Examensarbete

Low-PAR Precoding for Very-Large Multi-User MIMO Systems

Examensarbete i Kommunikationssystem utfört i samarbete med Ericsson Research vid Tekniska högskolan i Linköping av

Christopher Mollén
LiTH-ISY-EX--13/4671--SE
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Lågtoppvärdepräcodning för storskaliga fleranvändar-MIMO-system

Christopher Mollén

Very-large multi-user MIMO systems, with hundreds of base station antennas, are increasingly attracting attention from both academia and industry. One reason is that such systems can use multi-user precoding to simultaneously serve multiple single-antenna users over the same time-frequency resource. This implies increased data rates and improved spectral efficiency. Another reason is that the energy consumed by the base station is expected to decrease linearly with the number of antennas because of the increasing array gain. To enable the massive increase in the number of antennas, each antenna, together with its tranceiver chain, has to be cheap. If one could manufacture base station antennas using low-cost, mass-produced handset technology, including power amplifiers without advanced linearisation techniques, then very-large multi-user MIMO could become reality.

Handset power amplifiers generally aim to be power-efficient, and in doing so often have highly non-linear transfer characteristics. It is of benefit to transmit signals with low peak-to-average ratio (PAPR) through such power amplifiers, to avoid excessive distortion and to maximise the power efficiency by only having small operating back-offs. Conventionally precoded signals unfortunately have high PAPR (>10 dB). This work has investigated the low-PAPR precoding scheme for very-large MIMO proposed by Mohammed et al. (2013a). It is shown that, the transmit signals of this precoding scheme have 4 dB PAPR, and that by further limiting the phase variation, the PAPR can be made arbitrarily small. However, the more the phase is constrained, the smaller the array gain will be. For example, if the phase variation is limited to \( \frac{\pi}{2} \), the PAPR is lowered to 2.6 dB, but 2–3 dB more transmit power is needed to maintain the same performance or, equivalently, 1.6–2.0 times more antennas are needed at the base station. Continuous phase modulation has briefly been studied as a means of producing constant-envelope transmit signals. Low-PAPR precoding, where the transmit signals lie inside a circle, is suggested as a way to decrease the required transmit power without increasing PAPR noticeably (<4.5 dB) relative to scheme of Mohammed et al. The algorithm that was developed for this purpose, however got stuck in local minima, which degraded its performance. The transmit power could therefore only be slightly (<1 dB) lowered in the regime of high data rates.

A preliminary link budget analysis based on a simplistic model of the power amplifier has shown that, assuming perfect channel state information and frequency-flat fading, low-PAPR precoding can reduce energy consumption by 33 % compared to conventional linear precoding in a base station with 100 antennas. The analysis suggests that using unlinearised class AB handset power amplifiers might be a viable option for very-large multi-user MIMO base stations.

Nyckelord
MIMO, PAPR Reduction, Precoding, Multi-User
Sammanfattning


En preliminär länkbudget baserad på en enkel effektförstärkarmodell har visat att, med fullständig kanalkännedom och i frekvensplatt fädning, skulle lågtoppvärförrådning kunna minska energiförbrukningen med 33 % jämfört med konventionell, linjär förkodning i en basstation med 100 antenner. Analysen antyder att olineriserade klass AB mobiltelefonseffektförstärkare kan vara ett alternativ för storskalig fleranvändare-MIMO-basstationer.
Abstract

Very-large multi-user MIMO systems, with hundreds of base station antennae, are increasingly attracting attention from both academia and industry. One reason is that such systems can use multi-user precoding to simultaneously serve multiple single-antenna users over the same time-frequency resource. This implies increased data rates and improved spectral efficiency. Another reason is that the energy consumed by the base station is expected to decrease linearly with the number of antennae because of the increasing array gain. To enable the massive increase in the number of antennae, each antenna, together with its tranceiver chain, has to be cheap. If one could manufacture base station antennae using low-cost, mass-produced handset technology, including power amplifiers without advanced linearisation techniques, then very-large multi-user MIMO could become reality.

Handset power amplifiers generally aim to be power-efficient, and in doing so often have highly non-linear transfer characteristics. It is of benefit to transmit signals with low peak-to-average ratio (PAR) through such power amplifiers, to avoid excessive distortion and to maximise the power efficiency by only having small operating back-offs. Conventionally precoded signals unfortunately have high PAR (approx. 10 dB). This work has investigated the low-PAR precoding scheme for very-large MIMO proposed by Mohammed et al. (2013a). It is shown that, the transmit signals of this precoding scheme have 4 dB PAR, and that by further limiting the phase variation, the PAR can be made arbitrarily small. However, the more the phase is constrained, the smaller the array gain will be. For example, if the phase variation is limited to $\pi/2$, the PAR is lowered to 2.6 dB, but 2–3 dB more transmit power is needed to maintain the same performance or, equivalently, 1.6–2.0 times more antennae are needed at the base station. Continuous phase modulation has briefly been studied as a means of producing constant-envelope transmit signals. Low-PAR precoding, where the transmit signals lie inside a circle, is suggested as a way to decrease the required transmit power without increasing PAR noticeably (<4.5 dB) relative to scheme of Mohammed et al. The algorithm that was developed for this purpose, however got stuck in local minima, which degraded its performance. The transmit power could therefore only be slightly (<1 dB) lowered in the regime of high data rates.

A preliminary link budget analysis based on a simplistic model of the power amplifier has shown that, assuming perfect channel state information and frequency-flat fading, low-PAR precoding can reduce energy consumption by 33 % compared to conventional linear precoding in a base station with 100 antennae. The analysis suggests that using unlinearised class AB handset power amplifiers might be a viable option for very-large multi-user MIMO base stations.
摘要
大規模多用戶多輸入多輸出通信系統，即配備上百基站天線的系統，正吸引著學術界及工業界越來越多的關注。其中一個原因是通過多用戶預編碼，該系統可以在同一時頻資源上同時服務多個單天線用戶，有效地增加數據速率及頻譜效率。另一個原因是，隨著陣增益的增加，基站功耗將隨天線數線性遞減。爲了使天線數的極大化可行，每根天線與其收發機的成本必須非常廉價。只有在多天線基站的生產中使用低成本的手機配件，比如不包含複雜線性化技術的功率放大器，大規模多用戶多輸入多出系統才有可能真正實現。

手機功放通常爲了降低功耗而有著高度非線性傳輸特性。因此，通過這樣的功放更適合傳輸低峯均比的信號以避免過度失真，同時可以在小的運作功率下提高功耗效率。傳統預編碼的信號峯均比不巧很高（約10分貝）。本論文研究了由Mohammed等人（2013a）提出的低峯均比預編碼。表明該預編碼的信號有4分貝的峯均比，另外加上相位變化約束信號峯均比可以降到任意小。但相位約束越緊陣增益會隨之減小。譬如約束相位變化小於\(\frac{\pi}{2}\)，峯均比降低到2.6分貝，但需要增加2至3分貝的發射功率保持相同的性能，或增加天線數於1.6至2倍。本文也簡要地描述了恆定包絡信號的連續相位調制，並提出一個預編碼讓傳播信號在一個圓內，以便減少所需要的發射功率，而峯均比也不明顯比Mohammed等人的預編碼大（<4.5分貝）。時此設計的算法會陷入局部最優點，從而降低其性能。因此傳輸功率只有在高數據速率場景觀察到稍微減小（<1分貝）。

一個初步的基於簡單的功放模型的鏈路預算分析表明，假設收發端具有全部的信道狀態信息，並假設頻率平坦衰落，在一台以一百根天線的基站，低峯均比預編碼可以比普通的線性預編碼進一步降低33%功耗。也表示在大規模多用戶多入多出基站中使用非線性的手機功放應該是可行的。
Zusammenfassung


Mobiltelefonleistungsverstärker sind gewöhnlich eher auf hohen Wirkungsgrad angepasst, deren Übertragungseigenschaften sind daher stark nichtlinear. Es ist von Vorteil, Signale mit niedrigem Scheitelfaktor durch solche Leistungsverstärker zu übertragen, um übermäßige Verzerrung zu vermeiden und die Wirkungsgrad durch kleineren Backoff vom Arbeitspunkt zu maximieren. Leider haben herkömmlich vorkodierte Signale hohen Scheitelfaktor (ca. 10 dB). Diese Arbeit untersucht die Vorkodierungsmethode von Mohammed u.a. (2013a) zur Verringerung des Scheitelfaktors. Es wird gezeigt, dass die Signale dieser Vorkodierungsmethode einen Scheitelfaktor von 4 dB und dass der Scheitelfaktor durch eine zusätzliche Begrenzung der Phasenvariation beliebig klein gemacht werden kann. Je mehr die Phasen begrenzt werden, desto kleiner wird jedoch die Antennenerhöhung. Z.B. wenn die Phasenvariation auf $\pi/2$ begrenzt wird, wird der Scheitelfaktor auf 2,6 dB reduziert, aber 2–3 dB höhere Sendeleistung ist benötigt, um die gleiche Datenraten zu behalten, oder, entsprechend, muss die Antennenzahl um einen Faktor 1,6–2 erhöht werden. Modulation mit stetiger Phase, als eine Methode um Sendesignale mit konstanten Einhüllenden zu bekommen, wird kurz untersucht. Eine Vorkodierungsmethode, wo die Signale innerhalb eines Kreises liegen, wird vorgeschlagen, zur Verringerung der erforderlichen Sendeleistung, ohne den Scheitelfaktor (<4,5 dB), im Vergleich zur Vorkodierung von Mohammed u.a., erkennbar zu erhöhen. Der Algorithmus, der für diesen Zweck entwickelt wurde, fährt aber in lokale Minima fest, was dessen Leistung verringert. Die Sendeleistung kann deshalb nur im Bereich hohen Datenraten etwas (<1 dB) gesenkt werden.

Eine Kanalgewinnanalyse, die auf einem einfachen Leistungsverstärkergmodell beruht, zeigt ansatzweise im Fall perfekter Kanalzustandsinformation und Flachschwund, dass geeignete Scheitelfaktorvorkodierung in einem Basisstation mit 100 Antennen den Stromverbrauch im Vergleich zu herkömmlicher linearer Vorkodierung um 33 % verringern kann. Die Analyse deutet an, dass die Anwendung unlinearisierter Mobiltelefonleistungsverstärker Klasse AB eine Möglichkeit in Mehrbenutzer-MIMO-Basisstationen mit hunderten von Antennen ist.
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I want to thank my supervisors, but emphasize that any errors or mistakes in the report are the result of my own negligence. Without the great support and supervision, which I have received for this thesis, I would not have got far.

– Christopher Mollén, Kista, 2013-06-05
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NotaƟon

Scalars, Vectors, Matrices and Random Variables

Scalars are denoted with italic upper- or lower-case letters. Upper-case letters are treated as constants and lower case letters as indices, or numbers depending on the context. Vectors and matrices are both denoted with bold characters, vectors in lower case letters and matrices in upper case. Random variables are denoted with the same symbols as their deterministic counterparts, context has to tell them apart.

\[ M \quad \text{Integer, number of transmit antennae} \]
\[ m \quad \text{Integer, index of a generic transmit antenna, } m \in \{1, \ldots, M\} \]
\[ K \quad \text{Integer, number of users} \]
\[ k \quad \text{Integer, index of a generic user, } k \in \{1, \ldots, K\} \]
\[ N_0 \quad \text{Non-negative real number, power spectral density of white Gaussian noise} \]
\[ h_{km} \quad \text{Complex number, channel coefficient from antenna } m \text{ to user } k \]
\[ h_k \quad \text{Complex valued vector, channel vector from the antenna array of the transmitter to user } k \]
\[ H \quad \text{Complex-valued matrix, channel realization} \]
\[ x \quad \text{Complex-valued vector, channel input} \]
\[ r \quad \text{Complex-valued vector, channel output} \]
\[ s \quad \text{Complex-valued vector, symbols} \]
\[ g_k \quad \text{Complex-valued vector, precoding weights for the symbol intended for user } k \]
\[ G \quad \text{Complex-valued matrix, precoding matrix} \]
\[ \text{diag}(a_1, \ldots, a_n) \quad \text{Diagonal matrix with the elements } a_1 \text{ to } a_n \text{ on its diagonal} \]
\[ \text{CN}(\mu, \sigma^2) \quad \text{Circularly symmetric complex Gaussian random variable, where the real and imaginary parts are i.i.d. normally distributed with variance } \sigma^2 \text{ and mean } \text{Re}(\mu), \text{Im}(\mu) \text{ respectively} \]
Sets
Sets are denoted by capital letters in a cursive font, except for the standard sets of numbers, which are denoted with double-struck capital letters.

\[ Z \] The set of all integers
\[ \mathbb{R} \] The set of all real numbers
\[ \mathbb{R}^+ \] The set of all positive real numbers, \( 0 \notin \mathbb{R}^+ \)
\[ C \] The set of all complex numbers
\[ \mathcal{M}_{\epsilon}(\mathbf{H}) \] The set of all possible receive signals when the transmit signals are \( \epsilon \)-limited in amplitude below \( A \) for the channel realization \( \mathbf{H} \), see Definition 4.1
\[ \mathcal{M}_0(\mathbf{H}) \] The set of all possible receive signals when the transmit signals have constant amplitude for the channel realization \( \mathbf{H} \)
\[ [a, b] \] The closed interval from \( a \) to \( b \)
\[ [a, b[ \] The half open interval from \( a \) to, but not including, \( b \)
\[ \mathcal{N}(\mathbf{H}) \] The null space of the matrix \( \mathbf{H} \)
\[ \mathcal{O}(g(x)) \] The set of all functions that grows as fast as \( g(x) \) or approach 0 as fast as \( g(x) \)

Operators

\( \bar{z} \) Complex conjugate of a complex scalar, elementwise complex conjugate of complex vector or optimal value of real valued parameter
\( \mathbf{X}^T \) Transpose of a matrix or a vector
\( \mathbf{X}^H \) Complex-conjugate transpose, hermitian, of a matrix or a vector
\( \mathbf{X}^\dagger \) Pseudo-inverse of matrix
\( \text{Tr} \) The trace of a matrix
\( \| \mathbf{x} \| \) The two-norm, \( \| \mathbf{x} \| = \left( \sum_{m=1}^{\text{dim}(\mathbf{x})} |x_m|^2 \right)^{1/2} \)
\( \| \mathbf{x} \|_\infty \) The infinity norm, \( \| \mathbf{x} \|_\infty = \max \{|x_m|, m = 1, \ldots, \text{dim}(\mathbf{x})\} \)
\( \| \mathbf{x} \|_1 \) The one-norm, \( \| \mathbf{x} \|_1 = \sum_{m=1}^{\text{dim}(\mathbf{x})} |x_m| \)
\( \mathbb{E}[\cdot] \) Expectation of a random variable
\( \text{Pr}(\cdot) \) Probability of an event
\( \mu(X) \) Peak of a signal
\( \text{Re}(\cdot) \) Real part of a complex number
\( \text{Im}(\cdot) \) Imaginary part of a complex number
\( |\cdot| \) The modulus of a complex number
\( \text{arg}(\cdot) \) The phase of a complex number, \( \text{arg}(z) \in [0, 2\pi[ \)
\( a^+ = \max(a, 0) \)
\( a \mod b \) The modulo operator, \( b \in \mathbb{R}, (a \mod b) \in [0, b[ \)
\( \# \) The cardinality of a set, i.e. the number of elements in a finite set
\( \mu \) The outer measure of a set
\( \lg \) The logarithm with base 10
\( \sim \) Distributed in the same way as
Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
</tr>
<tr>
<td>BER</td>
<td>Bit-Error-Rate</td>
</tr>
<tr>
<td>bpcu</td>
<td>bits per channel use</td>
</tr>
<tr>
<td>DP</td>
<td>Digital Pre-Distortion</td>
</tr>
<tr>
<td>EIRP</td>
<td>Equivalent Isotropically Radiated Power</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>Identically and independently distributed</td>
</tr>
<tr>
<td>HSPA</td>
<td>High Speed Packet Access, a mobile broadband technology</td>
</tr>
<tr>
<td>LTE</td>
<td>Long Term Evolution, a telephone and mobile broadband communication standard</td>
</tr>
<tr>
<td>MIMO</td>
<td>Multiple-Input-Multiple-Output</td>
</tr>
<tr>
<td>MLSE</td>
<td>Maximum Likelihood Sequence Estimation</td>
</tr>
<tr>
<td>OFDM</td>
<td>Orthogonal Frequency Division Multiplexing</td>
</tr>
<tr>
<td>PA</td>
<td>Power Amplifier</td>
</tr>
<tr>
<td>PAR</td>
<td>Peak-to-Average Ratio, see General Definitions</td>
</tr>
<tr>
<td>PAPR</td>
<td>Peak-to-Average-Power Ratio, see General Definitions</td>
</tr>
<tr>
<td>Pd</td>
<td>Pre-Distorter</td>
</tr>
<tr>
<td>RF</td>
<td>Radio Frequency</td>
</tr>
<tr>
<td>RMS</td>
<td>Root-Mean-Square of a signal</td>
</tr>
<tr>
<td>RRC</td>
<td>Root-Raised Cosine</td>
</tr>
<tr>
<td>SINR</td>
<td>Signal-to-Interference-and-Noise Ratio</td>
</tr>
<tr>
<td>SIR</td>
<td>Signal-to-Interference Ratio</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
</tr>
</tbody>
</table>
General Definitions

Array Gain

The power gain that is achieved by using multiple antenna elements, compared to using a single antenna at the receiver and the transmitter, is called array gain. Array gains can be achieved by multiple antennae only at the transmitter, only at the receiver or both at the transmitter and receiver.

Complementary Cumulative Distribution Function, CCDF

The probability that a random variable $x$ is above a certain value $p$ is given by the complementary cumulative distribution function, which is defined as

$$
\text{CCDF}(p) = \Pr(x > p).
$$

The cumulative distribution function is given by $1 - \text{CCDF}(p)$.

Multi-User Interference

In a multi-user system, if $u$ is the symbol intended for reception by some user, and $r$ is the noise-free receive signal at that user, then the multi-user interference is denoted by $e$ and is defined as

$$
e = r - u.
$$

Peak Value

The peak value of a finite discrete-time signal $x[n]$, $n = 1, \ldots, N$, is defined* as the smallest real number that 99.99% of the samples have an amplitude less than, i.e. the smallest $p$, for which

$$
\frac{\# \{n : p \leq |x[n]| \}}{N} \leq 10^{-4}.
$$

*This is the way peak value and subsequently $\text{PAR}$ is defined by Ericsson AB.
The peak value of a finite continuous-time signal \( x(t), t \in [0, T] \), is defined as the smallest real number that the absolute value of the amplitude of the signal is less than for 99.99% of the time, i.e. the smallest \( p \), for which

\[
\frac{\mu[t : p \leq |x(t)|]}{T} \leq 10^{-4}.
\]

The peak value of a random process \( X \) is defined as the smallest real number \( p \), for which

\[
Pr(p \leq |X[n]|) \leq 10^{-4}.
\]

We denote the peak of any signal by \( p(x) \).

