Frequency-Domain Identification of Continuous-Time Output Error Models Part II - Non-uniformly Sampled Data and B-spline Output Approximation

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Abstract

This paper concerns the subject of identification of continuous-time output error (OE) models based on non-uniformly sampled output data. The exact method for doing this is well known in the time-domain, where the continuous-time system is discretized, simulated and the result is fitted in a mean square sense to measured data. The material presented here is based on a method proposed in a sister paper [5] which dealt with the same topic but for the case of uniformly sampled data. In this text it will be shown how that method entails that the output should be reconstructed using a B-spline with uniformly distributed knots. This representation can then be used to directly identify the continuous-time system without proceeding via discretization. Only the relative degree of the model is used to choose the order of the spline.

Keywords: System Identification

Frequency-Domain Identification of Continuous-Time Output Error Models

Part II - Non-uniformly Sampled Data

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Abstract

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Key words: continuous-time; parameter estimation; system identification; frequency-domain; splines; output-error

1 Introduction

The focus of this paper is the identification of continuous-time input output models from non-uniformly sampled data. That is, the estimation of the parameters of a transfer function $G_c(s, \theta)$ in Figure 1 when the output data is sampled non-uniformly $\{y(kT_s)\}_{k=1}^{N_t}$. For equidistantly

computed as follows

$$\begin{cases} U_d(e^{i\omega_n T_s}) = T_s \sum_{k=1}^{N_t} u(kT_s) e^{i\omega_n T_s}, \omega_n = \frac{2\pi}{T_s} n, \\ Y_d(e^{i\omega_n T_s}) = T_s \sum_{k=1}^{N_t} y(kT_s) e^{i\omega_n T_s}, n = 0, \dots, \lfloor \frac{N_t-1}{2} \rfloor. \end{cases}$$

Then, the parameters can be identified by minimizing the sum of the square of the difference of the measured and expected frequency response as follows [7]

$$\hat{\theta} = \arg \min_{\theta} \frac{1}{N_{\omega}} \sum_{k=1}^{N_{\omega}} |Y_d(e^{i\omega_k T_s}) - \hat{Y}_d(e^{i\omega_k T_s}, \theta)|^2 \quad (1)$$

$$\hat{Y}_d(e^{i\omega_k T_s}, \theta) = G_d(e^{i\omega_k T_s}, \theta) U_d(e^{i\omega_k T_s}), k = 1..N_{\omega} \quad (2)$$

where N_{ω} represented the number of frequency components.

In [5] the authors argued for a direct method which

Fig. 1. Input and sampling setup for a continuous-time system.

sampled input and output data $\{u(kT_s), y(t_k)\}_{k=1}^{N_t}$, the discrete-time Fourier transforms of the sampled data is

circumvents the use of the discrete-time representation $G_d(e^{i\omega T_s}, \theta)$ in the case of uniformly sampled data. The line of reasoning was that the frequency domain relationship between the sampled system input and output is governed by

$$Y_d(e^{i\omega T_s}) = G_d(e^{i\omega T_s}, \theta)U_d(e^{i\omega T_s}) + \text{transients}$$

if the input is assumed to be piecewise constant. In continuous-time, the analog expression would be

$$Y_c(i\omega) = G_c(i\omega, \theta)U_c(i\omega) = G_c(i\omega, \theta)H(i\omega)U_d(e^{i\omega T_s}),$$

since the connection between the continuous- and discrete-time Fourier transforms of the input u is

$$U_c(i\omega) = H(i\omega)U_d(e^{i\omega T_s})$$

due to the zero-order hold circuit H .

Now suppose that you knew the exact parameter values θ_0 . Then, it would be possible to compute the exact continuous-time Fourier transform of the output as

$$Y_c(i\omega) = \frac{G_c(i\omega, \theta_0)H(i\omega)}{G_d(e^{i\omega T_s}, \theta_0)}Y_d(e^{i\omega T_s}). \quad (3)$$

The crux is that knowledge of θ_0 is main objective. A result from the previous paper shows that

$$\frac{G_c(i\omega, \theta_0)H(i\omega)}{G_d(e^{i\omega T_s}, \theta_0)} \rightarrow F_{\ell+1, T_s}^c(i\omega) \quad (4)$$

where

$$F_{\ell+1, T_s}^c(i\omega) = \frac{T_s \left(\frac{e^{i\omega T_s} - 1}{i\omega T_s} \right)^{\ell+1}}{e^{i\omega T_s} \Pi_\ell(e^{i\omega T_s})} \quad (5)$$

where $F_{\ell+1, T_s}^c$ is independent of the true parameters and $\Pi_\ell(e^{i\omega T_s})$ are the Euler-Frobenius polynomials [4]. On the other hand, one has to assume that the system G_c is strictly proper, stable and of relative degree $\ell = n - m$, with a rate of sampling which is above the system bandwidth. Then, the result opens for the identification of the continuous-time parameters as

$$\hat{\theta} = \arg \min_{\theta} \sum_{k=1}^{N_\omega} \left| \hat{Y}_c(i\omega_k) - G_c(i\omega_k, \theta)U_c(i\omega_k) \right|^2. \quad (6)$$

where the continuous-time Fourier transform of the output is estimated as

$$\hat{Y}_c(i\omega) = F_{\ell+1, T_s}^c(i\omega)Y_d(e^{i\omega T_s}) \quad (7)$$

without involving the exact discrete-time frequency response $G_d(e^{i\omega T_s})$. In the time domain, the relationship above can be formulated as

$$Y_c(i\omega) = \int_{-\infty}^{\infty} \hat{y}_c(t) e^{-i\omega t} dt \quad (8)$$

where

$$\hat{y}_c(t) = \sum_{k=0}^{N_t-1} y(kT_s) F_{\ell+1, T_s}^c(t - kT_s). \quad (9)$$

