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Dynamic Lotsizing with a Finite Production Rate

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Abstract

The *dynamic lotsizing problem* concerns the determination of optimally produced/delivered batch quantities, when demand, which is to be satisfied, is distributed over time in different amounts at different times. The standard formulation assumes that these batches are provided instantaneously, i.e. that the production rate is infinite.

Using a cumulative geometrical representation for demand and production, it has previously been demonstrated that the *inner-corner condition* for an optimal production plan reduces the number of possible optimal replenishment times to a finite set of given points, at which replenishments can be made. The problem is thereby turned into choosing from a set of zero/one decisions; whether or not to replenish at each time there is a demand. If n is the number of demand events, this provides 2^{n-1} alternatives, of which at least one solution must be optimal. This condition applies, whether an Average Cost approach or the Net Present Value principle is applied, and the condition is valid in continuous time, and therefore in discrete time.

In this current paper, the assumption of an infinite production rate is relaxed, and consequences for the inner-corner condition are investigated. It is then shown that the inner-corner condition needs to be modified to a *tangency condition* between cumulative requirements and cumulative production.

Also, we have confirmed the additional restriction for feasibility in the finite-production case (provided by R.M. Hill, 1997), namely the *production rate restriction*. Furthermore, in the NPV case, one further necessary condition for optimality, *the distance restriction* concerning the proximity between adjacent production intervals, has been derived. In an example this condition has shown to reduce the number of candidate solutions for optimality still further. An algorithm leading to the optimal solution is presented.

Keywords: Dynamic lotsizing, finite production rate, net present value, economic order quantity, EOQ, economic production quantity, EPQ, binary approach.

1. Introduction and notation

The dynamic lotsizing problem is a variational problem, namely to determine cumulative production over time requiring cumulative production to be at least cumulative requirements at each point in time optimising an objective function. The issue of determining how much to produce (optimal lotsizes) has a history of close to a century (Harris 1913, Erlenkotter 1989). Half way through this history the dynamic lotsizing problem was formulated and solved by a dynamic programming approach (Wagner and Whitin 1958). Several other algorithms have later been suggested, such as (Silver and Meal 1973, Federgruen and Tzur 1991, Grubbström 1999, 2005). For an early overview, see (De Bodt et al. 1984).

The majority of approaches have used a discrete time framework in which times (periods), at which requirements appear, are integers. The current author has argued against this restriction for a long time, instead advocating the use of a continuous time framework (of which discrete timing is a special case), and has expressed the dynamic lot-sizing problem in continuous time, including continuous-time formulations of the Wagner-Whitin and Silver-Meal algorithms (Grubbström 1997, 1999, 2005). The *capacitated* dynamic lotsizing problem in continuous time was introduced by Khmelnitsky and Tzur (2004).

The overwhelming volume of literature on the subject of lotsizing is based upon an *Average Cost* (AC) approach, in particular balancing inventory holding costs against fixed setup costs for each batch produced. Against this stands the *Net Present Value* (NPV) principle, based on the monetary stream consequences of decisions (cash flows), compatible with financial methodology. Through the years attention has been given to the differences these two approaches (Hadley 1964, Trippi and Levin 1974, Grubbström 1980, Kim and Philippatos 1986) and in recent years, an increasing interest in the use of the NPV principle for lotsizing optimisation has appeared (Teunter and van der Laan 2002, Grubbström and Kingsman 2004, Beullens and Janssens 2011). For standard problems, the average cost is shown to correspond to a mixed zeroth and first order approximation in the interest rate of the NPV expression.

It has been recognised for a long time that the dynamic lotsizing problem may be viewed as a binary problem, whether or not to replenish at times when requirements appear (Veinott 1969). This binary view led to the “inner-corner” condition for a production plan to qualify as a candidate for optimality, and it holds even for complex multi-item production structures (Grubbström, Bogataj and Bogataj 2009, Grubbström and Tang 2012). The inner-corner condition is a geometrical statement, in the discrete time case equivalent with “Inventory is not to be carried into a period where production takes place” (Aryanezhad 1992, pp. 425, 427).

The only two papers in the literature we have found concentrating on the current problem of dynamic lotsizing with a finite production rate is that of Roger M. Hill (1997), and of Y. Song and Gin Hor Chan (2005). In the first of these references, the problem is stated exactly as here, but in both references the authors restrict the objective function to applying only the AC principle. Neither Hill nor Song and Chan use cumulative functions for requirements and production, and, at least not explicitly, they do not formulate the decision variables in binary form. Because cumulate functions are not used, the inner-corner condition cannot be recognised geometrically. In Hill, however, what below is called the *production rate restriction*, is formulated in a similar way, and the main theorem offered by Hill is the same result as ours, but its validity is limited to the AC objective assumption. Whereas Hill formulates and solves the problem in a continuous time framework, Song and Chan start with continuous time, but later split time into discrete periods. But Song and Chan generalise the problem of Hill to a case when AC includes backlogging costs, which is not considered here. Both of these papers end up with proposing a dynamic programming algorithm.

The original dynamic lotsizing problem assumes that replenishments take place instantaneously, irrespective of their size. In the current paper the goal is to relax this restriction and study the consequences for the inner-corner condition when the production rate is finite.

The following main notation is introduced (additional notation will be used as the need arises).

n	Given number of requirement events.
t_i	Given time at which the ith requirement appears, i = 1, 2, …, n.
$\tau_i = t_{i+1} - t_i$	Time interval between (i + 1)st and ith requirement events, i = 1 , 2, … , n - 1.
D_i	Given size of requirement at time t_i, i = 1, 2, …, n.
$\overline{D}_i = \sum_{j=1}^{i} D_j$	Cumulative requirements immediately after time t_i.
$\overline{P}_i$	Cumulative production immediately after time t_i.
c	Unit production cost (out-payment).
K	Setup cost for producing one batch (ordering cost).
h	Inventory holding cost per unit and time unit.
ρ	Continuous interest rate, per time unit.
q	Finite production rate, units per time unit.

The section following provides a summary of the classical dynamic lotsizing problem (infinite production rate) when using binary decision variables. Section 3 extends the theory to the case of a constant finite production rate and includes our main theorem and two corollaries. Here a new result is that for optimality two successive production batches may not differ in time more than by a specified expression of the given parameters. In Section 4 expressions are developed for the objective functions, followed by a section with two numerical examples. We finalise with conclusions and a list of references.

2. Brief description of Dynamic Lotsizing with infinite production rate

In the case of an infinite production rate (treated in essentially all papers hitherto) cumulative demand follows a staircase function and so does cumulative production when the production rate is infinite. A feasible production plan is any staircase function above or touches the cumulative requirements from above. The basic question is which production staircase to choose when the objective function is to be optimised.

It has been shown earlier that the only solutions that can be qualified as candidates for an optimal solution, whether the NPV or AC measure is the objective, are cumulative staircases that fit into inner corners as shown in Figure 1. This has the consequence that the set of optimal production solutions must be found from those either having an inner-corner contact at a requirement event, or no production. This restricts the number of feasible optimal solution to 2^{n-1}, since always there must be a contact at the first inner corner (start of process).

Figure 1 shows the given demand staircase (Curve A, bold) and three types of feasible cumulative production curves (no shortages). Curve B (dotted) illustrates the most general case of feasible production (any non-decreasing curve above or possibly touching cumulative demand), Curve C (dotted and dashed) the general case of feasible production taking place in batches, making cumulative production a staircase function, and Curve D (dashed) a batch-production case which candidates for optimality, since it meets the inner-corner condition.

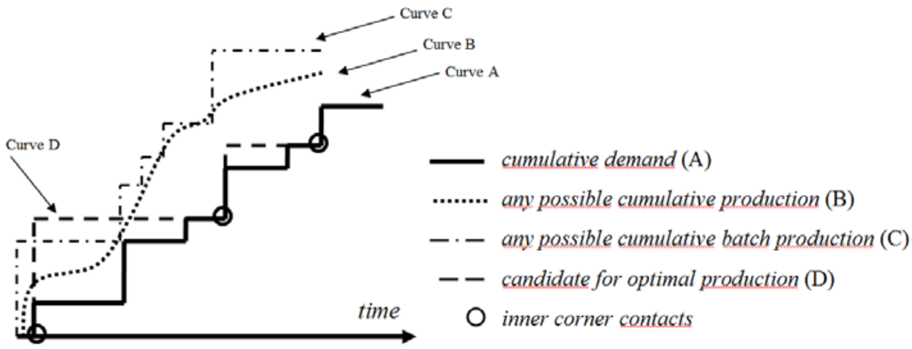

Figure 1. Illustration of inner-corner condition. From (Grubbström and Tang 2012).

Using the binary variables α_j, with $\alpha_j = 1$ meaning a setup at t_j, and $\alpha_j = 0$, if not, cumulative production immediately after t_{j-1} will either be $\overline{P}_{j-1} = \overline{D}_{j-1}$ or $\overline{P}_{j-1} = \overline{P}_j$, respectively. As earlier shown, (for instance in Grubbström, et al., 2009), this leads to a unique solution

$$\overline{P}_j = \left(\overline{D}_j + \sum_{k=j+1}^{n} D_k \prod_{l=j+1}^{k} (1-\alpha_l) \right), \tag{1}$$

showing how cumulative production depends on the binary decision variables α_i and the given requirements. Here, as in the following, the convention that $\prod_{l=j+1}^{j} (1-\alpha_l) = 1$ is adopted. The batch size of production when it takes place (the lotsize), is then given by

$$Q_j = \overline{P}_j - \overline{P}_{j-1} = \alpha_j \sum_{k=j}^{n} D_k \prod_{l=j+1}^{k} (1-\alpha_l). \tag{2}$$

If there is no setup at t_j, i.e. $\alpha_j = 0$, then $Q_j = 0$.

