} \left\{ \begin{bmatrix} B_w^T & 0 \end{bmatrix} \tilde{W}_o \begin{bmatrix} B_w \\ 0 \end{bmatrix} \right\} < \gamma^2, \end{array} \right. \quad (35)$$

then $\mathcal{X}(\omega)$ satisfies (29) and the system $(\Delta * M)$ defined in (11) has robust $\mathcal{H}_2$ norm less than γ^2 .

PROOF. See Paganini (1997). $\square$

Theorem 2 includes the problem with constant scaling matrices as a special case. Let

$$\hat{A} = A, \hat{B}_q = [B_q \quad 0], \hat{C} = C, \hat{D} = \begin{bmatrix} D & I_d \\ 0 \end{bmatrix}. \quad (36)$$

Then the following Corollary is a restatement of Theorem 2 for constant scaling matrices, i.e., for $\mathcal{X}(\omega) = X$.

Corollary 2. *If there exist matrix $X \in \mathbb{X}$ and symmetric matrices $P_-, P_+, Z \in \mathbb{R}^{n \times n}$ such that*

$$\left\{ \begin{array}{l} P_-, X \succ 0, \\ \begin{bmatrix} \hat{A}P_- + P_- \hat{A}^T & P_- \hat{C}^T \\ \hat{C}P_- & 0 \end{bmatrix} - \Pi(X, \hat{B}_q, \hat{D}) \prec 0, \\ \begin{bmatrix} \hat{A}P_+ + P_+ \hat{A}^T & P_+ \hat{C}^T \\ \hat{C}P_+ & 0 \end{bmatrix} - \Pi(X, \hat{B}_q, \hat{D}) \prec 0, \\ \begin{bmatrix} Z & I \\ I & P_+ - P_- \end{bmatrix} \succ 0, \\ \text{Tr} \{ B_w^T Z B_w \} < \gamma^2. \end{array} \right. \quad (37)$$

then $\mathcal{X}(\omega) = X$ satisfies (29) and the system $(\Delta * M)$ defined in (11) has robust $\mathcal{H}_2$ norm less than γ^2 .

PROOF. See Paganini (1997). $\square$

4. Gramian-based upper bound of the robust finite-frequency $\mathcal{H}_2$ norm

In this section the first method for calculating an upper bound of the robust finite-frequency $\mathcal{H}_2$ norm of the system in (6) is presented. The following theorem combines the ideas presented in Section 3.1, regarding the finite-frequency observability Gramians, with the results of Section 3.2.2, and computes an upper bound of the robust finite-frequency $\mathcal{H}_2$ norm for (6). Hereafter this method is referred to as *Method 1*.

Theorem 3. *Let P_-, P_+, X, Q and $\tilde{W}_o$ be a solution to (35), then*

$$\sup_{\Delta \in B_\Delta} \|\Delta * M\|_{2, \bar{\omega}}^2 \\ \leq \text{Tr} \left\{ \begin{bmatrix} B_w \\ 0 \end{bmatrix}^T \left(L(\tilde{\mathcal{A}}, \bar{\omega})^* \tilde{W}_o + \tilde{W}_o L(\tilde{\mathcal{A}}, \bar{\omega}) \right) \begin{bmatrix} B_w \\ 0 \end{bmatrix} \right\} \quad (38)$$

where $L(\tilde{\mathcal{A}}, \bar{\omega})$ is defined in (18) and $\tilde{\mathcal{A}} = \tilde{A} - (\Pi_{12} - P_- \tilde{C}^T) \Pi_{22}^{-1} \tilde{C}$.

PROOF. See Appendix C. $\square$

Remark 4. By Remark 2, the integral in (C.4) can be restated for a generic frequency range, e.g., $[\underline{\omega}, \bar{\omega}]$. This allows us to compute an upper bound of robust finite-frequency $\mathcal{H}_2$ norm for general frequency ranges.

As was mentioned in Section 3.2, by using dynamic scaling matrices and increasing the order of these scaling matrices, it is possible to reduce the conservativeness of the results. In order to further reduce the conservativeness of the bounds and improve the numerical properties of the optimization problems, it is useful to perform uncertainty partitioning. In this approach, for each of the uncertainty partitions, an upper bound of the robust finite-frequency $\mathcal{H}_2$ norm of the system is computed and the maximum of these bounds is considered as the final result.

5. Frequency gridding based upper bound of the robust finite-frequency $\mathcal{H}_2$ norm

In this section the second method to compute upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm is presented. The following corollary to Theorem 1, which is a straightforward extension of Corollary 1, plays a central role in the proposed algorithm.