**PAR—Peak-to-Average Ratio—or Crest Factor**

For the discrete-time signal \( x[n] \), defined for \( n = 1, \ldots, N \), PAR is defined as the ratio between its peak value and its RMS value:

\[
PAR = \frac{p(x)}{\sqrt{\frac{1}{N} \sum_{n=1}^{N} |x[n]|^2}}.
\]

For the continuous-time signal \( x(t) \), defined for \( t \in [0, T] \), PAR is defined similarly as

\[
PAR = \frac{p(x)}{\sqrt{\frac{1}{T} \int_{0}^{T} |x(t)|^2 dt}}.
\]

For the zero-mean weak-sense stationary random process \( x \), PAR is defined as the ratio between its peak and its standard deviation:

\[
PAR = \frac{p(x)}{\sqrt{\mathbb{E}(|x[n]|^2)}}.
\]

The decibel value of PAR is defined as:

\[
20 \log_{10}(\text{PAR}) \ [\text{dB}].
\]

**PAPR—Peak-to-Average-Power Ratio**

PAPR is the ratio between the peak instantaneous power—the peak value squared—and the average power of the signal. As such, it is the square of PAR.

\[
PAPR = \text{PAR}^2
\]

The decibel value of PAPR coincides with that of PAR and is defined as:

\[
10 \log_{10}(\text{PAPR}) \ [\text{dB}].
\]
Chapter 1

Introduction

1.1 Background

In the past decade, the number of wirelessly connected devices has exploded and is continuing to increase at an ever faster rate. This is changing the way we live, as increasingly many people rely on wireless connectivity in more and more aspects of daily life. To serve the future Networked Society, where everybody and everything is connected, and to handle the huge amount of data it will generate, industry, policy makers and academia are already trying to find out what the next generation of wireless standards—5G—will look like.

Much attention is paid to multi-antenna systems (Baldemair, 2013). The concept of very-large multi-user MIMO seems to be a good candidate for future standards, because it promises remarkable improvements in data rate and power savings through high-order spatial multiplexing and array gains (Rusek et al. 2013). It could also open up a whole new spectrum of frequencies—the millimetre waves (3-300 GHz)—for use in wireless communication by overcoming the strong path losses in these bands through enhanced robustness against fades and through increased signal strengths (Samsung, 2013).

To practically enable the huge increase in the number of antennae, which is required in a very-large multi-user MIMO system, it is vital that the cost of the hardware around each antenna is cheaper than in today’s single-antenna base stations. Otherwise, the cost of a multi-antenna base station would scale linearly with its massive number of antennae, which would inhibit its practical implementation.

One of the biggest single costs at the base station transmitter chain is the power amplifier (Brech, 2003), which has to be highly linear and use sophisticated linearisation techniques to handle the high-PAR* signals of modern transmission schemes. If a multi-antenna base station could use non-linear, inexpensive power amplifiers of the type used in mobile user equipment instead, this would be a

*Peak-to-Average Ratio (PAR) is a measure of how high the peaks are above the RMS amplitude of the signal, see General Definitions.
large step forward to the realisation of MIMO base stations with very-large antenna arrays.

To use non-linear power amplifiers, transmit signals with low PAR are needed, in order to avoid the signal distortion and out-of-band radiation that the non-linear region of the power amplifier gives rise to. In very large multi-user MIMO systems, the high-dimensional null space of the channel can be used to shape the transmit signals, for example to lower their PAR. Any vector $\hat{x} \in M(H)$ in the null space can be added to a transmit signal vector, without changing the receive vector. Hence, if $\hat{x}$ were the original transmit signal, we could just as well transmit $x = \hat{x} + \hat{x}$. If $\hat{x}$ is carefully chosen, it could lower the PAR of the signals.

Mohammed et al. (2013a) describe one way of utilising these extra degrees of freedom of the null space to produce discrete-time constant-envelope signals at the transmit antennae, signals that when viewed at one sample per symbol duration all lie on a circle. Such signals should exhibit low PAR, as opposed to signals from conventional precoding, which have high PAR.

If this is the case, the precoding scheme, which they describe, could enable the use of low-cost, handset power amplifiers in the base station. If the continuous-time signals still exhibit a too high PAR, it would be of interest to find a way to lower the PAR further, in order to enable the use of handset power amplifiers.

Handset power amplifiers cost little to produce and if it were possible to use them in the base station, they could make very-large multi-user MIMO a feasible solution to address future 5G requirements on high data rates and lowered energy consumption.

Another advantage of using non-linear power amplifiers and signals with low PAR is the increased power efficiency in the power amplifier. Signals with low PAR would therefore not only make very-large MIMO systems affordable, but also make them more power efficient than today’s base stations, which is important in order to lower the ecological footprint of the base station.

This work has investigated the scheme that Mohammed et al. propose and suggested some improvements. We have also evaluated the scheme in a link budget. This preliminary analysis showed how multi-antenna systems outperformed a single-antenna system in terms of power consumption and that discrete-time constant-envelope precoding, instead of conventional precoding, could further lower the power consumption of very-large multi-user MIMO systems.

1.2 Problem Statement

This study has investigated the possibility to use the excess degrees of freedom that are available in very-large multi-user MIMO to reduce the PAR of the transmit signal. It has also investigated what effects this PAR reduction might have on the energy consumption of the base station. The precoding scheme proposed by Mohammed et al. (2013a), which will be called discrete-time constant-envelope precoding in this report, has been the focus of the investigation, which can be divided
1.3 Assumptions and Limitations

Into two parts:

1. Investigate discrete-time constant-envelope precoding, see if the precoding scheme can be extended and see how it compares to conventional precoding.

2. Investigate whether handset technology could be used in very-large multi-user MIMO base stations.

1.3 Assumptions and Limitations

In our investigation, we will assume that the base station has perfect channel state information and that the fading between the base station and the users is frequency-flat, i.e. that the impulse response of the channel from one antenna of the base station to one user consists of only one complex tap.

When we analyse the precoding schemes in terms of sum rates, we assume that the channel coefficients are uncorrelated. Uncorrelated antennae is a best case scenario, where the resolution of the array is good and users can be distinguished easily. In a real world scenario, the coefficients are often correlated to some degree, which will incur a performance degradation relative to the uncorrelated case.

We do not consider the geometry of the array or any specific placement of the antenna elements. However, we assume that there is no mutual coupling between the antennae of the array.

In the link budget, we assume that the base station uses Gaussian signalling to convey the information bits.

1.4 Novel Contributions

To our knowledge, this work is the first to study phase constraints in addition to the discrete-time constant-envelope precoding proposed by Mohammed et al. (2012, 2013). The idea of continuous phase modulation in connection with discrete-time constant-envelope precoding is new but not yet fully explored. The relaxation of the amplitude constraints in the discrete-time constant-envelope precoding is first described in this report.

The complete characterisation of the region of possible receive signals in the single-user discrete-time constant-envelope case is given in the work of Pan (2013), where the missing lower bound on the amplitudes of the signals in the set is derived. This lower bound was independently derived in this work in a slightly different way. A method of exact phase recovery in the single-user discrete-time constant-envelope case was first given by Pan (2013). A new method of similar complexity is given in this work.

This work also proposes new ways of choosing the energy scaling factor $E$, which is the parameter that balances the array gain and multi-user interference.
However, because of channel hardening, the performance gain is negligible in very-large multi-user MIMO systems.

1.5 Structure of the Report

In Chapter 2, the concept of very-large multi-user MIMO is presented, our system model is explained and the conventional precoding schemes, zero-forcing and maximum ratio transmission, are studied. A brief introduction to power amplifiers is given in Chapter 3. There we discuss the most common classes of power amplifiers used in radio frequency devices, namely class A, B and AB.

The discrete-time constant-envelope precoding algorithm for very-large multi-user MIMO proposed by Mohammed et al. (2013a) is presented in Chapter 4. We characterise the set of all possible receive signals for the single-user discrete-time constant-envelope precoding. Further in this chapter, we derive a new exact phase recovery method for precoding in the single-user case, which is different from the approximative method proposed by Mohammed et al. In the last section of this chapter, we discuss the energy scaling factor $E$, which is an important parameter of the discrete-time constant-envelope precoding that balances the array gain and multi-user interference of the system.

In the following three chapters, my contributions to the field of low-PAR precoding are presented. In Chapter 5, an extension to the discrete-time constant-envelope precoding scheme is presented, which limits the phase variations of the transmit signals, something that further improves the PAR of the continuous-time signal. In Chapter 6, a continuous phase modulation scheme that results in transmit signals with 0 dB PAR in continuous time is suggested. In Chapter 7, the amplitude constraints are relaxed and the transmit signals are allowed to lie inside a circle to see if the performance increases.

Then the precoding schemes are evaluated in terms of ergodic sum rates and bit error rates in Chapter 8. As a final comparison, the low-PAR precoding schemes are compared to a single-antenna transmission scheme and a multi-antenna scheme that uses zero-forcing in a link budget analysis.

Conclusions are made in Chapter 9 and suggested further research work is presented in Chapter 10.
Chapter 2

Very-Large Multi-User MIMO

This chapter provides an overview of very-large multi-user MIMO (Multiple-Input-Multiple-Output). It aims at giving an introduction to its history, some practical implementation issues and the ongoing work in the scientific community. A very-large multi-user MIMO system model is set up and described. The conventional precoding methods zero-forcing and maximum ratio transmission are introduced and the PAR of their transmit signals are computed.

2.1 History

Multiple-input-multiple-output communication schemes, schemes using multiple antennae at both transmitter and receiver, are today well-known and commonly used transmission techniques. They are becoming increasingly popular because the multiplicity of antennae increases the diversity of the system and makes it robust against multi-path fading (Dahlman, 2010). Additionally, the multiple antennae enable multiple data streams to be sent over the same time-frequency resource, which translates into a multiplexing gain that improves the overall capacity of the system (ibid.). This is why MIMO is making its way into most of today’s wireless standards, including: LTE-Advanced, 802.16m (Mobile WiMAX Release 2), 802.11n (Wi-Fi), Evolved HSPA.

Recently, much attention has been given to multi-user MIMO—an off-shoot of point-to-point MIMO, in which an array of multiple antennae simultaneously serves a multiplicity of autonomous users over the same time-frequency resource (see for example Gesbert 2007, Marzetta 2010, Rusek 2013). The users could use cheap, single-antenna devices to share the multiplexing gains of the MIMO system, see Figure 2.1. This is advantageous in mobile environments, where the user equipment often is limited in physical size and by low cost requirements, and therefore only can support a single or a very limited number of antennae.

For point-to-point MIMO the multiplexing gain can disappear in certain scattering environments, e.g. line-of-sight. A multi-user MIMO system is more robust
against such events, since the users, most often, are further away from each other than the resolution of the array (Marzetta, 2010).

### 2.2 Very-Large Antenna Arrays

As the number of antennae at the base station is increased, the diversity order of the system increases and, due to the law of large numbers, the properties of the channel become less stochastic—this is sometimes termed *channel-hardening* (Hochwald, 2004). The consequence of this is that the effects of small-scale fading are averaged out, see Figure 2.2a. Channel vectors to different users also become pairwise orthogonal, see Figure 2.2b, and multi-user interference can efficiently be suppressed with simple linear signal processing (Marzetta, 2010). Multi-user MIMO systems with a sufficient number of antennae at the base station, such that these hardening phenomena are observed, we call very-large multi-user MIMO systems. Another commonly used name is massive multi-user MIMO systems.

There is no definition commonly agreed upon that states what is meant by *very large*. In this report, we study a systems with 100 antennae. Figure 2.2 shows how deep fades become rare and the correlation between channels decreases with growing number of transmit antennae. Already at 100 transmit antennae, the average path loss is more than 0.8 and the correlation between two users is less than 0.2 for 95 % of the channel realisations and the channel can be considered relatively well-behaved. A base station with 100 transmit antennae can therefore be classified as very large in an i.i.d. Rayleigh channel.

With $M$ antennae at the base station and $K$ single-antenna users, very-large MIMO can achieve a multiplexing gain* of $\min(M, K)$ and a diversity of order $M$, which will improve data rate, spectral efficiency and communication reliability compared to today’s single- or few-antenna systems. Apart from these advantages, compared to today’s base stations, a very-large MIMO base station will benefit from a high array gain, and will therefore consume less power. In the uplink, a

---

*This is usually equal to $K$ in a very-large multi-user MIMO system since $M$ is big.
Figure 2.2: When the number of transmit antennae grows, the properties of the channel become less and less stochastic, this is called channel hardening. In both figures a) and b) above, a Rayleigh fading channel with i.i.d. \( \mathcal{CN}(0, 1) \) coefficients is studied. Figure a) shows the array gain normalised with respect to the number of transmit antennae \( \| \mathbf{h} \|^2 / M \). In b) the correlation between two users \( \frac{\| \mathbf{h}_1^H \mathbf{h}_2 \|}{\| \mathbf{h}_1 \| \| \mathbf{h}_2 \|} \) is shown. In both cases, the dashed lines show the 5% and 95% percentiles of the random variable. The convergence of the dashed lines to the mean (fat line) is a sign of channel hardening.
high gain can be obtained by coherently combining the received signals. In the
downlink, the base station can focus the energy in a small physical area, where the
user is located. The transmit power can thus be reduced by an order of magnitude
or more (Ngo, 2012).

The decreased transmit power does not only decrease the operational cost
and environmental footprint of the base station, it will also decrease its spectral
contamination—when transmit power is concentrated to where the user is, the
operation of other users and base stations will suffer from little interference. This
could lead to increased spectral reuse and decreased intercell interference*.

2.3 Acquiring Channel State Information

In order to achieve a spatial multiplexing gain, the base station needs accurate and
up-to-date channel state information. The large number of antennae at the very-
large multi-user MIMO base station makes it impossible to send unique orthogonal
pilots from each of the antennae within one coherence time-frequency interval.
If each base station antennae sent their own orthogonal pilot, M pilots would be
needed, where M is the number of antennae at the base station, which is big in
very-large multi-user MIMO systems.

To gain channel state information, large antenna arrays have to rely on chan-
nel reciprocity—the assumption that the channel response looks the same in the
uplink and in the downlink. With this assumption, the K users can send the pilots
instead. Now only K pilots have to be sent, which makes it feasible to gain chan-
nel state information, even for a very-large multi-user MIMO transmission system.
(Rusek, 2013)

2.4 Antenna Array Geometry

As Rusek et al. (2013) point out, the geometry of the array is important. In order to
avoid major coupling and correlation between antennae, which is detrimental to
the performance of the spatial multiplexing of the system, each antenna element
has to be placed far enough apart from any other antenna element.

It is generally accepted that the smallest spacing between adjacent antenna
elements to avoid significant coupling is λ/2, where λ is the wavelength of the
signal (see for example Rusek 2013, Rajagopal 2011). The correlation, however,
depends on the scattering environment. Whether λ/2 is a big enough spacing in
a given scenario remains to be studied.

The correlation between antenna elements is strongly influenced by the geo-
metry of the array. With a linear horizontal array, the resolution in azimuth is
high but there is no resolution in elevation. To get a better vertical resolution,

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*As long as the pilot contamination that Marzetta mentions in his paper (2010) is negligible.
the array needs a second dimension. We can argue that in most cases the positions of the scatterers differ mostly in the horizontal direction and not much in the vertical, therefore a horizontally wide rectangular antenna array might be sensible.

The size of an antenna array thus depends on the wavelength of the transmit signal. The array could be made smaller by using a higher carrier frequency. However, a too high carrier frequency will suffer from an increased path loss, provide poorer non-line-of-sight performance and restrict the coverage of the base station. The sizes of different geometries of a 100-antenna array are given in Table 2.1. The two frequencies: the usual carrier frequency for mobile communication 2 GHz and the experimental millimetre-wave frequency 28 GHz, are considered in the table. From a practical point of view, these dimensions are reasonable.

Table 2.1: Sizes of Arrays with 100 Antennae

<table>
<thead>
<tr>
<th>Carrier Frequency</th>
<th>2 GHz</th>
<th>28 GHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear array, 1×100</td>
<td>7.4 m</td>
<td>0.53 m</td>
</tr>
<tr>
<td>wide rectangular, 2×50</td>
<td>3.7×0.075 m</td>
<td>0.26×0.0054 m</td>
</tr>
<tr>
<td>square array, 10×10</td>
<td>0.68×0.68 m</td>
<td>0.048×0.048 m</td>
</tr>
</tbody>
</table>

As long as coupling is avoided and good correlation properties are obtained, having a regularly-shaped array geometry is not of importance from a performance perspective. In principle, any placement of the antennae is possible. A façade-mounted antenna array could be installed according to the premises and shape of the building. A limiting design factor though is the local oscillators, which have to be synchronised and connected by equally long wires that cannot be too long because of signal attenuation. The design of a very-large antenna array is out of the scope of this thesis.

2.5 Positioning of the Base Station

The location of the base station influences the scattering environment that it sees. If it is placed outdoors on a rooftop, it might have free sight over the surroundings and the major scatter centres are all close to the user. In this scenario, the correlation between the antennae elements is expected to be high. The correlation between users, on the other hand, is expected to be low, because the location of the users normally differ in azimuth. In terms of the system model presented later in this chapter, elements of the same row in the channel matrix are correlated, but the elements from different rows are not.

If we place the base station below the rooftops, the scattering environment gets richer. Then, we can expect the antennae to be sufficiently uncorrelated. The range of the base station is however diminished due to the increased path
Figure 2.3: A functioning 8×8-array multi-user MIMO base station, the Argos Project, see Shepard, 2012. Courtesy of Mr. Shepard. http://argos.rice.edu/

loss. Indoors, we would also expect the scattering environment to be rich and the antennae to be uncorrelated.

### 2.6 Existing Very-Large Antenna Arrays

Very-large antenna arrays are no longer a mere theoretical being. The first prototypes of very-large multi-user MIMO base stations have been built. To convince the sceptical reader of the practicality of a very-large antenna array, a picture of a working 64-antennae base station from the Argos project (Shepard, 2012) is given in Figure 2.3. It is built in a collaboration between universities and, among others, Bell Laboratories and Alcatel Lucent. The same research group is now working on the next generation of their base station. One of the two base station towers is shown in Figure 2.4. This next generation base station will use an array of 96 antenna elements or more.

Samsung (2013) have also announced that they have built an adaptive array transceiver operating in the millimetre-wave Ka bands (26.5–40 GHz) and achieving speeds up to 1.056 Gbit/s up to a distance of 2 kilometres at a frequency of 28 GHz. Their press release, however, fails to enclose exact details of the circumstances and results.
2.7 Pre-Equalisation

In an environment with frequency-selective fading, very-large multi-user MIMO can make the need for equalisation at the users unnecessary by precoding in time. With multiple antennae at the base station, it is possible to equalise the impact of the channel before transmission, this is called pre-equalisation. After pre-equalisation, the users would see intersymbol-interference-free symbols. This could make complex equalisation techniques, such as OFDM, redundant (Rusek, 2013). Pre-equalisation would simplify user equipment and possibly make them cheaper and make them consume less power, which would increase their battery life.

2.8 System Model

A general multi-user MIMO system model is shown in Figure 2.5, and is described in the following subsections 2.8.1, 2.8.2 and 2.8.3. It is the model that we will use throughout the report and the variables defined here will frequently reappear.
2.8.1 Transmitter

We consider the general multi-user MIMO communication system in Figure 2.5, where the transmitter has $M$ antennae and each of the $K$ users has a single antenna. The symbol $s_k \in \mathbb{C}$, which is intended for user $k$, is taken from a constellation $\mathcal{S}_k$ with average energy 1:

$$\mathbb{E}[|s_k|^2] = 1.$$ 

The symbol vector $s = (s_1, \ldots, s_K)^T$ contains the symbols for all users. It is amplified by the energy matrix

$$E = \begin{pmatrix} \sqrt{E_1} & 0 \\ 0 & \sqrt{E_K} \end{pmatrix},$$

which effectively scales the constellation diagrams. The scaled symbols are given by $u = (u_1, \ldots, u_K)^T = Es$. The scaled symbol vector is fed to a precoder that is assumed to have perfect channel knowledge. The precoder outputs $x = (x_1, \ldots, x_M)^T = (A_1e^{j\theta_1}, \ldots, A_Me^{j\theta_M})^T$, $A_m \in \mathbb{R}^+$, $\theta_m \in [0, 2\pi]$, which has a total energy less than one.

$$\sum_{m=1}^{M} \mathbb{E}[A_m^2] \leq 1$$

The transmit signals $x_m$ are amplified by $\sqrt{P}$ and transmitted over the channel.
2.8.2 Channel

We consider a flat-fading broadcast channel, over which one base station serves a multitude of independent users, with channel matrix

$$\mathbf{H} = \begin{pmatrix} \mathbf{h}_1^\top \\ \vdots \\ \mathbf{h}_K^\top \end{pmatrix} \in \mathbb{C}^{K \times M},$$

where $M$ is the number of transmit antennas and $K$ the number of single-antenna users. The vector $\mathbf{h}_k$ describes the fading from the antenna array of the transmitter to user $k$. Each entry $h_{km}$ of $\mathbf{H}$ represents the fading coefficient between transmit antenna $m$ and user $k$. Only the small-scale fading is considered, therefore the channel matrix is normalised such that

$$\mathbb{E}[\text{Tr } \mathbf{HH}^\dagger] = MK.$$

In this study, we will assume that each entry of the channel matrix is i.i.d. Rayleigh fading $\mathcal{CN}(0, 1)$, which fulfils the normalisation criteria above. This would correspond to a base station placed outdoors below the rooftops or indoors in non-line-of-sight.