The interpretation of this is straightforward, and points towards interpolation with the kernel function $F_{\ell+1, T_s}^c(t)$ in order to reconstruct the intersample behavior of the function $y(t)$. In particular, interpolation in terms of B-splines [2,9]. Elucidating the mechanisms behind these statements and their consequences for the frequency domain estimation of deterministic continuous-time systems from non-uniformly sampled data, will be the objective of the remaining part of this paper.

2 Outline

The outline of the paper will be the following. First, in Section 3 there will be a brief introduction to polynomial interpolation. Mainly in order to prepare the reader for the basics of polynomial splines which are introduced in Section 4. The exposition found in these sections is mostly based on classical material, which can be found in basic textbooks on numerical analysis and on spline interpolation, i.g. [3], [2], [9] and [10]. For a more extensive empirical investigation on interpolation using splines, we refer to the paper by Rolain [8]. The discussion in Section 3 and Section 4 is quite general by nature and applies equally well to uniform and non-uniform sampling. However, the degree of generality found here will obscure some of the connections that can be made with the methods found in 6 and 7, where data is presented uniformly. Therefore, in Section 5 we will lend ourselves to the domain of equidistant sampling where the linear system techniques found in the papers by [12–14] can be used to uncover this connection. In Section 5.3, the true nature of the function $F_{\ell+1, T_s}^c(t)$ in (5) is revealed and the special interpolation form in (9) will have its explanation. Finally, in Section 7 a frequency domain method based on splines will be tested numerically.

3 Polynomial Interpolation

Assume that function values $y(t_k) = y_k$ are known at $(\ell + 1)$ different points $\{t_k\}_{k=1}^{\ell+1}$ and we seek a function P such that

$$P(t_k) = y_k, \quad k = 1, 2, \dots, \ell + 1. \quad (10)$$

That is, P should *interpolate* y at the points $\{t_k\}_{k=1}^{\ell+1}$. The first question that comes to ones mind after this definition, is which class of functions one should use. An obvious choice would be to use polynomials of degree ℓ , maybe represented in *Newton's form*

$$P_\ell(t) = c_0 + \sum_{k=1}^{\ell} c_k \prod_{l=1}^k (t - t_l) \quad (11)$$

because of its simple structure and easy evaluation.

For instance, let $\ell = 2$. Then the polynomial P_2 will have the representation

$$P_2(t) = c_0 + c_1(t - t_1) + c_2(t - t_1)(t - t_2). \quad (12)$$

and the interpolation conditions in (10) together with the polynomial representation in (12) will give the following system of equations for the extraction of the coefficients c_0 , c_1 and c_2 ,

$$\begin{cases} c_0 & = y_1 \\ c_0 + c_1(t_2 - t_1) & = y_2 \\ c_0 + c_1(t_3 - t_1) + c_2(t_3 - t_1)(t_3 - t_2) & = y_3. \end{cases} \quad (13)$$

Solving the particular triangular set of equations in (13) will then yield the coefficients

$$c_0 = y_1 \quad (14)$$

$$c_1 = \frac{y_2 - y_1}{t_2 - t_1} \quad (15)$$

$$c_2 = \frac{y_3 - y_1 - \frac{t_3 - t_1}{t_2 - t_1}(y_2 - y_1)}{(t_3 - t_1)(t_3 - t_2)}. \quad (16)$$

Generally, the system in (13) will have a triangular shape due to the interpolation conditions and the special structure of the polynomial in (11). This in turn will produce a recursive expression for an ℓ order polynomial P_ℓ that would interpolate at the points $\{t_k\}_{k=1}^{\ell+1}$. This formula that can be stated as

$$P_0(t) = c_0 \quad (17)$$

$$P_k(t) = P_{k-1}(t) + c_k \prod_{l=1}^k (t - t_l). \quad (18)$$

where P_k is the polynomial of order k which interpolates at the points $\{t_l\}_{l=1}^{k+1}$ and $k \leq \ell$. This means that c_k will only depend on the values $\{y(t_k)\}_{k=1}^{k+1}$ and we can therefore write

$$c_k = [t_1, t_2, \dots, t_{k+1}]y \quad (19)$$

where $[t_1, t_2, \dots, t_{k+1}]$ is known as the k th *divided difference* of y . By means of further trivial calculations, it

is possible to show that [3]

$$\begin{aligned} [t_k]y &= y(t_k) \\ [t_1, t_2, \dots, t_{k+1}]y &= \frac{[t_2, t_3, \dots, t_{k+1}]y - [t_1, t_3, \dots, t_k]y}{t_{k+1} - t_1} \end{aligned} \quad (20)$$