Corollary 3. *Let $\mathcal{I}(\omega_i)$ for $i = 1, \dots, m$ be disjoint frequency intervals such that $\mathcal{I}_d = [-\bar{\omega}, \bar{\omega}] = \bigcup_{i=1}^m \mathcal{I}(\omega_i)$. Also let the constant matrices X_i for $i = 1, \dots, m$ be the scaling matrices for which $M_{11}^* X_i M_{11} + M_{21}^* M_{21} - X_i \prec 0 \quad \forall \omega \in \mathcal{I}(\omega_i)$. Then, it holds that*

$$\begin{aligned} & \sup_{\Delta \in B_\Delta} \|\Delta * M\|_{2,\bar{\omega}}^2 \\ & \leq \sup_{\Delta \in B_\Delta} \sum_{i=1}^m \int_{\omega \in \mathcal{I}(\omega_i)} \text{Tr} \{(\Delta * M)^*(\Delta * M)\} \frac{d\omega}{2\pi} \quad (39) \\ & \leq \sum_{i=1}^m \int_{\omega \in \mathcal{I}(\omega_i)} \text{Tr} \{Y_i(\omega)\} \frac{d\omega}{2\pi}, \end{aligned}$$

where $Y_i(\omega)$ is defined as in (24), with $\mathcal{X}(\omega) = X_i$.

Corollary 3 provides a sketch for computing upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm via frequency gridding. However, calculating a suitable scaling matrix X_i requires checking $M_{11}^* X_i M_{11} + M_{21}^* M_{21} - X_i \prec 0$ for an infinite number of frequencies in $\mathcal{I}(\omega_i)$. Next a method is proposed to solve this issue. Consider the following two LMIs

$$M_{11}(j\omega)^* \mathcal{X}(\omega) M_{11}(j\omega) + M_{21}(j\omega)^* M_{21}(j\omega) - \mathcal{X}(\omega) \prec 0, \quad (40)$$

$$\begin{bmatrix} M_{11}(j\omega) & 0 \\ M_{21}(j\omega) & 0 \end{bmatrix}^* \bar{\mathcal{X}}(\omega) \begin{bmatrix} M_{11}(j\omega) & 0 \\ M_{21}(j\omega) & 0 \end{bmatrix} - \bar{\mathcal{X}}(\omega) \prec 0. \quad (41)$$

Then $\bar{X}_i = \begin{bmatrix} X_i & 0 \\ 0 & I_l \end{bmatrix}$ satisfies (41) for $\omega = \omega_i$, if and only if X_i satisfies (40) for the same frequency. The following theorem taken from Roos and Biannic (2010), solves the issue of infinite dimensionality of the problem in Corollary 3 by providing a way to extend the validity of a scaling matrix that satisfies $M_{11}^* X_i M_{11} + M_{21}^* M_{21} - X_i \prec 0$ for a single frequency, e.g., $\omega = \omega_i$, to a frequency interval, $\mathcal{I}(\omega_i)$.

Theorem 4. *Let $\tilde{M} = \begin{bmatrix} M_{11} & 0 \\ M_{21} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$, and let $D = \bar{X}_i^{\frac{1}{2}}$, where $\bar{X}_i$ satisfies the LMI in (41) for $\omega = \omega_i$. Define*

$$\mathcal{G} = A_X - B_X D_X^{-1} C_X, \quad (42)$$

where

$$\begin{aligned} A_X &= \begin{bmatrix} A_G & 0 \\ -C_G^* C_G & -A_G^* \end{bmatrix}, & B_X &= \begin{bmatrix} -B_G \\ C_G^* D_G \end{bmatrix}, \\ C_X &= [D_G^* C_G \quad B_G^*], & D_X &= I - D_G^* D_G, \end{aligned} \quad (43)$$

in which

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right] = \left[\begin{array}{c|c} \tilde{A} - j\omega_i I & \tilde{B} D^{-1} \\ \hline D \tilde{C} & D \tilde{D} D^{-1} \end{array} \right], \quad (44)$$

and define ω_{low} and ω_{high} as

$$\omega_{low} = \begin{cases} -\omega_i, & \text{if } j\mathcal{G} \text{ has no positive real eigenvalue} \\ \max\{\lambda \in \mathcal{R}_- : \det(\lambda I + j\mathcal{G}) = 0\}, & \text{otherwise} \end{cases}$$

$$\omega_{high} = \begin{cases} \infty, & \text{if } j\mathcal{G} \text{ has no negative real eigenvalue} \\ \min\{\lambda \in \mathcal{R}_+ : \det(\lambda I + j\mathcal{G}) = 0\}, & \text{otherwise} \end{cases}$$

Then $\bar{X}_i$ satisfies (41) for all ω such that $\omega \in \bar{\mathcal{I}}(\omega_i) = [\omega_i + \omega_{low}, \omega_i + \omega_{high}]$.

PROOF. See Appendix D. $\square$

Using Corollary 3 and Theorem 4, the following algorithm can be used for calculating an upper bound of the robust finite-frequency $\mathcal{H}_2$ norm. This algorithmic method is referred to as *Method 2*.

