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Abstract: The robust $\mathcal{H}_2$ norm plays an important role in analysis and design in many fields. However, for many practical applications, design and analysis is based on finite frequency range. In this paper we review the concept of the robust finite frequency $\mathcal{H}_2$ norm, and we provide an algorithmic method for calculating an upper bound for the mentioned quantity.

Keywords: Robust $\mathcal{H}_2$ norm, Uncertain systems, Robust control.

1. INTRODUCTION

The $\mathcal{H}_2$ norm has been one of the major analysis and design criteria in many fields of engineering, e.g. automatic control. Over the years different methods have been proposed for calculating $\mathcal{H}_2$ and robust $\mathcal{H}_2$ norms using Linear Matrix Inequalities (LMIs) and Ricatti equations, Doyle et al. (1989), Stoorvogel (1993), Boyd et al. (1994), Paganini (1997), Paganini (1999) and many more. Paganini and Feron (2000) provides a survey of advances and different methods in the field.

Despite the importance of $\mathcal{H}_2$ norms, it is sometimes unnecessary to compute the norm based on all frequencies, and it is beneficial to concentrate on only a finite frequency range of interest. Gawronski (2000) introduces a method for computing the finite frequency $\mathcal{H}_2$ norm for systems with no uncertainty by computing the finite frequency observability Gramian, while Masi et al. (2010), describes a method for calculating an upper bound for the robust finite frequency $\mathcal{H}_2$ norm for systems with structured uncertainty via formulating a set of Linear Matrix Inequalities (LMIs) and computing the finite frequency observability Gramian.

This paper introduces a computationally cheaper algorithmic method for calculating an upper bound for the robust finite frequency $\mathcal{H}_2$ norm of an uncertain system, which is also related to the method presented in Roos and Biannic (2010), for computing upper bounds on the structured singular value.

The paper is organized as follows. Next we present some of the notations used. Section 2 presents the problem formulation. Section 3 introduces a candidate for an upper bound for the robust finite frequency $\mathcal{H}_2$ norm, and Section 4 provides an algorithm for calculating the mentioned bound. In Section 5 we present numerical examples, and finally in Section 6 we finish the paper with some concluding remarks.

Notation The notation in this paper is standard. The min and max represent the minimum and maximum of a function or a set, and similarly sup represents the supremum of a function. A transfer matrix in terms of state-space data is denoted

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} := C(j\omega I - A)^{-1}B + D. \quad (1)$$

With $\|\cdot\|_2$, we denote the Euclidian or 2-norm of a vector or the norm of a matrix induced by the 2-norm. Except on rare occasions we omit the dimensions of all vectors or matrices and assume that the dimensions are compatible. Also for the sake of brevity of notation, unless necessary, we drop the dependence of functions on frequency.

2. PROBLEM FORMULATION

2.1 $\mathcal{H}_2$ norm of a system

Consider the following state space system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (2)$$

Given $G(s)$ as the transfer function for the system in (2), the $\mathcal{H}_2$ norm of this system is defined as below

$$\|G\|_2^2 = \int_{-\infty}^{\infty} \text{trace}(G(j\omega)^* G(j\omega)) \frac{d\omega}{2\pi}. \quad (3)$$

Similarly the finite frequency $\mathcal{H}_2$ norm of the system is defined as follows

$$\|G\|_{2,\bar{\omega}}^2 = \int_{-\bar{\omega}}^{\bar{\omega}} \text{trace}(G(j\omega)^* G(j\omega)) \frac{d\omega}{2\pi}. \quad (4)$$

Fig. 1. Uncertain system with structured uncertainty

2.2 Robust $\mathcal{H}_2$ norm of a system

Now consider the following uncertain system

$$\begin{cases} \dot{x} = Ax + B_q q + B_w w \\ p = C_p x + D_{pq} q \\ z = C_z x + D_{zq} q \\ q = \Delta p \end{cases} \quad (5)$$

where $x \in \mathcal{R}^n$, $w \in \mathcal{R}^m$, $z \in \mathcal{R}^p$, $q, p \in \mathcal{R}^d$. Also Δ has the following structure

$$\Delta = \text{diag}[\delta_1 I_{r_1} \cdots \delta_L I_{r_L} \Delta_{L+1} \cdots \Delta_{L+F}], \quad (6)$$

and it belongs to B_Δ , where B_Δ is the unit ball for the induced 2-norm. This structure of Δ can represent both real parametric uncertainties and unmodeled system dynamics.