We may also inquire as to how the timing of supply depends on the decision variables α_j. Let $\overline{T}_j$ denote the time that the supplies immediately after t_j last until. Then, if there is a setup at t_j, then $\alpha_j = 1$ and $\overline{T}_{j-1} = t_j$, and if not, $\alpha_j = 0$, and $\overline{T}_j = \overline{T}_{j-1}$. This leads to the solution

$$\overline{T}_j = t_{j+1} + \sum_{k=j+1}^{n-1} \tau_k \prod_{l=j+1}^{k} (1-\alpha_l), \tag{3}$$

and the difference between $\overline{T}_j$ and $\overline{T}_{j-1}$, is obtained as

$$T_j = \overline{T}_j - \overline{T}_{j-1} = \alpha_j \sum_{k=j}^{n-1} \tau_k \prod_{l=j+1}^{k} (1-\alpha_l). \tag{4}$$

If $\alpha_j = 1$, then $\overline{T}_j$ is the length of the horizontal production step following t_j.

Applying the AC objective, it is intuitively clear that for a sufficiently high setup cost/holding cost ratio *K*/*h*, this must result in only one setup, the All-At-Once solution, whereas for a sufficiently small such ratio, the holding cost will always dominate, and the resulting optimum must be the Lot-For-Lot solution. The dynamic lotsizing problem thus exists only for intermediate ratios.

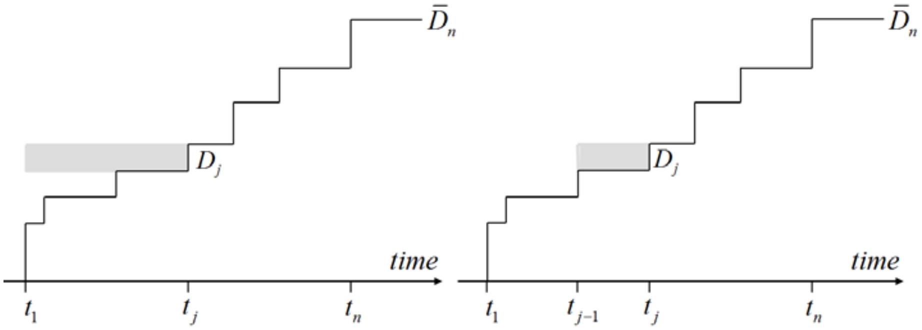

Figure 2. Illustration to extreme cases for setup/holding cost ratio.

Consider Figure 2. If *K*/*h* is larger than the shaded area in the left-hand diagram for all steps, no savings can be obtained by having more than one setup, i.e. the All-At-Once case is optimal. If instead this ratio is smaller than the smallest of the boxes illustrated in the right-hand diagram, then it can never be profitable having any inventory, i.e. the Lot-For-Lot case is obtained.

Instead using NPV as the objective, similar results obtain for parameters *K*, *c*, and ρ. For simplicity let $t_1 = 0$. In the right-hand part of the figure, if production of D_j is moved from t_j to t_{j-1} and the avoided discounted setup cost $Ke^{-\rho t_j}$ is smaller than the increase in NPV from moving production $cD_j\left(e^{-\rho t_{j-1}} - e^{-\rho t_j}\right)$ for all *j*, i.e. $K < c \operatorname*{Min}_{j \geq 2} D_j \left(e^{\rho(t_j - t_{j-1})} - 1\right)$, then the L4L solution must obtain. In the left part of the figure, if the discounted setup cost $Ke^{-\rho t_j}$ at t_j outweighs the increase in NPV from moving production of D_j from t_j to the beginning $t_1 = 0$, starting at *j* = 2, then *j* = 3, ... , *j* = *n*, i.e. $Ke^{-\rho t_j} > cD_j\left(1 - e^{-\rho t_j}\right)$ for all *j*, then all production should be allocated at t_1, i.e. the All-At-Once solution must obtain if $K > c \operatorname*{Max}_{j \geq 2} D_j \left(e^{\rho t_j} - 1\right)$. So,

$$\text{All-At-Once, if } K > c \operatorname*{Max}_{j \geq 2} D_j \left(e^{\rho t_j} - 1\right) \approx h \operatorname*{Max}_{j \geq 2} D_j t_j ,$$

$$\text{Lot-For-Lot, if } K < c \operatorname*{Min}_{j \geq 2} D_j \left(e^{\rho(t_j - t_{j-1})} - 1\right) \approx h \operatorname*{Min}_{j \geq 2} D_j \left(t_j - t_{j-1}\right), \tag{5}$$

where a first-order approximation in ρ has been used and *h* is interpreted as $c\rho$.

3. Extension to a finite production rate

We now turn our attention to the case that production (a replenishment) does not take place instantaneously, but rather at a finite production rate *q*. This is a similar generalisation as from the *Economic Order Quantity* (EOQ) to the *Economic Production Quantity* (EPQ).

Throughout we continue to assume that requirements are given finite amounts D_i at given times t_i, and that therefore cumulative requirements form a staircase function of time.

First, we may note that cumulative production no longer is a staircase function of time, but instead a set of adjoined ramp segments, separated by horizontal steps when there is no production. Figure 3 illustrates the basic form of the cumulative production function.

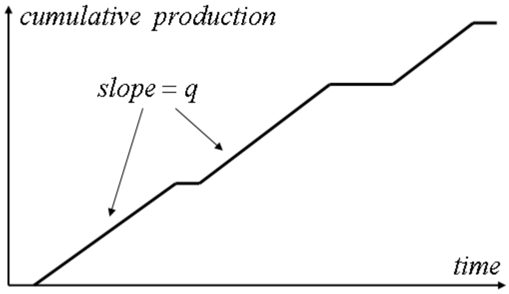

Figure 3. General form of cumulative production.

As before, the general condition for feasibility is that cumulative production always is at least as high as cumulative requirements. However, with a finite production rate, the conditions for feasibility become more complex than in the classical case.

As a first development, we find that there may be cases when subsets of requirement events do not have any influence on optimality considerations. As is illustrated by Figure 4, necessary for a requirement to have any influence on the choice of production is that each point on the requirements staircase can be reached. So the latest point to start initial production from is when the first production ramp touches the requirements staircase.

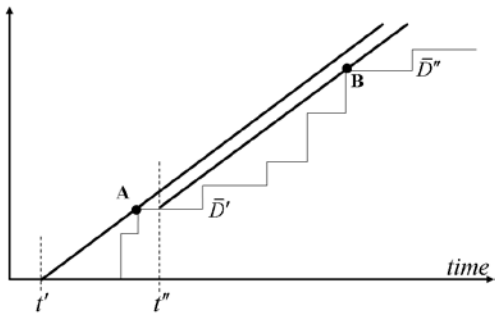

Figure 4. Production rate restriction.

Writing this point in time as t', this time must satisfy $t' \le \left(t_i - \bar{D}_i / q\right)$ for all *i*, so:

$$t' = \underset{i}{\text{Min}}\left(t_i - \bar{D}_i / q\right). \tag{6}$$

This implies that requirements that appear earlier than at the point of tangency marked **A** in Figure 4, are irrelevant to the problem. Let cumulative requirements at the point **A** be denoted $\bar{D}'$. Similarly, a point t'' may be defined as the latest a second batch may start to be produced, possibly resulting in further irrelevant requirement events and defining a second level $\bar{D}''$, as the figure shows:

$$t'' = \underset{i,t_i > t'}{\mathrm{Min}} \left(t_i - \left(\bar{D}_i - \bar{D}' \right) / q \right). \tag{7}$$

The further extensions to third, fourth, etc., times and levels are obvious.

We may regard requirement events that have no influence on the problem in this sense as "dominated" by the production rate *q*. Such events may be ignored, and the remaining requirement events renumbered accordingly. We assume that this is the case in the following. However, we may note that the size of the set that may be eliminated in this way depends strongly on the level of the production rate. With a higher rate, the less chance of dominance there is. This elimination rule is called the *production rate restriction*. It was probably observed first by Hill (1997).

If the production rate *q* is sufficiently small, all requirement events will be dominated except the last. This is easily found to occur when $q \le \left(\bar{D}_n - \bar{D}_j \right) / \left(t_n - t_j \right)$, for all $j < n$, i.e.

$$q \le \min_{j<n} \left(\bar{D}_n - \bar{D}_j \right) / \left(t_n - t_j \right) \tag{8}$$

and thus results in the All-At-Once solution.

We now develop our main theorem corresponding to the necessity of inner-corner contacts for optimality in the classical case.

Theorem 1
Necessary for optimality, either the NPV or total cost is the objective function, is that each production segment starts from a horizontal step of cumulative requirements.

The final production segment will end at the same level as the final horizontal step of cumulate requirements, since there is never any reason to produce too much, so by the theorem, all production ramps start and end on horizontal steps of cumulative requirements.

Since there are *n* steps in the requirements staircase, and there must be a setup before the first step, the number of solution alternatives is 2^{n-1}, where *n* possibly has been reduced by the production rate restriction. This number is the same as in the classical case, except for the possible reduction in *n* by the production rate restriction.

In the finite production rate case currently investigated, the timing of the setup cost within each segment has an influence, when adopting the NPV principle. This flexibility is new to NPV and does not exist when keeping to the AC objective. When applying the AC measure, also interest considerations concerning timing of setup costs are normally disregarded.

Here, we investigate two extreme cases, when either the setup cost is assigned to the start of the segment, or, alternatively, to the end of the segment. Deviations from these two cases could easily be analysed.

Proof

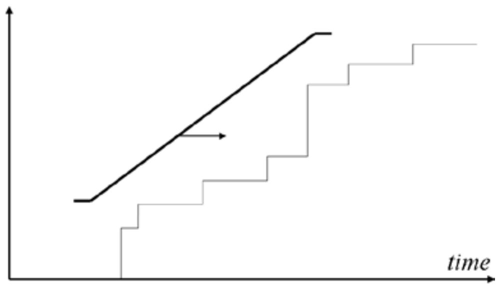

Figure 5. A segment not touching cumulative requirements may be moved to the right lowering costs (increasing NPV).

Consider Figure 5. If a production segment does not touch any corner of cumulative requirements, it is feasible to move the segment to the right, and any such move will postpone production out-payments as well as the setup cost (wherever it is timed) reducing the NPV of out-payments. Applying the AC measure instead, setup costs will not change, but inventory will be reduced, lowering holding costs. Therefore a segment belonging to an optimal production plan must touch a corner of cumulative requirements.

