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Variance reduction through robust design of boundary conditions for stochastic hyperbolic systems of equations

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Abstract

We consider a hyperbolic system in one space dimension with uncertainty in the boundary and initial data. Our aim is to show that different boundary conditions gives different convergence rates of the variance of the solution. This means that we can with the same knowledge of data get a more or less accurate description of the uncertainty in the solution. A variety of boundary conditions are compared and both analytical and numerical estimates of the variance of the solution is presented. As an application, we study the effect of this technique on a subsonic outflow boundary for the Euler equations

Keywords: uncertainty quantification, hyperbolic system, initial boundary value problems, well posed, stability, boundary conditions, stochastic data, variance reduction, robust design, summation-by parts

1. Introduction

In most real-world applications based on partial differential equations, data is not perfectly known, and typically varies in a stochastic way. There are essentially two different techniques to quantify the resulting uncertainty in the solution. Non-intrusive methods [1, 2, 3, 4, 5, 6, 7] use multiple runs of existing deterministic codes for a particular statistical input. Standard quadrature techniques, often in combination with sparse grid techniques [8] can be used to obtain the statistics of interest. Intrusive methods [9, 10, 11, 12, 13, 14, 15, 16] are based on polynomial chaos expansions leading to a systems of equations for the expansion coefficients. This implies that new specific non-deterministic codes must be developed. The statistical properties

are obtained by a single run for a larger system of equations. There are also examples of semi-intrusive methods [18, 17]. The different procedures are compared in [19, 20] and a review is found in [21].

In this paper we take a step back from the technical developments mentioned above and focus on fundamental questions for the governing initial boundary value problem, and in particular on the influence of boundary conditions. Our aim is to minimize the uncertainty or variance of the solution for a given stochastic input. The variance reduction technique in this paper is closely related to well-posedness of the governing initial boundary value problem. In particular it depends on the sharpness of the energy estimate, which in turn depend on the boundary conditions.

We begin by deriving general well posed boundary conditions for our hyperbolic problem [22, 23, 32, 33]. These boundary conditions are implemented in a finite difference scheme using summation-by-parts (SBP) operators [24, 29, 30, 31] and weak boundary conditions [38, 39, 40, 41]. The statistical moments are computed by using the non-intrusive methodology, where the problem is solved multiple times for different values of the stochastic variable [36, 37]. The statistical moments are calculated with quadrature formulas based on the probability density distribution [34, 35].

In section 2 the continuous problem is defined and requirements for well-posedness on the boundary operators for homogeneous and non-homogeneous boundary data are derived. In section 3 we present the semi-discrete formulation and derive stability conditions. Section 4 presents the stochastic formulation of the problem together with estimates of the variance of the solution. We illustrate and analyze the variance for a model problem in section 5. In section 6, we study the implications of this technique on subsonic outflow boundary conditions for the Euler equations. Finally in section 7 we summarize and draw conclusions.

2. The continuous problem

The hyperbolic system of equations with stochastic data that we consider is,

$$\begin{aligned} u_t + Au_x &= F(x, t, \xi) & 0 \leq x \leq 1, & t \geq 0 \\ H_0 u &= g_0(t, \xi) & x = 0, & t \geq 0 \\ H_1 u &= g_1(t, \xi) & x = 1, & t \geq 0 \\ u(x, 0, \xi) &= f(x, \xi) & 0 \leq x \leq 1, & t = 0, \end{aligned} \tag{1}$$

where $u = u(x, t, \xi)$ is the solution, and ξ is the variable describing the stochastic variation of the problem. In general ξ is a vector of multiple stochastic variables, but for the purpose in this paper, one suffice. H_0 and H_1 are boundary operators defined on the boundaries $x = 0$ and $x = 1$. A is a symmetric $M \times M$ matrix. $F(x, t, \xi) \in \mathbb{R}^M$, $f(x, \xi) \in \mathbb{R}^M$, $g_0(t, \xi) \in \mathbb{R}^M$ and $g_1(t, \xi) \in \mathbb{R}^M$ are data of the problem.

In this initial part of the paper, we do not focus on the stochastic part of the problem, that will come later in section 4. We will derive conditions for (1) to be well posed, and focus on the boundary operators H_0 and H_1 .