2.8.3 Receiver

If $\sqrt{P}\mathbf{x}$ is the transmitted vector, user $k$ will receive $r_k = \sqrt{P}\mathbf{h}_k^\top \mathbf{x}$ when noise is not considered. The vector $\mathbf{r} = (r_1, \ldots, r_K)^\top$ will denote the vector of received signals at all users. The noise-free input-output relation of the channel can then be written $\mathbf{r} = \sqrt{P}\mathbf{Hx}$.

We will also study the behaviour of this system when receiver noise is considered. If $\mathbf{w} = (w_1, \ldots, w_K)^\top$ is the noise vector, where $w_k$ is the noise term at receiver $k$, the input-output relation can be written $\mathbf{r} = \sqrt{P}\mathbf{Hx} + \mathbf{w}$. The systems will be evaluated in terms of the transmit power required to achieve a given ergodic sum rate. For this purpose, we define the normalised transmit power, where the transmit power is normalised with respect to the average noise power at the users.

Definition 2.1. Normalised transmit power is defined as

$$KP \frac{\mathbb{E}[||\mathbf{x}||^2]}{\mathbb{E}[||\mathbf{w}||^2]}.$$

2.9 Conventional Precoding

In this section, two linear precoding techniques for very-large MIMO—zero-forcing and maximum ratio transmission—are studied. The motivation for the study is to
see what peak-to-average ratio (PAR) the transmit signals of the two techniques have. The PAR will indirectly determine the cost and power efficiency of the power amplifier. We will look at the specific case, where the channel is modelled as an i.i.d. Rayleigh-fading channel, i.e. where the channel coefficients $h_{km}$ are i.i.d. CN(0, 1). Such a channel could be expected, given sufficient antenna spacing (greater than half the signal wavelength) and a sufficiently rich scattering environment (Rusek et al. 2013).

The parameter $E$ is not used in these conventional precoding schemes, in this chapter $E = I_K$, where $I_K$ is the $K \times K$-identity matrix. Hence, for now $s = u$.

### 2.9.1 Zero-Forcing Transmission

Zero-forcing, in multi-user MIMO contexts, is analogous to null-steering in a beamforming context. In null-steering, the radiated energy is nulled in certain directions. In zero-forcing, the multi-user interference is nulled at each user, who might be shadowed from the base station. This nulling involves solving a linear equation system of $K - 1$ equations. To achieve complete interference nulling, $K - 1$ has to be smaller than $M$. The remaining $M - K + 1$ degrees-of-freedom will be used to maximise the receive signal strength.

In very-large multi-user MIMO $M \gg K$ and interference nulling can be done and the cost in signal strength is usually not a problem. Only if two users are not sufficiently separable, for example if they are close to each other, zero forcing might result in inefficient use of power. This can be solved by scheduling such users in different time-frequency blocks.

The zero-forcing transmit vector is given by

$$x = \sum_{k=1}^{K} u_k g_k,$$

where $u_k$ is the symbol intended for user $k$ and $g_k$ is the precoding vector for user $k$. The precoding vectors are chosen such that $g_k \perp h_i, \forall i \neq k$, which will null the interference at all users, and such that the array gain \* $|h_k^T g_k|^2$ is maximised. This is achieved by choosing the precoding matrix to be the pseudo inverse of the channel matrix (Gesbert, 2007). If the channel matrix has full rank, which it has with probability one (Tulino, 2004), the transmit vector is given by:

$$x = \frac{1}{\sqrt{\text{Tr}((HH^H)^{-1})}} H^u.$$

Here $H^u = H^H (HH^H)^{-1}$.

\*Monte Carlo simulation analysis has shown that the array gain for a zero-forcing system with 100 transmit antennae and 10 single-antenna users is 9.55 dB.
2.9 Conventional Precoding

2.9.2 Maximum Ratio Transmission

In maximum ratio transmission, the receive power is maximised at each user by precoding in such a way that the signals from each transmit antenna element add up coherently at each user. This can be done locally at each antenna by weighting the transmit symbol with the conjugated channel coefficient: \( h^*_{lm}u_k \). This method does not consider what is sent to the other users and will therefore lead to interference between users—no effort is made to mitigate the resulting interference, which is treated as additional noise at the users.

The maximum ratio transmit vector is given by

\[
x = \frac{1}{\sqrt{\text{Tr}\mathbf{H}\mathbf{H}^H}} \mathbf{H}^H \mathbf{u}.
\]

As Rusek et al. (2013) note, this scheme becomes optimal in the limit of an infinite number of antennae at the transmitting base station, since the probability that the vectors \( \mathbf{h}_l \) and \( \mathbf{h}_k \) are close to orthogonal, when \( l \neq k \), goes to one as \( M \to \infty \). In other words, the multi-user interference term at any user \( l = 1, \ldots, K \) becomes arbitrarily small as the number of transmit antennae grows:

\[
e_l = \frac{1}{\sqrt{\text{Tr}\mathbf{H}\mathbf{H}^H}} \sum_{k=1}^{K} \sum_{k \neq l}^K \mathbf{h}_l^H \mathbf{h}_k u_k \xrightarrow{\text{prob.}} 0, \quad \text{as } M \to \infty.
\] (2.1)

Marzetta notes in his paper (2010) that, in the limit of infinite number of base station antennae, maximum ratio transmission will effectively suppress all fast fading and intra-cell interference, with the only limiting factor being a phenomenon that he calls **pilot contamination**, which leads to inter-cell interference.

This increasingly effective interference suppression can be seen from the increasing curve in Figure 2.6*, where a maximum ratio transmission scheme with 10 users and an increasing number of transmit antennae is shown. It can be observed that the SIR indeed grows without bound.

However, if the number of users grows too, along with the number of antennae, then the effect can disappear: The other, decreasing curve in the same figure shows the expected SIR when the number of users is a tenth of the number of transmitting antennae. We can see that instead of growing without limit, the curve converges to a fixed interference level. As a matter of fact, complete interference suppression can only be observed as long as the number of users grows slower than the number of transmitting antennae.

It should also be noted that, even though each term in the sum (Eq. 2.1) goes to zero as \( \mathcal{O}(M^{-1}) \), the sum also grows linearly with \( K \), which means that the variation in SIR does not depend so much on the number of transmitting antennae (or the channel hardening), but rather on the number of interfering users. Not

*Since perfect channel knowledge is assumed, the degenerated behaviour due to pilot contamination is not observed.
Figure 2.6: The signal-to-interference ratio when using maximum ratio transmission with different numbers of transmit antennae $M$ and different numbers of users $K$ in an i.i.d. Rayleigh fading scenario. The blue line shows the relationship between SIR and number of transmit antennae when there are 10 users; the red line, when there are 10 times more transmit antennae than users. The dashed lines show the 5% and 95% percentiles.

until the number of interferers becomes large does the law of large numbers lead to an aggregate maximum ratio transmission channel that is hard.

For extremely large antenna arrays, with a number of antenna elements that depends on the number of users, this simple precoding scheme becomes interesting because of the above property—good enough interference suppression can be done without the complex computations involved in zero forcing.

### 2.9.3 PAR of Conventional Precoding Schemes

To determine what PAR the transmit signals would have when the two precoding schemes zero-forcing and maximum ratio transmission are used, a simulation has been conducted. A system with $M = 100$ transmit antennae and $K = 10$ single antenna users that transmits i.i.d. random symbols taken from three different constellations: 4-QAM, 16-QAM and a Gaussian alphabet, all normalised so that the symbols at average had unit energy, has been simulated. In total, $10^7$ random symbol vectors were generated and precoded with respect to a new realisation of an i.i.d. Rayleigh-fading channel for each 1000’th symbol vector. Then the transmit signal at antenna 1 was recorded.

The PAR value for the discrete-time baseband signal, i.e. one sample per symbol duration, at antenna 1 was computed for the three constellations. The result is presented in Table 2.2. The distribution of the signal amplitudes, normalised with respect to the RMS of the signal, is illustrated in Figure 2.7.

The signals were also pulse shape filtered with a root-raised cosine with roll-
2.9 Conventional Precoding

Table 2.2: Signal PAR for different symbol alphabets

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4-QAM</td>
<td>9.727</td>
<td>10.11</td>
<td>4-QAM</td>
<td>9.646</td>
<td>10.04</td>
</tr>
<tr>
<td>16-QAM</td>
<td>10.08</td>
<td>10.40</td>
<td>16-QAM</td>
<td>10.00</td>
<td>10.32</td>
</tr>
<tr>
<td>Gauss</td>
<td>10.75</td>
<td>10.94</td>
<td>Gauss</td>
<td>10.76</td>
<td>10.95</td>
</tr>
</tbody>
</table>

(a) Zero Forcing  
(b) Maximum Ratio Transmission

off factor 0.3 and the continuous-time PAR was calculated*. The result is also presented in Table 2.2. Interestingly, the continuous-time PAR is not much different from the discrete-time PAR. At these high PAR†, pulse shape filtering does not seem to have a great impact on the PAR, because at the same time as the relative power decreases, the peaks get more scarce and the peak value of the signal decreases.

In Figure 2.8, two transmit signals, one from zero forcing and one from maximum ratio transmission, are shown in an i-q diagram to illustrate their signal properties.

---

*rather the PAR at 20 samples per symbol duration was calculated, which is a close approximation of the continuous-time PAR.

†High PAR cannot be the only explanation for the low impact of pulse shape filtering, c.f. Figure 7.1, where the pulse shape filtering has far more impact on the signal PAR than in this case. The reason for the low impact of pulse shape filtering in the case of conventional precoding is probably the nature of the signals; high signal amplitudes are scarce, whereas in Figure 7.1, all amplitudes within the circle have the same probability.
Figure 2.7: The complementary cumulative distribution function of the amplitude of the discrete-time signal at one transmit antenna when using conventional precoding for three different symbol alphabets, normalised by the rms of the signal. The PAR is given as the value, for which the complementary cumulative probability is $10^{-4}$. 
2.9 Conventional Precoding

(a) Zero Forcing

![Graph](image1)

PAR: 8.96 dB

(b) Maximum Ratio Transmission

![Graph](image2)

PAR: 7.37 dB

Figure 2.8: To the left: 250 discrete-time signal points, one sample per symbol duration, at an transmit antenna. To the right: The signal after pulse shape filtering with a root-raised-cosine filter with roll-off factor 0.3.
Chapter 3

Power Amplifiers

In this chapter, the basic properties of radio frequency power amplifiers are described. A general simplistic model of a power amplifier is presented and the three power amplifier types, A, B and AB, are studied in more detail in order to determine their power efficiency at different operating points. It is also explained why class AB power amplifiers are generally used in user equipment.

3.1 Modelling a Power Amplifier

One of the simplest power amplifier models is the polynomial input-output model, in which the response of the power amplifier is modelled as a polynomial. In such a model, usually, only three terms in the polynomial expansion are considered. If the input signal to the power amplifier has the following form

\[ x(t) = A(t) \cos(\omega_c t + \theta(t)), \]

where \( A(t) \) is the envelope of the signal and \( \theta(t) \) is the phase, then the output \( y \) of the power amplifier can be modelled as

\[ y(t) = \alpha_1 x(t - \tau_1) + \alpha_2 x(t - \tau_2)^2 + \alpha_3 x(t - \tau_3)^3, \]  

(3.1)

where \( \tau_i, i = 1, 2, 3 \), are different delays due to memory effects in the power amplifier. Inserting \( x \) in this model, yields the output

\[ y(t) = \frac{1}{2} \alpha_2 A(t - \tau_2)^2 \]

\[ + \alpha_1 A(t - \tau_1) \cos(\omega_c t + \theta(t - \tau_1) - \phi_1) + \frac{3}{4} \alpha_3 A(t - \tau_3)^3 \cos(\omega_c t + \theta(t - \tau_3) - \phi_3) \]

\[ + \frac{1}{2} \alpha_2 A(t - \tau_2)^2 \cos(2\omega_c t + 2\theta(t - \tau_2) - 2\phi_2) \]

\[ + \frac{1}{4} \alpha_3 A(t - \tau_3)^3 \cos(3\omega_c t + 3\theta(t - \tau_3) - 3\phi_3), \]
where $\varphi_i = \omega_i \tau_i$, $i = 1, 2, 3$. The second term in this sum is the desired power amplifier output. The other terms cause signal distortion. The first, fourth and fifth terms cause out-of-band radiation, since they have a frequency different from $\omega_c$. The third term causes inband distortion, because it has the same frequency as the desired output.

The inband distortion can be further divided into phase distortion and amplitude distortion. Usually, the delays $\tau_i$, $i = 1, 2, 3$, are small in comparison with the baud rate and can be neglected in the envelope and phase expressions. However, the term $\varphi_3$, which causes phase distortion, can usually not be neglected. The severity of the phase distortion depends on the amplitude, or rather the cube of the amplitude, therefore it is also called AM-AM distortion. Similarly, the cubed amplitude of the third term also causes amplitude distortion, which then is called AM-PM distortion.

Yet another kind of distortion—cross modulation—shows up if the input to the power amplifier contains a second tone. Study the input signal given by $x(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$.

The output of the power amplifier is then

$$y(t) = a_1 \left( A_1 \cos(\omega_1 t - \varphi_{11}) + A_2 \cos(\omega_2 t - \varphi_{21}) \right)$$

$$+ a_2 \left( \frac{1}{2} (A_1^2 + A_2^2) + A_1 A_2 \cos((\omega_1 - \omega_2)t - \varphi_{12} + \varphi_{22}) \right)$$

$$+ A_1 A_2 \cos((\omega_1 + \omega_2)t - \varphi_{12} - \varphi_{22})$$

$$+ \frac{1}{2} A_1^2 \cos(2\omega_1 t - 2\varphi_{12}) + \frac{1}{2} A_2^2 \cos(2\omega_2 t - 2\varphi_{22})$$

$$+ a_3 \left( \frac{3}{4} A_1^3 \cos(\omega_1 t - \varphi_{13}) + \frac{3}{4} A_2^3 \cos(\omega_2 t - \varphi_{23}) \right)$$

$$+ \frac{3}{2} A_1 A_2^2 \cos(\omega_1 t - \varphi_{13}) + \frac{3}{2} A_1^2 A_2 \cos(\omega_2 t - \varphi_{23})$$

$$+ \frac{3}{4} A_1^2 A_2 \cos(2\omega_1 - \omega_2)t - 2\varphi_{13} + \varphi_{23} + \frac{3}{4} A_1^2 A_2 \cos((2\omega_1 + \omega_2)t - 2\varphi_{13} - \varphi_{23})$$

$$+ \frac{3}{4} A_1^2 A_2 \cos(2\omega_2 - \omega_1)t + \varphi_{13} - 2\varphi_{23} + \frac{3}{4} A_1 A_2^2 \cos((2\omega_2 + \omega_1)t - \varphi_{13} + 2\varphi_{23})$$

$$+ \frac{1}{4} A_1^3 \cos(3\omega_1 t - 3\varphi_{13}) + \frac{1}{4} A_2^3 \cos(3\omega_2 t - 3\varphi_{23})$$

where $\varphi_{ij} = \omega_i \tau_j$.

If we filter away all frequency components except the one around $\omega_1$, we get

$$y_{BP}(t) = a_1 A_1 \cos(\omega_1 t - \varphi_{11}) + \frac{3}{4} a_3 A_1^3 \cos(\omega_1 t - \varphi_{13}) + \frac{3}{2} a_3 A_1 A_2^2 \cos(\omega_1 t - \varphi_{13})$$

The first term is the linear, desired, response, the second term is the gain compression or expansion term, which causes AM-AM and AM-PM distortion, and the last term is the cross modulation term—the distortion caused by other frequencies.
Without loss of generality, set $\varphi_{11} = 0$. The complex baseband representation of the error is then given by

$$e = \alpha_3 \left( \frac{3}{4} A_1^3 + \frac{3}{2} A_1 A_2^2 \right) e^{i \varphi_{13}}.$$

We note that the distortion only depends on the amplitude of the signal, it is therefore common to draw the characteristics of a power amplifier in terms of the signal amplitude or rather the instantaneous power of the signal envelope, which is the amplitude squared, see Figure 3.1.

Often $\alpha_3$ is small and negative, the amplitude distortion is therefore only observed when $A_1$ is large, then the gain compression will start to affect the output signal and make it saturate, i.e. the output power stops growing at a certain power level. Without memory, the terms $\tau_i$, $i = 1, 2, 3$, and $\varphi_{13}$ are zero and amplitude distortion is the only distortion we get. The phase is thus distorted as a result of non-zero $\tau_i$—delays due to the electrical components in the power amplifier.

**Definition 3.1.** When the amplitude is that big, that the amplitude distortion prevents any further amplification, the power amplifier is said to have saturated or passed its saturation point. The saturation point is defined as the input power, which would have resulted in the saturated power $P_{\text{sat}}$, if the amplifier had been perfectly linear. Another measure of saturation is the $3\,\text{dB}$ saturation point, defined as the input power, for which the output power is $3\,\text{dB}$ less than it would have been if the power amplifier had been perfectly linear.

The response of the amplifier will asymptotically approach the saturated power $P_{\text{sat}}$ after saturating. The simplistic polynomial model we have been studying will saturate in the sense that the response will deviate from the straight line seen in Figure 3.1. The polynomial (3.1), however, fails to model the response after saturation, where the amplitude first will decline and then grow without unbound, which is an unrealistic behaviour.

High frequency power amplifiers operate most efficiently close to their saturation point. However, there the distortion is too severe for any practical communication purpose, therefore the signal has to be backed off, i.e. the power amplifier has to be operated at a lower power than its saturation point.

The signal should first be backed off down to the linear part of the amplitude response. We call this back-off $b_{\text{NLR}}$, where NRL stands for Non-Linear Region. This back-off is taken as the width of the non-linear region.

The signal $\text{PAR}$ is a measure of how much higher the peaks are compared to the average power of the signal. It is important that the whole signal, up to its peaks, is linearly amplified, i.e. that the peaks do not enter the non-linear region. Therefore the operating point of the amplifier has to be chosen such that the peaks of the signal end up below the saturation point. An additional back-off equal to the signal $\text{PAR}$ is done for this purpose, we call this back-off $b_{\text{PAR}}$. 
Figure 3.1: The relation between input and output instantaneous envelope power and output power (top), input instantaneous envelope power and phase distortion (middle), and input power and efficiency of the power amplifier (bottom). We see that the additional back-off corresponds to the non-linear region of the AM-AM curve.
3.2 Class A Amplifiers

The most simple circuit diagram of a class A power amplifier is shown in Figure 3.2. The class A power amplifier is a simple construction with only one transistor. It has low power efficiency and the dissipated energy sometimes has to be led away by big cooling systems. The linearity of this class of power amplifiers is good.