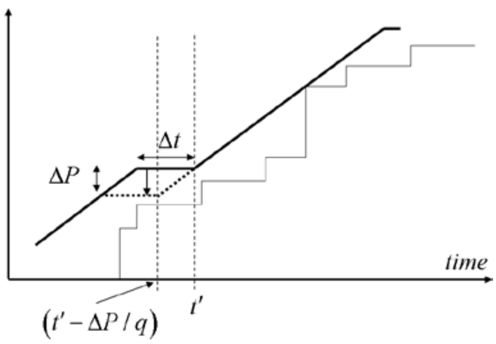

Figure 6. A segment not starting on a horizontal step of cumulative requirements is feasible to start earlier.

Consider two adjacent segments. If the later segment does not start on a horizontal step of cumulative requirements, it is feasible to start this segment earlier by moving production from the earlier segment to the later, as illustrated in Figure 6. Assume that the time difference between the segments is Δt and that ΔP of production is moved from the earlier to the later segment. The dotted lines in the figure show the situation after such a change. The left segment will then end $\Delta P / q$ earlier than before, and the right segment start $\Delta P / q$ earlier than before. We designate the start of the right segment as time t' before the change.

In the AC case, setup costs will not change, but inventories will be lowered making the change profitable. The change can be made until the right segment starts from a horizontal step of cumulative requirements. So optimality, applying the AC measure, requires that each production segment starts from a horizontal step of cumulative requirements.

In the NPV case, the reasoning becomes more intricate. The top part of the left segment (indicated by ΔP in Figure 6) accounts for a constant out-payment flow cq beginning at $t' - \Delta t - \Delta P / q$ and ending at $t' - \Delta t$. The original contribution to the NPV (before the move) discounted to the start of this part of the segment, $t' - \Delta t - \Delta P / q$, is thus

$-(cq/\rho)\left(1-e^{-\rho\Delta P/q}\right)$ and discounted to time zero it will be $-(cq/\rho)\left(1-e^{-\rho\Delta P/q}\right)e^{-\rho(t'-\Delta P/q-\Delta t)}$. After the move, the NPV from the segment part (dotted part of right ramp in Figure 6) will be $-(cq/\rho)\left(1-e^{-\rho\Delta P/q}\right)e^{-\rho(t'-\Delta P/q-\Delta t)}e^{-\rho\Delta t}$, since the payment flow is uniformly moved Δt time units into the future. The consequence in NPV from postponing production out-payments therefore amounts to $(cq/\rho)\left(1-e^{-\rho\Delta P/q}\right)e^{-\rho(t'-\Delta P/q-\Delta t)}\left(1-e^{-\rho\Delta t}\right)$ = $(cq/\rho)e^{-\rho t'}\left(e^{\rho\Delta P/q}-1\right)\left(e^{\rho\Delta t}-1\right)$.

But there is also an NPV consequence from the changed timing of setups. If the setup cost is allocated to the beginning of each production segment, this timing will be moved from t' to $(t'-\Delta P/q)$ and the NPV consequence will be $-Ke^{-\rho(t'-\Delta P/q)}+Ke^{-\rho t'}$. Adding these consequences together, the total change in NPV becomes:

$$\Delta\text{NPV} = e^{-\rho t'}\left(e^{\rho\Delta P/q}-1\right)\left(\frac{cq}{\rho}\left(e^{\rho\Delta t}-1\right)-K\right). \tag{9}$$

This change is not necessarily positive (as the AC reasoning would have indicated), since it depends on the sign of the factor $\left((cq/\rho)\left(e^{\rho\Delta t}-1\right)-K\right)$. If this factor is positive, it is profitable to make the change, otherwise not. The change could then be made until the right segment starts from a horizontal step of cumulative requirements (as with the AC measure). Rewriting this condition slightly, profitability of the downward move requires

$$e^{\rho\Delta t} > 1+\frac{\rho K}{cq}, \text{ or, equivalently, } \Delta t > \frac{1}{\rho}\ln\left(1+\frac{\rho K}{cq}\right) \approx \frac{K}{cq}. \tag{10}$$

It might be interesting to note that this condition, using the last first-order approximation, is independent of the interest rate ρ.

Keeping to the assumption of setup costs being allocated to the start of each segment, we investigate making an upward move, handing over production from the right to the left segment, as is depicted in Figure 7. Such a move is always feasible.

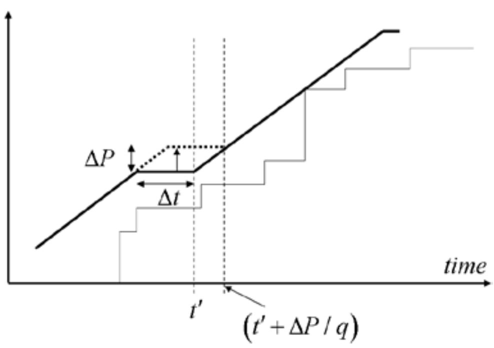

Figure 7. A segment is always feasible to start later.

With the notation from this figure, the NPV consequences for production out-payments become $-(cq/\rho)\left(1-e^{-\rho\Delta P/q}\right)e^{-\rho(t'-\Delta t)}+(cq/\rho)\left(1-e^{-\rho\Delta P/q}\right)e^{-\rho t'}$ and the setup consequences

$-Ke^{-\rho(t'+\Delta P/q)}+Ke^{-\rho t'}$. Adding these together to form ΔNPV, the move upwards will be profitable, if

$$\Delta\text{NPV}=e^{-\rho t'}\left(1-e^{-\rho\Delta P/q}\right)\left(K-\frac{cq}{\rho}\left(e^{\rho\Delta t}-1\right)\right) \tag{11}$$

is positive (which depends solely on the last factor). If ΔNPV happens to be zero, there is no consequence from a small move, but the move can be made all the way up until the segment disappears. At that point a setup cost is eliminated, so the condition for a move upwards to be profitable may be written

$$e^{\rho\Delta t}\le 1+\frac{\rho K}{cq}\text{, or, equivalently } \Delta t\le\frac{1}{\rho}\ln\left(1+\frac{\rho K}{cq}\right)\approx\frac{K}{cq}. \tag{12}$$

In either the downward or upward case, our conclusion is that it cannot be profitable to have a segment not starting on a horizontal step of cumulative requirements.

Now, we turn to the alternative location of setup costs, namely at the end of each segment. With either an upward or a downward move, the NPV consequences from production out-payments will be the same as reported above. For the NPV consequences from setup changes, for the downward move, these will be $-Ke^{-\rho(t'-\Delta t-\Delta P/q)}+Ke^{-\rho(t'-\Delta t)}$ and for the upward move $-Ke^{-\rho(t'-\Delta t+\Delta P/q)}+Ke^{-\rho(t'-\Delta t)}$. When setup costs are assigned to the start of segments, the economic consequences concern the setup moving along the right segment, and vice versa. The total NPV change for a downward move will be

$$\Delta\text{NPV}=e^{-\rho(t'-\Delta t)}\left(e^{\rho\Delta D/q}-1\right)\left(\frac{cq}{\rho}\left(1-e^{-\rho\Delta t}\right)-K\right), \tag{13}$$

and for an upward move

$$\Delta\text{NPV}=e^{-\rho(t'-\Delta t)}\left(1-e^{-\rho\Delta D/q}\right)\left(K-\frac{cq}{\rho}\left(1-e^{-\rho\Delta t}\right)\right). \tag{14}$$

The profitability of a move is then determined by the sign of the last factor, an upward move being profitable, if

$$K\ge\frac{cq}{\rho}\left(1-e^{-\rho\Delta t}\right)\text{, or, equivalently, } \Delta t\le-\frac{1}{\rho}\ln\left(1-\frac{\rho K}{cq}\right)\approx\frac{K}{cq}, \tag{15}$$

and a downward move profitable otherwise. It might be noted that if $\rho K\ge cq$, then (15) is always valid and an upward move with eventually cancelling the segment in question is desirable. But this conclusion is restricted to setup costs being assigned to end points of segments.

From our developments above, we conclude that for optimality, each production segment must start from a horizontal step of cumulative requirements. This finalises our proof of Theorem 1. ■

As two corollaries we state:

Corollary 1
If two adjacent segments are separated in time less than or equal to $(1/\rho)\ln\left(1+\left(\rho K/(cq)\right)\right)\approx K/(cq)$ *(setup costs at start of segments) or less than or equal to* $-(1/\rho)\ln\left(1-\rho K/(cq)\right)\approx K/(cq)$ *(setup costs at end of segments), the inclusion of both of these segments is unprofitable.*

This is a (new) necessary condition for optimality, which we call the *distance restriction*.

Proof. The proof is given by Eqs (12) and (15). ■

Corollary 2
If the production rate restriction has been applied eliminating all unnecessary requirement events, a segment starting on a horizontal step of cumulative requirements will necessarily touch the nearest corner of cumulative requirements.

Proof. If this were not the case, further unnecessary requirements could be eliminated by the production rate restriction. ■

The distance restriction may be interpreted in the following way. Postponing a portion of a production ramp by Δt would save the amount $cq\Delta t$ at the expense of an additional setup costing K. Hence, disregarding interest effects, profitability of the move would require $\Delta t > K/(cq)$.

As a final exemplification in this section, Figure 8 illustrates the two extreme ordering policies Lot-For-Lot (L4L) and All-At-Once ($\forall @1$) for our current case of dynamic lotsizing with a finite production rate.

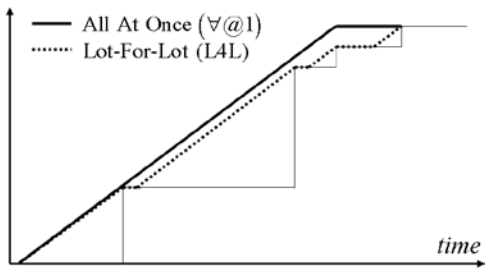

Figure 8. The All-At-Once ($\forall @1$*) and Lot-For-Lot (L4L) solutions to the dynamic lotsizing problem with a finite production rate.*

The L4L solution implies a policy that keeps inventories over time at a level as low as ever possible, whereas the $\forall @1$ solution keeps the number of setups to a minimum, which is always one (this setup covering the whole horizon). But the L4L solution also plays other roles concerning the timing of internal demand in MRP systems (Grubbström, Bogataj, and Bogataj 2009, Grubbström and Tang 2012) and the long-run average balance of flows between different parts of a production-inventory system (Grubbström 2012).