2.1. Well-posedness

Letting $F = 0$, we multiply (1) by u^T and integrate in space. By rearranging and defining $\|u\|^2 = \int_{\Omega} u^T u dx$ we get,

$$\|u\|_t^2 = (u^T A u)_{x=0} - (u^T A u)_{x=1}. \quad (2)$$

Due to the fact that A is symmetric, we have

$$A = X \Lambda X^T, \quad X = [X_+, X_-], \quad \Lambda = \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix}. \quad (3)$$

In (3), X_+ and X_- are the eigenvectors related to the positive and negative eigenvalues respectively. The eigenvalue matrix Λ is divided into diagonal block matrices Λ^+ and Λ^- containing the positive and negative eigenvalues respectively. Using (2) and (3) we get,

$$\|u\|_t^2 = (X^T u)_0^T \Lambda (X^T u)_0 - (X^T u)_1^T \Lambda (X^T u)_1. \quad (4)$$

The boundary conditions we consider are of the form

$$\begin{aligned} H_0 u &= (X_+^T - R_0 X_-^T) u_0 = g_0, & x &= 0 \\ H_1 u &= (X_-^T - R_1 X_+^T) u_1 = g_1, & x &= 1, \end{aligned} \quad (5)$$

where R_0 and R_1 are matrices of appropriate size and dimension. The relation (5) means that we specify the ingoing characteristic variables in terms of the outgoing ones' and given data, see [22, 23].

2.1.1. The homogeneous case

By using (5) in (4) and neglecting the data, we obtain,

$$\begin{aligned} \|u\|_t^2 &= (X_-^T u)_0^T [R_0^T \Lambda^+ R_0 + \Lambda^-] (X_-^T u)_0 \\ &\quad - (X_+^T u)_1^T [R_1^T \Lambda^- R_1 + \Lambda^+] (X_+^T u)_1. \end{aligned} \quad (6)$$

From (6) we conclude that the energy is bounded for homogeneous boundary conditions if,

$$\begin{aligned} R_0^T \Lambda^+ R_0 + \Lambda^- &\leq 0, \\ R_1^T \Lambda^- R_1 + \Lambda^+ &\geq 0. \end{aligned} \quad (7)$$

Consequently (7) puts a restriction on R_0 and R_1 . Note that with $R_0 = R_1 = 0$ we have the so called characteristic boundary conditions.

We summarize the result in

Proposition 1. *The problem (1) with the boundary conditions (5), subject to condition (7) and zero boundary data is well-posed.*

Proof. By integration of (6), subject to (7) we get

$$\|u\|^2 \leq \|f\|^2. \quad (8)$$

The estimate (8) implies uniqueness. Existence is guaranteed by the fact that we use the correct number of boundary conditions in (5). $\square$

2.1.2. The non-homogeneous case

We now consider the case with non-zero data. Using (5) in (4) and keeping the data gives us,

$$\begin{aligned} \|u\|_t^2 &= \begin{bmatrix} (X_-^T u)_0 \\ g_0 \end{bmatrix}^T \begin{bmatrix} R_0^T \Lambda^+ R_0 + \Lambda^- & R_0^T \Lambda^+ \\ \Lambda^+ R_0 & \Lambda^+ \end{bmatrix} \begin{bmatrix} (X_-^T u)_0 \\ g_0 \end{bmatrix} \\ &\quad - \begin{bmatrix} (X_+^T u)_1 \\ g_1 \end{bmatrix}^T \begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & R_1^T \Lambda^- \\ \Lambda^- R_1 & \Lambda^- \end{bmatrix} \begin{bmatrix} (X_+^T u)_1 \\ g_1 \end{bmatrix}. \end{aligned} \quad (9)$$

We can now add and subtract $g_0^T G_0 g_0$ and $g_1^T G_1 g_1$ where $G_{0,1}$ are positive semi-definite matrices since g_0 and g_1 are bounded. The result is,

$$\begin{aligned} \|u\|_t^2 &= \begin{bmatrix} (X_-^T u)_0 \\ g_0 \end{bmatrix}^T \underbrace{\begin{bmatrix} R_0^T \Lambda^+ R_0 + \Lambda^- & R_0^T \Lambda^+ \\ \Lambda^+ R_0 & -G_0 \end{bmatrix}}_{M_0} \begin{bmatrix} (X_-^T u)_0 \\ g_0 \end{bmatrix} \\ &\quad - \begin{bmatrix} (X_+^T u)_1 \\ g_1 \end{bmatrix}^T \underbrace{\begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & R_1^T \Lambda^- \\ \Lambda^- R_1 & G_1 \end{bmatrix}}_{M_1} \begin{bmatrix} (X_+^T u)_1 \\ g_1 \end{bmatrix} \\ &\quad + g_0^T (\Lambda^+ + G_0) g_0 - g_1^T (\Lambda^- - G_1) g_1. \end{aligned} \quad (10)$$