An ideal, linear class A power amplifier consumes the same amount of power, \(2P_{\text{sat}}\), regardless of its delivered output power. This gives a maximum theoretical efficiency of \(\eta_{\text{sat}} = 50\%\) and also a rapid degradation in efficiency as back-off is increased. The efficiency of a class A power amplifier is given by (Walker, 2011):

\[
\eta = \eta_{\text{sat}} \frac{P_{\text{op}}}{P_{\text{sat}}} = \frac{\eta_{\text{sat}}}{b}.
\]

3.3 Class B Amplifiers

The most simple circuit diagram of a push-pull class B power amplifier is shown in Figure 3.3. It consists of two bipolar junction transistors of reversed types: one NPN type and one PNP type bipolar transistor, one of which conducts half of the
time and the other conducts the other half of the time. The two currents are added up at the output. This arrangement has good efficiency—the maximum theoretical efficiency is $\eta_{\text{sat}} = 78.5\%$. But it can suffer from a small mismatch in the cross-over region—that the two halves do not match perfectly—this is called cross-over distortion.

The efficiency of a class B power amplifier is approximately given by (Walker, 2011):

$$\eta = \eta_{\text{sat}} \sqrt{\frac{P_{\text{op}}}{P_{\text{sat}}}} = \eta_{\text{sat}} \sqrt{\frac{1}{b}}.$$  

### 3.4 Class AB Amplifiers

One way to make the two signal halves of the class B power amplifier to match better and to get rid of the cross-over distortion is to make the two amplifying elements of the class B power amplifier conduct a little even during the time when they are supposed to be off, this extra current is called quiescent current. This way, the two amplifying elements works somewhat like two class A power amplifiers and a push-pull class B power amplifier at the same time, thereby the name AB. This construction greatly decreases the problem with cross-over distortion but at the cost of lowered efficiency. The power efficiency is still better than that of a class A power amplifier and the linearity is better than that of class B. This is why class AB power amplifiers are used in user equipment.

The efficiency of a class AB depends on the quiescent current but can be approximated by:

$$\eta = \eta_{\text{sat}} \left( \frac{P_{\text{op}}}{P_{\text{sat}}} \right)^{\frac{1}{2}} = \eta_{\text{sat}} \left( \frac{1}{b} \right)^{\frac{1}{2}}.$$
where $\eta_{\text{sat}} = 55\%$.

### 3.5 Amplifier Linearisation

High amplitudes will not see the same gain as lower amplitudes in a non-linear power amplifier; this leads to distortion, as we have seen in previous sections. To linearise the power amplifier, i.e. to mitigate the signal distortion, the signal can be fed to a predistorter prior to amplification. The predistorter will amplify the high amplitudes in such a way that, when amplified by the non-linear power amplifier, they will get the same gain as low amplitudes and the whole system, predistorter and power amplifier, can be seen as a perfectly linear power amplifier.

The relation between the input and output amplitudes of the predistorter is illustrated in Figure 3.4. It can be seen that the whole system, predistorter and power amplifier together, has an almost perfectly linear response up to the saturation limit of the power amplifier, which is the highest amplitude that a predistorter can correct.

Distortions due to memory in the power amplifier, such as phase distortion, can be compensated for in a similar way.

Linearisation through predistorters is feasible in power amplifiers for high power signals, such as in the power amplifiers used in today’s base stations, because the power consumption of the predistorter compared to that of the power amplifier is small.

Because the response of the power amplifier changes with temperature and time, the predistorter has to be trained regularly. To train the predistorter a known signal shape is passed through the power amplifier and the output is measured and fed back to the predistorter. With this information, the predistorter can compute the an up-to-date predistortion curve. For this there has to exist a feedback-loop as well as computational circuits. The required feedback loop and computational circuits add a lot to the complexity of the radio front end, which might make adaptive predistorters infeasible in a very-large antenna array.

A power amplifier for low power signals, such as those found in handsets, cannot afford the linearisation of an advanced predistorter system, either from a cost- or from an efficiency point of view. Simpler linearisation, such as static analogue predistortion without memory compensation, might be used. However, a backed-off, sufficiently linear amplifier without linearisation might be the best option in a very-large antenna array.
Figure 3.4: The predistorter amplifies different envelope amplitudes $P_{in}$ such that the output of the predistorter $P_{out}$ will be amplified by the non-linear power amplifier to the desired power level $\alpha P_{in}$, where $\alpha$ is the gain of the power amplifier.
Chapter 4

Discrete-Time Constant-Envelope Precoding

This chapter describes the discrete-time constant-envelope precoding scheme that was proposed by Mohammed et al. (2013a), which is a way of doing PAR reduction and precoding jointly.

4.1 Precoding Algorithm

The idea of Mohammed et al. is to only consider transmit signals $x$, where each signal $x_m = Ae^{j\theta_m}$ has constant envelope $A = M^{-1/2}$ when viewed in its discrete-time baseband representation. The precoder modulates the phases of the discrete-time constant-envelope transmit signals, in a way that minimises the multi-user interference at all users.

$$x_{CE} = \arg\min_x \|Hx - u\|, \text{ subject to } |x_m| = \frac{1}{\sqrt{M}} \forall m.$$ 

This minimisation problem is non-convex and hard to solve explicitly. The minimisation method that Mohammed et al. describe is an iterative search method, which first initialises all phases to zero and then updates one phase at a time. All phases, except one, are fixed and the multi-user interference is minimised with respect to this non-fixed phase; this is called one subiteration. One by one, each phase $\theta_1$ to $\theta_M$ is updated, and upon completing all $M$ subiterations, one iteration is finished.

This method converges fast and is computationally simple, since the minimization problem of a subiteration has an explicit solution. Consider the multi-user interference power $\|Hx - u\|^2$, where all phases, except $\theta_n$, are fixed. The optimal
value of $\theta_n$ is then given by:

$$
\theta^*_n = \arg \min_{\theta_n} \| \mathbf{Hx} - \mathbf{u} \|^2 = \arg \min_{\theta_n} \left| \sum_{k=1}^{K} (u_k - h_{kn}Ae^{j\theta_n} - \sum_{m=1 \atop m \neq n}^{M} h_{km}Ae^{j\theta_m}) \right|^2 
$$  

(4.1)

$$
= \arg \min_{\theta_n} \sum_{k=1}^{K} (u_k - h_{kn}Ae^{j\theta_n} - \sum_{m=1 \atop m \neq n}^{M} h_{km}Ae^{j\theta_m})(u^*_k - h^*_{kn}Ae^{-j\theta_n} - \sum_{m=1 \atop m \neq n}^{M} h^*_{km}Ae^{-j\theta_m}) 
$$  

(4.2)

$$
= \arg \min_{\theta_n} \sum_{k=1}^{K} \left( h_{kn}Ae^{j\theta_n} \sum_{m=1 \atop m \neq n}^{M} h^*_{km}Ae^{-j\theta_m} + h^*_{kn}Ae^{-j\theta_n} \sum_{m=1 \atop m \neq n}^{M} h_{km}Ae^{j\theta_m} - u_k h^*_{kn}Ae^{-j\theta_n} - u^*_k h_{kn}Ae^{j\theta_n} \right) 
$$  

(4.3)

$$
= \arg \min_{\theta_n} \sum_{k=1}^{K} 2 \Re \left( h_{kn}Ae^{j\theta_n} \sum_{m=1 \atop m \neq n}^{M} h^*_{km}Ae^{-j\theta_m} - u^*_k h_{kn}Ae^{j\theta_n} \right) 
$$  

(4.4)

$$
= \arg \min_{\theta_n} \Re \left( e^{j\theta_n} \sum_{k=1}^{K} h_{kn} \left( \sum_{m=1 \atop m \neq n}^{M} h^*_{km}Ae^{-j\theta_m} - u^*_k \right) \right) 
$$  

(4.5)

$$
= \arg \left( \sum_{k=1}^{K} h^*_n (u_k - \sum_{m=1 \atop m \neq n}^{M} h_{km}Ae^{j\theta_m}) \right) 
$$  

(4.6)

In (4.2), the absolute value is factorised as the product of two conjugates. In (4.3), the product is expanded and constant terms that do not depend on $\theta_n$ are discarded. We note that the remaining terms pair up in conjugates and can be written as the real part of a complex number. In (4.4), the finite outer summation is moved inside the real-part operator and the term $e^{j\theta_n}$, which does not depend on any of the summation variables, is factored out. Finally, we see that the optimal $\theta_n$ is the phase that makes the complex number inside the real-part operator purely real and negative, i.e. the phase has to be chosen such that the argument of that complex number is $\pi$.

If we give the complex number in (4.6) a name,

$$
\zeta_n = \sum_{k=1}^{K} h^*_n (u_k - \sum_{m=1 \atop m \neq n}^{M} h_{km}Ae^{j\theta_m}), 
$$  

(4.7)

we can use the following pseudo code to describe the algorithm of Mohammed et al.
The symbol constellation diagrams, from which the input symbols are taken, have to be chosen such that they lie within the region of possible receive signals, i.e. the set of signals that can be received when the transmit signals are constrained in amplitude, or acceptably close to the region. It is therefore of interest to study the properties of this set of signals.

First we look at the discrete-time constant-envelope single-user case, for which we will establish a full characterisation of the set. Most of the work in characterising the set was done by Mohammed et al. (2012). The lower bound on the set, which is given in Lemma 4.5, remained to be established—it was done by Pan et al. (2013). I independently derived the same lower bound, unaware of their work, in a slightly different way.

Then we discuss the implications of adding more than one user, and give a rough description of what symbol vectors that can be expected to lie inside the higher dimensional region of possible receive signals.

We start by defining the region of possible receive signals, at this point $\epsilon = 0$.

**Definition 4.1.** Given a channel realisation $H \in \mathbb{C}^{K \times M}$ and two non-negative real numbers $\epsilon \geq 0$ and $A \geq 0$, the region of possible receive signals $\mathcal{M}_\epsilon(H)$ is defined as:

$$\mathcal{M}_\epsilon(H) = \{ r \in \mathbb{C}^K : r = H \begin{pmatrix} A_1 e^{i \theta_1} \\ \vdots \\ A_M e^{i \theta_M} \end{pmatrix}, \theta_m \in [0, 2\pi], A_m \in [A - \epsilon, A], \sum_{m=1}^M A_m^2 \leq 1 \}.$$ 

As a special case, given a channel realisation $h = (h_1, \ldots, h_M)^T \in \mathbb{C}^M$ and two non-negative real numbers $\epsilon \geq 0$ and $A \geq 0$, the region of possible receive signals $\mathcal{M}_\epsilon(h)$

----

*The distance to the set is the multi-user interference that the user will see, this multi-user interference has to be acceptably small.*
is defined as:
\[ \mathcal{M}_\varepsilon(h) = \{ r \in \mathbb{C} : r = \sum_{m=1}^{M} A_m e^{i \theta_m} h_m, \theta_m \in [0, 2\pi], A_m \in [A - \varepsilon, A], \sum_{m=1}^{M} A_m^2 \leq 1 \}. \]

The set \( \mathcal{M}_\varepsilon(H) \) contains all receive vectors that are possible to produce at the users when the amplitude is constrained to lie between \( A - \varepsilon \leq |x_m| \leq A \) at the transmitter. Note that \( \mathcal{M}_\varepsilon(H) \subseteq \mathcal{M}_\varepsilon(h_1) \times \cdots \times \mathcal{M}_\varepsilon(h_K) \) but that the reversed inclusion only holds true when \( K = 1 \).

We note two general properties of the region of possible receive vectors, summarised in the next two lemmata. Mohammed (2012) states and proves Lemma 4.2 and 4.3 for the specific case, where there is one user and \( \varepsilon = 0 \).

**Lemma 4.2.** Given a complex matrix \( H \in \mathbb{C}^{K \times M} \), a complex vector \( r \in \mathcal{M}_\varepsilon(H) \) and any angle \( \varphi \in [0, 2\pi] \), the uniform phase shift \( re^{i\varphi} \) also belongs to \( \mathcal{M}_\varepsilon(H) \).

**Proof.** Assume \( r \in \mathcal{M}_\varepsilon(H) \), then there exists \( M \) phases \( (\theta_1, \ldots, \theta_M) \in [0, 2\pi]^M \) and \( M \) amplitudes \( A_1, \ldots, A_M \in [A - \varepsilon, A] \) for which \( \sum_{m=1}^{M} A_m^2 \leq 1 \), such that \( H(A_1 e^{i \theta_1}, \ldots, A_M e^{i \theta_M})^T = r \). Consider an angle \( \varphi \in [0, 2\pi] \) and the complex vector \( re^{i\varphi} \).

\[
re^{i\varphi} = e^{i\varphi}H \begin{pmatrix} A_1 e^{i \theta_1} \\ \vdots \\ A_M e^{i \theta_M} \end{pmatrix} = H \begin{pmatrix} A_1 e^{i (\theta_1 + \varphi)} \\ \vdots \\ A_M e^{i (\theta_M + \varphi)} \end{pmatrix} = H \begin{pmatrix} \varphi \theta_1 \\ \vdots \\ \varphi \theta_M \end{pmatrix} \in \mathcal{M}_\varepsilon(H),
\]

where \( \varphi \theta_m = \theta_m + \varphi \mod 2\pi, m = 1, \ldots, M \). Thus, any uniform phase shift \( re^{i\varphi} \) of a complex vector \( r \in \mathcal{M}_\varepsilon(H) \) also belongs to the set \( \mathcal{M}_\varepsilon(H) \). \( \square \)

**Lemma 4.3.** Given a complex matrix \( H \in \mathbb{C}^{K \times M} \), the set \( \mathcal{M}_\varepsilon(H) \) is path-connected, i.e. any two points in \( \mathcal{M}_\varepsilon(H) \) can be connected by a path entirely inside the set.

**Proof.** Consider two points \( r_1, r_2 \in \mathcal{M}_\varepsilon(H) \). Since they belong to \( \mathcal{M}_\varepsilon(H) \), there exist phases \( \theta_1, \ldots, \theta_M, \omega_1, \ldots, \omega_M \in [0, 2\pi] \) and amplitudes \( A_1, \ldots, A_M, B_1, \ldots, B_M \in [A - \varepsilon, A] \) such that \( H(A_1 e^{i \theta_1}, \ldots, A_M e^{i \theta_M})^T = r_1 \) and \( H(B_1 e^{i \omega_1}, \ldots, B_M e^{i \omega_M})^T = r_2 \). Since both the sets \([0, 2\pi]\) and \([A - \varepsilon, A]\) are convex, any convex combination of phases and amplitudes in these sets will still be in the sets. Because of this and because the power of the convex combination still is less than 1\(^*\), the map

\[
f : t \mapsto H \begin{pmatrix} (1-t)A_1 + tB_1 e^{i((1-t)\theta_1 + t\omega_1)} \\ \vdots \\ (1-t)A_M + tB_M e^{i((1-t)\theta_M + t\omega_M)} \end{pmatrix}
\]

will always take values in \( \mathcal{M}_\varepsilon(H) \) as the domain is limited to \([0, 1]\). Since \( f : [0, 1] \to \mathcal{M}_\varepsilon(H) \) is continuous, it is a path inside \( \mathcal{M}_\varepsilon(H) \) that starts at \( f(0) = r_1 \) and ends at \( f(1) = r_2 \). \( \square \)

\*Due to the Schwartz inequality \( \sum_{m=1}^{M} A_m B_m \leq 1 \) and we get \( \sum_{m=1}^{M} ((1-t)A_m + tB_m)^2 = (1 - t)^2 \sum A_m^2 + t^2 \sum B_m^2 + 2t(1-t) \sum A_m B_m \leq (1-t)^2 + t^2 + 2t(1-t) = 1 \)
4.2 The Region of Possible Receive Signals

In this chapter, we are studying the case where the envelope of the signal at each antenna is constant, i.e., where $\varepsilon = 0$. In this setting, the phases of the transmit signal fully determine the emitted wave field. To easily describe the received signal at each user, we use the following definition.

**Definition 4.4.** Given a complex matrix $H \in \mathbb{C}^{K \times M}$ with entry $h_{km}$ on position $(k,m)$, for any $M$-tuple $\theta = (\theta_1, \ldots, \theta_M)$, the complex vector $r(\theta)$ is defined:

$$r(\theta) = \frac{1}{\sqrt{M}} \left( \sum_{m=1}^{M} e^{j\theta_m} h_{1m} \right)$$

As a special case, given a complex vector $h \in \mathbb{C}^M$, for any $M$-tuple $\theta = (\theta_1, \ldots, \theta_M)$, the complex number $r(\theta)$ is defined:

$$r(\theta) = \sum_{m=1}^{M} \frac{1}{\sqrt{M}} e^{j\theta_m} h_m.$$  

When the $M$-tuple $\theta = (\theta_1, \ldots, \theta_M)$ of phases is used to modulate the transmit signal, the received signal is given by $r = r(\theta)$. Using this new notation, the region of possible receive vectors in the constant-envelope case can easily be expressed as:

$$\mathcal{M}_0(H) = \{ r \in \mathbb{C}^K : r = r(\theta), \theta \in [0, 2\pi[^M \}$$

or in the special case of only one user

$$\mathcal{M}_0(h) = \{ r \in \mathbb{C} : r = r(\theta), \theta \in [0, 2\pi[^M \}.$$  

4.2.1 The Single-User Region

In the single-user case, the region of possible receive signals is easily characterised. The following two lemmata, which give an upper and a lower bound on the modulus of any complex number in the set, will allow us to fully characterise the set. This characterisation is summarised in Theorem 4.7. Lemma 4.5 was derived independently in a slightly different way by Pan (2013) and Lemma 4.6 was derived by Mohammed (2012).

**Lemma 4.5.** Given a complex vector $h \in \mathbb{C}^M$, $r_{\text{min}} = M^{-\frac{1}{2}}(2\|h\|_\infty - \|h\|_1)^+ \in \mathcal{M}_0(h)$ and all $r \in \mathcal{M}_0(h)$ have moduli $|r| \geq r_{\text{min}}$. Here $\|h\|_\infty$ is the infinity-norm of $h$, $\|h\|_1$ the one-norm and $(a)^+ = \max(a, 0)$.

**Proof.** If the dimension $M = 1$ or 2, the proof is trivial. Assume $M \geq 3$ and, without loss of generality, consider the reordered vector $h = (h_1, \ldots, h_M)^T$, where $|h_1| = \|h\|_\infty$. Note that $\sum_{m=2}^{M} |h_m| = \|h\|_1 - \|h\|_\infty$. There are two cases to study, in order to establish the lower bound on the modulus:
1. If \(|h_1| \geq \sum_{m=2}^{M} |h_m|\), we get the smallest modulus of any \(r \in \mathcal{M}_0(h)\) by aligning all the other channel coefficients in the opposite direction of \(h_1\), its modulus is therefore lower bounded by

\[
|r| \geq r_{\min} = r(\theta_{\min}) = \frac{1}{\sqrt{M}} (|h_1| - \sum_{m=2}^{M} |h_m|),
\]

where \(\theta_{\min} = (- \arg(h_1), \pi - \arg(h_2), \ldots, \pi - \arg(h_M)) \mod 2\pi\), which is equal to the non-negative number \(M^{-\frac{1}{2}}(2\|h\|_{\infty} - \|h\|_1)\).