4. Expressions for the Net Present Value and Total Costs and a Solution Algorithm

Let us now examine the economic consequence of production decisions in terms of the binary production decisions α_j when the production rate is at a finite level q.

First, we define $\bar{P}_j$ as the cumulate production level reached from a production segment that started in the interval $\left(t_{j-1}, t_j\right)$ at the horizontal step of cumulative requirements with the level $\bar{D}_{j-1}$. If a batch is started in this interval we have $\alpha_j = 1$, and if not, then $\alpha_j = 0$. Although cumulative production now follows sloped segments, the same basic conditions as for the classical case (1)-(2) still apply, If $\alpha_j = 1$, then $\bar{P}_{j-1} = \bar{D}_{j-1}$, and if $\alpha_j = 0$, then $\bar{P}_j = \bar{P}_{j-1}$. This gives the difference equation $\left(\left(1-\alpha_j\right)\bar{P}_j - \bar{P}_{j-1} + \alpha_j \bar{D}_{j-1}\right) = 0$ and the solution for $\bar{P}_j$ has the same form as in the classical problem (1). Defining the batch size as $Q_j = \bar{P}_j - \bar{P}_{j-1}$ for the segment starting in interval $\left(t_{j-1}, t_j\right)$ this must then obey (2). Whereas the batch Q_j in the classical case started at t_j, it now must start at $t_j - D_j / q$, since this segment touches the corner of cumulative requirements at the coordinate $\left(t_j, \bar{D}_j\right)$ according to Corollary 2.

The net present value of production out-payments for the batch Q_j discounted to its start at $t_j - D_j / q$, will be $-\left(cq/\rho\right)\left(1-e^{-\rho Q_j/q}\right)$ and discounted to time 0, it will be $-\left(cq/\rho\right)\left(1-e^{-\rho Q_j/q}\right)e^{-\rho(t_j - D_j/q)}$, and the NPV of the setup cost discounted to time 0 if it is located at the segment starting point $\alpha_j K e^{-\rho(t_j - D_j/q)}$, and if located at the end point $\alpha_j K e^{-\rho(t_j - D_j/q + Q_j/q)}$. Adding together, we have the NPV contribution from the batch Q_j as $-\left(cq/\rho\right)\left(1-e^{-\rho Q_j/q}\right) - \alpha_j K e^{-\rho(t_j - D_j/q + \beta Q_j/q)}$, where $\beta = 0$ for setups at start and $\beta = 1$ for setups at end, which all together add up to the net present value discounted to time zero:

$$\mathrm{NPV} = -\sum_{j=1}^{n}\left(\left(cq/\rho\right)\left(1-e^{-\rho Q_j/q}\right)e^{-\rho(t_j - D_j/q)} + \alpha_j K e^{-\rho(t_j - D_j/q + \beta Q_j/q)}\right), \tag{16}$$

where the batch quantities Q_j are given by (2).

In order to find the corresponding total cost expression, we take a first-order Maclaurin expansion of NPV in (16). For the setup cost contributions, we disregard the interest rate effect, but first we add the discounted cost value of requirements $c\sum_{j=1}^{n} D_j e^{-\rho t_j}$.

These are standard procedures in the AC approach. To illustrate this, we may choose the simplest EPQ sawtooth model with Q as the batch quantity and D as the given constant demand rate. Assuming the setup cost K to be located at the beginning of each cycle, the NPV (of out-payments) is $\left(K + cq\left(1-e^{-\rho Q/q}\right)/\rho\right)/\left(1-e^{-\rho Q/D}\right)$. A first-order differentiation with respect to Q leads to the optimality condition $c\left(D\left(e^{\rho Q/D}-1\right) - q\left(e^{\rho Q/q}-1\right)\right)/\rho = Ke^{\rho Q/q}$. The first-order Maclaurin expansion in ρ of the coefficient belonging to the unit production cost c will be $\rho Q^2\left(\left(1/D\right)-\left(1/q\right)\right)/2$ (the zeroth-order term vanishes), and the similar expansion of the coefficient belonging to K is $1+\rho Q/q$.

Solving the equation using these first-order approximations gives the solution $Q = KD/\left(c\left(q-D\right)\right) \quad \pm\sqrt{\left(KD/\left(c\left(q-D\right)\right)\right)^2 + 2KDq/\left(c\rho\left(q-D\right)\right)}$, whereas solving the equation using a zeroth-order approximation for the K term, leads to $Q = \pm\sqrt{2KDq/\left(c\rho\left(q-D\right)\right)} = \pm\sqrt{2KDq/\left(h\left(q-D\right)\right)}$ with h interpreted as $c\rho$, the latter being the standard result of the AC approach. So the AC approach neglects the interest dependence of the setup cost term. The two solutions coincide in the limit when $\rho \to 0$ while keeping h finite (so $c = h/\rho \to \infty$).

The NPV expression is based on out-payments for setups and for production. Whereas the former belong to the category of inventory-related costs, the latter includes more. Investigating the *annuity stream* (Grubbström 1980) corresponding to the NPV of production out-payments, which is $\rho\left(cq\left(1-e^{-\rho Q/q}\right)/\rho\right)/\left(1-e^{-\rho Q/D}\right)$ for this infinite-horizon process, the zeroth-order term will be $\lim_{\rho\to 0}\rho\left(cq\left(1-e^{-\rho Q/q}\right)/\rho\right)/\left(1-e^{-\rho Q/D}\right) = cD$, which accounts for average production costs (per time unit). In the AC approach, this term does not belong to "inventory-related costs" so in order to make a comparison between the AC and NPV, production costs need to be corrected for this.

In the finite-horizon case treated here, the discounted cost value of requirements $c\sum_{j=1}^{n} D_j e^{-\rho t_j}$ is not inventory-related, and it is therefore added to the NPV expression in (16).

Choosing a positive sign for costs, from (16) total inventory-related costs TC will amount to

$$\begin{aligned}\mathrm{TC} = c\sum_{j=1}^{n}&\left[\left(q/\rho\right)\left(1-e^{-\rho Q_j/q}\right)e^{-\rho(t_j - D_j/q)} - D_j e^{-\rho t_j}\right]_{\text{1st approximation in }\rho} \\ &+K\sum_{j=1}^{n}\alpha_j\left[e^{-\rho(t_j - D_j/q+\beta Q_j/q)}\right]_{\text{0th approximation in }\rho} = \\ &= h\sum_{j=1}^{n}\left(Q_j D_j/q - Q_j t_j - Q_j^2/(2q) + D_j t_j\right) + K\sum_{j=1}^{n}\alpha_j\ ,\end{aligned} \quad (17)$$

in which $\overline{P}_n = \sum_{j=1}^{n} Q_j$ and $\overline{D}_n = \sum_{j=1}^{n} D_j$ cancel, and where the inventory holding cost parameter h is interpreted as $c\rho$. The zeroth-order term of the coefficient of c vanished due to the subtraction of the discounted cost value of requirements.

In cases when events have been dominated by the production rate restriction, with the AC approach, when computing the resulting total inventory-related costs, these need to be adjusted for the avoided time-weighted inventory areas joining the requirements staircase (exemplified below in Figure 12). Similarly with the NPV principle, the discounted cost value of requirements for neglected steps should be adjusted for, using original amounts and their timing in order to reach the resulting NPV measure.

We finally provide a forward algorithm leading to optimality for solving the problem based on maximising the savings in NPV compared to the Lot-For-Lot solution, modified from

(Grubbström 2005). Let $l \geq 0$ be the hypothetical number of future steps that a production ramp (starting at $t_{j-1} - D_{j-1}/q$) covers apart from the step D_{j-1}, i.e. steps $D_j, .D_{j+1}, \ldots, D_{j-1+l}$. The NPV of production out-payments for this ramp beyond t_{j-1} (discounted to time zero) is $(cq/\rho)\left(1-e^{-\rho(\bar{D}_{j+l}-\bar{D}_{j-1})/q}\right)e^{-\rho t_{j-1}}$. If this section of the ramp is moved forward to the next possible location, starting at $(t_j - D_j/q)$, the savings in NPV of production out-payments will be $(cq/\rho)\left(1-e^{-\rho(\bar{D}_{j+l}-\bar{D}_{j-1})/q}\right)\left(e^{-\rho t_{j-1}}-e^{-\rho(t_j-D_j/q)}\right)$. On the other hand, a new setup must be generated at $(t_j - D_j/q)$ with a consequence $Ke^{-\rho\left(t_j-\left(D_j-\beta\left(\bar{D}_{j+l}-\bar{D}_{j-1}\right)\right)/q\right)}$, where $\beta = 0$ if located at the beginning of production, or $\beta = 1$ if located at end.

We define two triangular arrays $W_j(l)$ and $\gamma_j(l)$, $j = 1, \ldots, n$, $l = 0, \ldots, n - j$. The element $W_j(l)$ denotes the NPV of saving out-payments compared to Lot-For-Lot, if given decisions for steps 1, 2, ..., j - 1, where l is the hypothetical number of steps covered beyond D_{j-1}.

We let $W_1(l)$ represent the NPV of the necessary first setup cost. If this cost is located at the beginning, $\beta = 0$, $W_1(l)$ will be independent of l, but in the opposite case it will decrease in l. For a higher index j, $W_j(l)$ will be the opportunity cost for the NPV of setup and production payments, if l is the number of future steps covered, compared to the comparable NPV of production payments, if the Lot-For-Lot solution were followed.

So the choice of moving the ramp concerns which of $Ke^{-\rho\left(t_j-\left(D_j-\beta\left(\bar{D}_{j+l}-\bar{D}_{j-1}\right)\right)/q\right)} + W_{j-1}(0)$ and $(cq/\rho)\left(1-e^{-\rho(\bar{D}_{j+l}-\bar{D}_{j-1})/q}\right)\left(e^{-\rho t_{j-1}}-e^{-\rho(t_j-D_j/q)}\right) + W_{j-1}(l+1)$ is the smaller. If the former is smaller, we set $\gamma_j(l)$ to unity (a conditional setup), otherwise to zero.