and hence, all divided differences can be computed in the recursive manner above. In case the distance between the data points is uniform such that $\Delta t_k = T_s$ the particularly simple form

$$[t_k, t_{k+1}, \dots, t_{k+\ell+1}]y = \frac{\Delta^{\ell+1}y(t_k)}{(\ell+1)!} \quad (22)$$

for the $(\ell+1)$ th divided difference at the point t_k can be derived. Here, we define the *forward difference operator* Δ (or *delta operator*) [6] acting on a function as

$$\Delta y(t) = \frac{y(t + T_s) - y(t)}{T_s}. \quad (23)$$

This means, that in general we will have the binomial expansion [1]

$$\Delta^{\ell+1}y(t_k) = \sum_{l=0}^{\ell+1} \frac{(-1)^l}{T_s^{\ell+1}} \binom{\ell+1}{l} y(t_k + (\ell+1-l)T_s) \quad (24)$$

and the k th divided difference can be computed as

$$\begin{aligned} [t_k, t_{k+1}, \dots, t_{k+\ell+1}]y &= \\ \sum_{l=0}^{\ell+1} \frac{(-1)^l}{(\ell+1)!T_s^{\ell+1}} \binom{\ell+1}{l} y(t_k + (\ell+1-l)T_s). \end{aligned} \quad (26)$$

4 Spline Interpolation

Unfortunately, interpolation with very high order polynomials suffer from a serious problem. Imagine that we would like to interpolate the function

$$f(t) = \frac{1}{1 + 25t^2} \quad (27)$$

at the ℓ equidistantly distributed points

$$t_k = -1 + (k-1)\frac{2}{\ell} \quad i = 1 \dots \ell+1 \quad (28)$$

with a polynomial P_ℓ of degree ℓ . Then, near the end-points, the polynomial solution will oscillate wildly and the error

$$\lim_{\ell \rightarrow \infty} \max_{-1 \leq t \leq 1} |f(t) - P_\ell(t)| = \infty \quad (29)$$

will be out of control. This effect is known as *Runge's phenomenon* and indicates that interpolation by a high order polynomial is not advisable. A way to resolve this issue is to allow piecewise polynomial functions such as splines.

Let therefore $\{\tau_k\}_{k=1}^{N_t+1}$ be a strictly increasing sequence of points, which are called the *breakpoints* of the function f . Also define a sequence of polynomials $\{P_k^\ell\}_{k=1}^{N_t+1}$ each of order ℓ . Then, we can define the corresponding piecewise polynomial $\hat{y}_c$ of order k by the expression

$$\hat{y}_c(t) = \begin{cases} P_1^\ell(t) & \text{if } t < \tau_1 \\ P_k^\ell(t) & \text{if } \tau_k < t < \tau_{k+1} \quad \forall k = 1, \dots, N_t \\ P_{N_t+1}^\ell(t) & \text{if } \tau_{N_t+1} < t. \end{cases} \quad (30)$$

The class of all such piecewise polynomial functions of order ℓ with the breakpoint sequence $\{\tau_k\}_{k=1}^{N_t+1}$ is then denoted with $\mathbb{P}_{\ell,\tau}$.

The values $\{\hat{y}_c(\tau_k)\}_{k=1}^{N_t+1}$ of the piecewise polynomial $\hat{y}_c$ at the breakpoints are actually not defined by expression (30). Therefore we treat the function $\hat{y}_c$ as right continuous and let

$$\hat{y}_c(\tau_k) = \hat{y}_c(\tau_k^+) \quad k = 1, \dots, N_t + 1 \quad (31)$$

where

$$\hat{y}_c(\tau^+) = \lim_{h \rightarrow 0, h > 0} \hat{y}_c(\tau + h) \quad (32)$$

$$\hat{y}_c(\tau^-) = \lim_{h \rightarrow 0, h < 0} \hat{y}_c(\tau + h). \quad (33)$$

Now, assume again that we are given function values at the points $t_1 < t_2 < \dots < t_{N_t+1}$. This time, we wish to interpolate these points using a piecewise polynomial function $\hat{y}_c$ of order 2 with a continuous first derivative. Also, we wish to choose the breakpoint sequence such that

$$\tau_k = t_k, \quad k = 1, \dots, N_t + 1. \quad (34)$$

Since each individual polynomial in $\hat{y}_c$ has 3 individual degrees of freedom we will get $3(N_t + 1)$ degrees of freedom in total that has to be reduced in order to produce uniqueness. The interpolation conditions together with the continuity requirements would then yield the following $3(N_t + 1)$ equations

$$\begin{cases} \hat{y}_c(\tau_k^+) = y_k & k = 1, \dots, N_t + 1 \\ \hat{y}_c(\tau_k^-) - \hat{y}_c(\tau_k^+) = 0 \\ \dots \\ \hat{y}_c(\tau_k^-) - D\hat{y}_c(\tau_k^+) = 0. \end{cases} \quad (35)$$

From (35) it is evident that the $2(N_t + 1)$ homogenous equations do not directly depend on the data $\{y_k\}_{k=1}^{N_t+1}$. Therefore it would be possible to use these conditions in order to reduce the number of degrees of freedom in (35) to an absolute minimum $N_t + 1$. Such a construction is equivalent to constructing a linear basis $\{\phi_k(t)\}_{k=1}^{N_t+1}$ for the linear mapping from the coefficients c_k , $k = 1, \dots, N_t + 1$ to the interpolations points $\hat{y}_c(t_k)$, $k = 1, \dots, N_t + 1$. That interpretation means that the interpolation can be represented as

$$\hat{y}_c(t) = \sum_{k=1}^{N_t+1} c(k) \phi_k(t). \quad (36)$$

One can therefore say that the homogenous equations will define a subspace of the space of functions $\mathbb{P}_{k,\tau}$ satisfying these homogenous conditions.