For (10) to lead to an estimate, M_0 and M_1 must be negative respectively positive semi-definite. We start by considering M_0 ,

$$M_0 = \begin{bmatrix} R_0^T \Lambda^+ R_0 + \Lambda^- & (\Lambda^+ R_0)^T \\ \Lambda^+ R_0 & -G_0 \end{bmatrix}. \quad (11)$$

To prove that (11) is negative semi-definite we multiply M_0 with a matrix from left and right, and obtain

$$\hat{M}_0 = \begin{bmatrix} I & C_0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} R_0^T \Lambda^+ R_0 + \Lambda^- & (\Lambda^+ R_0)^T \\ \Lambda^+ R_0 & -G_0 \end{bmatrix} \begin{bmatrix} I & C_0 \\ 0 & I \end{bmatrix}. \quad (12)$$

Evaluating (12), requiring that $R_0^T \Lambda^+ R_0 + \Lambda^-$ is strictly negative definite and choosing $C_0 = -(R_0^T \Lambda^+ R_0 + \Lambda^-)^{-1} (\Lambda^+ R_0)^T$, gives us the diagonal matrix

$$\hat{M}_0 = \begin{bmatrix} R_0^T \Lambda^+ R_0 + \Lambda^- & 0 \\ 0 & -(\Lambda^+ R_0)^T (R_0^T \Lambda^+ R_0 + \Lambda^-)^{-1} (\Lambda^+ R_0) - G_0 \end{bmatrix}. \quad (13)$$

With the *Schur complement* $-(\Lambda^+ R_0)^T (R_0^T \Lambda^+ R_0 + \Lambda^-)^{-1} (\Lambda^+ R_0) - G_0$ being negative semi-definite, $\hat{M}_0$ is negative semi-definite. This means that the choice,

$$G_0 \geq -(\Lambda^+ R_0)^T (R_0^T \Lambda^+ R_0 + \Lambda^-)^{-1} (\Lambda^+ R_0), \quad (14)$$

and the requirement that $R_0^T \Lambda^+ R_0 + \Lambda^-$ is negative definite, see (7), makes $\hat{M}_0$ negative semi-definite, and hence also M_0 .

Similarly we look at the matrix M_1 ,

$$M_1 = \begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & (\Lambda^- R_1)^T \\ \Lambda^- R_1 & G_1 \end{bmatrix}. \quad (15)$$

To prove that (15) is positive semi-definite we multiply M_1 with a matrix from left and right, which yields

$$\hat{M}_1 = \begin{bmatrix} I & C_1 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & (\Lambda^- R_1)^T \\ \Lambda^- R_1 & G_1 \end{bmatrix} \begin{bmatrix} I & C_1 \\ 0 & I \end{bmatrix}. \quad (16)$$

As before we perform the multiplication and choose $C_1 = -(R_1^T \Lambda^- R_1 + \Lambda^+)^{-1} (\Lambda^- R_1)^T$, which requires $R_1^T \Lambda^- R_1 + \Lambda^+$ to be positive definite, such that we get a diagonal matrix. This particular choice of C_1 gives us,

$$\hat{M}_1 = \begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & 0 \\ 0 & -(\Lambda^- R_1)^T (R_1^T \Lambda^- R_1 + \Lambda^+)^{-1} (\Lambda^- R_1) + G_1 \end{bmatrix}. \quad (17)$$

With the *Schur complement* $-(\Lambda^- R_1)^T (R_1^T \Lambda^- R_1 + \Lambda^+)^{-1} (\Lambda^- R_1) + G_1$ being positive semi-definite, $\hat{M}_1$ is positive semi-definite. Choosing G_1 as,

$$G_1 \geq (\Lambda^- R_1)^T (R_1^T \Lambda^- R_1 + \Lambda^+)^{-1} (\Lambda^- R_1), \quad (18)$$

ensures positive semi-definiteness of M_1 .