2. If \(|h_1| < \sum_{m=2}^{M} |h_m|\) (\(*\)), consider the three positive real numbers

\[
a_1 = \sum_{m=1}^{N-1} |h_m|, \quad a_2 = |h_N|, \quad a_3 = \sum_{m=N+1}^{M} |h_m|,
\]

where \(N\) is the greatest integer such that \(\sum_{m=1}^{N-1} |h_m| < \sum_{m=N}^{M} |h_m|\). Note that \(N\) will be greater than 2, because of (\(*\)), and strictly smaller than \(M\), because \(h\) is reordered. Due to the definition of the three numbers: \(a_1 \leq a_2 + a_3\) and \(a_1 + a_2 \geq a_3\) and due to (\(*\)) and the reordering of \(h\): \(a_1 + a_3 \geq a_2\). Hence \(a_1, a_2, a_3\) can be seen as the sides of a triangle, see Figure 4.1. If we picture the sides of that triangle as the complex numbers

\[
z_1 = a_1, \quad z_2 = a_2 e^{j\alpha}, \quad z_3 = a_3 e^{j\beta},
\]

where

\[
\alpha = \arccos \frac{a_2^2 - a_1^2 - a_3^2}{2a_1a_3}, \quad \beta = \pi + \arccos \frac{a_1^2 - a_2^2 + a_3^2}{2a_1a_3},
\]

the sum of those numbers would be 0. Thus, if we choose angles in the following manner:

\[
\theta_{\min} = (- \arg(h_1), \ldots, - \arg(h_{N-1}), a - \arg(h_N), \beta - \arg(h_{N+1}), \ldots, \beta - \arg(h_M)) \mod 2\pi,
\]

we get a complex number \(r(\theta_{\min})\) inside \(\mathcal{M}_0(h)\) equal to 0, thus \(r_{\min} = 0 \in \mathcal{M}_0(h)\).

**Lemma 4.6.** Given a complex vector \(h \in \mathbb{C}^M\), \(r_{\max} = M^{-\frac{1}{2}}\|h\|_1 \in \mathcal{M}_0(h)\) and all \(r \in \mathcal{M}_0(h)\) have moduli \(|r| \leq r_{\max}\).

**Proof.** The triangle inequality gives us the following inequality for any \(r(\theta) \in \mathcal{M}_0(h)\):

\[
|r(\theta)| \leq \frac{1}{\sqrt{M}} \sum_{m=1}^{M} |h_m| = \frac{1}{\sqrt{M}} \|h\|_1.
\]
with equality if $\theta = \theta_{\text{max}} = (- \arg(h_1), \ldots, - \arg(h_M)) \mod 2\pi$. Thus $r_{\text{max}} = r(\theta_{\text{max}}) \in \mathcal{M}_0(h)$ and no element in $\mathcal{M}_0(h)$ has modulus greater than $r_{\text{max}}$. \hfill \Box

**Theorem 4.7.** Given a complex vector $h$ of dimension $M$, the set $\mathcal{M}_0(h)$ is an annulus in the complex plane characterised by:

$$\mathcal{M}_0(h) = \{ r : \frac{1}{\sqrt{M}} (2\|h\|_{\infty} - \|h\|_1)^+ \leq |r| \leq \frac{1}{\sqrt{M}} \|h\|_1 \},$$

where $\|h\|_{\infty}$ is the infinity-norm of $h$, $\|h\|_1$ the one-norm and $(x)^+ = \max(x, 0)$.

**Proof.** According to Lemma 4.5, 4.6 and 4.3, for any modulus in the interval

$$[M^{-\frac{1}{2}}(2\|h\|_{\infty} - \|h\|_1), M^{-\frac{1}{2}}\|h\|_1],$$

the set $\mathcal{M}_0(h)$ contains at least one complex number with this modulus and no complex number with modulus outside this interval. Lemma 4.2 then concludes the proof by telling us that any rotation $re^{i\varphi}, \varphi \in [0, 2\pi]$, of $r \in \mathcal{M}_0(h)$ also belongs to $\mathcal{M}_0(h)$. \hfill \Box

### 4.2.2 The Multi-User Region

When there are multiple users, the set $\mathcal{M}(H)$ is harder to characterise. It is bounded by $\mathcal{M}_k(h_1) \times \cdots \times \mathcal{M}_k(h_K)$ but high energy points close to the boundary of this product set do not necessarily belong to $\mathcal{M}_k(H)$ when $K$ is big. A symbol $s_k$ close to the upper bound of $\mathcal{M}_k(h_k)$ will consume most of the degrees of freedom available in our very-large MIMO system in order to reproduce $s_k$ at user $k$, so not many degrees of freedom are left for the other users, to which only a small choice of symbols then can be transmitted.

However, for large $M$ of any interesting size, modest-energy points, not close to the upper bound, lie within the region of possible receive signals with high probability. In the words of Mohammed et al. (2013a): for a fixed number of users and a fixed total transmit power constraint, any fixed symbol point can be reproduced at a user with arbitrarily small multi-user interference for a large enough number of transmit antennae.
4.3 Exact Phase Recovery in the Single-User Case

When the transmitter only serves a single user, the optimal discrete-time constant-envelope precoding phases can be computed explicitly. One method to recover the transmit phases is given by Pan et al. (2013). I have developed another method, unaware of their findings. In complexity, the methods are similar.

To alleviate notation, in this section we assume that the transmit signal have moduli $A = 1$, this is no restriction on generality, since everything can be scaled by $\frac{1}{\sqrt{M}}$.

Our idea is to line the channel coefficients up into three concatenated line segments in the complex plane. These three line segments are then either folded back or folded forward to form a triangle, where the last line segment ends up right on the symbol, which we intend to transmit to the user, see Figure 4.2.

**Definition 4.8.** For any symbol $u \in \mathbb{C}$, we say that we fold at coefficient $N$, if we choose the phases of the transmit signal in this fashion:

$$\theta_{\alpha,\beta} = (\text{arg}(\frac{u}{h_1}), \ldots, \text{arg}(\frac{u}{h_{N-1}}), \alpha + \text{arg}(\frac{u}{h_N}),$$

$$\beta + \text{arg}(\frac{u}{h_{N+1}}), \ldots, \beta + \text{arg}(\frac{u}{h_M})) \mod 2\pi,$$

for some angles $\alpha$ and $\beta \in [0, \pi]$ such that $\text{arg}(r(\theta_{\alpha,\beta})) = \text{arg}(u)$.

Thus, we fold at $N$, if we choose the phase of the $N-1$ first antennae so that all their channel coefficients add up coherently with the same phase as the transmitted symbol, we choose the phase of the $N$-th antenna such that its channel coefficient adds up with a slightly bigger phase than the symbol and we choose the phase of the remaining antennae such that the receive signal gets the same phase as the symbol.

The transmit signal from the first $N-1$ antennae then forms the first line segment, the transmit signal from the $N$-th antenna the second segment and the transmit signal from the rest of the antennae forms the third segment. The receive
signal, which is a complex addition of these transmit signal, can be illustrated in the complex plane, see Figure 4.2.

We note that we can fold forward of fold backwards at \( N \) and that either way the transmit signals form a triangle in the complex plane. Two of the sides in this triangle will have the lengths:

\[
a_1(N) = |h_N|, \quad a_2(N) = \sum_{m=N+1}^{M} |h_m|.
\]

If it is possible to fold in such a way that the receive signal equals the symbol \( u \), then the third length

\[
a_3(N) = \begin{cases} 
|u| - \sum_{m=1}^{N-1} |h_m|, & \text{if } N > 1 \\
|u|, & \text{if } N = 1
\end{cases}
\]

will form a triangle together with the two first lengths. This is the foundation of the test, which we will use when we look for an index, at which we can fold.

**Lemma 4.9.** For any symbol \( u \in \mathbb{C} \), it will be possible to fold at the index \( N \) so that \( r(\theta_{\alpha,\beta}) = u \), if

\[
\begin{align*}
a_1(N) &\leq a_2(N) + a_3(N) \\
a_2(N) &\leq a_1(N) + a_3(N) \\
a_3(N) &\leq a_1(N) + a_2(N)
\end{align*}
\]

the angles \( \alpha \) and \( \beta \) are then given by:

\[
\begin{align*}
&\text{if } \sum_{m=1}^{N-1} |h_m| \leq |u|: \\
&\begin{cases} 
\alpha = \arccos \frac{a_1^2 + a_2^2 - a_3^2}{2a_1a_2}, \\
\beta = \alpha + \arccos \frac{a_1^2 + a_2^2 - a_3^2}{2a_1a_2}
\end{cases} \\
&\text{if } \sum_{m=1}^{N-1} |h_m| > |u|: \\
&\begin{cases} 
\alpha = \arccos \frac{a_1^2 - a_2^2 - a_3^2}{2a_1a_2}, \\
\beta = \alpha + \arccos \frac{a_1^2 - a_2^2 - a_3^2}{2a_1a_2}
\end{cases}
\end{align*}
\]

where \( a_1 = a_1(N), a_2 = a_2(N), a_3 = a_3(N) \).

**Proof.** If the three numbers \( a_1, a_2, a_3 \) satisfy the three inequalities, there exists a triangle with side lengths equal to these three numbers. If we illustrate the summation of the complex numbers \( r(\theta_{\alpha,\beta}) = \sum_{m=1}^{M} h_m e^{j\theta_m} \) as a vector summation in the plane, we get Figure 4.2. The right illustration shows the case \( \sum_{m=1}^{N-1} |h_m| \leq |u| \), i.e. where we fold forward to reach \( u \), the left the case \( \sum_{m=1}^{N-1} |h_m| > |u| \), where we fold back.

An angle in any triangle is given by \( \theta_1 = \arccos \frac{l_1^2 + l_2^2 - l_3^2}{2l_1l_2} \), where \( l_1 \) is the length of the side opposite to the angle \( \theta_1 \) and \( l_2, l_3 \) are the lengths of the sides adjacent to the angle. Using this formula, the angles \( \alpha \) and \( \beta \) can easily be derived in the two cases. The resulting angles are those given in the theorem.

By using the phases \( \theta_{\alpha,\beta} \) and the triangle argument illustrated in Figure 4.2, we see that the summation \( r(\theta_{\alpha,\beta}) = \sum_{m=1}^{M} h_m e^{j\theta_m} \) equals \( u \). \( \square \)
To find the channel coefficient, at which the folding will be done, we will start by trying the first coefficient \( N = 1 \). If the inequalities of Lemma 4.9 do not hold, we increment the index by one and try \( N = 2 \). We continue to increment until we find a coefficient, at which the folding is successful, i.e. where three inequalities of Lemma 4.9 hold.

If we reorder the channel coefficients such that \( |h_1| = \|h\|_\infty \), we will always reach an index \( N \), where we can do the folding and successfully transmit the given symbol, as we will see from the following lemma. The reordering does not in any way restrict the generality of the method, since addition of complex numbers is associative.

**Lemma 4.10.** Given a channel realisation \( h \in \mathbb{C}^M \), where \( |h_1| = \|h\|_\infty \), and a symbol \( u \in \mathbb{C} \cap \mathcal{M}(h) \), there exist an index \( N \in [1, M - 1] \cap \mathbb{Z} \), at which folding will be successful, i.e. there exists an index at which \( r(\theta_{a,b}) = u \).

**Proof.** Without loss of generality, we will study a real symbol \( u \in \mathbb{R}^+ \cap \mathcal{M}(h) \). The proof immediately applies also to complex symbols by just adding \( \arg(u) \) to the phases that reached its real counterpart \( |u| \).

Consider the intervals \( \mathcal{I}_n^{(-)} \) and \( \mathcal{I}_n^{(+)}) \), which are defined as follows and illustrated in Figure 4.3.

\[
\mathcal{I}_n^{(-)} = \left[ \sum_{m=1}^{n-1} |h_m| - \sum_{m=n}^{M} |h_m|, \sum_{m=1}^{n-1} |h_m| - \sum_{m=n+1}^{M} |h_m| - |h_n| \right]
\]

\[
\mathcal{I}_n^{(+)}) = \left[ \sum_{m=1}^{n-1} |h_m| + \sum_{m=n+1}^{M} |h_m| - |h_n|, \sum_{m=1}^{M} |h_m| \right]
\]

The interval \( \mathcal{I}_n^{(-)} \) contains all points on the real axis that can be reached by folding back at position \( n \), i.e. choosing the phases in the sum \( \sum_{m=1}^{M} h_m e^{j\theta_m} \) such that the first \( m = 1, \ldots, n - 1 \) terms form a trail along the real axis, the \( n \)-th term points up into the first quadrant and the last \( m = n + 1, \ldots, M \) terms form a straight line back to a point on the real axis before the trail of complex numbers first left the real axis. The interval \( \mathcal{I}_n^{(+)}) \) contains all points on the real axis that can be reached by folding forward at position \( n \), i.e. choosing the phases in the sum \( \sum_{m=1}^{M} h_m e^{j\theta_m} \) such that the first \( m = 1, \ldots, n - 1 \) terms form a trail along the real axis, the \( n \)-th term points up into the first quadrant and the last \( m = n + 1, \ldots, M \) terms form a straight line back to a point on the real axis after the trail of complex numbers first left the real axis.

If \( u \in \mathcal{I}_N^{(-)} \cup \mathcal{I}_N^{(+)}, \) for some \( N \in \{1, \ldots, M - 1\} \), then folding is successful at \( N \) and \( r(\theta_{a,b}) = u \). Since \( u \in \mathcal{M}(h) \) implies that \( |u| \in [2\|h\|_\infty - \|h\|_1, \|h\|_1] \), folding is successful at some index, if the whole interval \( [2\|h\|_\infty - \|h\|_1, \|h\|_1] \subseteq \bigcup_{n=1}^{M-1} (\mathcal{I}_n^{(-)} \cup \mathcal{I}_n^{(+)}) \).

Denote the smallest \( n \) such that \( |h_n| > \sum_{m=n+1}^{M} |h_m| \) with \( \hat{n} \). If no such \( \hat{n} \) exists, \( \hat{n} = M - 1 \).
4.4 Choosing Symbol Energies

The parameter $E$ can be seen as an energy scaling factor that determines the array gain and thus the receive power. It scales the unit energy symbols $s$ to suitable size—big, but not too far outside the region of possible receive signals. If the diagonal elements of $E$ are chosen too small, the receive power at the users will be too poor to sustain a good throughput. If it is chosen too big, the symbols will lie outside the region of possible receive signals and the high multi-user interference at the users will significantly decrease the throughput. The preferable choice of $E$ would be the matrix that maximises the throughput.
Figure 4.4: The signal-to-interference ratio in a system with 10 users and 100 transmit antennae decreases with larger $\mathbf{E} = \text{diag}(E, \ldots, E)$ (the diagonal elements chosen to be identical). The big drop at $E = 4$ must mark the border, between where it is possible to reproduce the symbols exactly or not. For $E < 4$ the symbols can be reproduced exactly at the users, without any multi-user interference; values above 100 dB must be treated as numerical inaccuracies.
How the multi-user interference varies as a function of the choice of the parameter $E$ can be seen in Figure 4.4, where the signal-to-interference ratio at the user is shown.

### 4.4.1 Optimal Symbol Energies

The receive signal at user $k$ can be expressed as:

$$r_k = s_k \sqrt{P} \sqrt{E_k} + e_k \sqrt{P} + w_k, \quad \forall k = 1, \ldots, K,$$

where $w_k \sim \mathcal{CN}(0, \sigma^2)$ is the noise term of the receive signal at user $k$ and

$$e_k = \sum_{m=1}^{M} h_{km} A_m e^{i \theta_m} - s_k \sqrt{E_k}, \quad \forall k = 1, \ldots, K,$$

is the interference term.

The optimal choice of energies $E_1, \ldots, E_K$ would be the choice that maximises the sum-rate, i.e.

$$(E_1^*, \ldots, E_K^*) = \arg \max_{(E_1, \ldots, E_K)} C_{\text{inst}}(E_1, \ldots, E_K),$$

where $C_{\text{inst}} = \max_P \sum_{k=1}^{K} I(s; r_k)$ is the maximum achievable instantaneous sum-rate for a given channel realisation—the largest sum of mutual information between the transmit symbols $s$ and the receive signals $r_k$—where $P$ is the probability density function of the random vector $s$.

### 4.4.2 Near-Optimal Symbol Energies

Since we lack an explicit expression for the mutual information, it is not possible to compute the optimal parameter $E$. A possibility is to use the lower bound on the maximum achievable instantaneous sum-rate that was derived by Mohammed et al. (2013a). It was derived for a specific channel realisation and by using a Gaussian input alphabet. The derivation is reproduced in Appendix A. An achievable sum-rate is given by:

$$R_{\text{inst}}(E_1, \ldots, E_K) = \sum_{k=1}^{K} \log_2 \left( \frac{PE_k}{P E[|e_k|^2] + \sigma^2} \right) \text{ [bits]}, \quad (4.8)$$

which can be determined by Monte Carlo simulations, where a Gaussian input alphabet is used and the mean multi-user interference is determined. This will give us a lower bound on the maximum achievable sum-rate

$$C_{\text{inst}}(E_1, \ldots, E_K) \geq R_{\text{inst}}(E_1, \ldots, E_K).$$
Based on this lower bound, the energies can be chosen in this way:

\[(E_1, \ldots, E_K) = \arg \max_{(E_1, \ldots, E_K)} R_{\text{inst}}(E_1, \ldots, E_K).\]

This optimisation problem is possible to solve numerically, but its practicality is limited due to two reasons: 1. It has to be computed anew for every new channel realisation. 2. It is a \(K\)-dimensional optimisation problem.

**Static Choice**

By arguing that the channel of a very-large MIMO system has hardened and therefore the optimal choices of \(E_1, \ldots, E_K\) should be roughly the same both for different channel realisations and among users \(E_k \approx E_l, \forall k, l\), the optimisation problem can be simplified to:

\[E_1 = \cdots = E_K = \arg \max_E E_H R_{\text{inst}}(E, \ldots, E)\]  
(Choice 1)

This was the choice of the parameter \(E\) that Mohammed et al. (2013a) used in their article. To set all symbol energies equal, i.e. \(E_k = E_l, \forall k, l = 1, \ldots, K\), is suboptimal in most cases but usually quite close to optimal when \(h_k \sim h_l\). To support the claim that maximising over the average gives nearly the same result as maximising over each and every channel realisation, which in practice would be impractical, a simulation has been run, see Figure 4.5. It can be seen from the figure that optimising anew for each channel realisation does not increase the sum rate noticeably.

**Adaptable Choice**

As a parenthesis, I would like to suggest a middle way: a user specific choice that depends on the present channel realisation but is easy to compute. We know that the density of the set of possible receive signals roughly correlates to the distance from its boundaries, which are given by the single-user sets \(M_j(h_k)\). Assuming that the lower bound of the region of possible receive signals is zero—that the single-user sets are disks and not annuli—another way to choose the energy matrix based on the present channel realisation could be to choose each scaling factor as a fraction of the upper bound of that particular single-user set:

\[\sqrt{E_k} = \alpha \frac{1}{\sqrt{M}} \|h_k\|_1, \forall k,\]  
(Choice 2)

where \(\alpha \in [0, 1]\). Because the term \(\|h_k\|_1\) at average grows linearly with the number of antennae, the overall array gain grows linearly: \(E_k \propto \Theta(M)\).

This scheme has no performance advantage over the static choice in an i.i.d. Rayleigh fading scenario, as can be seen in Figure 4.5. The scheme might have an advantage, if the maximum array gain—the sum of the moduli of the channel coefficients—was more stochastic, i.e. in a system with few transmit antennae, or if the path loss to different users differed a lot.
4.4 Choosing Symbol Energies

Figure 4.5: The ergodic sum rate for the three different choices of array gains. Here 100 transmit antennae serve 10 users. The blue line shows the static choice 1 of the parameter $E$, the green line the case, where the parameter $E$ was optimised for each new channel realisation. The red line shows choice 2, where $E$ is chosen as a fraction of the coherently summed channel vector. With 100 transmit antennae the difference of the choices is negligible.