This gives the recursive relation:

$$W_j(l) = \underset{\gamma_j(l)\in\{0,1\}}{\text{Min}} \Big\{ \left(1-\gamma_j(l)\right)\left((cq/\rho)\left(1-e^{-\rho(\bar{D}_{j+l}-\bar{D}_{j-1})/q}\right)\left(e^{-\rho t_{j-1}}-e^{-\rho(t_j-D_j/q)}\right) + W_{j-1}(l+1)\right)$$
$$+\gamma_j(l)\left(Ke^{-\rho\left(t_j-(D_j-\beta\left(\bar{D}_{j+l}-\bar{D}_{j-1}\right)/q\right)} + W_{j-1}(0)\right)\Big\}, j = 2, 3, \ldots, n. \quad (18)$$

Introducing the sequence l_j, $j = n$ - 1, n - 2, ... , as the optimal value of l at Step j, the optimal sequence of binary decisions is then generated as

$$\alpha_n = \gamma_n(0),\ l_n = 0,$$
$$l_{j-1} = \left(1-\alpha_j\right)\left(l_j+1\right),$$
$$\alpha_{j-1} = \left(1-\alpha_j\right)\gamma_{j-1}\left(l_{j-1}\right) + \alpha_j\gamma_{j-1}(0),\ j = n, n-1, \ldots, 1. \quad (19)$$

If α_j is zero, then l at the earlier stage j - 1 is one step higher than at step j, so the optimal decision at j - 1 will be $\gamma_{j-1}\left(l_j+1\right)$, whereas if α_j is unity, then the earlier stage must have a zero level $l_{j-1}=0$ and the optimal decision is $\gamma_{j-1}\left(0\right)$.

The optimal value of the NPV of all payments will then be the NPV of production out-payments of the Lot-For-Lot solution from which the optimal adjustment $W_n\left(0\right)$ is subtracted, i.e. ,

$$\text{NPV}_{\text{optimum}} = -\sum_{i=1}^{n}\left(cq/\rho\right)\left(1-e^{-\rho D_i/q}\right)e^{-\rho(t_i-D_i/q)} - W_n(0) \ . \tag{20}$$

For a similar algorithm in the AC case, the recursive equation (18) is easily modified by replacing the coefficient of c by a first order approximation in ρ and the coefficient of K by a zeroth order approximation. To obtain an expression for the NPV of inventory-related consequences, the discounted cost value of requirements $c\sum_{j=1}^{n}D_je^{-\rho t_j}$ is added to $\text{NPV}_{\text{optimum}}$. This algorithm is applied to an example at the end of the next section. It may be speeded up somewhat by recognising that once $\gamma_j\left(l\right)=1$, it will remain so for greater values of l.

5. Two numerical examples

A first simple numerical example is provided in Figure 9. Requirements appear in amounts of $D_j=1$ at times $t_j=1, 3, 6, 10, 15$ and the finite production rate is q = 1 (units per time unit). It is clear that the production rate requirement (cf. Figure 4) will not eliminate any of the demand events.

We illustrate this example by choosing one of the solutions having the decision variable values $\alpha_j=1, 0, 1, 0, 0$. The shaded areas show the inventory level integrated over time.

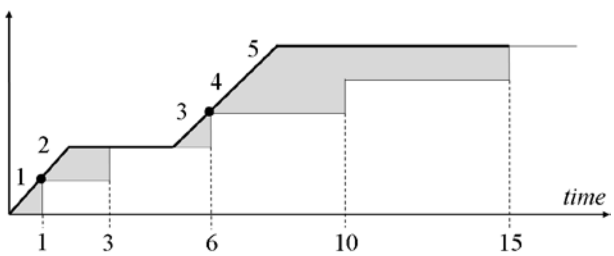

Figure 9. Cumulative production and requirements in first example.

With this cumulative requirements staircase and values of decision variables α_j, the batch quantities are obtained from (2) as $Q_1=1\sum_{k=1}^{5}1\prod_{l=2}^{k}(1-\alpha_l)=2$, $Q_2=0$, $Q_3=3$, and $Q_4=Q_5=0$. From (17) we obtain total costs as

$$\begin{aligned}\text{TC} &= h\sum_{j=1}^{n}\left(Q_jD_j/q-Q_jt_j-Q_j^2/(2q)+D_jt_j\right)+K\sum_{j=1}^{n}\alpha_j \\ &= 13.5\cdot h+2K \ .\end{aligned}$$

By visually inspecting the shaded areas in Figure 9 (time-weighted inventory), starting from the left, we first have two triangles each of size ½, then a square of size 1, then an additional triangle of ½, followed by a larger triangle of 2, a square of 4 and a rectangle of 5, all adding up to 13.5, as the formula also tells us.

Applying Theorem 1, there are altogether $2^{5-1} = 16$ solution alternatives. Evaluations of these solutions are shown in Table 1 applying the AC approach. The All-At-Once ($\forall$@1) solution is shown on the first line, and the Lot-For-Lot (L4L) solution on the bottom line. Solutions No 4 and No 6 happen to have identical consequences. Writing the number of setups as *x*, and time-weighted inventory as *y* for each solution, we have TC = *Kx* + *hy*.

Solution No	**Decisions** α_j	**Lot sizes** Q_j	**Time-weighted inventory** y	**Number of setups** $x = \sum_{j=1}^{5} \alpha_j$
1	**1,0,0,0,0**	**5,0,0,0,0**	**22.5**	**1**
2	1,0,0,0,1	4,0,0,0,1	12.5	2
3	**1,0,0,1,0**	**3,0,0,2,0**	**10.5**	**2**
4	**1,0,0,1,1**	**3,0,0,1,1**	**6.5**	**3**
5	1,0,1,0,0	2,0,3,0,0	13.5	2
6	**1,0,1,0,1**	**2,0,2,0,1**	**6.5**	**3**
7	1,0,1,1,0	2,0,1,2,0	7.5	3
8	**1,0,1,1,1**	**2,0,1,1,1**	**3.5**	**4**
9	1,1,0,0,0	1,4,0,0,0	18.5	2
10	1,1,0,0,1	1,3,0,0,1	9.5	3
11	1,1,0,1,0	1,2,0,2,0	8.5	3
12	1,1,0,1,1	1,2,0,1,1	4.5	4
13	1,1,1,0,0	1,1,3,0,0	12.5	3
14	1,1,1,0,1	1,1,2,0,1	5.5	4
15	1,1,1,1,0	1,1,1,2,0	6.5	4
16	**1,1,1,1,1**	**1,1,1,1,1**	**2.5**	**5**

Table 1. Evaluation of 16 solutions of first example.

A scatter diagram for the economic consequences is shown in Figure 10 with number of setups along the horizontal axis and time-weighted inventory along the vertical axis. The diagram shows that only five of the solutions would qualify for cost-minimum optimality (marked in bold in Table 1). The isocost line with a slope of minus *K/h* = 5, shows solution No 3 to be optimal for that ratio.

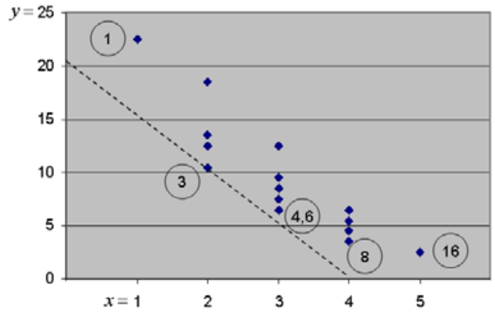

Figure 10. Scatter diagram of solution consequences in first example.
Dashed isocost line represents ratio K/h = 5.

In Table 2 is demonstrated how the optimal solution depends on the ratio *K/h*.

Ratio	$K/h < 1$	$1 < K/h < 3$	$3 < K/h < 4$	$4 < K/h < 12$	$K/h > 12$
Optimal solution	16	8	4, 6	3	1
	L4L				∀@1

Table 2. Optimal solution depending on ratio K/h.

As our second case, we choose the example from (Hill 1997). In this article, the following values are assumed, $K = 36$, $h = 1$, $q = 5$, and the requirements and their timing are reproduced in Table 3:

j	1	2	3	4	5	6	7	8	9	10
t_j	3	4	6	8	9	10	14	15	19	20
D_j	8	6	8	4	6	7	8	5	9	7

Table 3. Requirements and their timing of second example, from (Hill 1997).

In Figure 11 the cumulative requirements staircase is depicted together with dotted lines having the slope $q = 5$. Examining consequences from the production rate restriction, we find that events at times 3, 8, 9, 14 and 19 are dominated, and therefore can be taken away from the problem, which leaves us with the simpler staircase in Figure 12. From a binary complexity point of view, the number of solutions to choose among has been reduced drastically from $2^{10-1} = 512$ to $2^{5-1} = 16$.

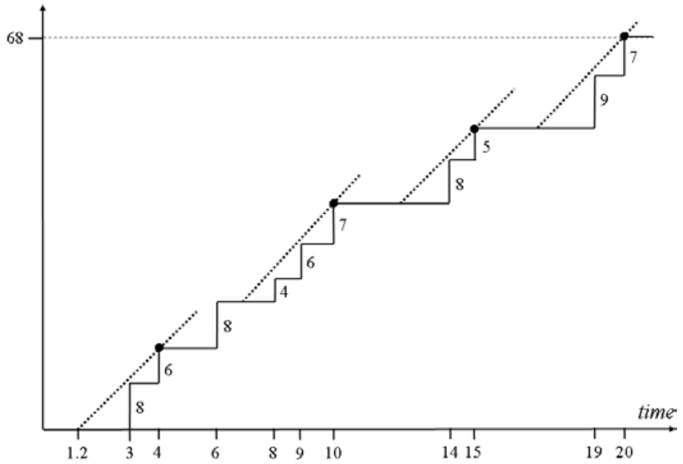

Figure 11. Original requirements staircase in second example. Dotted lines illustrate events that are dominated according to production rate restriction.