Algorithm 1. *Let $\mathcal{I}_d = [-\bar{\omega}, \bar{\omega}]$ denote the frequency range of interest. Then*

- (I) *Divide $\mathcal{I}_d$ into a desired number of disjoint partitions, $\mathcal{I}(\omega_i)$, where ω_i is the center of the respective partition.*
- (II) *For each of the partitions, compute X_i such that it satisfies (40) for $\omega = \omega_i$. If there is a partition for which there exists no feasible solution, exit the algorithm.*
- (III) *Construct $\bar{X}_i$ from the achieved X_i in (II).*
- (IV) *Using Theorem 4 calculate the valid frequency range, $\bar{\mathcal{I}}(\omega_i)$, for the LMIs in (41). If $\mathcal{I}(\omega_i) \not\subseteq \bar{\mathcal{I}}(\omega_i)$, go back to (I) and choose a finer partitioning for $\mathcal{I}_d$.*
- (V) *Define $Y_i(\omega)$ using (24) with $\mathcal{X}(\omega) = X_i$.*
- (VI) *Use numerical integration to calculate*

$$\int_{\omega \in \mathcal{I}(\omega_i)} \text{Tr} \{Y_i(\omega)\} \frac{d\omega}{2\pi}.$$
- (VII) *By Corollary 3, compute the upper bound by summing up the integrals computed in (VI).*

Remark 5. As can be seen from Step (II) of Algorithm 1, in case there exists a partition for which it is impossible to find X_i that satisfies (40) for $\omega = \omega_i$ this algorithm fails to produce an upper bound. Note that this can stem either from the fact that the system is not robustly stable or from conservatism of the robust stability condition based on (40).

The second step of Algorithm 1, requires computation of constant scaling matrices that satisfy (40) for $\omega = \omega_i$ for each of the partitions. This can be accomplished through different approaches. Considering the expression in (39) and the importance of $\text{Tr}\{Y_i(\omega)\}$ in the tightness of the proposed upper bound, it seems intuitive to calculate the scaling matrices while aiming at minimizing $\text{Tr}\{Y_i(\omega_i)\}$. The following two approaches utilize this in the process of computing suitable scaling matrices.

Approach 1. Compute X_i in Step (II) of Algorithm 1 as the solution of the following optimization problem

$$\begin{aligned} & \underset{X_i, Y_i}{\text{minimize}} && \text{Tr}\{Y_i\} \\ & \text{subj. to} && (21) \text{ with } \omega = \omega_i. \end{aligned} \quad (45)$$

Remark 6. The idea of frequency gridding was also presented in Paganini (1999b), where the authors consider the $\mathcal{H}_2$ performance problem for discrete time systems. In that paper, an optimization problem similar to (45) for frequencies $0 = \omega_0 \dots \omega_N = 2\pi$ is formulated and then the integral $\int_0^{2\pi} \text{trace}(Y(\omega)) \frac{d\omega}{2\pi}$ is approximated by the following Riemann sum expression

$$\frac{1}{2\pi} \sum_{i=1}^N \text{Tr}\{Y_i\}(\omega_i - \omega_{i-1}), \quad (46)$$

where $0 = \omega_0 \dots \omega_N = 2\pi$. However, this approach does not necessarily provide a guaranteed upper bound of the robust $\mathcal{H}_2$ norm of the system.

For any X_i satisfying the LMI in (40) for $\omega = \omega_i$ let

$$\begin{aligned} f(\alpha) = & \text{Tr}\{M_{12}^* \alpha \mathcal{X}_i M_{12} + M_{22}^* M_{22} - \\ & (M_{12}^* \alpha \mathcal{X}_i M_{11} + M_{22}^* M_{21}) \times \\ & (M_{11}^* \alpha \mathcal{X}_i M_{11} + M_{21}^* M_{21} - \alpha \mathcal{X}_i)^{-1} \times \\ & (M_{12}^* \alpha \mathcal{X}_i M_{11} + M_{22}^* M_{21})^*\}. \end{aligned} \quad (47)$$

This function is convex with respect to α . Next, following the same objectives as in Approach 1, an alternative method for calculating suitable scaling matrices is introduced.

Approach 2. Compute X_i in Step (II) of Algorithm 1 using the following sequential method

- (I) Find X_i such that it satisfies the LMI in (40) for $\omega = \omega_i$.

- (II) Minimize $f(\alpha)$, in (47), with the achieved X_i with respect to all α such that αX_i still satisfies the LMI in (40) for $\omega = \omega_i$.