In summary, we can choose the matrices R_0 , R_1 , G_0 and G_1 such that M_0 and M_1 are negative respectively positive semi definite. By doing this we get an energy estimate also in the non homogeneous case. Note that we only need to prove that there exist a certain choice of G_0 and G_1 which bounds (9), the particular choice of matrix is irrelevant.

We summarize the result in

Proposition 2. *The problem (1) with boundary conditions (5), condition (7) with strict definiteness and the choices G_0 , G_1 in (14), (18) respectively, is strongly well-posed.*

Proof. By integrating (10), with the conditions (7), (14), (18) leads to,

$$\|u\|^2 \leq \|f\|^2 + \int_0^T g_0^T (\Lambda^+ + G_0) g_0 + g_1^T (|\Lambda^-| + G_1) g_1 dt. \quad (19)$$

The relation (7) guarantees uniqueness. Existence is guaranteed by the correct number of boundary conditions in (5). $\square$

Remark 1. *If we can estimate the solution in all forms of data, the solution is strongly well-posed, if zero boundary data is necessary for obtaining an estimate, it is well posed see [43, 44, 45] for more details on well-posedness.*

Remark 2. *The fact that we can get a more or less sharp energy estimate depending on the choice of R_0 and R_1 will have implications for the variance in the solution.*

3. The semi-discrete problem

It is convenient to introduce the *Kronecker product*, defined for any $M \times N$ matrix A and $P \times Q$ matrix B ,

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1N}B \\ \vdots & \ddots & \vdots \\ a_{M1}B & \cdots & a_{MN}B \end{bmatrix}.$$

We now consider a finite difference approximation of (1) using SBP operators, see [24, 25, 26]. The boundary conditions are implemented using the weak penalty technique using simultaneous approximation terms (SAT) described in [30, 31, 38, 39, 40]. The semi-discrete SBP-SAT formulation of (1) is,

$$\begin{aligned} u_t + (D \otimes A)u &= (P^{-1}E_0 \otimes \Sigma_0)((I_N \otimes H_0)u - e_0 \otimes \tilde{g}_0) \\ &\quad + (P^{-1}E_N \otimes \Sigma_N)((I_N \otimes H_1)u - e_N \otimes \tilde{g}_1). \\ u(0) &= f \end{aligned} \quad (20)$$

in (20), $D = P^{-1}Q$ is the difference operator. P is a positive definite matrix and Q satisfies $Q + Q^T = \text{diag}[-1, 0, \dots, 0, 1]$. E_0 and E_N are zero matrices except for element $(1, 1) = 1$ and $(N, N) = 1$ respectively. Difference operators of this form exist for order 2, 4, 6, 8 and 10, see [24, 25, 26, 28]. The boundary data $\tilde{g}_0$ and $\tilde{g}_1$ are defined as,

$$\tilde{g}_0 = [g_0, 0]^T, \quad \tilde{g}_1 = [0, g_1]^T,$$

where g_0 and g_1 are the original boundary data of problem (1).

The numerical solution u is a vector organized as,

$$u = [u_0, \dots, u_i, \dots, u_N]^T, \quad u_i = [u_0, \dots, u_j, \dots, u_M]_i^T,$$

where u_{ij} approximates $u_j(x_i)$. The vectors e_0 and e_N are zero except for the first and last element respectively which is one. Σ_N and Σ_0 are penalty matrices of appropriate size. H_0 and H_1 are defined as

$$H_0 = \begin{bmatrix} I^+ & -R_0 \\ 0 & 0 \end{bmatrix} X^T, \quad H_1 = \begin{bmatrix} 0 & 0 \\ -R_1 & I^- \end{bmatrix} X^T. \quad (21)$$

In (21), I^+ and I^- is the identity matrix with the same size as Λ^+ and Λ^- and R_0, R_1 are the matrices in the boundary conditions (5). Note that H_0 and H_1 in (21) are expanded versions of H_0 and H_1 in (5).

3.1. Stability

In this section we prove stability using the discrete energy-method. We follow essentially the same route as in section 2 for the continuous problem.