Bit-Error-Rate Choice

Since we will evaluate the system performance, not in terms of achievable sum rates, but in terms of bit-error-rates of a 16-QAM symbol constellation diagram, the energy scaling factor $E$ has simply been chosen to be the parameter that gives the lowest probability of error $P_e$ in Chapter 8.2:

$$E_1 = \cdots = E_K = \arg\min_E P_e(E, \ldots, E).$$

4.4.3 Influence of System Parameters

When we increase $E$ in the parameter $E$, the receive power will increase but also the multi-user interference. Ngo (2013) notes in his paper that, in the choice between low multi-user interference and high receive power, it is optimal to balance the both, such that the multi-user interference is of the same order of magnitude as the noise power. If the interference power were significantly larger than the noise variance, it would correspond to operating at high SNR—increasing the transmit power would increase both the desired signal and the interference power equally much and the SNR would remain the same, which means that the system performance will not improve.

This dependence can be seen in Figure 4.6b—when the relative noise power is lowered, the optimal choice of the symbol energies decreases. This is to expect, because keeping the interference low becomes more important at high normalised
transmit powers—we get the best performance when the multi-user interference roughly is of the same magnitude as the noise.

The optimal choice of the energy scaling factor $E$ depends on three parameters: normalised transmit power, number of transmit antennae and number of users.

In Figure 4.7, the ergodic sum rate for different choices of the parameter $E$ is shown. In the left figure, the parameter is normalised with respect to the number of transmit antennae. We note that the sum rate has its maximum at about the same value of $E/M$ independently of the number of transmit antennae. This shows that the optimal choice of $E$ grows linearly with the number of transmit antennae and that the maximum possible array gain grows at least linearly with the number of antennae. This can also be seen in Figure 4.6a.

The optimal choice of the parameter $E$ also varies with the number of users. In Figure 4.7b, it can be seen that the optimal choice of the parameter $E$ decreases with increasing number of users. In Figure 4.8, we see how the ergodic sum rate increases with increased number of transmit antennae and with increased normalised transmit power.

Figure 4.6: Optimal static array gain with 10 users.
4.4 Choosing Symbol Energies

(a) Sum rate for 10 users and different numbers of transmit antennae

(b) Rate per user for 100 transmit antennae and different numbers of users

Figure 4.7: The sum rate averaged over different channel realisations for a fixed normalised transmit power of 5 dB. In 4.7a we see that the optimal symbol energy grows linearly with the number of transmit antennae. In 4.7b we see that the optimal symbol energy decreases as the number of users increases.

(a) 5 dB normalised transmit power

(b) 100 transmit antennae

Figure 4.8: Ergodic sum-rates with 10 users.
The precoding scheme of Mohammed et al. (2013a) results in discrete-time constant-envelope signals—signals with discrete-time PAR of 0 dB. Even if the discrete-time signals have constant envelope, it does not imply that their continuous-time counterparts will have constant envelope. Study the “constant-envelope” signal in Figure 5.1. This signal has, after pulse shape filtering, a non-zero PAR of 3.96 dB—higher than the 0 dB of the discrete-time signal. The PAR is, however, better than that of conventional precoding, which normally results in a PAR of 10 dB, see Chapter 2.9.

If we stick to pulse shape filtering as the means of transmitting the discrete-time signals, one way of lowering the continuous-time PAR further would be to avoid excessive phase variation between consecutive symbol durations. In this way the signal will avoid passing close to zero, which would lower the power of the signal and increase PAR. This idea was briefly mentioned in the article of Mohammed et al. (2013a) but not investigated.

Denote the largest allowed difference between the phase $\theta_n$ of the present signal at antenna $n$ and the phase $\omega_n$ of the previous signal at the same antenna by $\Delta\theta$. Denote the difference between two phases $\Delta(\theta, \omega)$:

$$\Delta(\theta, \omega) = \left| (\theta - \omega + \pi) \mod 2\pi - \pi \right|.$$ 

Then we can formulate a phase constraint that will limit the phase variation between consecutive symbol durations: $\Delta(\theta_n, \omega_n) \leq \Delta\theta$. Now the optimisation problem of a subiteration in the discrete-time constant-envelope precoding of Mohammed et al. turns into:

$$\theta_n^* = \arg\min_{\theta_n : \Delta(\theta_n, \omega_n) \leq \Delta\theta} \Re \left( e^{j\theta_n} \zeta_n \right),$$

where $\zeta_n$ is defined in Equation (4.7). Compare this expression with its non-
The optimal phase is now given by:

$$\theta_n^* = \begin{cases} 
\arg(\zeta_n), & \text{if } \Delta(\arg(\zeta_n), \omega_n) \leq \Delta \theta \\
\alpha_{\min}^*, & \text{otherwise} 
\end{cases}$$

where $\alpha_{\min}$ is the end point of the phase interval, which lies closest to $\arg(\zeta_n)$:

$$\alpha_{\min} = \arg \min_{\alpha \in \{\omega_n - \Delta \theta, \omega_n + \Delta \theta\}} \Delta(\arg(\zeta_n), \alpha).$$

The result of limiting the phase variation can be seen in Figure 5.2. The PAR decreases as we constrain the phase variation more. At, for example, $\Delta \theta = \pi/2$, the PAR of the transmit signal is lowered to 2.7 dB—down 1.3 dB from the 4.0 dB of the unconstrained case. How the PAR varies with phase constraint can be seen in Figure 5.3.

The consequence of constraining the phase is a performance degradation—as the phase is more and more constrained, it will become more and more difficult to find precoding weights, which produces high energy symbol points with low multi-user interference at the users. This can be seen in Figure 5.4, where the transmit power needed to uphold the same performance increases as the maximum phase variation $\Delta \theta$ decreases.

To evaluate the performance of this phase constrained scheme, we need an expression for its achievable sum rate. The derivation in Appendix A of the achievable sum rate (A.9) of the discrete-time constant-envelope precoding scheme still holds true when we constrain the phases. We will therefore use this lower bound on the achievable sum rate to evaluate this scheme in Chapter 8.
Figure 5.2: The root-raised-cosine filtered transmit signal with roll-off factor 0.3, where the phase variation between consecutive symbols has been constrained. Upper left: The phase variation is less than $\pi/2$. Upper right: The phase variation is less than $\pi/3$. Bottom left: The phase variation is less than $\pi/4$. Bottom right: The phase variation is less than $\pi/6$.

Figure 5.3: The continuous-time PAR of the transmit signals for different choices of maximum phase variation.
Figure 5.4: The required normalised transmit power $P\sigma^2$ to achieve a bit error rate of $10^{-2}$ in a 16-QAM system with 100 transmit antennae and 10 users for different maximum phase variations $\Delta\theta$; $\Delta\theta = \pi$ represents the case without phase constraints.
Chapter 6

Continuous-Time Constant-Envelope Precoding

To preserve the discrete-time constant-envelope property when modulating a signal, continuous phase modulation can be done. Continuous phase modulation is a modulation scheme, where the constant-envelope signal points are modulated by changing their phases constantly. The result is a continuous-time signal with constant envelope, i.e. PAR 0 dB, which is the ideal signal from a power amplifier point of view.

If \( x[n] \) is the discrete-time constant-envelope signal that we want to transmit on some antenna, one mapping to a continuous-time continuous-phase signal \( x(t) \) could be:

\[
x(t) = A e^{j \int_{-\infty}^{t} \sum_{n=-\infty}^{\infty} \arg(x[n]) h_f(\tau-nT) d\tau},
\]

where \( T \) is the symbol duration and \( h_f(\tau) \) the frequency shaping pulse. The simplest choice of shaping pulse would be the cog-wheel pulse:

\[
h_f(\tau) = \begin{cases} 
\frac{1}{T}, & \text{if } \tau \in [-T, 0] \\
-\frac{1}{T}, & \text{if } \tau \in [0, T] \\
0, & \text{otherwise}
\end{cases}
\]

This choice of shaping pulse, which is not constrained to one symbol duration, will result in a partial-response continuous phase modulation signal, which will cause inter-symbol interference. Maximum Likelihood Sequence Estimation (MLSE) has to be used to optimally demodulate the signal. The structure of this optimal MLSE demodulator is not known.

The demodulation of a continuous phase modulated discrete-time constant-envelope precoded signal has to be investigated further, in order to determine the power penalty that continuous phase modulation incurs. This will be left outside this study.
When we look at the power spectral density of the continuous phase modulated signal in Figure 6.1, we see that the bandwidth of the signal is unacceptably wide. The bandwidth of a constant phase modulated signal can be derived through the instantaneous frequency $f_{dev}(t)$ of the signal. If

$$\varphi(t) = \int_{-\infty}^{t} \sum_{n=-\infty}^{\infty} \text{arg}(x[n]) h_f(\tau - nT) d\tau$$

is the phase of the signal, then the instantaneous frequency is defined as:

$$f_{dev}(t) = \frac{1}{2\pi} \frac{d}{dt} \varphi(t)$$

and given by:

$$f_{dev}(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \text{arg}(x[n]) h_f(t - nT).$$

The bandwidth is strongly influenced by the *peak frequency deviation*, which is given by

$$k_f = \max_{t} \left| f_{dev}(t) \right|.$$  

If we use the cog-wheel pulse, then the peak frequency deviation is

$$k_f = \max_{t} \left| \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \text{arg}(x[n]) h_f(t - nT) \right|.$$
Figure 6.2: Same as Figure 6.1 but now with the phase variation limited to below $\pi/2$, which has made the main lobe narrower.

$$= \frac{1}{2\pi T_n} \max_n |\Delta(\theta[n], \theta[n + 1])|.$$ 

We can see that the only two ways to make the bandwidth narrower is to either increase the symbol duration or to limit the phase variation between consecutive discrete-time signals. A way to constrain the phase variation was presented in Chapter 5.

In Figure 6.2, we see that the bandwidth is narrowed when the phase variation is limited to $\pi/2$, i.e. $\Delta \theta = \pi/2$. The spectrum then resembles that of a minimum shift keying signal and after Gaussian lowpass filtering the power spectral density looks similar to a Gaussian minimum shift keying signal, which is the signal used in GSM.

If we measure bandwidth as the frequencies, for which the signal strength is above -20 dB of its peak value, the normalised bandwidth of the signal without phase constraints is 2 times the Nyquist bandwidth ($\frac{1}{2T}$) and the normalised bandwidth of the signal, whose phase variation is constrained to below $\pi/2$, is 1.375 times the Nyquist bandwidth. This is an improvement of 31% in terms of used bandwidth.
Chapter 7

Precoding with Disk Constraints

Thus far, we have constrained the amplitudes of the transmit signal severely in order to achieve low $\text{PAR}$. The upper bound on the amplitudes is necessary to avoid peaks and to ensure a low $\text{PAR}$. The lower bound on the other hand is not as important—the power amplifier does not distort small amplitudes. To get a low $\text{PAR}$, it is important to have a signal $\text{RMS}$ just below the peak value of the signal. This could indirectly be achieved by a lower bound on the signal amplitudes. We have, however, observed that, even if the lower bound on the signal amplitudes is relaxed, the $\text{RMS}$ of the signal does not drop significantly and a low $\text{PAR}$ is still achieved. Relaxed signal amplitudes do, however, improve the system performance somewhat.

In this chapter, we will study the impact of allowing the discrete-time transmit signals to lie inside an annulus in the complex plane. We will call this precoding scheme annulus-constrained precoding and the special case, where $\epsilon = A$, which we will study in greater detail, we will call disk-constrained precoding. What the region of possible receive signals looks like when the amplitude constraint is relaxed is discussed in Appendix B.

7.1 Precoding Algorithm

We now turn to the problem of finding an iterative search method for optimal precoding weights, whose amplitudes are constrained to lie inside an interval

$$A_m \in [A - \epsilon, A], \quad \forall m = 1, \ldots, M,$$

for some* $A \in \mathbb{R}^+$.  

Building on the idea used in the search algorithm for the discrete-time constant-envelope multi-user case, we develop a method based on subiterations, during

*The value of $A$ is not important, rather the ratio $E/A$ is, where $E$ is the power of the symbols. Without loss of generality, we could fix $A = M^{-\epsilon}$. 
which one precoding weight is updated at a time. All precoding weights are initialised to $A - \epsilon$. We fix all precoding weights except the $n$-th, which is updated by optimising it with respect to the multi-user interference $\|Hx - u\|_2$; one such step is called a subiteration. We then move on to update the next $(n + 1)$-th precoding weight and continue thus until all $M$ weights have been updated once, then one iteration is completed.

The optimisation problem of a subiteration, where all, except one precoding weight, are fixed, has an explicit solution. In the $n$-th subiteration, we wish to optimise the $n$-th weight with all other weights fixed, thus find a weight, for which $A - \epsilon \leq A_n \leq A$, that minimises the multi-user interference. The optimal weight is given by:

$$A_n^* e^{j \theta_n^*} = \arg\min_{A_n, \theta_n} \|Hx - u\|^2 = \arg\min_{A_n, \theta_n} \sum_{k=1}^K |u_k - \sum_{m=1}^M h_{km} A_m e^{j \theta_m}|^2$$

(7.1)

$$= \arg\min_{A_n, \theta_n} \sum_{k=1}^K (u_k - \sum_{m=1}^M h_{km} A_m e^{j \theta_m}) (u_k^* - \sum_{m=1}^M h_{km}^* A_m e^{-j \theta_m})$$

(7.2)

$$= \arg\min_{A_n, \theta_n} \sum_{k=1}^K (-u_k h_{kn}^* A_n e^{-j \theta_n} - u_k^* h_{kn} A_n e^{j \theta_n} + \sum_{m=1}^M \sum_{l=1}^M h_{km} h_{kl}^* A_m A_l e^{j (\theta_m - \theta_l)})$$

(7.3)

$$= \arg\min_{A_n, \theta_n} \sum_{k=1}^K \left( -2 \Re (u_k h_{kn}^* A_n e^{-j \theta_n}) + \sum_{m=1}^M h_{km} h_{kn}^* A_m A_n e^{j (\theta_m - \theta_n)} + \sum_{m=1}^M h_{km}^* h_{kn} A_m A_n e^{j (\theta_m - \theta_n)} \right)$$

(7.4)

$$= \arg\min_{A_n, \theta_n} \sum_{k=1}^K \left( |h_{kn}|^2 A_n^2 - 2 \Re \left( u_k h_{kn}^* A_n e^{-j \theta_n} - \sum_{m=1}^M h_{km}^* h_{kn} A_m A_n e^{j (\theta_m - \theta_n)} \right) \right)$$

(7.5)

$$= \arg\min_{A_n, \theta_n} \left( A_n^2 \sum_{k=1}^K |h_{kn}|^2 - 2 A_n \Re \left( e^{-j \theta_n} \sum_{k=1}^K h_{kn} \left( u_k - \sum_{m \neq n}^M h_{km} A_m e^{j \theta_m} \right) \right) \right)$$

(7.6)

In (7.2), the absolute value is factorised as the product of two conjugates. In (7.3), the product is expanded and some constant terms, which neither depend on $A_n$ nor on $\theta_n$, are discarded. Whereupon, in (7.4), we realise that many of the terms in the double sum likewise do not depend on $A_n$ or $\theta_n$—they are subsequently discarded too. In the same step, we see that the two first terms form a conjugated pair and can be written as the real part of a complex number. The same observation can be made again: the two sums, except for one term in the first sum, can be seen as the
7.1 Precoding Algorithm

real part of a complex number. In the last step (7.6), the finite summation over $k$ is broken up and moved inside the real-part operator, whereupon the terms that depend on $A_n$ and $\theta_n$ can be factored out.

The problem of finding the optimal phase $\theta_n$ now decouples from the problem of finding an optimal amplitude $A_n$. If we introduce the complex number

$$\zeta_n = \sum_{k=1}^{K} h^*_kn \left(u_k - \sum_{m=1, m \neq n}^{M} h_{km}A_me^{j\theta_m} \right),$$

then the optimal phase is given by:

$$\theta_n^* = \arg \max_{\theta_n} \Re(e^{j\zeta_n}e^{-j\theta_n}) = \arg(\zeta_n).$$

Using this optimal value of the phase in (7.6), we get a second degree polynomial expression in $A_n$, which is easy to minimise with respect to $A_n$:

$$A_n^* = \arg \min_{A_n \in [A-\epsilon, A]} \left(A_n^2 \sum_{k=1}^{K} |h_{kn}|^2 - 2A_n|\zeta_n| \right)$$

This real-valued second-degree expression has a minimum, which we can compute by differentiating with respect to $A_n$ and setting the derivative to zero. This would result in the following optimal value of $A_n$:

$$A_n^0 = \frac{|\zeta_n|}{\sum_{k=1}^{K} |h_{kn}|^2}.$$  

However, $A_n$ is constrained to lie in the interval $[A-\epsilon, A]$. Since the second degree expression is monotone in any interval not containing the minimum $A_n^0$, the optimal value is given by:

$$A_n^* = \begin{cases} 
A - \epsilon, & \text{if } A_n^0 < A - \epsilon \\
A_n^0, & \text{if } A - \epsilon \leq A_n^0 \leq A \\
A, & \text{if } A_n^0 > A 
\end{cases}$$

We note that, in the case $\epsilon = 0$, this search algorithm reduces to the same algorithm proposed by Mohammed et al. (2013a) for the discrete-time constant-envelope precoding.

The signal power of the transmit signal $x_p$ obtained through the above proposed precoding scheme will be less than 1 (with $A = M^{-1/2}$). To compensate for this, we need to multiply with a power compensation factor $c \in \mathbb{R}^+$. The unit power transmit signal is given by

$$x = cx_p$$
The power compensation factor $c$ depends on the choice of the symbol energies $E$ and the ratio between the number of transmit antennae and users $M/K$. It has to be empirically determined and tabulated.

The amplitude-constrained precoding algorithm is summarised in the following pseudo code.

```
Pseudo Code for Amplitude Relaxed Search Algorithm:
# initialization
θ = zeros(nr_antennae)
A = (A_max - ε)×ones(nr_antennae)
# iteration
for rep_id in range(1, nr_reps):
    # subiteration
    for n in range(1, nr_antennae):
        compute ζ_n
        θ[n] = arg(ζ_n)
        A[n] = abs(ζ_n)/sum(sq(abs(H[:,n])))
        if A[n] < A_max - ε:
            A[n] = A_max - ε
        else if A[n] > A_max:
            A[n] = A_max
        end if
    end for
end for
# power compensation
A = c×A
```

It has been observed in simulations that it is best to do the optimisation of the subiterations in a pseudo random order, i.e. update the transmit signals in a pseudo random order. Otherwise, the transmit signals updated last will stay around $A_ε$, where they were initialised and their PAR will be bad in comparison to the rest of the transmit signals.

### 7.2 PAR and Performance

We will study the case where $ε = A$, i.e. where the transmit signals are constrained to a disk in the complex plane with radius $A$.

It has been observed in simulations, for high symbol energies $E$, the PAR is close to that of discrete-time constant-envelope precoding. As a matter of fact, the PAR of disk-constrained precoding will approach that of discrete-time constant-envelope precoding when the symbol energies grow. On the other hand, for small symbol energies, the PAR of the disk-constrained precoding will be of the same
magnitude as that of conventional precoding. The PAR of the resulting transmit signal varies with the choice of the symbol energies $E$. This is illustrated in Figure 7.1.

It has also been observed that the choice of initialisation has a great impact on the performance and on the PAR. If the search algorithm is initialised by setting all $A_m = A$, the disk-constrained precoding scheme will look more or less like the discrete-time constant-envelope precoding scheme, with almost the same performance and almost the same PAR. The reason for this is that the disk-constrained search algorithm contains no means of minimising the transmit energy, therefore the transmit signals will lie close to their initial values if possible, which then would be on the circle.