A particularly useful basis is the one made up of the so called *B-splines* ("basis splines")

$$B_{k,\ell+1,\tau}^c(t) = (\tau_{k+\ell+1} - \tau_k) [\tau_k, \dots, \tau_{k+\ell+1}] (\cdot - t)_+^\ell \quad (37)$$

of order ℓ for the *knot sequence* $\tau_1, \tau_2, \dots, \tau_{n+1}$. Here, we have

$$(\tau - t)_+^\ell = \begin{cases} (\tau - t)^\ell & \text{if } t \leq \tau \\ 0 & \text{if } \tau < t \end{cases}. \quad (38)$$

It should be noted that the knots are not equivalent to the breakpoints. There could be several knots at the same point which thereby dictate the number of continuous derivatives found there. We will however not go deeper into this issue and from now on we will assume that there is one knot at each break point. A cubic B-spline is illustrated in Figure 4.

Since these B-splines constitute a finite basis such that

$$\hat{y}_c(t) = \sum_{k=1}^{N_t+1} c(k) B_{k,\ell,\tau}^c(t), \quad (39)$$

finding a function which satisfies the interpolation conditions $\hat{y}_c(t_k) = y_k, k = 1, \dots, N_t + 1$, is just a matter of solving the linear system of equations

$$\mathbf{B}\mathbf{c} = \mathbf{y} \quad (40)$$

where

$$\mathbf{B} = \begin{pmatrix} B_{1,\ell+1,\tau}^c(t_1) & \dots & B_{N_t+1,\ell+1,\tau}^c(t_1) \\ \vdots & \ddots & \vdots \\ B_{1,\ell+1,\tau}^c(t_{N_t+1}) & \dots & B_{N_t+1,\ell+1,\tau}^c(t_{N_t+1}) \end{pmatrix}$$

and

$$\mathbf{c} = \begin{pmatrix} c_1 & \dots & c_{N_t+1} \end{pmatrix}^T, \mathbf{y} = \begin{pmatrix} y(t_1) & \dots & y(t_{N_t+1}) \end{pmatrix}^T.$$

Since the functions B_k are known to have a compact support, i.e.

$$B_{k,\ell+1,\tau}^c(t) = 0 \quad x \notin [t_k, t_k + \ell] \quad (41)$$

the matrix $\mathbf{B}$ will actually have a banded structure, which makes the computation of $\mathbf{c}$ less demanding. It is well known that, if the interpolation and knot points coincide and are equidistantly distributed, the system in (41) can be solved at lightning speed by means of linear filtering methods.

5 Linear Filtering and Spline Functions

As mentioned above, the knot sequence and data points in this section are always equidistantly distributed, i.e.

$$\tau_k = t_k = kT_s, \quad k = 1, \dots, n+1. \quad (42)$$

Consequently, an $\ell + 1$ order spline basis with that knot sequence will be denoted by $B_{k,\ell+1,T_s}^c$ and each individual basis function indexed by k will only be a translation of the others such that

$$B_{k,\ell+1,T_s}^c(t) = B_{0,\ell+1,T_s}^c(t - kT_s). \quad (43)$$

From now on we will therefore define

$$B_{\ell+1,T_s}^c(t) = B_{0,\ell+1,T_s}^c(t). \quad (44)$$

Let us also define the sampled version of a spline function as

$$B_{\ell+1,\tau}^d(k) = B_{\ell+1,\tau}^c(kT_s) \quad (45)$$

Then, the expression

$$y(lT_s) = \hat{y}_c(lT_s) \quad (46)$$

$$= \sum_{k=-\infty}^{\infty} c(k) B_{0,\ell+1,\tau}^c(T_s l - T_s k), \quad (47)$$

$$l = -\infty \dots \infty$$

will contain the interpolation conditions for a bi-infinite sequence $\{y(kT_s)\}_{k=-\infty}^{\infty}$ of equidistantly distributed data points of some ℓ times continuously differentiable function $y(t)$. What this means is, that in order to find a spline function of order $\ell + 1$ with the interpolation property, it is unnecessary to solve a system of equations as the one in (41) in order to find the coefficients $c(k)$.

Fig. 2. Cubic spline basis function ($\ell = 3$).