Denote α^* as the minimizing α . Then $\alpha^* X_i$ will be used within the remaining steps of Algorithm 1. In order to assure that $\alpha^* X_i$ satisfies (40) the search for α should be subject to the constraint $\alpha > \alpha_{\min}$, where

$$\alpha_{\min} = \frac{1}{\min \left\{ \text{eig} \left(\begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} U(-M_{11}^* \mathcal{X}_i M_{11} + \mathcal{X}_i) U^* \begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \right) \right\}}, \quad (48)$$

in which U , a unitary matrix, and Λ , are defined by a singular value decomposition $M_{21}^* M_{21} = U^* \begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} U$.

It is important to note that for some problems it might be required to perform many iterations between the first and the fourth steps of Algorithm 1. One of the ways to alleviate this issue and even compute better upper bounds, is to modify the proposed approaches by augmenting new constraints for other frequencies from the partition under investigation. In this case the cost function can also be modified accordingly. As an example, Approach 1 can be modified as follows

$$\begin{aligned} & \underset{X_i, Y_i}{\text{minimize}} && \text{Tr}\{Y_i\} \\ & \text{subj. to} && M(j\omega)^* \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \preceq 0 \\ & && \text{for } \omega = \omega_j \in \mathcal{I}(\omega_i), j = 1, \dots, N_i, \end{aligned} \quad (49)$$

or alternatively as

$$\begin{aligned} & \underset{X_i, Y_i^j, j=1, \dots, N_i}{\text{minimize}} && \sum_{j=1}^{N_i} \text{Tr}\{Y_i^j\} \\ & \text{subj. to} && M(j\omega)^* \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} X_i & 0 \\ 0 & Y_i^j \end{bmatrix} \preceq 0 \\ & && \text{for } \omega = \omega_j \in \mathcal{I}(\omega_i), j = 1, \dots, N_i. \end{aligned} \quad (50)$$

Remark 7. In case we use either of the formulations in (49) and (50) for the second step of Algorithm 1 and fail to find a feasible solution, it does not necessarily mean that we cannot use this algorithm. In this case it is possible to return to the first step of the algorithm and try a finer partitioning for the frequency range of interest.

Similar to Method 1, uncertainty partitioning improves the quality of the calculated upper bound using this method too.

Remark 8. Although the calculated value for the upper bound using Algorithm 1 has usually a decreasing trend

with respect to the number of partitions, this trend is not necessarily monotonic. This is due to the fact that the calculated upper bound not only is dependent on the number of partitions but also on the quality of the calculated scaling matrices and how they affect the numerical integration procedure.

Remark 9. Note that in Corollary 3, we can choose $\mathcal{I}_D = [\underline{\omega}, \bar{\omega}]$. This allows us to use this method for computing upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm over general frequency ranges.

6. Numerical example

In this section the proposed methods are tested on a theoretical example. The example has been chosen deliberately simple, so that the exact robust $\mathcal{H}_2$ norm can be computed via routine calculations. All the computations are conducted by using the Yalmip toolbox, Löfberg (2004), with the SDPT3 solver, Toh et al. (1999). The platform used for the simulations uses a Dual Core AMD OpteronTM Processor 270 as CPU and 4 GB of RAM.

Consider the uncertain system in (6) with the following system matrices

$$A = \begin{bmatrix} -2.5 & 0.5 & 0 & -50 & 0 \\ 0 & -1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 100 \\ 0 & 0 & 0 & -100 & 0 \end{bmatrix}, \quad B_q = \begin{bmatrix} 0.25 & -0.5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 5 \end{bmatrix}, \quad C = \begin{bmatrix} C_p \\ C_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} D_{pq} \\ D_{zq} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (51)$$

In this example $\Delta(\delta) = \delta I_2$ with $-1 \leq \delta \leq 1$. This system is known to have robust $\mathcal{H}_2$ norm, as defined in (12), equal to 1.5311 which is attained for $\delta = 0.25$. Figure 2 illustrates the gain plots of the system for different values of the uncertain parameter. The aim is to calculate the robust finite-frequency $\mathcal{H}_2$ norm of the system and avoid the peak occurring at 100 rad/s. This is motivated by Figure 3 which presents the calculated finite-frequency $\mathcal{H}_2$ norm of the system in (51), with respect to different values for the uncertain parameter and frequency bounds. As can be seen from this figure and the jump at $\bar{\omega} = 100$ rad/s, the contribution of this peak to the robust finite-frequency $\mathcal{H}_2$ norm cannot be neglected. In order to avoid this peak, the frequency bound that has been considered for this example is $\bar{\omega} = 50$ rad/s. The actual value for the robust finite-frequency $\mathcal{H}_2$ norm for (51) with this frequency bound is 0.8919.

Figure 2: Gain plot over frequency for different values for the uncertain parameter.