Hence, we have decided to initialise the transmit signals to $A_m = A - \epsilon$, which to some degree will minimise the transmit power. By doing this, we see a small improvement in performance compared to the discrete-time constant-envelope scheme. We also see a slight increase in PAR.

The optimal choice of symbol energies varies with the relative noise level in the receiver. For normalised transmit powers of -1 to 15 dB, the optimal choice of $E$ is between 5 to 9, which corresponds to a PAR of 4.5 dB in the worst case.

To evaluate the performance of this scheme we can use the same lower bound (A.9) on the sum rate as in the discrete-time constant-envelope precoding scheme, since its derivation still holds true when we relax the amplitudes, see Appendix A. An achievable rate curve and a discussion on the performance is given in Chapter 8.

Figure 7.1: The PAR for different choices of symbol energies, $E$. 
Chapter 8

Evaluation of Low-PAR Precoding Techniques

In this chapter, we will evaluate the low-PAR precoding schemes that have been discussed in the previous chapters. First, we compare them in terms of information theoretic bounds on their sum rates, to see what transmit power is needed to achieve a certain ergodic sum rate. Then, we simulate a very-large multi-user MIMO communication system, that uses 16-QAM symbols to transmit bits over a flat-fading AWGN channel with different precoding schemes: zero-forcing, discrete-time constant-envelope precoding, phase-constrained precoding and disk-constrained precoding. Finally, we set up a simple link budget to compare the different schemes in terms of how much power their power amplifiers consume. In the link budget, the very-large multi-user MIMO precoding schemes are also compared to a single-antenna system.

8.1 Relative Required Transmit Power

Information-theoretic bounds on the achievable sum rates for a given channel realisation have been used to calculate the ergodic sum rates through Monte Carlo simulations. The relative transmit powers required to achieve a certain ergodic sum rate for a base station with 100 transmit antennae and 10 users can be seen in Figure 8.1.

The extra power, relative to zero-forcing, needed to achieve certain sum rates are given in Table 8.1. The numbers in this table are computed by linear interpolation of the values obtained in the Monte Carlo simulations.

Gaussian signalling has been used to provide an upper bound on what data rates that we can expect. With appropriate modulation order, error correcting codes and block lengths, this upper bound can be approached arbitrarily close. It should be kept in mind, however, that in any practical system a small degradation in performance has to be expected.
Figure 8.1: The normalised transmit power needed to achieve different sum rates in a system with 100 transmit antennae and 10 users.

Table 8.1: Required Transmit Powers Relative to Zero-Forcing

<table>
<thead>
<tr>
<th>sum rate</th>
<th>Dtce</th>
<th>Disk</th>
<th>Δθ=π/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 bpcu</td>
<td>1.13 dB</td>
<td>1.46 dB</td>
<td>3.08 dB</td>
</tr>
<tr>
<td>20 bpcu</td>
<td>1.37 dB</td>
<td>1.33 dB</td>
<td>4.12 dB</td>
</tr>
<tr>
<td>40 bpcu</td>
<td>2.32 dB</td>
<td>1.89 dB</td>
<td>5.31 dB</td>
</tr>
<tr>
<td>60 bpcu</td>
<td>3.17 dB</td>
<td>2.31 dB</td>
<td>6.34 dB</td>
</tr>
</tbody>
</table>
In the paper of Mohammed et al. (2013a), it is stated that the extra transmit power required for discrete-time constant-envelope precoding is 1.7 dB at an ergodic rate of 2 bpcu\(^*\) per user in the limit of infinite number of transmit antennae. In our simulations, we see that the extra transmit power needed to achieve 2 bpcu per user with discrete-time constant-envelope precoding, compared to the practical linear precoding scheme zero-forcing, is merely 1.37 dB with a finite number of 100 transmit antennae. The difference 1.7 − 1.37 = 0.33 dB is the cost of using zero-forcing instead of an optimal transmission scheme.

We also note that disk-constrained precoding performs worse than the discrete-time constant-envelope precoding for low normalised transmit powers. In the regime of high noise levels and with disk constraints on the transmit signals, the optimal transmit signals all have maximal amplitude—lie on the circle circumference. Despite the fact that transmit signals on the circle are admissible, the disk-constrained precoding results in transmit signals that sometimes lie inside the circle circumference, even in the high noise level regime. This is due to the search algorithm, which is suboptimal and gets stuck in local minima. The difference in required transmit power quickly turns in the favour of disk-constrained precoding at higher normalised transmit powers however.

### 8.2 Bit Error Rates

A communication system with 100 transmit antennae and 10 users that uses a 16-QAM symbol constellation to modulate its bits has been simulated with different precoding techniques. The signals were sent over a frequency-flat i.i.d. Rayleigh fading channel with AWGN.

The uncoded bit error rate is shown in Figure 8.2. In the figure, it can be seen that zero-forcing performs better than any of the other precoding schemes at these noise levels. This is expected, because zero-forcing nulls all interference at the users. Maximum ratio transmission has a huge error floor in this uncoded case, since a too high modulation order is used. The 16-QAM implies a data rate of 4 bpcu, which is higher than the highest sustainable rate that maximum ratio transmission can support, even without noise.

The low-\(\text{PAR}\) precoding schemes behave as we would expect with the sum rate plot in the previous section in mind. They require more transmit power to give the same bit-error-rate as zero-forcing. Disk-constrained precoding is a little better than discrete-time constant-envelope precoding and phase constrained precoding performs worst of the low-\(\text{PAR}\) precoding schemes.

The relative required transmit powers for a BER of \(10^{-4}\) compared to zero-forcing are given in Table 8.2. The figures in Table 8.2 are close to the required

---

\(*\)Bits-per-channel-use, the number of encoded bits that can be transmitted per symbol period, a measure of the spectral efficiency of the system. 2 bpcu per user gives a total system spectral efficiency of 20 bpcu. This can be compared to an LTE system, which can deliver a total system spectral efficiency of 16 bpcu.
transmit powers predicted by the sum rate calculations, see Table 8.1, row 40 bpcu.

Table 8.2: Required Transmit Powers Relative to Zero-Forcing

<table>
<thead>
<tr>
<th>BER</th>
<th>DT-CE</th>
<th>Disk</th>
<th>Δθ=π/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^{-4}</td>
<td>2.78 dB</td>
<td>2.00 dB</td>
<td>6.62 dB</td>
</tr>
</tbody>
</table>

8.3 Link Budget Comparison

In this section, we compare five different communication systems by estimating the total input power they need to provide a data rate of 4.5 Mbit/s per user to 10 users simultaneously over a bandwidth of 5 MHz. The five systems are numbered accordingly:

1. A single-antenna system with a linearised power amplifier that serves 10 users through some orthogonal multiplexing scheme*.

2. A multi-antenna system with 100 antennae and unlinearised power amplifiers that serves 10 users by doing zero forcing precoding.

3. A multi-antenna system with 100 antennae and unlinearised power amplifiers that serves 10 users by discrete-time constant-envelope precoding.

---

*such as time division multiplexing, frequency division multiplexing or code division multiplexing.
4. A multi-antenna system with 100 antennae and unlinearised power amplifiers that serves 10 users by discrete-time constant-envelope precoding with phase constraints, $\Delta \theta = \frac{\pi}{2}$.

5. A multi-antenna system with 100 antennae and unlinearised power amplifiers serving 10 users by disk-constrained precoding.

All systems use class $\text{ab}$ power amplifiers of the sort used in handsets. The power amplifiers operate with a conduction angle of around 200°, which is required to reach the higher power efficiencies but requires an additional back-off of $b_{\text{NLR}} = 3 \text{ dB}$ (González, 2003).

It should be emphasised that in this link budget, we have done several simplifications and assumptions. The results should be seen as indications of what relative gains and losses in performance that are to be expected from different precoding schemes. A refined study will have to conclude any absolute figures in the result.

In the link budget, Gaussian signalling is used and an ideal Nyqvist bandwidth is assumed in the computation of the noise power. Furthermore, the fading margin is merely a qualitative guess. A very simple model of the power amplifier has been used in the link budget; the amount of back-off needed should be verified against real measurements and the relation between back-off and power efficiency should be studied closer.

### 8.3.1 Required Receive Power

We assume that the noise at the users is white over the bandwidth in question. The noise power $\sigma^2$ at one user is then given by its noise figure $NF$ and the bandwidth $W$.

$$\sigma^2 = N_{0h} W = k T_0 NF W,$$

here $k$ is Boltmann’s constant and $T_0 = 290 \text{ K}$ the standard noise temperature. $N_{0h} = N_0 NF$ is the noise density in the handset and $N_0 = k T_0$ is the spectral density of the thermal noise.

With the values given in Table 8.3, this gives us a noise power of -99 dBm at the user.

If we assume that we are using Gaussian signalling to transmit the information, Shannon’s channel coding theorem gives us a lower bound on the receive power $P_R$ needed to achieve the desired data rate in System 1 and 2.

$$R = \frac{W}{N} \log_2 \left( 1 + \frac{P_R}{\sigma^2} \right) \quad \text{(Cover, 2012)}$$

$$\implies P_R = \sigma^2 \left( 2^{RN/W} - 1 \right),$$

here $N$ stands for the number of users who are not multiplexed in space and have to use the same time-frequency resource, $N = 10$ for System 1 and $N = 1$ for
### Table 8.3: User Equipment Properties

<table>
<thead>
<tr>
<th>UE HANDSET</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Users, $K$</td>
<td>10</td>
</tr>
</tbody>
</table>

**Noise Power Estimation:**
- Boltzmann’s constant, $k$: -229 dBm/kgs$^{-2}$
- Temperature, $T_0$: 290 K
- Thermal Noise Density, $N_0$: -174 dBm/Hz
- Handset Noise Figure, $NF$: 8 dB
- Handset Noise Density, $N_{0h}$: -166 dBm/Hz
- Coded Data Rate per User, $R$: 4500000 bits/s
- Noise Effective Bandwidth, $W$: 5000000 Hz
- Noise Effective Bandwidth, $W$: 67 dBHz
- Handset Noise Power, $\sigma^2 = N_{0h}W$: -99 dBm

**Gains and Losses:**
- Rx Antenna Gain, $G_R$: 0 dBi
- Body Loss, $L_B$: 3 dB

### Table 8.4: Required Receive Power

<table>
<thead>
<tr>
<th></th>
<th>System 1</th>
<th>System 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Required Signal-to-Noise Ratio, $P_R/\sigma^2$</td>
<td>27</td>
<td>-0.6 dB</td>
</tr>
<tr>
<td>Required Receive Power, $P_R$</td>
<td>-72</td>
<td>-100 dBm</td>
</tr>
<tr>
<td>Required Receive Power, $P_R$</td>
<td>$6.45 , \times , 10^{-11}$</td>
<td>$1.09 , \times , 10^{-13}$ W</td>
</tr>
</tbody>
</table>
8.3 Link Budget Comparison

System 2. Now, we can calculate how much power each user needs to receive, in order to ensure the desired data rate, see Table 8.4.

Because of the multi-user interference of System 3 and 4, we cannot determine how much receive power we require in these systems. Luckily, this is not necessary, instead we will rely on the relative transmit power requirement between low-PAR precoding and zero-forcing that was determined in Section 8.1 and shown in Figure 8.1, see Subsection 8.3.3.

8.3.2 Path Loss

We use the Cost-Hata model (Damosso, 1999) to estimate the path loss of the channel. In this model the median path loss is given by:

\[
L_{\text{NLOS}} = 46.3 + 33.9 \log f_c - 13.82 \log h_b - a(h_R) + (44.9 - 6.55 \log h_b) \log d + C_M, \quad \text{[dB]}
\]

where the different parameters are given in Table 8.5. In a medium urban setting without line-of-sight, at a distance of 500 m and a base station that is placed 30 m above the roof tops, the path loss is around 127 dB.

<table>
<thead>
<tr>
<th>Channel</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency, ( f_c )</td>
<td>2000 MHz</td>
</tr>
<tr>
<td>Wavelength, ( \lambda )</td>
<td>0.15 m</td>
</tr>
<tr>
<td>Link Distance, ( d )</td>
<td>500 m</td>
</tr>
<tr>
<td>(Example: LOS Path Loss, ( L_{\text{LOS}} ))</td>
<td>92 dB</td>
</tr>
</tbody>
</table>

NLOS Median Path Loss Model (Cost-Hata):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base Station Height, ( h_b )</td>
<td>30 m</td>
</tr>
<tr>
<td>UE Height, ( h_R )</td>
<td>1.6 m</td>
</tr>
<tr>
<td>Terrain Correction Factor, ( C_M ) (Medium City)</td>
<td>0 dB</td>
</tr>
<tr>
<td>Mobile Antenna Height Correction Factor, ( a(h_R) )</td>
<td>0.34 dB</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLOS Median Path Loss, ( L_{\text{NLOS}} )</td>
<td>127 dB</td>
</tr>
<tr>
<td>Fading Margin, ( \Phi_M )</td>
<td>10 dB</td>
</tr>
</tbody>
</table>

We have frequency-selective fading, if the product between the bandwidth and the delay spread is much larger than one half: \( \sigma_c W \ll 1/2 \) (Tse, 2008). In a semi-urban setting a typical delay spread is \( \sigma_c = 0.2 \mu s \). In our case, the product equals one and we have frequency-flat fading. To protect ourselves from deep fades, we add a 10 dB fading margin to the link budget.

Since our base stations are located above roof level, the scattering in azimuth is
small and we expect the channel coefficients to each user to be highly correlated*, we therefore still face the risk of deep fades and should add a fading margin, even to our very-large multi-user MIMO systems.

8.3.3 Required Radiated Power

We are now ready to compute how much power each base station needs to radiate in order to provide the desired data rate.

The EIRP (Equivalent Isotropically Radiated Power) tells us how much an isotropical antenna has to radiate in order for the user to receive the required signal power $P_R$. It depends on the path loss $L_{\text{NLOS}}$, the receiver antenna gain $G_R$, the fading margin $\Delta$ and the body loss† $L_B$, see Table 8.3 and 8.5.

$$\text{EIRP} = P_R - G_R + L_B + L_{\text{NLOS}} + \Delta \quad \text{[dB]}$$

The required radiated power $P$ for each system is then given by

$$P = \text{EIRP} - G_T \quad \text{[dB]}$$

where $G_T$ is the antenna gain of the transmitter, see Table 8.6. The antenna gain of the single-antenna system is due to the use of a highly directive high end antenna. The very-large MIMO systems use cheap patch antennas, which are close to isotropic, to keep costs down. The array gain of zero-forcing, which is the equivalent of the antenna gain in the single-antenna system, was derived through Monte Carlo simulations in Chapter 2.9.1 and determined to 9.55 dB for a system with 100 transmit antennas and 10 users. The relative transmit power to achieve the same data rates with discrete-time constant-envelope precoding as with zero-forcing was determined in Table 8.1. To achieve the same data rate, the discrete-time constant-envelope precoding scheme needs 1.13 dB more transmit power than zero-forcing‡.

The output power of the power amplifier also has to compensate for losses $L_c$ in the cables to the $M$ transmit antennae. The output power of the power amplifier is therefore given by

$$P_{\text{out}} = P + L_c - 10 \log(M) \quad \text{[dB]}$$

The required output power of each power amplifier is given in Table 8.6.

8.3.4 Required Input Power

We can now establish how much input power the power amplifiers will consume in the different base stations. We have computed the PAR of all systems except the

*The correlation between different users is expected to be uncorrelated though, which is a requirement for multi-user precoding.
†The loss due to the human body standing in the way of the receiver.
‡The sum rate is close to 10 bpcu.
8.3 Link Budget Comparison

Table 8.6: Required Transmitter Powers

<table>
<thead>
<tr>
<th>System</th>
<th>System 2</th>
<th>System 3</th>
<th>System 4</th>
<th>System 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Required EIRP (dBm)</td>
<td>68</td>
<td>40</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Tx Antenna Gain, $G_T$ (dBi)</td>
<td>18</td>
<td>9.55</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Required Total Radiated Power, $P$ (dBm)</td>
<td>49.9</td>
<td>30.7</td>
<td>31.8</td>
<td>33.8</td>
</tr>
<tr>
<td>Required Total Radiated Power, $P$ (W)</td>
<td>99</td>
<td>1.17</td>
<td>1.52</td>
<td>2.38</td>
</tr>
<tr>
<td>Cable Loss per Antenna, $L_c$ (dB)</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Required Output Power per PA, $P_{out}$ (dBm)</td>
<td>46</td>
<td>6.7</td>
<td>7.8</td>
<td>9.8</td>
</tr>
<tr>
<td>Required Output Power per PA, $P_{out}$ (W)</td>
<td>39</td>
<td>$4.7 \cdot 10^{-3}$</td>
<td>$6.0 \cdot 10^{-3}$</td>
<td>$9.5 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

The single-antenna system. If we assume it is using a discrete-time constant-envelope modulation scheme, such as QPSK, the continuous-time $\text{PAR}$ will be the same as that of discrete-time constant-envelope precoding, i.e. 3.96 dB. This is a lower bound on the $\text{PAR}$ that we can expect from this system.

Using the power efficiency formulae in Chapter 3, we get the input power for each system required to provide the desired data rate, see Table 8.7. To compensate for the non-linearities that this mode of operation causes, we do an additional back-off of $b_{\text{NLR}} = 3$ dB (González, 2013). This back-off is additional to the back off done to accommodate the $\text{PAR}$ of the signal. In the single-antenna case, we do not do any additional back-off, because it employs linearisation. Instead, we add 10 W, which is the power consumed by the digital predistorter.

Table 8.7: Power Amplifier Requirements

<table>
<thead>
<tr>
<th>System</th>
<th>System 2</th>
<th>System 3</th>
<th>System 4</th>
<th>System 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal PAPR (RRC $\beta = 0.3$) (dB)</td>
<td>4.0</td>
<td>10</td>
<td>4.0</td>
<td>2.6</td>
</tr>
<tr>
<td>Minimum PA Backoff, $b_{\text{PAR}}$ (dB)</td>
<td>4.0</td>
<td>10</td>
<td>4.0</td>
<td>2.6</td>
</tr>
<tr>
<td>Additional PA Backoff, $b_{\text{NLR}}$ (dB)</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>DPW Input Power (W)</td>
<td>10</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Maximum PA Efficiency, $\eta_{\text{sat}}$</td>
<td>0.55</td>
<td>0.55</td>
<td>0.55</td>
<td>0.55</td>
</tr>
<tr>
<td>Efficiency of PA, $\eta$</td>
<td>0.35</td>
<td>0.13</td>
<td>0.26</td>
<td>0.30</td>
</tr>
<tr>
<td>Required Input Power per PA (W)</td>
<td>111</td>
<td>0.04</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>Total Input Power, $P_{\text{in}}$ (W)</td>
<td>121</td>
<td>3.53</td>
<td>2.36</td>
<td>3.19</td>
</tr>
</tbody>
</table>

We see that the multi-antenna systems outperform the single-antenna system in terms of power consumption. They only consume roughly 2–3% of the power that the single-antenna system consumes. Admittedly, the single-antenna system is operating in its bandwidth limited region and the relative power difference would be smaller if more bandwidth were used or if the desired data rate were lowered.
On the other hand, the multi-antenna systems are operating in their power limited regimes and the data rate could easily be made significantly larger than 4.5 Mbit/s per user with little extra power.

The discrete-time constant-envelope system consumes 33% less energy than the system that uses zero-forcing. Doing PAR reduction in conjunction with precoding is thus worthwhile.

Further, we see that constraining the phase variation costs too much in terms of increased transmit power. It consumes less power than the zero-forcing system but still more power than the discrete-time constant-envelope scheme. Phase constraints are therefore not a feasible option alone. However, together with continuous phase modulation, it could still make sense, if a proper receiver structure could be developed.