Later we also introduce a class of positive definite matrices, χ , which commute with Δ , i.e. they have the following structure, Fan et al. (1991),

$$\chi = \text{diag}[X_1 \cdots X_L \ x_{L+1} I_{m_1} \cdots x_{L+F} I_{m_F}]. \quad (7)$$

The transfer matrix for this system is defined as below, also illustrated in Figure 1,

$$M(j\omega) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} A & B_q & B_w \\ C_p & D_{pq} & 0 \\ C_z & D_{zq} & 0 \end{bmatrix}, \quad (8)$$

and the transfer function matrix from w to z is given by the upper LFT

$$(\Delta * M) = M_{22} + M_{21} \Delta (I - M_{11} \Delta)^{-1} M_{12}. \quad (9)$$

Having defined (9), the robust $\mathcal{H}_2$ norm for system in (5) is defined as

$$\sup_{\Delta \in B_\Delta} \|\Delta * M\|_2^2 = \sup_{\Delta \in B_\Delta} \int_{-\infty}^{\infty} \text{trace}((\Delta * M)^*(\Delta * M)) \frac{d\omega}{2\pi}, \quad (10)$$

and the respective robust finite frequency $\mathcal{H}_2$ norm for the system is defined as

$$\sup_{\Delta \in B_\Delta} \|\Delta * M\|_{2,\bar{\omega}}^2 = \sup_{\Delta \in B_\Delta} \int_{-\bar{\omega}}^{\bar{\omega}} \text{trace}((\Delta * M)^*(\Delta * M)) \frac{d\omega}{2\pi}. \quad (11)$$

In this paper we aim to find an upper bound for the robust finite frequency $\mathcal{H}_2$ norm of system in (5).

3. AN UPPER BOUND FOR THE ROBUST $\mathcal{H}_2$ NORM

Consider the following condition:

Condition 1. There exists $\mathcal{X}(\omega) \in \chi$ and $Y(\omega) = Y(\omega)^* \in \mathcal{C}^{m \times m}$ such that for $\epsilon > 0$

$$M(j\omega)^* \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. \quad (12)$$

This condition can be restated as follows:

Lemma 2. If there exists $\mathcal{X}(\omega) \in \chi$ such that $M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega) \prec 0$, then Condition 1 is satisfied if and only if, there exists a $Y(\omega) = Y(\omega)^*$ such that

$$\begin{aligned} & M_{12}^* \mathcal{X}(\omega) M_{12} + M_{22}^* M_{22} - (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21}) \times \\ & (M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega))^{-1} \times \\ & (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21})^* \preceq Y(\omega). \end{aligned} \quad (13)$$

Proof. Consider the left hand side of Condition 1

$$\begin{aligned} & M^*(j\omega) \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & I \end{bmatrix} M(j\omega) - \begin{bmatrix} \mathcal{X}(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} = \\ & \begin{bmatrix} M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega) & M_{11}^* \mathcal{X}(\omega) M_{12} + M_{21}^* M_{22} \\ M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21} & M_{12}^* \mathcal{X}(\omega) M_{12} + M_{22}^* M_{22} - Y(\omega) \end{bmatrix}. \end{aligned} \quad (14)$$

Now if we assume that there exists $\mathcal{X}(\omega) \in \chi$ such that $M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega) \prec 0$, then Lemma 2 is the direct outcome of Schur's lemma.

The following theorem, extracted from Paganini (1997) and Masi et al. (2010), utilizes this condition to provide an upper bound for $\|\Delta * M\|_2^2$ for any frequency and any $\Delta \in B_\Delta$.

Theorem 3. If there exists $\mathcal{X}(\omega) \in \chi$ such that $M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega) \prec 0 \quad \forall \omega$ and we define $Y(\omega)$ as below

$$\begin{aligned} Y(\omega) = & M_{12}^* \mathcal{X}(\omega) M_{12} + M_{22}^* M_{22} - \\ & (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21}) \times \\ & (M_{11}^* \mathcal{X}(\omega) M_{11} + M_{21}^* M_{21} - \mathcal{X}(\omega))^{-1} \times \\ & (M_{12}^* \mathcal{X}(\omega) M_{11} + M_{22}^* M_{21})^*, \end{aligned} \quad (15)$$

then $(\Delta * M)(j\omega)^*(\Delta * M)(j\omega) \preceq Y(\omega) \quad \forall \omega$.