Instead, one can concentrate on the discrete convolution equation

$$y(lT_s) = \sum_{k=-\infty}^{\infty} c(k) B_{\ell+1,\tau}^d(l - k), \quad (48)$$

$$l = -\infty \dots \infty, \quad (49)$$

which captures the Toeplitz structure of the matrix in $\mathbf{B}$ in (41) and takes advantage of methods from linear filtering and linear systems. Applying the z-transform to the expression above we will then yield

$$Y(z) = \sum_{k=-\infty}^{\infty} y(kT_s) z^{-k} = C(z) B_{\ell+1,\tau}^d(z), \quad (50)$$

where

$$C(z) = \sum_{k=-\infty}^{\infty} c(k) z^{-k}, \quad (51)$$

$$B_{\ell+1,\tau}^d(z) = \sum_{k=-\infty}^{\infty} B_{\ell+1,\tau}^d(k) z^{-k}. \quad (52)$$

Extracting the z-transform of the coefficients is then just a matter of inversion, and we will get

$$C(z) = [B_{\ell+1,\tau}^d(z)]^{-1} Y(z). \quad (53)$$

What remains is to compute the Z-transform of $B_{\ell+1,\tau}^d(z)$.

5.1 Z-transform of B-Splines

Because of what we know from (25) about the properties of the divided difference in (19) for uniformly distributed

data points with inter-point distance T_s , it is possible to show that [13]

$$\begin{aligned} B_{0,\ell+1,T_s}^c(t) &= (\tau_{\ell+1} - \tau_0)[\tau_0, \dots, \tau_{\ell+1}](\cdot - t)_+^\ell \\ &= (\ell+1)T_s[0, \dots, (\ell+1)T_s](\cdot - t)_+^\ell \\ &= (\ell+1)T_s \frac{\Delta^{\ell+1}(\cdot - t)_+^\ell}{(\ell+1)!} \\ &= \sum_{l=0}^{\ell+1} \frac{(-1)^l}{\ell! T_s^\ell} \binom{\ell+1}{l} ((\ell+1-l)T_s - t)_+^\ell. \end{aligned}$$

In turn, this means that the transformed version of the spline basis function can be computed as [13]

$$\begin{aligned} B_{\ell+1,T_s}^d(z) &= \sum_{k=-\infty}^{\infty} B_{\ell+1,T_s}^d(k) z^{-k} \\ &= z^{-(\ell+1)} \sum_{l=0}^{\ell+1} \frac{(-1)^l}{\ell!} \binom{\ell+1}{l} z^\ell \sum_{k=0}^{\infty} k^\ell z^k \\ &= z^{-(\ell+1)} \frac{(1-z)^{\ell+1}}{\ell!} \sum_{k=0}^{\infty} k^\ell z^k. \end{aligned} \quad (54)$$

Now, because of the relationship [9]

$$\sum_{k=0}^{\infty} k^\ell z^k = z \frac{\Pi_\ell(z)}{(1-z)^{\ell+1}}, \quad |z| \leq 1 \quad (55)$$

between the Euler-Frobenius polynomials $\Pi_\ell(z)$ and the summation in (54), the following explicit expression for $B_{\ell+1,T_s}^d(z)$ is possible,

$$B_{\ell+1,T_s}^d(z) = \frac{z^{-\ell} \Pi_\ell(z)}{\ell!}. \quad (56)$$

Using the following expression [11] for $\Pi_\ell(z)$

$$\Pi_\ell(z) = b_1^\ell z^{\ell-1} + b_2^\ell z^{\ell-2} + \dots + b_\ell^\ell, \quad \ell \geq 1 \quad (57)$$

where

$$b_k^\ell = \sum_{l=1}^k (-1)^{k-l} l^\ell \binom{\ell+1}{k-l}, \quad k = 1, \dots, \ell \quad (58)$$

we can then show that

$$B_{2,T_s}^d(z) = \frac{1}{z} \quad (59)$$

$$B_{3,T_s}^d(z) = \frac{z+1}{2z^2} \quad (60)$$

$$B_{4,T_s}^d(z) = \frac{z^2+4z+1}{6z^3} \quad (61)$$

$$B_{5,T_s}^d(z) = \frac{z^3+11z^2+11z+1}{24z^4}. \quad (62)$$

5.2 Fourier Transform of B-Splines

Altogether, the discussion above means that when a function is represented as a spline in (39), its continuous time Fourier transform will be

$$\begin{aligned} \hat{Y}_c(i\omega) &= \int_{-\infty}^{\infty} \hat{y}_c(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} c(k) B_{k,\ell+1,T_s}^c(t) \right) e^{-i\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} c(k) \int_{-\infty}^{\infty} B_{0,\ell+1,T_s}^c(t - T_s k) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} B_{0,\ell+1,T_s}^c(t) e^{-i\omega t} dt \sum_{k=-\infty}^{\infty} c(k) e^{-i\omega T_s k}. \end{aligned} \quad (63)$$