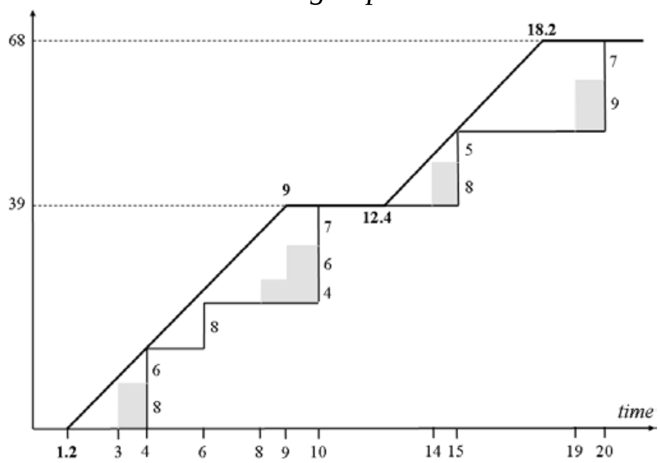

Figure 12. Resulting requirements staircase after applying production rate restriction. Shaded areas show time-weighted inventory in total of 39 later to be added back. Optimal production ramps and their timing are also depicted.

In Figure 12 is also shown optimal cumulative production and the corresponding start and end times (in bold) of the two batches using the AC approach. The shaded areas show portions of time-weighted inventory that have been avoided when applying the production rate restriction. These areas need to be added back when determining the inventory-related costs using this approach (but do not affect the optimal solution).

The optimum values for the economic consequences differ slightly from those given in Hill (1997), because of a minor numerical fault in the article (the time of the last horizontal step should be $20-18.2=1.8$, and not 1.6 as reported). The minimum costs are TC = $107.4 \cdot h + 2 \cdot K$ $=107.4+72=179.4$ with these two setups. Data for the 16 solutions are given in Table 4 and in a scatter diagram in Figure 13. Table 5 provides the relation between the AC-optimal solution for each ratio *K*/*h*.

Solution No	**Decisions** α_j	**Lot sizes** Q_j	**Time-weighted inventory** y	**Number of setups** $x=\sum_{j=1}^{5}\alpha_j$
1	**1,0,0,0,0**	**68, 0, 0, 0, 0**	**206.0**	**1**
2	1,0,0,0,1	52, 0, 0, 0, 16	122.8	2
3	**1,0,0,1,0**	**39, 0, 0, 29, 0**	**107.4**	**2**
4	**1,0,0,1,1**	**39, 0, 0, 13, 16**	**78.6**	**3**
5	1,0,1,0,0	22, 0, 46, 0, 0	160	2
6	1,0,1,0,1	22, 0, 30, 0, 16	92.8	3
7	1,0,1,1,0	22, 0, 17, 29, 0	90.4	3
8	**1,0,1,1,1**	**22, 0, 17, 13, 16**	**61.6**	**4**
9	1,1,0,0,0	14, 54, 0, 0, 0	184.4	2
10	1,1,0,0,1	14, 38, 0, 0, 16	107.6	3
11	1,1,0,1,0	14, 25, 0, 29, 0	97.4	3
12	1,1,0,1,1	14, 25, 0, 13, 16	68.6	4
13	1,1,1,0,0	14, 8, 46, 0, 0	156.8	3
14	1,1,1,0,1	14, 8, 30, 0, 16	89.6	4
15	1,1,1,1,0	14, 8, 17, 29, 0	87.2	4
16	**1,1,1,1,1**	**14, 8, 17, 13, 16**	**58.4**	**5**

Table 4. Solutions for example in Hill(1997). Time-weighted inventory has been adjusted for shaded areas from Figure12.

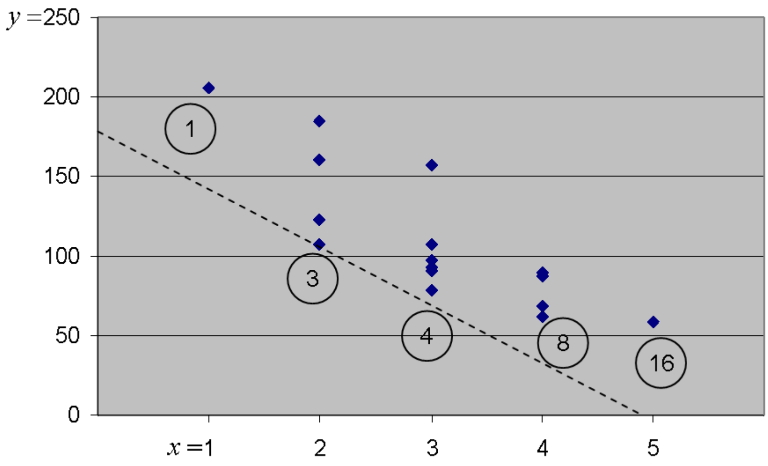

Figure 13. Scatter diagram of solution consequences in second example. Dashed isocost line represents ratio K/h = 36.

Ratio	$K/h < 3.2$	$3.2 < K/h < 17$	$17 < K/h < 28.8$	$28.9 < K/h < 98.6$	$K/h > 98.6$
Optimal solution	16	8	4	3	1
	L4L				$\forall$@1

Table 5. Optimal AC solution depending on ratio K/h in second example.

We now turn our attention to the case of NPV being the objective, not treated in Hill (1997). For enabling a comparison with AC, we choose four different values of the two parameters ρ (continuous interest rate) and c (unit production cost) keeping their product equal to a unit holding cost of $\rho c = h = 1$. With a choice of four values of ρ as displayed in Table 6, where also the discounted cost value of originally timed requirements $c\sum_{j=1}^{n} D_j e^{-\rho t_j}$ and the minimum time difference between production segments (Corollary 1) is shown.

ρ	0.0001	0.001	0.01	0.1
c	10000	1000	100	10
Discounted cost value of requirements	679250.5	67255.3	6100.4	265.3
$NPV_{L4L\ prod}$	- 679308.9	- 67313.1	- 6153.0	- 289.0
$K/(cq) = \rho K/(hq)$	0.00072	0.0072	0.072	0.72

Table 6.Consequences from four values of ρ, keeping holding cost h at unit level.

No	Decisions α_j	AC	$\rho = 0.0001$					$\rho = 0.001$				
		TC=Holding and Setup Costs	$NPV_{production}$	NPV_{setup}		NPV_{total}		$NPV_{production}$	NPV_{setup}		NPV_{total}	
				$\beta = 0$	$\beta = 1$	$\beta = 0$	$\beta = 1$		$\beta = 0$	$\beta = 1$	$\beta = 0$	$\beta = 1$
1	1,0,0,0,0	242.0	-205.7	-36.0	-35.9	-241.7	-241.7	-203.4	-36.0	-35.5	-239.4	-238.9
2	1,0,0,0,1	194.8	-122.7	-71.9	-71.9	-194.6	-194.6	-121.5	-71.4	-70.9	-192.9	-192.4
3	1,0,0,1,0	179.4	-107.3	-72.0	-71.9	-179.2	-179.2	-106.1	-71.5	-71.0	-177.6	-177.2
4	1,0,0,1,1	186.6	-78.5	-107.9	-107.8	-186.4	-186.4	-77.8	-106.9	-106.4	-184.7	-184.3
5	1,0,1,0,0	232.0	-159.8	-72.0	-71.9	-231.8	-231.7	-157.9	-71.7	-71.2	-229.6	-229.1
6	1,0,1,0,1	200.8	-92.7	-107.9	-107.9	-200.6	-200.6	-91.8	-107.1	-106.6	-198.9	-198.4
7	1,0,1,1,0	198.4	-90.3	-107.9	-107.9	-198.2	-198.2	-89.3	-107.3	-106.8	-196.5	-196.1
8	1,0,1,1,1	205.6	-61.5	-143.9	-143.8	-205.4	-205.4	-61.0	-142.7	-142.2	-203.6	-203.2
9	1,1,0,0,0	256.4	-184.2	-72.0	-71.9	-256.1	-256.1	-182.0	-71.8	-71.3	-253.8	-253.3
10	1,1,0,0,1	215.6	-107.5	-107.9	-107.9	-215.4	-215.4	-106.4	-107.2	-106.7	-213.6	-213.1
11	1,1,0,1,0	205.4	-97.3	-107.9	-107.9	-205.2	-205.2	-96.2	-107.4	-106.9	-203.6	-203.1
12	1,1,0,1,1	212.6	-68.5	-143.9	-143.8	-212.4	-212.4	-67.9	-142.8	-142.3	-210.7	-210.2
13	1,1,1,0,0	264.8	-156.6	-108.0	-107.9	-264.5	-264.5	-154.7	-107.6	-107.1	-262.3	-261.8
14	1,1,1,0,1	233.6	-89.5	-143.9	-143.8	-233.4	-233.3	-88.6	-143.0	-142.5	-231.6	-231.1
15	1,1,1,1,0	231.2	-87.1	-143.9	-143.9	-231.0	-230.9	-86.1	-143.1	-142.6	-229.2	-228.7
16	1,1,1,1,1	238.4	-58.3	-179.9	-179.8	-238.2	-238.1	-57.8	-178.5	-178.0	-236.3	-235.8