Proof. If the assumptions of the theorem are satisfied, then by Lemma 2, Condition 1 is valid, i.e. (12) holds. Define

$$\hat{M} = \begin{bmatrix} \mathcal{X}(\omega)^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} \mathcal{X}(\omega)^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}. \quad (16)$$

Then (12) can be rewritten as

$$\hat{M}^* \hat{M} - \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix} \preceq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}. \quad (17)$$

As a result

$$\hat{M}^* \hat{M} \preceq \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix}. \quad (18)$$

Define $\bar{q}(j\omega) = \mathcal{X}(\omega)^{\frac{1}{2}} q(j\omega)$ and $\bar{p}(j\omega) = \mathcal{X}(\omega)^{\frac{1}{2}} p(j\omega)$. By pre and post multiplying both sides of (18) by $[\bar{q}(j\omega)^* \ w(j\omega)^*]$ and $\begin{bmatrix} \bar{q}(j\omega) \\ w(j\omega) \end{bmatrix}$, respectively, we have

$$|z(j\omega)|^2 + |\bar{p}(j\omega)|^2 \leq |\bar{q}(j\omega)|^2 + w(j\omega)^* Y(\omega) w(j\omega). \quad (19)$$

For all frequencies Δ commutes with $\mathcal{X}(\omega)^{-\frac{1}{2}}$, and hence $\bar{q} = \mathcal{X}^{\frac{1}{2}} q = \mathcal{X}^{\frac{1}{2}} \Delta \mathcal{X}^{-\frac{1}{2}} \bar{p} = \Delta \bar{p}$. Considering the fact that $\Delta \in B_\Delta$, it now follows from (9) and (19) that

$$\begin{aligned} |z(j\omega)|^2 &= w(j\omega)^* (\Delta * M)(j\omega)^* (\Delta * M)(j\omega) w(j\omega) \\ &\leq w(j\omega)^* Y(\omega) w(j\omega), \end{aligned} \quad (20)$$

which completes the proof.

The following results are direct consequences of Theorem 3:

Corollary 4. If there exists a constant $\mathcal{X} \in \chi$ and a frequency interval $\mathcal{I}(\omega_i)$ such that

$$M_{11}^* \mathcal{X} M_{11} + M_{21}^* M_{21} - \mathcal{X} \prec 0 \quad \forall \omega \in \mathcal{I}(\omega_i), \quad (21)$$

and we consider $Y(\omega)$ as defined in (15) for the mentioned frequency interval then

$$\int_{\omega \in \mathcal{I}(\omega_i)} \text{trace}((\Delta * M)^* (\Delta * M)) \frac{d\omega}{2\pi} \leq \int_{\omega \in \mathcal{I}(\omega_i)} \text{trace}(Y(\omega)) \frac{d\omega}{2\pi}, \quad (22)$$

for all $\Delta \in B_\Delta$

Corollary 5. Let $\mathcal{I}(\omega_i)$ for $i = 1, \dots, m$ be disjoint frequency intervals such that $\bigcup_{i=1}^m \mathcal{I}(\omega_i) = [-\bar{\omega} \ \bar{\omega}]$. Also let $\mathcal{X}_i$ for $i = 1, \dots, m$ be the multipliers for which $M_{11}^* \mathcal{X}_i M_{11} + M_{21}^* M_{21} - \mathcal{X}_i \prec 0 \quad \forall \omega \in \mathcal{I}(\omega_i)$. Then, it holds that

$$\begin{aligned} &\sup_{\Delta \in B_\Delta} \|\Delta * M\|_{2, \bar{\omega}}^2 \\ &\leq \sup_{\Delta \in B_\Delta} \sum_{i=1}^m \int_{\omega \in \mathcal{I}(\omega_i)} \text{trace}((\Delta * M)^* (\Delta * M)) \frac{d\omega}{2\pi} \\ &\leq \sum_{i=1}^m \int_{\omega \in \mathcal{I}(\omega_i)} \text{trace}(Y_i(\omega)) \frac{d\omega}{2\pi}. \end{aligned} \quad (23)$$

where $Y_i(\omega)$ is defined as in (15), with $\mathcal{X}(\omega) = \mathcal{X}_i$.

Corollary 5 provides an upper bound for the finite frequency robust $\mathcal{H}_2$ norm of the system defined in (5). Next we present an algorithm for calculating this upper bound for arbitrary frequency intervals.