If we use the relationship [9]

$$B_{\ell+1,T_s}^c(i\omega) = \int_{-\infty}^{\infty} B_{\ell+1,T_s}^c(t) e^{-i\omega t} dt = \left(\frac{1 - e^{-i\omega T_s}}{i\omega} \right)^{\ell+1},$$

and if this expression is substituted into (63) one will get

$$\hat{Y}_c(i\omega) = F_{\ell+1,T_s}^c(i\omega) Y_d(e^{i\omega T_s}) \quad (64)$$

$$= \frac{B_{\ell+1,T_s}^c(i\omega)}{B_{\ell+1,T_s}^d(e^{i\omega T_s})} Y_d(e^{i\omega T_s}), \quad (65)$$

where

$$F_{\ell+1,T_s}^c(i\omega) = \frac{\left(\frac{e^{i\omega T_s} - 1}{i\omega} \right)^{\ell+1}}{\frac{e^{i\omega T_s} \Pi_\ell(z)}{\ell!}}, \quad (66)$$

is identical to 5. This means, that estimating the continuous-time Fourier transform as in 7 is equivalent to interpolating your uniformly sampled data with an $\ell+1$ order B-spline function with inter-knot distance T_s

5.3 Fundamental Polynomial B-Spline Function

An interesting consequence of the above line of reasoning is that if we interpret $F_{\ell+1,T_s}^c(i\omega)$ as the continuous-time Fourier transform of some function $F_{\ell+1,T_s}^c(t)$ such that

$$F_{\ell+1,T_s}^c(i\omega) = \frac{B_{0,\ell+1,T_s}^c(i\omega)}{B_{0,\ell+1,T_s}^d(e^{i\omega T_s})} \quad (67)$$

$$= \int_{-\infty}^{\infty} F_{\ell+1,T_s}^c(t) e^{-i\omega t} dt. \quad (68)$$

Then, $F_{\ell+1,T_s}^c(t)$ is the so called *fundamental spline function* of order $\ell+1$ (see for instance Lecture 4 in [9]) which

corresponds to the solution of the interpolation problem

$$\delta(l) = \sum_{k=-\infty}^{\infty} c(k) \beta^{(\ell)}(T_s l - T_s k) \quad l = -\infty, \dots, \infty \quad (69)$$

where

$$\delta(l) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases} \quad (70)$$

is the *Kronecker delta* function. This means that we can also write our interpolation function in (39) as

$$\hat{y}_c(t) = \sum_{k=-\infty}^{\infty} y(T_s k) F_{\ell+1, T_s}^c(t - T_s k) \quad (71)$$

which is called the *Lagrange form* [2] of the spline representation. This explains the particular appearance of the expression found in (9).

The functions $F_{\ell+1, T_s}^c(t - kT_s)$ can be thought of as an orthogonal basis for the linear mapping from the interpolation points to the spline of order $\ell + 1$ with equidistantly distributed knots. In Figure 3 we have portrayed the fundamental spline function $F_{\ell+1, T_s}^c(t - kT_s)$ of cubic order $\ell + 1 = 4$.

Fig. 3. Fundamental cardinal spline basis function $f^{(\ell)}$ of cubic order $\ell = 3$

6 Non-Uniform Sampling and Splines

Assume again that we have the setting described in Figure 1. That is, a continuous-time system fed with a piecewise constant input signal $u(t)$. Then, the conclusion of the discussion held in the sections above is that,

estimating the continuous-time Fourier transform from sampled data $\{y(kT_s)\}_{k=1}^{N_t+1}$ as

$$\hat{Y}_c(i\omega) = F_{\ell+1, T_s}^c(i\omega) Y_d(e^{i\omega T_s}) \quad (72)$$

where

$$F_{\ell+1, T_s}^c(i\omega) = \frac{\left(\frac{e^{i\omega T_s} - 1}{i\omega}\right)^{\ell+1}}{\frac{e^{i\omega T_s} \Pi_{\ell}(z)}{\ell!}} \quad (73)$$

is equivalent to interpolating the data using polynomial splines of order $\ell + 1$ with equidistant knots, such that

$$\hat{y}_c(t) = \sum_{k=-\infty}^{\infty} c(k) B_{k, \ell+1, T_s}^c(t) \quad (74)$$

in a B-spline basis or

$$\hat{y}_c(t) = \sum_{k=-\infty}^{\infty} y(T_s k) F_{k, \ell+1, T_s}^c(t) \quad (75)$$

in the fundamental spline basis. The reason for this is mainly that the input is a known piecewise constant function produced by running a discrete signal $u(kT_s)$ through a zero-order hold circuit. Thereby producing an input signal $u(t)$ such that

$$u(t) = \sum_{k=0}^{\infty} u(kT_s) (H(t - kT_s) - H(t - (k+1)T_s)) \quad (76)$$

where H is the Heaviside step function. The sampling is then fast enough to allow the intersample behavior to be approximated by a gain followed by ℓ integrators.