In Method 1, presented in Section 4, we consider the following structure for scaling matrices

$$\psi(s) = \begin{bmatrix} \frac{(s-p)^{n_\psi-1}}{(s+p)^{n_\psi}} I_2 & \frac{(s-p)^{n_\psi-2}}{(s+p)^{n_\psi-1}} I_2 & \dots & \frac{1}{(s+p)} I_2 & I_2 \end{bmatrix}, \quad (52)$$

where n_ψ is the order of the scaling matrix and we have chosen $p = 150$. This choice of scaling matrices has been inspired by Laguerre basis functions, Wahlberg (1991) (similar approaches has also been considered in Scherer and Köse (2006, 2008)). For this particular example dynamic scaling matrices with order higher than 3 do not produce any better upper bounds, so only scaling matrices up to order 3 are considered.

Method 2, presented in Section 5, has been applied to the example with Approaches 1 and 2. The number of frequency partitions is increased until either the performance matches the performance of Method 1 or the improvement in the computed upper bound is not discernible anymore. Figures 4 and 5 illustrate the achieved upper bounds for different frequency bounds, $\bar{\omega}$, using methods 1 and 2. In both figures, the curve marked with the solid line reports the actual values for the robust finite-frequency $\mathcal{H}_2$ norm of the system. In Figure 4, the dashed lines present the achieved upper bounds using Method 1. As the order of the dynamic scaling matrices increases, the computed upper bound becomes tighter. These upper bounds have been computed for $n_\psi = 0, 1, 2, 3$. Note that the upper bounds computed using scaling matrices with $n_\psi \geq 1$ are practically indistinguishable. Hence, in Figure 4, the dashed line furthest from the solid line represents the upper bound computed with $n_\psi = 0$ and the ones closest to the solid line are those computed with $n_\psi = 1, 2, 3$. In Figure 5, the bounds presented with the dashed lines are results achieved by applying Method 2 to this example. The dashed curve furthest from the solid line correspond to the bound computed using Approach 2. Hence, as can

Figure 3: Finite-frequency $\mathcal{H}_2$ norm versus different values of the uncertain parameter and frequency bounds.

be seen from Figure 5, Method 2 with Approach 1 can produce better upper bounds than the second approach and can match the performance of Method 1 (the plotted curve refers to the case $n_{par} = 40$). Table 1 presents a summary of the achieved results.

Table 1: Numerical example: comparison of Methods 1 and 2.

Method	Estimated Bound	Elapsed Time[sec]
M.1, $n_\psi = 0$	1.2609	11
M.1, $n_\psi = 1$	1.1972	10
M.1, $n_\psi = 2$	1.1944	12
M.1, $n_\psi = 3$	1.1911	13
M.2, App.1, $n_{par} = 40$	1.189	44
M.2, App.1, $n_{par} = 200$	1.186	144
M.2, App.2, $n_{par} = 200$	1.3184	552

So far the presented results are achieved without any uncertainty partitioning. In order to illustrate the effect of uncertainty partitioning on the performance of the proposed methods, Method 1 and Method 2 with Approach 1 are applied to this example with uncertainty partitioning. Figures 6 and 7 present the achieved upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm of the system with $\bar{\omega} = 50$ rad/s using Methods 1 and 2, respectively. These figures illustrate the upper bound with respect to the number of uncertainty partitions and the order of dynamic scaling matrices, for Method 1, and the number of frequency grid points, for Method 2. As can be seen from the figures and considering the actual robust finite-frequency $\mathcal{H}_2$ norm of the system, the computed upper bounds using both methods are extremely tight. A summary of the results from this analysis is presented in tables 2 and 3. As it can be observed, although both methods produce equally

Figure 4: Robust finite-frequency $\mathcal{H}_2$ norm and the computed upper bounds for different frequency bounds, using Method 1. The solid line illustrates the actual value of the robust finite-frequency $\mathcal{H}_2$ norm. The dashed lines represent the achieved upper bounds using Method 1 for different orders of the scaling matrix.

Table 2: Numerical example: effect of partitioning on Method 1.

n_ψ	No. Uncer. Par.	Estimated Bound	Elapsed Time[sec]
2	1	1.1944	12
2	20	0.8928	434

tight upper bounds, Method 1 achieves this goal with lower computational time.

7. Application to flight comfort analysis

As a practical application, a comfort analysis problem for a civil aircraft model is considered. The derivation of uncertainty models from aircraft physical models is generally a hard task; the resulting models are usually high dimensional and therefore difficult to handle through standard robust analysis tools (see e.g., Poussot-Vassal and Roos (2012)). In this paper we refer to the model developed in Roos (2009). Due to the model size, the problem is computationally much more challenging than the one addressed in Section 6. The objective is to provide an estimate of the energy of the oscillations induced by disturbances like wind gusts or turbulences at different positions along the fuselage of the aircraft. The considered problem

Table 3: Numerical example: effect of partitioning on Method 2.