4. AN ALGORITHM TO CALCULATE AN UPPER BOUND FOR THE ROBUST FINITE FREQUENCY $\mathcal{H}_2$ NORM

Consider the following two LMIs,

$$M_{11}(j\omega)^* \mathcal{X}(\omega) M_{11}(j\omega) + M_{21}(j\omega)^* M_{21}(j\omega) - \mathcal{X}(\omega) \prec 0 \quad (24)$$

$$\begin{bmatrix} M_{11}(j\omega) & 0 \\ M_{21}(j\omega) & 0 \end{bmatrix}^* \bar{\mathcal{X}}(\omega) \begin{bmatrix} M_{11}(j\omega) & 0 \\ M_{21}(j\omega) & 0 \end{bmatrix} - \bar{\mathcal{X}}(\omega) \prec 0. \quad (25)$$

Then $\bar{\mathcal{X}}_i = \begin{bmatrix} \mathcal{X}_i & 0 \\ 0 & I \end{bmatrix}$ satisfies (25) for $\omega = \omega_i$, if and only if $\mathcal{X}_i$ satisfies (24) for the same frequency.

Although Corollary 5 introduces a way for producing an upper bound, it also requires checking $M_{11}^* \mathcal{X}_i M_{11} + M_{21}^* M_{21} - \mathcal{X}_i \prec 0 \quad \forall \omega \in \mathcal{I}(\omega_i)$ for an infinite number of frequencies in $\mathcal{I}(\omega_i)$. Next we present a theorem, from Roos and Biannic (2010), that introduces a method for extending the validity of such LMIs, e.g. (25) from a single frequency point to a frequency interval.

Theorem 6. Let $\tilde{M} = \begin{bmatrix} M_{11} & 0 \\ M_{21} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$, and let $D = \mathcal{X}_i^{\frac{1}{2}}$, where $\bar{\mathcal{X}}_i$ satisfies the LMI in (25). Define

$$\mathcal{G} = A_X - B_X D_X^{-1} C_X, \quad (26)$$

where

$$\begin{aligned} A_X &= \begin{bmatrix} A_G & 0 \\ -C_G^* C_G & -A_G^* \end{bmatrix}, & B_X &= \begin{bmatrix} -B_G \\ C_G^* D_G \end{bmatrix}, \\ C_X &= [D_G^* C_G \ B_G^*], & D_X &= I - D_G^* D_G \end{aligned} \quad (27)$$

in which

$$G = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix} = \begin{bmatrix} \tilde{A} - j\omega_i I & \tilde{B} D^{-1} \\ \tilde{C} & \tilde{D} D D^{-1} \end{bmatrix}, \quad (28)$$

and define ω_{min} and ω_{max} as

$$\omega_{min} = \begin{cases} -\omega_i & \text{if } j\mathcal{G} \text{ has no positive real eigenvalue} \\ \max\{\lambda \in \mathcal{R}_- : \det(\lambda I + j\mathcal{G}) = 0\} & \text{otherwise} \end{cases} \quad (29)$$

$$\omega_{max} = \begin{cases} \infty & \text{if } j\mathcal{G} \text{ has no negative real eigenvalue} \\ \min\{\lambda \in \mathcal{R}_+ : \det(\lambda I + j\mathcal{G}) = 0\} & \text{otherwise} \end{cases} \quad (30)$$

Then $\bar{\mathcal{X}}_i$ satisfies (25) $\forall \omega \in \mathcal{I}(\omega_i) = [\omega_i + \omega_{min} \ \omega_i + \omega_{max}]$.

Proof. Consider the LMI in (25) with $\bar{\mathcal{X}}(\omega) = \bar{\mathcal{X}}_i$. This LMI can be rewritten as

$$\bar{\mathcal{X}}_i^{-\frac{1}{2}} \tilde{M}^* \bar{\mathcal{X}}_i^{\frac{1}{2}} \bar{\mathcal{X}}_i^{-\frac{1}{2}} \tilde{M} \bar{\mathcal{X}}_i^{-\frac{1}{2}} - I \prec 0. \quad (31)$$

Let $G(j\omega) = \bar{\mathcal{X}}_i^{\frac{1}{2}} \tilde{M}(j(\omega + \omega_i)) \bar{\mathcal{X}}_i^{-\frac{1}{2}}$. It now follows that $G = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix}$. In this theorem we are looking for the largest frequency interval, for which the LMI in (31) is valid. On the boundary of this interval $I - G(j\omega)^*G(j\omega)$ becomes singular, i.e. $\det(I - G(j\omega)^*G(j\omega)) = 0$.