3.1.1. The homogeneous case

To prove stability for the homogeneous case ($g_0 = g_1 = 0$) we multiply equation (20) by $u^T(P \otimes I_M)$ from the left, and add the transpose. By defining the discrete norm $\|u\|_{P \otimes I}^2 = u^T(P \otimes I)u$, using $Q + Q^T = \text{diag}[-1, 0, \dots, 0, 1]$ and $A = X \Lambda X^T$ we obtain,

$$\begin{aligned} \frac{d}{dt} \|u\|_{P \otimes I}^2 &= u_0^T (X \Lambda X^T + \Sigma_0 H_0 + (\Sigma_0 H_0)^T) u_0 \\ &\quad - u_N^T (X \Lambda X^T - \Sigma_N H_1 - (\Sigma_N H_1)^T) u_N. \end{aligned} \quad (22)$$

In (22), u_0 and u_N is the vector u at grid point zero and N respectively.

Rewriting (22) and using the fact that $XX^T = I$ yields,

$$\begin{aligned} \frac{d}{dt} \|u\|_{P \otimes I}^2 &= (X^T u)_0^T (\Lambda + (X^T \Sigma_0 H_0 X) + (X^T \Sigma_0 H_0 X)^T) (X^T u)_0 \\ &\quad - (X^T u)_N^T (\Lambda - (X^T \Sigma_N H_1 X) - (X^T \Sigma_N H_1 X)^T) (X^T u)_N. \end{aligned} \quad (23)$$

Let $\hat{\Sigma}_0 = X^T \Sigma_0$ and $\hat{\Sigma}_N = X^T \Sigma_N$, where we choose Σ_0 and Σ_N such that $\hat{\Sigma}_0$ and $\hat{\Sigma}_N$ are diagonal matrices, that is

$$\hat{\Sigma}_N = \begin{bmatrix} \hat{\Sigma}_N^+ & 0 \\ 0 & \hat{\Sigma}_N^- \end{bmatrix}, \quad \hat{\Sigma}_0 = \begin{bmatrix} \hat{\Sigma}_0^+ & 0 \\ 0 & \hat{\Sigma}_0^- \end{bmatrix}.$$

The sizes of $\hat{\Sigma}_N^+$, $\hat{\Sigma}_N^-$, $\hat{\Sigma}_0^+$, $\hat{\Sigma}_0^-$ corresponds to the sizes of Λ^+ and Λ^- respectively. By the following split,

$$(X^T u)_0 = \begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \end{bmatrix}, \quad (X^T u)_N = \begin{bmatrix} (X_+^T u)_N \\ (X_-^T u)_N \end{bmatrix}, \quad (24)$$

we can rewrite (23) as,

$$\begin{aligned} \frac{d}{dt} \|u\|_{P \otimes I}^2 &= \begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \end{bmatrix}^T \underbrace{\begin{bmatrix} \Lambda^+ + 2\hat{\Sigma}_0^+ & -\hat{\Sigma}_0^+ R_0 \\ -R_0^T \hat{\Sigma}_0^+ & \Lambda^- \end{bmatrix}}_{B_0} \begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \end{bmatrix} \\ &\quad - \begin{bmatrix} (X_+^T u)_N \\ (X_-^T u)_N \end{bmatrix}^T \underbrace{\begin{bmatrix} \Lambda^+ & \hat{\Sigma}_N^- R_1 \\ R_1^T \hat{\Sigma}_N^- & \Lambda^- - 2\hat{\Sigma}_N^- \end{bmatrix}}_{B_N} \begin{bmatrix} (X_+^T u)_N \\ (X_-^T u)_N \end{bmatrix}. \end{aligned} \quad (25)$$

To obtain an estimate, the matrices B_0 and B_N must be negative and positive semi-definite respectively.

We first consider the left boundary. Note that the boundary term at $x = 0$ in the continuous energy estimate (6) can be written as,

$$\begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & R_0^T \Lambda^+ R_0 + \Lambda^- \end{bmatrix} \begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \end{bmatrix}. \quad (26)$$

We know from (7) that (26) is negative semi-definite. To take advantage of that, we rewrite B_0 in the following way,

$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & R_0^T \Lambda^+ R_0 + \Lambda^- \end{bmatrix} + \begin{bmatrix} \Lambda^+ + 2\hat{\Sigma}_0^+ & -\hat{\Sigma}_0^+ R_0 \\ -R_0^T \hat{\Sigma}_0^+ & -R_0^T \Lambda^+ R_0 \end{bmatrix}. \quad (27)$$

We consider the second matrix in (27), which we rewrite as,

$$\begin{bmatrix} I & 0 \\ 0 & R_0 \end{bmatrix}^T \begin{bmatrix} \Lambda^+ + 2\hat{\Sigma}_0^+ & -\hat{\Sigma}_0^+ \\ -\hat{\Sigma}_0^+ & -\Lambda^+ \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R_0 \end{bmatrix}. \quad (28)$$