No. freq. Grids	No. Uncer. Par.	Estimated Bound	Elapsed Time[sec]
20	1	1.1945	30
20	20	0.8924	532

Figure 5: Robust finite-frequency $\mathcal{H}_2$ norm and the computed upper bounds for different frequency bounds, using Method 2. The solid line illustrates the actual value of the robust finite-frequency $\mathcal{H}_2$ norm. The lines represent the bounds obtained using Method 2 with the two different approaches considered.

involves a model of a civil aircraft, including both rigid and flexible modes, along with parametric uncertainty.

Such a problem can be reformulated as an $\mathcal{H}_2$ performance analysis problem for an extended system, including the model of the aircraft, a so-called Von Karman filter (modeling the wind spectrum), and an output filter, accounting for the turbulence field, Papageorgiou et al. (2011). In this aircraft model the uncertain parameter δ corresponds to the level of fullness of the fuel tanks and it is normalized to vary within the range $[-1, 1]$. The overall extended system is presented in LFT form, as in (6), with $n = 21$ states and an uncertainty block size of $d = 14$.

The aircraft model is valid for frequencies up to 15 rad/s, and beyond that it does not have any physical meaning, Roos (2009). This motivates performing finite-frequency $\mathcal{H}_2$ performance analysis, limited to this frequency range. Figure 8, illustrates the gain plots of the system as a function of frequency. Different curves in this figure correspond to different uncertainty values. As can be seen from the figure, the frequency bound at 15 rad/s is necessary to avoid the peak at approximately 20 rad/s which is outside the validity range of the model.

The methods considered for performing comfort analysis are methods 1 and 2 with the use of constant scaling matrices and Approach 1, respectively. Tables 4 and 5 summarize the achieved results using methods 1 and 2, respectively. As can be seen from the tables, both methods perform equally accurate in estimating the robust finite-frequency $\mathcal{H}_2$ norm of the system. However, in contrast to the example in Section 6, Method 2 is faster in calculating the upper bound with equal accuracy. Similar to Section 6, it is possible to improve the computed upper bounds via uncertainty partitioning. This can be observed from tables 4 and 5. A possible way to further reduce the computational times could be to apply adaptive partition-

Figure 6: The achieved upper bounds of robust finite-frequency $\mathcal{H}_2$ norm for $\bar{\omega} = 50$ rad/s with respect to the number of uncertainty partitions and the order of dynamic scaling matrices.

ing techniques, like those proposed for example in Oishi and Fujioka (2009), Garulli et al. (2011).

8. Discussion and General remarks

This section highlights the advantages and disadvantages of the proposed methods and provides insight on how to improve the performance of these methods.

8.1. The observability Gramian based method

This method considers the whole frequency interval and calculates an upper bound of the robust finite-frequency $\mathcal{H}_2$ norm of the system in one shot or one iteration by solving an SDP (SemiDefinite Program). However, the dimension of this optimization problem grows rapidly with the number of states and/or size of the uncertainty block. This limits the capabilities of this method in handling medium or large sized problems, i.e., analysis of systems with high number of states and/or large uncertainty blocks.

The most apparent possibility to improve the accuracy of the computed upper bound using this method is to increase the order of the dynamic scaling matrices. This comes at the cost of higher number of optimization variables in the underlying SDP and affects the computational tractability of the method.

Table 4: Numerical results for the flight comfort application using Method 1.

n_ψ	No. Uncer. Par.	Estimated Bound	Elapsed Time[h]
0	50	1.2434	8.62
0	450	0.7970	59.24

Figure 7: The achieved upper bound of the robust finite-frequency $\mathcal{H}_2$ norm for $\bar{\omega} = 50$ rad/s with respect to the number of uncertainty partitions and the number of frequency grid points.

Table 5: Numerical results for the flight comfort application using Method 2.

No. freq. Grids	No. Uncer. Par.	Estimated Bound	Elapsed Time[h]
80	1	1.2382	0.5611
80	10	0.7911	4.25

Another way of improving the computed upper bound is to perform uncertainty partitioning, which proved to be effective for the examples presented in sections 6 and 7. However, this improvement comes at the cost of a much higher computational burden, see Table 4.

8.2. The frequency gridding based method

This method starts with an initial partitioning of the desired frequency interval and calculates an upper bound of the robust finite-frequency $\mathcal{H}_2$ norm by solving the corresponding SDP for each of the partitions. The sizes of the underlying SDPs in this method are smaller than the ones of the previous method and are mainly dependent on the size of the uncertainty block. Consequently, this method can handle larger problems. However for large problems, the algorithm might require some iterations between steps IV and I of the algorithm to be able to produce consistent results. Another issue with this method is the requirement to perform numerical integration on a rational function in step VI of the algorithm. This can become slightly problematic for high order systems.