No	Decisions α_j	AC	$\rho = 0.01$					$\rho = 0.1$				
		TC=Holding and setup costs	$NPV_{production}$	NPV_{setup}		NPV_{total}		$NPV_{production}$	NPV_{setup}		NPV_{total}	
				$\beta = 0$	$\beta = 1$	$\beta = 0$	$\beta = 1$		$\beta = 0$	$\beta = 1$	$\beta = 0$	$\beta = 1$
1	1,0,0,0,0	242.0	-181.6	-35.6	-31.0	-217.2	-212.6	-64.3	-31.9	-8.2	-96.2	-72.5
2	1,0,0,0,1	194.8	-110.5	-66.0	-61.5	-176.5	-172.1	-46.9	-38.6	-16.2	-85.5	-63.1
3	1,0,0,1,0	179.4	-95.5	-67.4	-62.9	-162.9	-158.4	-38.5	-42.3	-20.5	-80.9	-59.0
4	1,0,0,1,1	186.6	-71.3	-97.8	-93.4	-169.1	-164.7	-33.5	-49.1	-27.5	-82.6	-61.0
5	1,0,1,0,0	232.0	-140.2	-69.3	-64.8	-209.5	-205.0	-48.0	-50.5	-28.0	-98.5	-75.9
6	1,0,1,0,1	200.8	-83.1	-99.7	-95.3	-182.8	-178.4	-34.6	-57.2	-35.6	-91.9	-70.3
7	1,0,1,1,0	198.4	-79.8	-101.1	-96.6	-180.9	-176.4	-30.7	-61.0	-39.6	-91.6	-70.3
8	1,0,1,1,1	205.6	-55.6	-131.5	-127.1	-187.1	-182.7	-25.7	-67.7	-46.7	-93.3	-72.4
9	1,1,0,0,0	256.4	-161.9	-70.0	-65.5	-232.0	-227.5	-55.6	-55.1	-32.0	-110.7	-87.6
10	1,1,0,0,1	215.6	-96.5	-100.5	-96.0	-196.9	-192.5	-39.9	-61.8	-39.8	-101.7	-79.8
11	1,1,0,1,0	205.4	-86.1	-101.8	-97.4	-188.0	-183.5	-33.4	-65.5	-44.0	-98.9	-77.4
12	1,1,0,1,1	212.6	-62.0	-132.3	-127.8	-194.2	-189.8	-28.3	-72.2	-51.1	-100.6	-79.4
13	1,1,1,0,0	264.8	-137.2	-103.7	-99.2	-240.9	-236.4	-46.0	-73.7	-51.3	-119.7	-97.3
14	1,1,1,0,1	233.6	-80.1	-134.2	-129.7	-214.2	-209.8	-32.7	-80.4	-59.0	-113.1	-91.7
15	1,1,1,1,0	231.2	-76.7	-135.5	-131.1	-212.3	-207.8	-28.8	-84.1	-63.0	-112.9	-91.7
16	1,1,1,1,1	238.4	-52.6	-166.0	-161.5	-218.5	-214.1	-23.7	-90.8	-70.0	-114.6	-93.8

Table 7. Numerical calculations of economic consequences in example from Hill(1997).

Table 7 summarises the main economic consequences of the second example when applying the NPV as objective function. Here, $NPV_{production}$ refers to discounted production costs adjusted for the discounted cost value of requirements, β=0 refers to case of setup costs being assigned to beginning of production intervals, and β=1 to end of intervals. NPV_{total} is the sum of $NPV_{production}$ and NPV_{setup}, the latter depending on the time location of the setup cost β. The sum TC of holding and setup costs are obtained from Table 4 using h = 1 and K = 36.

It is seen that there are only small differences between TC and NPV_{total} for small values of ρ, but that the differences increase substantially when ρ grows, also concerning the time assignment of the setup costs. But the former differences are also consequences of the simultaneous change in unit production cost c. However, the optimal solution (No 3) remains the same throughout. The latter insensitivity to changes in parameter values is typical for inventory optimisation problems, usually accredited to the "flat" minimum of the inventory-related total cost curve. In this example, also the relatively wide width of the fourth step (five time units) makes this step dominating. The wider a particular step is, the more the demand staircase becomes separated into two independent sub-staircases (just as for ordinary staircases in real life).

In Table 8 we highlight the difference between the economic consequences when applying the AC vs the NPV objectives.

No	Decisions α_j	$TC - \lvert NPV_{total} \rvert$							
		$\beta = 0$				$\beta = 1$			
		$\rho = 0.0001$	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$	$\rho = 0.0001$	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$
1	1,0,0,0,0	0.27	2.64	24.85	145.76	0.32	3.13	29.37	169.49
2	1,0,0,0,1	0.20	1.94	18.27	109.25	0.24	2.43	22.74	131.73
3	1,0,0,1,0	0.18	1.76	16.52	98.53	0.23	2.24	20.99	120.40
4	1,0,0,1,1	0.19	1.86	17.47	104.05	0.24	2.34	21.91	125.56
5	1,0,1,0,0	0.24	2.39	22.49	133.50	0.29	2.88	26.99	156.05
6	1,0,1,0,1	0.19	1.91	17.96	108.91	0.24	2.39	22.41	130.51
7	1,0,1,1,0	0.19	1.86	17.55	106.75	0.24	2.35	22.00	128.07
8	1,0,1,1,1	0.20	1.96	18.49	112.28	0.25	2.45	22.93	133.23
9	1,1,0,0,0	0.26	2.60	24.43	145.65	0.31	3.08	28.94	168.76
10	1,1,0,0,1	0.20	1.98	18.65	113.86	0.25	2.46	23.11	135.84
11	1,1,0,1,0	0.19	1.85	17.43	106.51	0.23	2.33	21.88	128.02
12	1,1,0,1,1	0.20	1.95	18.37	112.03	0.24	2.43	22.81	133.18
13	1,1,1,0,0	0.25	2.53	23.89	145.06	0.30	3.02	28.38	167.47
14	1,1,1,0,1	0.21	2.05	19.35	120.47	0.25	2.53	23.80	141.92
15	1,1,1,1,0	0.20	2.00	18.94	118.31	0.25	2.49	23.39	139.48
16	1,1,1,1,1	0.21	2.10	19.89	123.83	0.26	2.59	24.32	144.65

Table 8. Difference between total cost and numerical value of Net Present Value for the 16 solutions.

Table 8 demonstrates that the difference between the numerical value of the total cost and of the NPV of inventory-related payments NPV_{total} increases substantially when the interest rate increases. In all cases, the total cost overestimates the economic consequences according to NPV. This effect is pronounced for the case that the setup cost is allocated to the end of the production segment ($\beta = 1$). A major factor underlying this effect is the discounting of setup costs in the NPV case. The maximum deviation 169.49 in this example, which is obtained for the All-At-Once solution when $\rho = 0.1$ and $\beta = 1$, corresponds to an overestimation of more than three times. This effect is further illustrated in Figure 14.

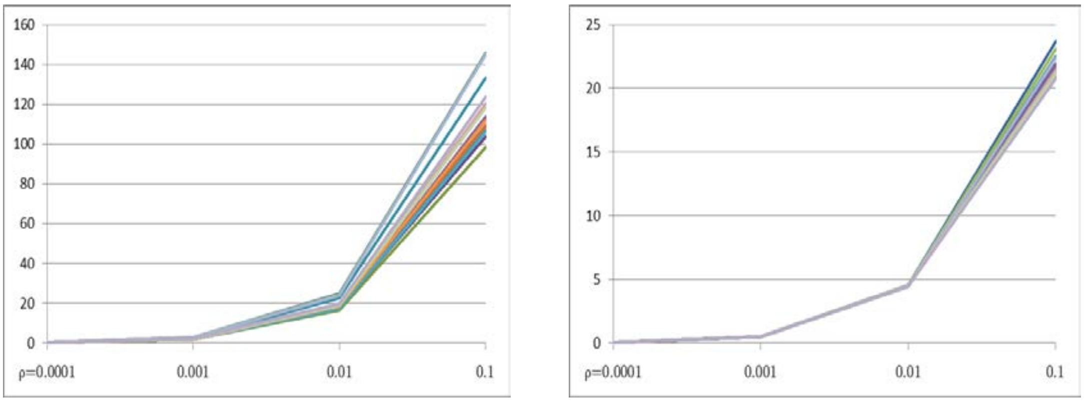

Figure 14. Left part of figure shows overestimation of TC *compared to* NPV_{total} *for the four interest rates when* $\beta = 0$ *(16 solutions). Right part shows the additional difference, when* β *changes from zero to unity, i.e. when setup costs are moved to the end.*

Each entry in Tables 7-8 rests on the same ratio $K/h = K/(c\rho) = 36$ and all optimal solutions in α_j are the same (No 3). In order to investigate the robustness of this solution, we choose to vary the unit production cost *c*. Keeping to the high interest alternative ($\rho = 0.1$) and keeping the value of *K* = 36, we change the value of *c* sufficiently to make a different solution optimal.

If $\beta = 0$ (setup cost at beginning) and *c* increases by 33.4%, a third setup is created at Step 3 and NPV_{tot} becomes - 93.74, which is a 15.9% increase in absolute value. Instead for $\beta = 1$, the necessary increase in *c* for a third setup to appear (also at Step 3) is 40.6%, resulting in $\text{NPV}_{\text{tot}} = -74.6$, which is an increase in absolute value by 26.5%. If, with $\beta = 0$, *c* instead is lowered to 38.5% of its former value, this avoids the second setup at Step 4 (making $\forall$@1), and NPV_{tot} becomes - 56.7, which is 70.0% of its former value. And with $\beta = 1$, *c* needs to be lowered to 45.7% for the second setup to disappear, giving $\text{NPV}_{\text{tot}} = -37.6$, which is 63.7% of its former value. It might appear counter-intuitive that *c* needs a percentage-wise higher increase in the $\beta = 1$ case for an additional setup to appear, since the effects of setup costs there are dampened by discount factors not present in the $\beta = 0$ case.

One may also readily check the *distance restriction* for all adjacent production ramps. In the optimum (Corollary 2), they need to be separated by approximately $\Delta t \leq K/(cq) = \rho K/(hq)$. At the high interest rate of 0.1, the minimum interval between two production ramps is thus 0.72. Quite naturally, in the optimum solution No 3, the interval between the two ramps is larger (3.4). However, all solutions including a start on the second horizontal step would violate this condition, since the distance between the first ramp and one beginning on the second horizontal step is 0.4 time units, cancelling the eight solutions 11000, 11001, 11010, 11011, 11100, 11101, 11110 and 11111 (including L4L). One may also note that the distance between starts on the second and third horizontal steps is 0.6, so all such solutions would also have been banned (but they already are). So, for $\rho = 0.1$, the step at *t* = 6, could have been dismissed from the offset.

Applying our algorithm Eqs (18)-(19) to this second problem (after having used the production rate restriction) provides us with the triangular matrices listed in Table 9 for $\rho = 0.0001, 0.001, 0.01, 0.1$ (with *c* adjusted making *h* = 1) and for β equal to zero or unity.