By (27) and (28), $I - G(j\omega)^*G(j\omega) = \begin{bmatrix} A_X & B_X \\ C_X & D_X \end{bmatrix}$. Using Sylvester's determinant lemma and some simple matrix manipulation we have

$$\begin{aligned} \det(I - G(j\omega)^*G(j\omega)) &= 0 \Leftrightarrow \\ \det(I + D_X^{-\frac{1}{2}} C_X (j\omega I - A_X)^{-1} B_X D_X^{-\frac{1}{2}}) &= 0 \Leftrightarrow \\ \det(I + (j\omega I - A_X)^{-1} B_X D_X^{-1} C_X) &= 0. \end{aligned} \quad (32)$$

By using the matrix determinant lemma and the definition of $\mathcal{G}$ it is also straight forward to establish equivalence between the following expressions

$$\begin{aligned} \det(I + (j\omega I - A_X)^{-1} B_X D_X^{-1} C_X) &\Leftrightarrow \\ \det(j\omega I - (A_X - B_X D_X^{-1} C_X)) &= 0 \Leftrightarrow \det(\omega I + j\mathcal{G}) = 0, \end{aligned} \quad (33)$$

which completes the proof.

Now considering Theorem 3 the following algorithm is proposed for calculating an upper bound on the robust finite frequency $\mathcal{H}_2$ norm of system in (5).

Algorithm 1. (Computation of an upper bound on the robust finite frequency $\mathcal{H}_2$ norm)

- (I) Divide the frequency interval of interest into desired number of disjoint partitions, $\mathcal{I}(\omega_i)$, where ω_i represents the center of the respective partition.
- (II) For each of the partitions, compute $\mathcal{X}_i$ such that it satisfies (24) with $\omega = \omega_i$ for all $\omega \in \mathcal{I}(\omega_i)$. In case there exist a partition for which there exists no feasible solution, return to Step I and choose a finer partitioning for the initial interval of interest.
- (III) Construct $\bar{\mathcal{X}}_i$ from the achieved $\mathcal{X}_i$ in (II).
- (IV) Using Theorem 3 calculate the validity frequency range for the mentioned LMIs in (II). If the achieved frequency range does not cover the respective frequency partition, go back to (I) and choose a finer partitioning for the frequency interval of interest.
- (V) Define $Y_i(\omega)$ using (15) with $\mathcal{X}(\omega) = \mathcal{X}_i$.
- (VI) Using numerical integration methods calculate $\int_{\omega \in \mathcal{I}(\omega_i)} \text{trace}(Y_i(\omega)) \frac{d\omega}{2\pi}$.
- (VII) By Corollary 5, calculate the upper bound by summing up the integrals computed in (VI).

Considering the proposed algorithm it goes without saying that the calculated value for $\text{trace}(Y_i(\omega))$ directly affects the resulting upper bound for the robust finite frequency $\mathcal{H}_2$ norm. As a result, in order to achieve good upper bounds, it seems intuitive to consider approaches for calculating $\mathcal{X}_i$ that aim at minimizing $\text{trace}(Y_i(\omega_i))$.

Next we present different approaches for solving the second step of the proposed algorithm.

Approach 7. This approach considers minimizing $\text{trace}(Y_i(\omega_i))$ and calculating $\mathcal{X}_i$ simultaneously using

$$\begin{aligned} &\text{minimize}_{\mathcal{X}_i, Y_i} && \text{trace}(Y_i) \\ &\text{subj. to} && (12) \quad \text{with } \omega = \omega_i. \end{aligned} \quad (34)$$

This results in an SDP (Semi Definite Programming) problem with dimension of $d^2 + m^2$, where d is the dimension of the uncertainty block and m is the dimension of the input to the system.

Remark 8. For any $\mathcal{X}_i$ satisfying the LMI in (24) with $\omega = \omega_i$

$$\begin{aligned} f(\alpha) &= \text{trace}(\alpha M_{12}^* \mathcal{X}_i M_{12} + M_{22}^* M_{22} - \\ &(\alpha M_{12}^* \mathcal{X}_i M_{11} + M_{22}^* M_{21}) \times \\ &(\alpha M_{11}^* \mathcal{X}_i M_{11} + M_{21}^* M_{21} - \alpha \mathcal{X}_i)^{-1} \times \\ &(\alpha M_{12}^* \mathcal{X}_i M_{11} + M_{22}^* M_{21})^*), \end{aligned} \quad (35)$$

is a convex function of α .