There are two main ways to improve the computed upper bounds using this method, namely increasing the number of partitions, and augmenting the SDP for each partition with more constraints for other frequency points in the partition and/or adding more variables to the SDPs

Figure 8: Gain plots for different values of the uncertain parameter.

corresponding to the partitions. This proved to scale better considering the computational time, as compared to Method 1, see tables 4 and 5.

9. Conclusion

This paper has provided two methods for calculating upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm. Through the paper different guidelines for improving the performance of the proposed methods have been presented and their effectiveness has been illustrated using both a theoretical and a practical example.

The proposed methods consider different formulations for calculating upper bounds of the robust finite-frequency $\mathcal{H}_2$ norm. Both methods can produce equally tight upper bounds, but they have different computational properties. While Method 1 is more suitable for small-sized problems and produce results faster than the second method for this type of problems, Method 2 can handle larger problems and produce results more rapidly for this type of problems.

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Appendix A. Proof of Lemma 2

Let

$$\begin{aligned} C_{11} &= M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega), \\ C_{12} &= M_{11}^* \mathcal{X}(\omega) M_{12} + M_{21}^* M_{22}, \\ C_{21} &= M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21}, \\ C_{22} &= M_{12}^* \mathcal{X}(\omega) M_{12} + M_{22}^* M_{22}. \end{aligned} \quad (\text{A.1})$$

Then the left hand side of Condition 1 can be written as

$$M^*(j\omega) \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} - Y(\omega) \end{bmatrix}. \quad (\text{A.2})$$

Now if we assume that there exists $\mathcal{X}(\omega) \in \mathbb{X}$ such that $C_{11} \prec 0$, then Lemma 2 is the direct outcome of Schur's lemma. $\square$

Appendix B. Proof of Theorem 1

If the assumptions of the theorem are satisfied, then by Lemma 2, Condition 1 is valid, i.e., (21) holds. Define

$$\hat{M} = \begin{bmatrix} \mathcal{X}(\omega)^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} \mathcal{X}(\omega)^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}. \quad (\text{B.1})$$

Then (21) can be rewritten as

$$\hat{M}^* \hat{M} - \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix} \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. \quad (\text{B.2})$$

As a result

$$\hat{M}^* \hat{M} \preceq \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix}. \quad (\text{B.3})$$

Define $\bar{q}(j\omega) = \mathcal{X}(\omega)^{\frac{1}{2}} q(j\omega)$ and $\bar{p}(j\omega) = \mathcal{X}(\omega)^{\frac{1}{2}} p(j\omega)$. By pre and post multiplying both sides of (B.3) by $\begin{bmatrix} \bar{q}(j\omega) \\ w(j\omega) \end{bmatrix}^*$ and $\begin{bmatrix} \bar{q}(j\omega) \\ w(j\omega) \end{bmatrix}$, respectively, we have

$$|z(j\omega)|^2 + |\bar{p}(j\omega)|^2 \leq |\bar{q}(j\omega)|^2 + w(j\omega)^* Y(\omega) w(j\omega). \quad (\text{B.4})$$

For all frequencies Δ commutes with $\mathcal{X}(\omega)^{-\frac{1}{2}}$, and hence $\bar{q} = \mathcal{X}^{\frac{1}{2}} q = \mathcal{X}^{\frac{1}{2}} \Delta \mathcal{X}^{-\frac{1}{2}} \bar{p} = \Delta \bar{p}$. Considering the fact that $\Delta \in B_\Delta$, it now follows from (11) and (B.4) that

$$\begin{aligned} |z(j\omega)|^2 &= w(j\omega)^* (\Delta * M)(j\omega)^* (\Delta * M)(j\omega) w(j\omega) \\ &\leq w(j\omega)^* Y(\omega) w(j\omega), \end{aligned} \quad (\text{B.5})$$

which completes the proof. $\square$

Appendix C. Proof of Theorem 3

Let us first introduce a technical lemma, taken from Paganini (1997), which is instrumental for proving Theorem 3.

Lemma 4. *Let P_- , P_+ , X , Q and $\tilde{W}_o$ satisfy (35), and $\mathcal{C}_{11}$ be defined as in (30). Then, $\mathcal{C}_{11} \succ 0$ and there exists a spectral factor $\tilde{N}$ such that $\tilde{N}$ and $\tilde{N}^{-1}$ are stable, $\mathcal{C}_{11} = \tilde{N} \tilde{N}^*$, and $\|\tilde{N}^{-1} M_2\|_2^2 < \gamma^2$, where M_2 is defined in (9). A state space realization for $\tilde{N}^{-1} M_2$ is given by*

$$\tilde{N}^{-1} M_2 = \left[\begin{array}{c|c} \tilde{A} - (\Pi_{12} - P_- \tilde{C}^T) \Pi_{22}^{-1} \tilde{C} & \begin{bmatrix} B_w \\ 0 \end{bmatrix} \\ \hline \Pi_{22}^{-\frac{1}{2}} \tilde{C} & 0 \end{array} \right]. \quad (\text{C.1})$$

Moreover, $\tilde{W}_o$ is the observability Gramian of $\tilde{N}^{-1} M_2$.