By choosing $\hat{\Sigma}_0^+ = -\Lambda^+$ in (28) we get

$$\begin{bmatrix} I & 0 \\ 0 & R_0 \end{bmatrix}^T \underbrace{\begin{bmatrix} -\Lambda^+ & \Lambda^+ \\ \Lambda^+ & -\Lambda^+ \end{bmatrix}}_{M_0} \begin{bmatrix} I & 0 \\ 0 & R_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & R_0 \end{bmatrix}^T \underbrace{(C \otimes \Lambda^+)}_{M_0} \begin{bmatrix} I & 0 \\ 0 & R_0 \end{bmatrix}.$$

The matrix,

$$C = \begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}, \quad (29)$$

has the eigenvalues $-2, 0$ which means that M_0 and hence B_0 is negative semi-definite. This choice of $\hat{\Sigma}_0$ and $\Sigma_0 = X\hat{\Sigma}_0$ makes B_0 negative semi-definite. We finally let $\hat{\Sigma}_0^- = 0$ since it has no importance for stability.

Next, consider the right boundary matrix

$$B_N = \begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -R_1^T \Lambda^- R_1 & \hat{\Sigma}_N^- R_1 \\ R_1^T \hat{\Sigma}_N^- & \Lambda^- - 2\hat{\Sigma}_N^- \end{bmatrix}. \quad (30)$$

The first matrix is positive semi-definite by (7), and we only need to consider the second matrix. As before we can factor out R_1 and choose $\hat{\Sigma}_N^- = \Lambda^-$ which gives us,

$$\begin{bmatrix} R_1 & 0 \\ 0 & I \end{bmatrix}^T \underbrace{\begin{bmatrix} -\Lambda^- & \Lambda^- \\ \Lambda^- & -\Lambda^- \end{bmatrix}}_{M_N} \begin{bmatrix} R_1 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & I \end{bmatrix}^T \underbrace{(C \otimes \Lambda^-)}_{M_N} \begin{bmatrix} R_1 & 0 \\ 0 & I \end{bmatrix},$$

where C is given in (29). This means B_N is positive semi-definite with $\Sigma_N = X\hat{\Sigma}_N$. Finally we let $\hat{\Sigma}_N^+ = 0$ since it has no importance for stability.

We summarize the result in

Proposition 3. *The approximation (20) with the boundary operators (21) and penalty coefficients*

$$\Sigma_0 = X\hat{\Sigma}_0, \quad \Sigma_N = X\hat{\Sigma}_N, \quad (31)$$

where,

$$\hat{\Sigma}_0 = - \begin{bmatrix} \Lambda^+ & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\Sigma}_N = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda^- \end{bmatrix}, \quad (32)$$

is stable.

Proof. Time-integration of (25) with the penalty coefficients (31),(32) lead to

$$\|u\|_{P \otimes I}^2 \leq \|f\|_{P \otimes I}^2. \quad (33)$$

□

Remark 3. *Note the similarity between the continuous estimate (8) and the discrete estimate (33).*

3.1.2. The non-homogeneous case

By using the same technique as in the homogeneous case but keeping the data we obtain the following quadratic form, where we recognize the top 2×2 blocks from the homogeneous case (25).

$$\begin{aligned} \frac{d}{dt} \|u\|_{P \otimes I}^2 &= \begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \\ g_0 \end{bmatrix}^T \underbrace{\begin{bmatrix} \Lambda^+ + 2\hat{\Sigma}_0^+ & -\hat{\Sigma}_0^+ R_0 & -\hat{\Sigma}_0^+ \\ -R_0^T \hat{\Sigma}_0^+ & \Lambda^- & 0 \\ -\hat{\Sigma}_0^+ & 0 & 0 \end{bmatrix}}_{B_0} \begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \\ g_0 \end{bmatrix} \\ &- \begin{bmatrix} (X_+^T u)_N \\ (X_-^T u)_N \\ g_1 \end{bmatrix}^T \underbrace{\begin{bmatrix} \Lambda^+ & \hat{\Sigma}_N^- R_1 & 0 \\ R_1^T \hat{\Sigma}_N^- & \Lambda^- - 2\hat{\Sigma}_N^- & \hat{\Sigma}_N^- \\ 0 & \hat{\Sigma}_N^- & 0 \end{bmatrix}}_{B_N} \begin{bmatrix} (X_+^T u)_N \\ (X_-^T u)_N \\ g_1 \end{bmatrix}. \end{aligned} \quad (34)$$

To obtain an estimate we must prove that the matrices B_0 and B_N are negative and positive semi-definite respectively.