Approach 9. This approach, calculates a suitable multiplier for Step II of Algorithm 1, using the following sequential method

- (I) Find $\mathcal{X}_i$ such that it satisfies the LMI in (24) for $\omega = \omega_i$.
- (II) Minimize $f(\alpha)$ with the achieved $\mathcal{X}_i$ with respect to all α such that $\alpha \mathcal{X}_i$ still satisfies the LMI in (24).

Denote α^* as the minimizing α . Then $\alpha^* \mathcal{X}_i$ can be used within the remaining steps of Algorithm 1. It is also worth mentioning that, this approach requires solving an SDP with dimension d^2 .

Remark 10. Considering Remark 8, Step II of the method in Approach 9, can be solved using bisection. Define

$$\alpha_{min} = \frac{1}{\min\{\text{eig}\left(\begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} U(-M_{11}^* \mathcal{X}_i M_{11} + \mathcal{X}_i) U^* \begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix}\right)\}}, \quad (36)$$

in which U , a unitary matrix, and Λ , are defined by the singular value decomposition $M_{21}^* M_{21} = U^* \begin{bmatrix} \Lambda^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} U$.

Then if the bisection is performed over the interval $\alpha > \alpha_{min}$, the resulting multiplier, $\alpha \mathcal{X}_i$, will always satisfy the LMI in (24) for the frequency under consideration.

Approach 11. In this approach, we simply solve the LMI in (24) and use the resulting multiplier in the remaining steps of the proposed algorithm without any modification.

Approach 12. This approach uses the same sequential method proposed in Approach 9. However, it utilizes a more computationally efficient method used for μ analysis and implemented in the Matlab μ analysis and synthesis toolbox, Balas et al. (1998), to calculate the multiplier $\mathcal{X}_i$ in Step (II) of Approach 9. The method is as follows:

- (I) Solve $M_{11}^* \mathcal{X}_i M_{11} - \beta^2 \mathcal{X}_i \prec 0$ using the Matlab μ analysis and synthesis toolbox. If the calculated β is less than 1, then continue with the second step of the method. This assures robust stability of the

system with respect to the considered uncertainty structure presented in (6). In order to avoid numerical problems, it is also highly recommended to normalize the achieved $\mathcal{X}_i$ with $\text{trace}(\mathcal{X}_i)$ before continuing to the next step.

- (II) Use the achieved $\mathcal{X}_i$ to define $f(\alpha)$ as in (35), and minimize $f(\alpha)$ with respect to all $\alpha > \alpha_{min}$, where α_{min} is defined in (36).

Remark 13. Algorithm 1 can be divided into two major parts, namely, finding a suitable multiplier and performing the numerical integration. Depending on the dimension of the problem and the structure of the calculated multiplier, either of these methods can occupy the major portion of the computational time. As a result, using a faster method for computing a suitable multiplier does not necessarily lead to shorter computational time for the whole algorithm. Moreover, the tightness and accuracy of the computed upper bound can vary depending on the system under consideration and the calculated multipliers within the algorithm.

Remark 14. For Approaches 7, 9 and 11, it is possible to compute a multiplier that is valid not only for the center frequency of the partition under consideration, but also for other frequencies within the same partition. This can be achieved by simply augmenting the existing constraints with similar constraints for other frequencies. This increases the chances that the computed multiplier will be valid for the whole partition under investigation.

In the next section we discuss the results achieved by using the proposed approaches on two examples with different uncertainty block size.

5. NUMERICAL EXAMPLES

In this section, we consider two numerical examples taken from Masi et al. (2010). The first one is an academic example for which the robust $\mathcal{H}_2$ norm can be computed analytically, while the second example concerns a model for a civil aircraft with a quite involved parametric uncertainty structure. All the simulations are conducted using the Yalmip toolbox Löfberg (2004) with the SDPT3 solver Toh et al. (1999).