The disk-constrained precoding scheme does not perform better than the discrete-time continuous-envelope scheme in this scenario, where the sum rate is low. It consumes 14% more power than the discrete-time constant-envelope precoding. If the desired data rate were higher, the disk-constrained precoding would get a bigger advantage over discrete-time constant-envelope precoding in terms of required transmit power. If we increase the desired data rate to 20 Mbit/s per user, then the disk-constrained precoding gains a small advantage over discrete-time constant-envelope precoding, see Table 8.8. The difference in total input power between the two schemes is, however, too small to be considered reliable.

<table>
<thead>
<tr>
<th>System 1</th>
<th>System 2</th>
<th>System 3</th>
<th>System 4</th>
<th>System 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Input Power, $P_{in}$</td>
<td>—</td>
<td>61.1</td>
<td>53.7</td>
<td>92.3</td>
</tr>
</tbody>
</table>

We also see in Table 8.8 that the gain of low-PAR precoding compared to zero-forcing is now only 12%. The gain of low-PAR precoding gets smaller at higher data rates, where the gap between the curves of required transmit power becomes wider, see Figure 8.1. The power required from the single-antenna system is impractically large at this high data rate and is left out.
Chapter 9

Conclusions

In the Problem Statement, the topics to be covered in this report were stated. Two areas of research were established on page 3 as the focus of this investigation. Let us now return to these two areas of investigation to give some conclusions.

1. Investigate discrete-time constant-envelope precoding, see if the precoding scheme can be extended and see how it compares to conventional precoding.

   • Thanks to the large number of antennae in a very-large multi-user MIMO system, it is possible to shape the transmit signals and lower their PAR in the process of precoding. This however comes at the price of lowered array gain, or, in other words, increased required transmit power.

     For example, a base station with 100 antennae that serves 10 users with zero-forcing precoding will have transmit signals with 10 dB PAR and an array gain of 9.55 dB. If, instead, discrete-time constant-envelope precoding was used, the PAR will be lowered to only 4.0 dB, but the array gain will drop to 6.4–8 dB, depending on the relative noise level of the receiver. The decreased PAR of the transmit signals will however enable more efficient operation of the power amplifiers, which might compensate for the increased transmit power and even decrease the total input power of a very-large multi-user MIMO base station.

   • A simulation of a practical 16-QAM transmission scheme has shown that the computations involved in the iterative search algorithm of discrete-time constant-envelope precoding are of relative low complexity and comparable to other linear precoding schemes, such as zero-forcing.

   • To optimise the discrete-time constant-envelope precoding, the transmitter needs to know the noise level of the user, which might or might not be available at the base station. For optimal operation, the energy scaling factor $E$, which depends on the noise level of the users, should be computed anew each time the channel or noise levels change; this is impractical. We suggest to compute a static value...
of the energy scaling factor, which works for the average channel realisation, a priori and store it in a look-up table. In a system with a very-large amount of antennae, this static choice does not degrade the performance significantly.

- By adding phase constraints to the discrete-time constant-envelope precoding scheme and limit the amount the phases of the transmit signals can vary between consecutive symbol durations, the PAR can be lowered further. It has been shown that the PAR can be made arbitrarily small by decreasing the allowed phase variation. The more the phases are constrained, though, the lower the array gain of the transmission will be. To achieve the same SINR at the users as without constraints, the transmit energy or the number of antennae has to be increased.

For example, if the phase variation is limited to $\pi/2$, the PAR is lowered to 2.6 dB, but 2–3 dB more transmit power is needed to maintain the same performance or, equivalently, the number of antennae has to be increased 1.6–2 times.

- Continuous phase modulation would make the PAR of the transmit signal 0 dB; we would thus get continuous-time signals with constant envelope, which are the ideal signals from a power amplifier point of view. It could allow us to operate the power amplifier at saturation, which is the most power efficient way of using a power amplifier. It has been shown that phase constraints can be used to lower the bandwidth of a continuous phase modulated signal. Since continuous phase modulation introduces inter-symbol interference, the optimal receiver structure is yet unknown. Until a receiver has been developed, it is hard to evaluate the practicality of this idea.

- By relaxing the amplitude constraints in the discrete-time constant-envelope precoding scheme and allowing the amplitudes to vary inside a circle, the PAR is not significantly increased; for any practical noise level, the PAR is at most 4.5 dB. At the same time, the performance, in terms of the transmit power required to achieve a given data rate, is improved. However, in the regime of high relative noise variance, this disk-constrained precoding scheme performed worse than the discrete-time constant-envelope precoding, because the search algorithm that was used failed to find the optimal precoding weights. In the regime of low relative noise variance, some improvements were seen; the required transmit power was decreased by 1 dB.

2. **Investigate whether handset technology could be used in very-large multi-user MIMO base stations.**

- The low power of each transmit signal in a very-large multi-user MIMO system might make the use of simple amplifiers without power-hungry linearisation techniques necessary, because doing linearisation would consume order of magnitudes more power than the power amplifier itself consumes.
A simplistic model of unlinearised, class AB power amplifiers—the type used in user equipment—has been developed and the practicality of using such amplifiers in a base station with 100 antennae serving 10 users has been tested in a simple link budget under some assumptions: frequency-flat fading, Gaussian signalling and perfect channel state information at the base station. The outcome of the link budget suggests that handset power amplifiers could be used in base stations with hundreds of antennae.

From this preliminary link budget the following conclusions can be drawn:

• Discrete-time constant-envelope precoding could save 33% of energy compared to conventional zero-forcing precoding at low data rates and 12% at high data rates. The energy savings are made possible by operating the power amplifier closer to its point of saturation, which can be done because the PAR of the transmit signals of discrete-time constant-envelope precoding is lower than for conventional precoding schemes.

• Multi-antenna systems consumes significantly less power than conventional base stations with only one transmit antenna; with all the assumptions of the link budget, a multi-antenna base station consumes 98% less power than the single-antenna base station.

• Constraining the phase variation of the transmit signals is not worthwhile. The improved power efficiency of the power amplifiers does not compensate for the increased required transmit power. Relative to discrete-time constant-envelope precoding, the phase-constrained precoding with $\Delta \theta = \pi/2$ consumed 14% more input power, even if the increased efficiency of the power amplifier due to the improved PAR of the phase-constrained transmit signals was taken into account.

• Only in the regime of high data rates, there seems to be a small improvement (4%) in required input power for the disk-constrained precoding compared to the discrete-time constant-envelope precoding.
Chapter 10

Further Research

We have shown that it is possible to lower the linearisation requirements of the power amplifier in very-large multi-user MIMO, but still much remains to be studied. Most interesting would be the possibility of continuous-time constant-envelope transmit signals by means of continuous phase modulation. This would enable the use of the cheapest and most power efficient power amplifiers on the market in the base station, which would make very-large multi-user MIMO an attractive technology for the future 5G standard.

In order to get more reliable estimates of the base station power consumption from the link budget, a more thorough study of real world power amplifiers has to be conducted, where distortion data from hardware measurements are taken into account, when determining the required back-off and power efficiency of the power amplifier. An investigation of handset power amplifiers would be essential before any definite numbers could be established. If such a study could indicate that the use of mass-produced handset power amplifiers were feasible in very-large multi-user MIMO systems, this would mean a paradigm shift in base station development, where high-end specialised hardware components no longer are necessary. This would make the new very-large multi-user MIMO base stations cheap and flexible, in production and maintenance as well as in operation.

The relaxed amplitude precoding technique proposed in Chapter 7 seems to provide a way of adjusting the level of PAR reduction to be made in the precoding. This would be of interest for high normalised transmit powers, where noise level at the user is low in comparison to the transmit power, there it is not optimal to limit the transmit signal to lie on the circle. An increased performance could be gained by relaxing the amplitude constraints. Maybe this precoding scheme could be developed into something that computes the optimal transmit signals given a PAR constraint, not by limiting the signal amplitudes from below, but by limiting the signal power from below. Then the power efficiency gain of low-PAR precoding can be balanced against the induced performance loss.

The use of wider bandwidths for increased data rates will mean that future communication systems have to be able to handle frequency-selective fading.
How low-PAR precoding should be done in a frequency-selective scenario remains to be studied. The modulation scheme OFDM is one possible way to handle frequency-selective channels, but its inherent high PAR makes it undesirable. In the yet unpublished paper of Mohammed et al. (2013b), the natural extension of the discrete-time constant-envelope precoding scheme to frequency-selective channels is made. In the paper, it is claimed that frequency-selective fading not only may be overcome but it might even be helpful in doing discrete-time constant-envelope precoding; helpful in the sense that the gap between the required transmit power of only power-constrained precoding and that of discrete-time constant-envelope precoding is narrower than it was in the scenario with frequency-flat fading.

How low-PAR precoding works in other fading environments than i.i.d. Rayleigh fading should also be studied. A channel model, where the fading coefficients in a row of the channel matrix are correlated and the coefficients of a column are uncorrelated, would be of special interest, because it would reflect the scenario, where the base station is mounted on a rooftop, which often is the case.

Another unresolved issue is the impact of having imperfect channel state information at the base station. To see how the performance of low-PAR precoding is degraded, and how this impact could be mitigated.
Appendix A

Derivation of the Achievable Sum Rate for Low-PAR Precoding

In this appendix, we will derive an achievable sum rate for discrete-time constant-envelope precoding, it will serve as a lower bound on the capacity of the transmission method. As we will see, the bound can be applied to many different transmission methods. We will use it when we evaluate discrete-time constant-envelope precoding, but also when we evaluate its phase constrained version and the disk-constrained precoding.

Let \( h \) denote the differential entropy of a random variable. Then the mutual information between the symbol vector \( s \) and the received signal \( r_k \) at user \( k \) can be lower bounded in the following way.

\[
I(s; r_k) \geq I(s_k; r_k) = h(s_k) - h(s_k | r_k) = h(s_k) - h\left(s_k - \frac{r_k}{\sqrt{P_E}} \right | r_k)
\]

(A.1)

\[
\geq h(s_k) - h\left(s_k - \frac{r_k}{\sqrt{P_E}}\right)
\]

(A.2)

\[
= h(s_k) - h\left(\frac{e_k}{\sqrt{E_k}} + \frac{w_k}{\sqrt{P_E}}\right)
\]

(A.3)

\[
= \log_2(\pi e) - h\left(\frac{e_k}{\sqrt{E_k}} + \frac{w_k}{\sqrt{P_E}}\right)
\]

(A.4)

\[
\geq \log_2(\pi e) - \log_2\left(\pi e \text{Var}\left(\frac{e_k}{\sqrt{E_k}} + \frac{w_k}{\sqrt{P_E}}\right)\right)
\]

(A.5)

\[
\geq - \log_2\left(\mathbb{E}\left[\frac{e_k}{\sqrt{E_k}} + \frac{w_k}{\sqrt{P_E}}\right]^2\right)
\]

(A.6)

\[
= - \log_2\left(\frac{\sigma^2}{P_E}\right)
\]

(A.7)
Derivation of the Achievable Sum Rate for Low-PAR Precoding

\[ R_{\text{inst}}(E_1, \ldots, E_K) = \sum_{k=1}^{K} \log_2 \left( \frac{P E_k}{P \mathbb{E}[|e_k|^2] + \sigma^2} \right) \quad \text{[bits]} \]  

(A.8)

In (A.1), the maximum instantaneous sum-rate is lower bounded by the sum-rate that can be achieved without doing interference cancellation, and then this sum-rate is expanded into a difference of two entropies. The next step (A.2) follows from the fact that conditioning reduces entropy. In (A.3) the full expression for \( r_k \) is used. Then we assume that the input alphabet is Gaussian. In (A.4) and (A.5), we use the fact that the entropy of a complex Gaussian random variable with variance \( \nu \) is \( \log(\pi e \nu) \). Since the entropy of a circular symmetric Gaussian random variable has the largest entropy among all random variables with the same variance (Cover, 2012), the expression can be lower bounded in (A.5). For any complex scalar random variable \( z \), \( \text{Var}(z) = \mathbb{E}[|z|^2] - |\mathbb{E}[z]|^2 \leq \mathbb{E}[|z|^2] \). This gives us the lower bound in (A.6). If we expand the square in (A.6) and move the expectation operator in, the cross terms will equal zero, because \( e_k \) and \( w_k \) are independent and \( w_k \) is zero mean.

Thus, for any choice of the parameter \( E \), we get an achievable sum-rate

\[ C_{\text{inst}}(E_1, \ldots, E_K) \geq R_{\text{inst}}(E_1, \ldots, E_K). \]

This lower bound is applicable in all the low-PAR precoding schemes that we have investigated in this report.
Appendix B

The Region of Possible Receive Signals with Disk Constraints

In this appendix, we will discuss how the region of possible receive signals looks like when we relax the constant-envelope constraint to allow a slight variation in the envelopes of the transmit signals, and allow the modulus of each transmit signal to vary within the interval \([A - \epsilon, A]\), where \(\epsilon > 0\). Additional consideration has to be paid to the power constraint when the amplitudes are allowed to vary. This makes the characterisation not so straightforward as in the constant-envelope case. Due to the path-connectedness (see Lemma 4.3) and the uniform-phase-shift property (see Lemma 4.2), the set is still an annulus in the complex plane.

If the power constraint would be relaxed, i.e. \(A\) and \(\epsilon\) chosen such that \(MA^2 \leq 1\), then the set of possible receive signals in the single-user case can be characterised similarly to the constant-envelope single-user case. Compare the following lemma to Theorem 4.7.

**Lemma B.1.** Given a channel realisation \(h\) of dimension \(M\) and two non-negative real numbers \(\epsilon > 0\) and \(A \geq 0\) such that \(MA^2 \leq 1\), the set \(\mathcal{M}_\epsilon(h)\) is an annulus in the complex plane characterised by:

\[
\mathcal{M}_\epsilon(h) = \left\{ r : (2(A - \epsilon)\|h\|_\infty - A\|h\|_1)^+ \leq |r| \leq A\|h\|_1 \right\},
\]

where \(\|h\|_\infty\) is the infinity-norm of \(h\), \(\|h\|_1\) the one-norm and \((x)^+ = \max(x, 0)\).

**Proof.** We note that we can move the transmit amplitudes to the channel coefficients and thus transform the problem to a class of constant-envelope precoding problems. Define the set of all modified channel vectors thus:

\[
\mathcal{H} = \{ \tilde{h} = (A_1h_1, \ldots, A_Mh_M)^T : (A_1, \ldots, A_M) \in [A - \epsilon, A]^M \},
\]

The \(\epsilon\)-relaxed set of possible receive signals can then be seen as the union of sets of possible receive signals with constant-envelope constraints and with modified...
channel vectors:

\[ \mathcal{M}_\epsilon(h) = \bigcup_{\tilde{h} \in \mathcal{F}} \mathcal{M}_0(\tilde{h}) \]

These constant-envelope constrained sets are characterised by Theorem 4.7. The moduli of the elements of \( \mathcal{M}_\epsilon(h) \) can therefore not be bigger than the radius of the widest \( \mathcal{M}_0(\tilde{h}) \), which is \( A\|h\|_1 \). Nor can they be smaller than the inner radius of the narrowest \( \mathcal{M}_0(\tilde{h}) \), which is \( (2(A - \epsilon)\|h\|_\infty - A\|h\|_1)^+ \).

According to Lemma 4.3, the set \( \mathcal{M}_\epsilon(h) \) is path connected. The union of annuli therefore covers all complex numbers with moduli between \( (2(A - \epsilon)\|h\|_\infty - A\|h\|_1)^+ \) and \( A\|h\|_1 \).

We now study the upper bound \( D(h) \) of the set with the proper power constraint. The following optimisation problem will give the outer radius of the annulus of possible receive signals.

\[
D(h) = \max_{\{(A_m, \theta_m)\}_{m=1}^M : \sum A_m^2 \leq P} \left| \sum_{m=1}^M A_m e^{j\theta_m} \hat{h}_m \right| \leq \max_{\sum A_m^2 \leq P} \sum_{m=1}^M A_m |h_m|
\]

The triangle inequality simplifies the optimisation; equality above is given by choosing \( \theta_m = -\arg h_m \). This problem can be solved using convex optimisation algorithms. If we relax the problem further, we will see how the solution can be recovered in a finite number of iterations.

\[
D(h) \leq \max_{\sum A_m^2 \leq P} \sum_{m=1}^M A_m |h_m| \leq \|h\| \sqrt{\sum_{m=1}^M A_m^2} \leq \|h\| \sqrt{P}.
\]

where the second inequality follows from Schwartz’s inequality. Equality, and thus maximum, is achieved by choosing

\[
A_m = \sqrt{P} \frac{|h_m|}{\|h\|}.
\]

Now we remember that we originally had confined \( A_m \) to lie inside the interval \([A - \epsilon, A]\). Since the this additional constraint is convex, we are still able to compute the optimal amplitudes through the following iterative method. Consider the choice of amplitudes, where we have truncated the amplitudes that are outside the interval:

\[
A_m^{(n+1)} = \begin{cases} 
A_m^{(n)}, & \text{if } m \in \mathcal{F}_n \\
A, & \text{if } \sqrt{P_{n+1} \frac{|h_m^{(n+1)}|}{\|h_m^{(n+1)}\|}} > A \\
\sqrt{P_{n+1} \frac{|h_m^{(n+1)}|}{\|h_m^{(n+1)}\|}}, & \text{if } \sqrt{P_{n+1} \frac{|h_m^{(n+1)}|}{\|h_m^{(n+1)}\|}} \in [A - \epsilon, A] \\
A - \epsilon, & \text{if } \sqrt{P_{n+1} \frac{|h_m^{(n+1)}|}{\|h_m^{(n+1)}\|}} < A - \epsilon \text{ and } m \notin \mathcal{F}_n
\end{cases}, \quad \forall m = 1, \ldots, M.
\]
where $\mathcal{F}_0 = \emptyset$, $P_1 = P$, $h_1 = h$. We call the amplitudes that have been set to one of the endpoints of the interval fixed and define the index set $\mathcal{F}_n = \{ m : A_m^{(n)} = A \text{ or } A - \epsilon \}$ as the indices of all the fixed amplitudes after iteration $n$. In each iteration we readjust the non-fixed amplitudes, by considering a new power constraint\footnote{This new power constraint will only make sense if it is non-negative, which it will be if $M(A - \epsilon)^2 \leq 1$.} $P_{n+1} = P_n - \sum_{m \in \mathcal{F}_n} A_m^{2}$ and a channel realisation $h_{n+1} = (h_{1}^{(n+1)} \ldots h_{M}^{(n+1)})^T$. where

$$h_{m}^{(n+1)} = \begin{cases} h_{m}^{(n)}, & \text{if } m \notin \mathcal{F}_n \\ 0, & \text{if } m \in \mathcal{F}_n \end{cases}$$

which consists of the channel coefficients “left”. Because the number of antennae is finite, the algorithm is bound to terminate after a certain index $N$, in the sense $A_m^{(n)} = A_m^{(n+1)}$, $\forall m$, for all $n \geq N$. The optimal amplitudes are then given by $A_m^{(\max)} = A_m^{(N)}$, $\forall m = 1, \ldots, M$.

If $\mathcal{F}_N^{(-)}$ are the indices of the amplitudes fixed to the lower limit of the interval and $\mathcal{F}_N^{(+)}$ the indices of the amplitudes fixed to the upper limit, the pre-coding weights $(A_1^{(\max)} e^{-j \arg h_1}, \ldots, A_M^{(\max)} e^{-j \arg h_M})^T$ will thus give the maximum amplitude:

$$D(h) = \sum_{m \in \mathcal{F}_N^{(-)}} (A - \epsilon)|h_m| + \sum_{m \in \mathcal{F}_N^{(+)}} (A)|h_m| + \sqrt{P_N}||h_N||.$$
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Bibliography