PROOF. See Paganini (1997). $\square$

Now, we are ready to prove Theorem 3. Let P_- , P_+ , X , Q and $\tilde{W}_o$ satisfy (35). Define

$$Y = (\tilde{N}^{-1} M_2)^* (\tilde{N}^{-1} M_2) = M_2^* \mathcal{C}_{11}^{-1} M_2 \succeq 0. \quad (\text{C.2})$$

where $\tilde{N}$ and M_2 are defined in Lemma 4. From (C.2) and $\mathcal{C}_{11} \succ 0$, by Schur's lemma it follows that

$$\begin{bmatrix} -\mathcal{C}_{11} & M_2 \\ M_2^* & -Y \end{bmatrix} \preceq 0 \quad (\text{C.3})$$

which corresponds to (29). By using Lemma 3, it turns out that (C.3) is equivalent to (21). By applying the same reasoning as in the proof of Theorem 1, (B.1)-(B.3) hold and one gets

$$(\Delta * M)(j\omega)^* (\Delta * M)(j\omega) \preceq Y(\omega) \quad \forall \omega, \forall \Delta \in B_\Delta.$$

As a result, by using (C.2), Lemma 4 and Lemma 1, one gets

$$\begin{aligned} \sup_{\Delta \in B_\Delta} \|\Delta * M\|_{2, \bar{\omega}}^2 &\leq \int_{-\bar{\omega}}^{\bar{\omega}} \text{Tr} \{Y(\omega)\} \frac{d\omega}{2\pi} \\ &= \int_{-\bar{\omega}}^{\bar{\omega}} \text{Tr} \left\{ (\tilde{N}^{-1}(j\omega) M_2(j\omega))^* (\tilde{N}^{-1}(j\omega) M_2(j\omega)) \right\} \frac{d\omega}{2\pi} \\ &= \text{Tr} \left\{ \begin{bmatrix} B_w \\ 0 \end{bmatrix}^T \left(L(\tilde{\mathcal{A}}, \bar{\omega})^* \tilde{W}_o + \tilde{W}_o L(\tilde{\mathcal{A}}, \bar{\omega}) \right) \begin{bmatrix} B_w \\ 0 \end{bmatrix} \right\} \end{aligned} \quad (\text{C.4})$$

which corresponds to (38). $\square$

Appendix D. Proof of Theorem 4

Consider the LMI in (41) with $\bar{\mathcal{X}}(\omega) = \bar{X}_i$. This LMI can be rewritten as

$$\bar{X}_i^{-\frac{1}{2}} \tilde{M}^* \bar{X}_i^{\frac{1}{2}} \bar{X}_i^{\frac{1}{2}} \tilde{M} \bar{X}_i^{-\frac{1}{2}} - I \prec 0. \quad (\text{D.1})$$

Let $G(j\omega) = \bar{X}_i^{\frac{1}{2}} \tilde{M}(j(\omega + \omega_i)) \bar{X}_i^{-\frac{1}{2}}$. It now follows that $G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right]$. In this theorem we are looking for the largest frequency interval, for which the LMI in (D.1) is valid. On the boundary of this interval $I - G(j\omega)^* G(j\omega)$ becomes singular, i.e., $\det(I - G(j\omega)^* G(j\omega)) = 0$.

By (43) and (44), $I - G(j\omega)^* G(j\omega) = \left[\begin{array}{c|c} A_X & B_X \\ \hline C_X & D_X \end{array} \right]$. Using Sylvester's determinant theorem and some simple matrix manipulations we have

$$\begin{aligned} \det(I - G(j\omega)^* G(j\omega)) &= 0 \Leftrightarrow \\ \det(I + D_X^{-\frac{1}{2}} C_X (j\omega I - A_X)^{-1} B_X D_X^{-\frac{1}{2}}) &= 0 \Leftrightarrow \\ \det(I + (j\omega I - A_X)^{-1} B_X D_X^{-1} C_X) &= 0. \end{aligned} \quad (\text{D.2})$$

By using the matrix determinant lemma and the definition of $\mathcal{G}$ it is also straight forward to establish equivalence between the following expressions

$$\begin{aligned} \det(I + (j\omega I - A_X)^{-1} B_X D_X^{-1} C_X) &= 0 \Leftrightarrow \\ \det(j\omega I - (A_X - B_X D_X^{-1} C_X)) &= 0 \Leftrightarrow \det(\omega I + j\mathcal{G}) = 0, \end{aligned} \quad (\text{D.3})$$

which completes the proof. $\square$