In the case of irregular sampling, it seems sensible to assume that the user is still in command of the input, which is chosen as in (76). On the other hand, there is no control of the sampling of the output which will occur at the time instances $\{t_k\}_{k=1}^{N_t+1}$. This means that the right way to do the interpolation would be to use an $\ell + 1$ order polynomial spline function as in (39) with equidistantly distributed knots with sample distance T_s . The coefficients would be computed as

$$\mathbf{B}_n \mathbf{c} = \mathbf{y}_n \quad (77)$$

where

$$\mathbf{B}_n = \begin{pmatrix} B_{1,\ell+1,T_s}^c(t_1) & \cdots & B_{N_t+1,\ell+1,T_s}^c(t_1) \\ B_{1,\ell+1,T_s}^c(t_2) & \cdots & B_{N_t+1,\ell+1,T_s}^c(t_2) \\ \vdots & \ddots & \vdots \\ B_{1,\ell+1,T_s}^c(t_{N_t+1}) & \cdots & B_{N_t+1,\ell+1,T_s}^c(t_{N_t+1}) \end{pmatrix} \quad (78)$$

and

$$\mathbf{c} = (c_1 \dots c_{N_t+1}), \quad \mathbf{y}_n = (y(t_1) \dots y(t_{N_t+1}))^T.$$

The reconstructed values at the sampling point could then be found as

$$\hat{\mathbf{y}}_u = \mathbf{B}_u \mathbf{c} \quad (79)$$

where

$$\mathbf{B}_u = \begin{pmatrix} B_{1,\ell+1,T_s}^c(0) & \cdots & B_{N_t+1,\ell+1,T_s}^c(0) \\ B_{1,\ell+1,T_s}^c(T_s) & \cdots & B_{N_t+1,\ell+1,T_s}^c(T_s) \\ \vdots & \ddots & \vdots \\ B_{1,\ell+1,T_s}^c(nT_s) & \cdots & B_{N_t+1,\ell+1,T_s}^c(N_tT_s) \end{pmatrix}$$

and

$$\mathbf{c} = (c_1 \dots c_{N_t+1})^T, \quad \hat{\mathbf{y}}_u = (\hat{y}_c(0) \dots \hat{y}_c(N_tT_s))^T.$$

From these values an estimate of the discrete-time Fourier transform can be produced as

$$\hat{Y}_d(e^{i\omega T_s}) = \sum_{k=0}^{N_t} \hat{y}_c(kT_s) e^{i\omega T_s} \quad (80)$$

and the continuous-time Fourier transform is estimated using

$$\hat{Y}_c(i\omega) = F_{\ell+1,T_s}^c(i\omega) \hat{Y}_d(e^{i\omega T_s}). \quad (81)$$

The parameter estimates are then found using the method in (6). Reconstruction to an equidistant grid can also be accomplished using the fundamental spline function such that

$$\hat{\mathbf{y}}_u = \mathbf{F}^{-1} \mathbf{y}_n$$

where

$$\mathbf{F} = \begin{pmatrix} F_{1,\ell+1,T_s}^c(t_1) & \cdots & F_{N_t+1,\ell+1,T_s}^c(t_1) \\ F_{1,\ell+1,T_s}^c(t_2) & \cdots & F_{N_t+1,\ell+1,T_s}^c(t_2) \\ \vdots & \ddots & \vdots \\ F_{1,\ell+1,T_s}^c(t_{N_t+1}) & \cdots & F_{N_t+1,\ell+1,T_s}^c(t_{N_t+1}) \end{pmatrix}$$

and

$$\hat{\mathbf{y}}_u = (\hat{y}_c(0) \dots \hat{y}_c(N_tT_s)), \quad \hat{\mathbf{y}}_u (y(t_1) \dots y(t_{N_t+1}))$$

7 Numerical Examples

In the following section we will illustrate how one can identify continuous-time deterministic models from non-uniformly sampled output data using spline interpolation. The example systems

$$G_1(s) = \frac{s + 0.5}{s^2 + 2s + 3} \quad (82)$$

$$G_2(s) = \frac{1}{s^3 + 2s^2 + 3s + 4} \quad (83)$$

$$G_3(s) = \frac{a}{s^3 + as^2 + as + a}, \quad a = 4 \quad (84)$$

and the input signal is uniform piecewise constant with randomly distributed amplitudes. The output $y(t)$ will be sampled at uniform time instances subject to jitter such that

$$t_k = kT_s + \delta_k, \quad k = 1, \dots, N_t - 1$$

where

$$\delta_k \in \mathcal{U}(-\delta_0, \delta_0) \quad k = 1, \dots, N_t - 1.$$

The parameter value $\delta_0 = T_s/3$ will be used in order to generate enough non-uniformity in the sampling. For the sake of simplicity, the initial and endpoint time instances have been chosen such that $t_0 = 0$ and $t_{N_t} = N_tT_s$. The output $y(t_k)$, $k = 1, \dots, N_t$ is then interpolated by a polynomial a spline of order $\ell + 1$ using uniformly distributed knots τ on the grid

$$\tau_k = kT_s, \quad k = -1, \dots, N_t + \ell. \quad (85)$$

The continuous-time Fourier transform is then estimated using the method found in (7) and parameter estimates are found using the method in (6).

Illustrations of parameter estimates versus the nominal sampling time T_s are illustrated in Figure 4 and Table 1 for the system in (82), in Figure 5 and Table 2 for the system in (83) and in Figure 6 and Table 3 for the system in (84)

Fig. 4. Parameter estimates for the model $G_1(s) = \frac{s+0.5}{s^2+2s+3}$ versus the sampling time T_s . Sampling instances have been generated such that $t_k = kT_s + \delta_k$ where $\delta_k \sim \mathcal{U}(-T_s/3, T_s/3)$. The number of samples used are $N_t = 10000$. Relative degree of this system is $\ell = 3$ and reconstruction to an uniform grid has been accomplished using an $\ell + 1 = 2$ order polynomial spline.