Let the following matrices describe the state space representation of the system under consideration.

$$A = \begin{bmatrix} -2.5 & 0.5 & 0 & -50 & 0 \\ 0 & -1 & 0.5 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 100 \\ 0 & 0 & -100 & 0 & 0 \end{bmatrix}, \quad B_q = \begin{bmatrix} 0.25 & -0.5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_w = [0 \ 5 \ 0 \ 0 \ 5], \quad C = \begin{bmatrix} C_p \\ C_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} D_{pq} \\ D_{zq} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (37)$$

The uncertainty block for this system only includes real parametric uncertainty with $\Delta = \delta I_2$, where $-1 \leq \delta \leq 1$. The robust $\mathcal{H}_2$ norm of this system is $\sup_{\Delta \in B_\Delta} \|\Delta * M\|_2 = 1.5311$ and can be computed analytically. In

Fig. 2. Numerical example 1: The solid line illustrates different magnitudes of $\|\Delta * M\|_2^2$ for different uncertainty values, and the dashed line represents the computed upper-bound.

this example we consider computing an upper bound for the robust finite frequency $\mathcal{H}_2$ norm of the system for $\bar{\omega} = 50 \text{ rad/sec}$. The actual value for this quantity is $\sup_{\Delta \in B_\Delta} \|\Delta * M\|_{2,50}^2 = 0.8919$.

Table 5 summarizes the performance of the different approaches presented in Section 4 for this specific example. We notice that Approach 7 provides the tightest bound, and that Approach 11 is the fastest. Approach 12 provides both the worst bound and takes the longest time to compute the bound.

Table 1. Numerical results for Example 1

Approach	Estimated Upper bound	Elapsed Time[sec]	Number of Partitions
Approach 7	1.1879	55	49
Approach 9	1.3564	33	49
Approach 11	1.4341	17	49
Approach 12	2.0193	78	49

Figure 2 illustrates the achieved bound $\text{trace} Y_i(\omega)$ using Approach 7, and $\|\Delta * M\|_2^2$ for different values of the uncertainty. We notice that the bound is not tight, however, the proposed algorithm proves to provide tighter bounds in comparison to recent results, e.g. Masi et al. (2010).

As the next example a model for a civil aircraft has been considered, which includes both rigid and flexible body dynamics. This model can be used for studying the effects of the wind turbulence on different points of the aircraft. The model is provided in the form (5) with 21 states. The uncertainty block of this model also only includes real parametric uncertainty which represents the fullness of the fuel tank with $\Delta = \delta I_{14}$, and it has been normalized such that $-1 \leq \delta \leq 1$.

Considering the fact that the model is only valid for frequencies up to $\bar{\omega} = 15 \text{ rad/sec}$, the calculation of the upper bound for the robust finite frequency $\mathcal{H}_2$ norm is also performed up to the mentioned frequency. We remark that the peak in the frequency response at about 20 rad/sec illustrates the need for finite frequency $\mathcal{H}_2$ norm calculations, see Figure 3. It would not have been possible to remove the peak with a low-pass filter and use infinite frequency $\mathcal{H}_2$ norm calculations. Figure 3 presents the resulting upper bound $\text{trace} Y_i(\omega)$ using Approach 7,

Fig. 3. Numerical example 2: The solid line illustrates different magnitudes of $\|\Delta * M\|_2^2$ for different uncertainty values, and the dashed line represents the computed upper-bound.

together with $\|\Delta * M\|_2^2$ for different uncertainty values. In order to be able to compute valid multipliers for all partitions without being forced to increase the number of partitions drastically, the constraints in (34) has been augmented with similar constraints for the end points of the partition under consideration, cf. Remark 14. The computed upper bound using this approach is 1.2567 which was achieved by dividing the frequency interval into 210 partitions. The elapsed time for calculating this upper bound is 57 minutes, which shows great computational improvement with respect to the previous approach, Masi et al. (2010), which it took approximately 11 hours to calculate the upper bound, 1.2553, for the same example.

6. CONCLUSIONS

In this paper we provided a new method for computing an upper bound for the robust finite frequency $\mathcal{H}_2$ norm, for systems affected by structured parametric uncertainty. The proposed algorithm and its variations, due to different approaches for computing a suitable multiplier, provide a wide range of choices for different system structures and requirements. The new method shows a great potential with respect to previous approaches, as it scales much better with the size of the uncertainty model under consideration.

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Sammanfattning Abstract <p>The robust $\mathcal{H}_2$ norm plays an important role in analysis and design in many fields. However, for many practical applications, design and analysis is based on finite frequency range. In this paper we review the concept of the robust finite frequency $\mathcal{H}_2$ norm, and we provide an algorithmic method for calculating an upper bound for the mentioned quantity.</p>					
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