Setup cost located at beginning $(\beta = 0)$				
	$\left[\gamma_j(l)\right]$	$\left[W_j(l)\right]$	$\text{NPV}_{\text{optimum}} = \\ =\text{NPV}_{\text{L4L prod}} - W_n(0)$	$\text{NPV}_{\text{total}}$
$\rho = 0.0001$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 36.0 & 39.2 & 56.2 & 92.1 & 120.9 \\ 36.0 & 46.0 & 69.2 & 92.1 & 0 \\ 36.0 & 51.2 & 75.2 & 0 & 0 \\ 36.0 & 57.6 & 0 & 0 & 0 \\ 36.0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-679308.9 - 120.9 = \\ = -679429.8$	$-679429.8 + 679250.5 = \\ = -179.2$
$\rho = 0.001$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 36.0 & 39.1 & 56.0 & 91.6 & 119.9 \\ 36.0 & 45.9 & 68.9 & 91.6 & 0 \\ 36.0 & 51.0 & 74.9 & 0 & 0 \\ 36.0 & 57.4 & 0 & 0 & 0 \\ 36.0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-67313.1 - 119.9 = \\ = -67432.9$	$-67432.9 + 67255.3 = -177.6$
$\rho = 0.01$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 35.6 & 38.6 & 54.3 & 86.1 & 110.3 \\ 35.6 & 44.9 & 66.0 & 86.1 & 0 \\ 35.6 & 49.6 & 72.3 & 0 & 0 \\ 35.6 & 55.2 & 0 & 0 & 0 \\ 35.6 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-6153.0 - 110.3 = -6263.3$	$-6263.3 + 6100.4 = -162.9$
$\rho = 0.1$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 31.9 & 33.9 & 41.7 & 52.1 & 57.2 \\ 31.9 & 37.1 & 46.1 & 52.1 & 0 \\ 31.9 & 38.9 & 50.2 & 0 & 0 \\ 31.9 & 40.6 & 0 & 0 & 0 \\ 31.9 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-289.0 - 57.2 = -346.2$	$-346.2 + 265.3 = -80.9$
Setup cost located at end $(\beta = 1)$				
	$\left[\gamma_j(l)\right]$	$\left[W_j(l)\right]$	$\text{NPV}_{\text{optimum}} = \\ =\text{NPV}_{\text{L4L prod}} - W_n(0)$	$\text{NPV}_{\text{total}}$
$\rho = 0.0001$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 36.0 & 39.2 & 56.2 & 92.1 & 120.8 \\ 36.0 & 46.0 & 69.1 & 92.1 & 0 \\ 36.0 & 51.1 & 75.1 & 0 & 0 \\ 36.0 & 57.5 & 0 & 0 & 0 \\ 35.9 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-679308.9 - 120.8 = \\ = -679429.7$	$-679429.7 + 679250.5 = \\ = -179.2$
$\rho = 0.001$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 35.9 & 39.0 & 55.7 & 91.2 & 119.4 \\ 35.8 & 45.6 & 68.5 & 91.1 & 0 \\ 35.7 & 50.7 & 74.4 & 0 & 0 \\ 35.6 & 56.9 & 0 & 0 & 0 \\ 35.5 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-67313.1 - 119.4 = \\ = -67432.4$	$-67432.4 + 67255.3 = -177.2$
$\rho = 0.01$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 34.6 & 37.1 & 51.7 & 82.7 & 105.9 \\ 34.0 & 42.3 & 62.5 & 81.7 & 0 \\ 32.9 & 46.1 & 67.8 & 0 & 0 \\ 32.1 & 50.7 & 0 & 0 & 0 \\ 31.0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-6153.0 - 105.9 = -6258.9$	$-6258.9 + 6100.4 = -158.4$
$\rho = 0.1$	$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 24.1 & 22.5 & 24.4 & 32.4 & 35.3 \\ 20.6 & 19.8 & 25.5 & 30.2 & 0 \\ 14.6 & 18.3 & 26.5 & 0 & 0 \\ 11.3 & 16.9 & 0 & 0 & 0 \\ 8.2 & 0 & 0 & 0 & 0 \end{bmatrix}$	$-289.0 - 35.3 = -324.3$	$-324.3 + 265.3 = -59.0$

Table 9. Optimum solution generated by algorithm in Eqs (18)-(19).
In the matrices j is column number and $l+1$ row number.
Operations are made with greater accuracy than shown in table.

It might be interesting to note that for smaller values of ρ the algorithm generates a hypothetical setup at Step 3, which it later discards.

6. Conclusions

In the foregoing, we have relaxed the assumption of instantaneous replenishments in the dynamic lotsizing problem, allowing for production to take place at a finite rate q. In our main theorem, it was shown that the inner-corner property for an optimal solution needs to be replaced by a condition for tangency between the cumulative production function and the cumulative requirements staircase. However, the resulting problem could still be formulated in binary terms, and certain results from earlier findings were still applicable.

We also confirmed one condition for simplifying the problem, the production rate restriction, by which certain requirement events can be eliminated. Secondly, when applying the NPV principle, a (new) condition for optimality was revealed that production segments need to be separated by more than a time interval depending on certain parameter values, the distance restriction (Corollary 2). This second necessary condition can be applied at an introductory stage of analysing the solution opportunities, and may thereby reduce the number of solutions candidating for optimality still further, which the second numerical example also illustrated.

An algorithm for finding the optimal solution in the NPV case was formulated and applied to one of the two examples illustrating our findings. As in several previous studies differences in consequences from applying the AC and NPV objectives tend to vanish as the interest rate is chosen to be smaller.

7. References

Aryanezhad, M.-B.Q., (1992). An algorithm based on a new sufficient condition of optimality in dynamic lot size model, *European Journal of Operational Research* 59(3), 425-433.

Beullens, P., Janssens, G.K., 2011. Holding costs under push or pull conditions – The impact of the Anchor Point, *European Journal of Operational Research*, 215, 115–125.

De Bodt, M.A., Gelders, L.F., Van Wassenhove, L.N., (1984). Lot sizing under dynamic demand conditions: A review, *Engineering Costs and Production Economics*, 8(3), 165-187.

Erlenkotter, D., 1989. An Early Classic Misplaced: Ford W. Harris's Economic Order Quantity Model of 1913, *Management Science*, 35(7), 898-900.

Federgruen, A., Tzur, M., 1991. A Simple Forward Algorithm to solve General Dynamic Lot Sizing Models with *n* Periods in 0(*n* log *n*) or 0(*n*) Time, *Management Science*, 37, 909-925.

Grubbström, R.W., 1980. A principle for determining the correct capital costs of work-in-progress and inventory, *International Journal of Production Research*, 18, 259-271.

Grubbström, R.W., 1997. *The Silver-Meal Algorithm in the Net Present Value Case – A Greeting to Ed Silver on His 60th Birthday*, Working Paper WP-243, Linköping Institute of Technology, August.

Grubbström, R.W., 1999. An Optimal Forward Dynamic Lot-Sizing Policy for the NPV and Average Cost Cases in Continuous Time, in Hillery, M. T., Lewis, H. J., (Eds), *The 15th International Conference on Production Research. ICPR 15 Manufacturing for a Global Market,* Limerick, Ireland, August 9-13, 381-384.

Grubbström, R.W., 2005. Algorithms for Optimal Dynamic Lotsizing in Discrete and Continuous Time, *International Journal of Agile Manufacturing*, 8(2), 85-96.

Grubbström, R.W., 2012. The Time-Averaged L4L Solution - a Condition for Long-Run Stability Applying MRP Theory, *International Journal of Production Research*, 50(21), 6099-6110.

Grubbström, R.W., Bogataj, M., Bogataj, L., 2009. Optimal Lotsizing within MRP Theory, *Annual Reviews in Control*, 34(1), 2010, 89–100.

Grubbström, R.W., Kingsman, B.G., 2004. Ordering and Inventory Policies for Step Changes in the Unit Item Cost: A Discounted Cash Flow Approach, *Management Science*, 50(2), 253–267.
Grubbström, R.W., Tang, O., 2012. The space of solution alternatives in the optimal lotsizing problem for general assembly systems applying MRP theory, *International Journal of Production Economics*, 140(2), 765-777.
Hadley, G., 1964. A comparison of order quantities using the average annual costs and the discounted costs, *Management Science*, 10(3), 472-476.
Harris, F.W., 1913. "How Many Parts to Make at Once," *Factory, The Magazine of Management*, 10(2), 135-136, 152.
Hill, R.M., 1997. Note: Dynamic Lot Sizing for a Finite Rate Input Process, *Naval Logistics Research*, 44(2), 221-228.
Khmelnitsky E., Tzur, M., 2004. Parallelism of discrete- and continuous-time production planning, *IIE Transactions*, 36(7), 611-628.
Kim, Y.H., Philippatos, G.C., Chung, K.H., 1986. Evaluating Investment in Inventory Policy: A Net Present Value Framework, *The Engineering Economist*, 31(2), 119-136.
Silver, E.A., Meal, H.C., 1973. A Heuristic for Selecting Lot Size Quantities for the Case of a Deterministic Time Varying Demand Rate and Discrete Opportunities for Replenishment, *Production and Inventory Management*, 14, 64-74.
Song, Y., Chan, G.H., 2005. Single item lot-sizing problems with backlogging on a single machine at a finite production rate, *European Journal of Operational Research*, 161(1), 191–202.
Teunter, R., van der Laan, E., 2002. On the non-optimality of the average cost approach for inventory models with remanufacturing, *International Journal of Production Economics*, 79(1), 67-73.
Trippi, R.R., Levin, D.E., 1974. A present value formulation of the classical EOQ problem, *Decision Sciences*, 5(1), 30-35.
Veinott Jr., A.F., 1969. Minimum concave-cost solution of Leontief substitution models of multifacility inventory systems. *Operations Research*, 17(2), 262–291.
Wagner, H.M., Whitin, T.M., 1958. A Dynamic Version of the Economic Lot Size Model, *Management Science*, 5(1), 89-96.