Table 1

Parameter estimates for the model $G_1(s) = \frac{s+0.5}{s^2+2s+3}$ versus the sampling time T_s . Sampling instances have been generated such that $t_k = kT_s + \delta_k$ where $\delta_k \sim \mathcal{U}(-T_s/3, T_s/3)$. The number of samples used are $N_t = 10000$. Relative degree of this system is $\ell = 3$ and reconstruction to an uniform grid has been accomplished using an $\ell + 1 = 2$ order polynomial spline.

T_s	$a_1 = 2$	$a_2 = 3$	$b_1 = 1$	$b_2 = 0.5$
0.1	2.0023	3.0071	0.9996	0.5029
0.2	1.9921	3.0098	0.9959	0.5056
0.3	1.9778	3.0232	0.9896	0.5129
0.4	1.9514	3.0425	0.9766	0.5239
0.5	1.9006	3.0459	0.9525	0.5325
0.6	1.8389	3.0653	0.9186	0.5429
0.7	1.7690	3.0734	0.8842	0.5520
0.8	1.6818	3.0696	0.8314	0.5586
0.9	1.5941	3.0486	0.7733	0.5690
1.0	1.4963	2.9653	0.7196	0.5525

8 Summary

The conclusion of this paper, is that interpolation in terms of polynomial spline functions can be a feasible mean of reconstructing the output of continuous-time input-output models from sampled data. Splines of the order ℓ , as the relative degree of the system could be used. The continuous-time Fourier transform is then rapidly computed and the parameters are identified using a direct continuous-time frequency domain

Fig. 5. Parameter estimates for the model $G_2(s) = \frac{1}{s^3+2s^2+3s+4}$ versus the sampling time T_s . Sampling instances have been generated such that $t_k = kT_s + \delta_k$ where $\delta_k \sim \mathcal{U}(-T_s/3, T_s/3)$. The number of samples used are $N_t = 1000$. Relative degree of this system is $\ell = 3$ and reconstruction to an uniform grid has been accomplished using an $\ell + 1 = 4$ order polynomial spline.

Table 2

Parameter estimates for the model $G_2(s) = \frac{1}{s^3+2s^2+3s+4}$ versus the sampling time T_s . Sampling instances have been generated such that $t_k = kT_s + \delta_k$ where $\delta_k \sim \mathcal{U}(-T_s/3, T_s/3)$. The number of samples used are $N_t = 1000$. Relative degree of this system is $\ell = 3$ and reconstruction to an uniform grid has been accomplished using an $\ell + 1 = 4$ order polynomial spline.

T_s	$a_1 = 2$	$a_2 = 3$	$a_3 = 4$	$b_1 = 1$
0.1	2.0069	3.0015	4.0117	1.0012
0.2	2.0001	3.0013	3.9995	0.9995
0.3	2.0057	3.0047	4.0104	1.0027
0.4	1.9992	3.0009	3.9991	0.9979
0.5	2.0000	3.0007	4.0003	0.9999
0.6	1.9983	3.0006	3.9962	0.9988
0.7	1.9955	2.9986	3.9910	0.9972
0.8	1.9920	2.9980	3.9827	0.9942
0.9	1.9810	2.9926	3.9599	0.9880
1.0	1.9646	2.9867	3.9250	0.9770

identification method

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Fig. 6. Parameter estimates for the model $G_3(s) = \frac{a}{s^3 + as^2 + as + a}$ versus the sampling time T_s . Sampling instances have been generated such that $t_k = kT_s + \delta_k$ where $\delta_k \sim \mathcal{U}(-T_s/3, T_s/3)$. The number of samples used are $N_t = 1000$. Relative degree of this system is $\ell = 3$ and reconstruction to an uniform grid has been accomplished using an $\ell + 1 = 4$ order polynomial spline.

Table 3

Parameter estimates for the model $G_3(s) = \frac{a}{s^3 + as^2 + as + a}$ versus the sampling time T_s . The sampling instances have been generated such that $t_k = kT_s + \delta_k$ where $\delta_k \sim \mathcal{U}(-T_s/3, T_s/3)$. The number of samples used are $N_t = 1000$. Relative degree of this system is $\ell = 3$ and reconstruction to an uniform grid has been accomplished using an $\ell + 1 = 4$ order polynomial spline.

T_s	0.1	0.2	0.3	0.4	0.5
a	4.0125	4.0171	4.0015	4.0114	4.0172
T_s	0.6	0.7	0.8	0.9	1.0
a	3.9900	3.9797	3.9635	3.9486	3.9303

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Sammanfattning Abstract <p>This paper concerns the subject of identification of continuous-time output error (OE) models based on non-uniformly sampled output data. The exact method for doing this is well known in the time-domain, where the continuous-time system is discretized, simulated and the result is fitted in a mean square sense to measured data. The material presented here is based on a method proposed in a sister paper [5] which dealt with the same topic but for the case of uniformly sampled data. In this text it will be shown how that method entails that the output should be reconstructed using a B-spline with uniformly distributed knots. This representation can then be used to directly identify the continuous-time system without proceeding via discretization. Only the relative degree of the model is used to choose the order of the spline.</p>		
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