We first consider the left boundary. Note that the boundary term at $x = 0$ in the continuous energy estimate (10) involves,

$$\begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \\ g_0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_0^T \Lambda^+ R_0 + \Lambda^- & R_0^T \Lambda^+ \\ 0 & \Lambda^+ R_0 & -G_0 \end{bmatrix} \begin{bmatrix} (X_+^T u)_0 \\ (X_-^T u)_0 \\ g_0 \end{bmatrix} \quad (35)$$

Proposition 2 implies that the matrix in (35) is negative semi-definite. To take advantage of that, we rewrite B_0 in the following way,

$$\begin{aligned} B_0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_0^T \Lambda^+ R_0 + \Lambda^- & R_0^T \Lambda^+ \\ 0 & \Lambda^+ R_0 & -G_0 \end{bmatrix} + \begin{bmatrix} \Lambda^+ + 2\hat{\Sigma}_0^+ & -\hat{\Sigma}_0^+ R_0 & -\hat{\Sigma}_0^+ \\ -R_0^T \hat{\Sigma}_0^+ & -R_0^T \Lambda^+ R_0 & -R_0^T \Lambda^+ \\ -\hat{\Sigma}_0^+ & -\Lambda^+ R_0 & -\Lambda^+ \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda^+ + G_0 \end{bmatrix}. \end{aligned}$$

We focus on the second matrix above and rewrite it as,

$$B_0^{(2)} = \begin{bmatrix} I & 0 & 0 \\ 0 & R_0 & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} \Lambda^+ + 2\hat{\Sigma}_0^+ & -\hat{\Sigma}_0^+ & -\hat{\Sigma}_0^+ \\ -\hat{\Sigma}_0^+ & -\Lambda^+ & -\Lambda^+ \\ -\hat{\Sigma}_0^+ & -\Lambda^+ & -\Lambda^+ \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & R_0 & 0 \\ 0 & 0 & I \end{bmatrix}. \quad (36)$$

By choosing $\hat{\Sigma}_0^+ = -\Lambda^+$ in (36) we get

$$B_0^{(2)} = \begin{bmatrix} I & 0 & 0 \\ 0 & R_0 & 0 \\ 0 & 0 & I \end{bmatrix}^T \underbrace{(C \otimes \Lambda^+)}_{M_0} \begin{bmatrix} I & 0 & 0 \\ 0 & R_0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

The matrix,

$$C = \begin{bmatrix} -1 & +1 & +1 \\ +1 & -1 & -1 \\ +1 & -1 & -1 \end{bmatrix}, \quad (37)$$

has the eigenvalues $-3, 0, 0$ which means that M_0 and hence B_0 is negative semi-definite. This choice of $\hat{\Sigma}_0$ and $\Sigma_0 = X\hat{\Sigma}_0$ makes B_0 negative semi-definite. We finally let $\hat{\Sigma}_0^- = 0$ since it has no importance for stability.

Next, consider the right boundary matrix, and proceed in the same way,

$$B_N = \begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & 0 & R_1^T \Lambda^- \\ 0 & 0 & 0 \\ \Lambda^- R_1 & 0 & G_1 \end{bmatrix} + \begin{bmatrix} -R_1^T \Lambda^- R_1 & \hat{\Sigma}_N^- R_1 & -R_1^T \Lambda^- \\ R_1^T \hat{\Sigma}_N^- & \Lambda^- - 2\hat{\Sigma}_N^- & \hat{\Sigma}_N^- \\ -\Lambda^- R_1 & \hat{\Sigma}_N^- & -\Lambda^- \end{bmatrix} \\ + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda^- - G_1 \end{bmatrix}.$$

The first matrix is positive semi-definite by (10), and again we only consider the second matrix where we factor out R_1 , choose $\hat{\Sigma}_N^- = \Lambda^-$. This gives us,

$$B_N^{(2)} = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \underbrace{(\tilde{C} \otimes \Lambda^-)}_{M_N} \begin{bmatrix} R_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

where,

$$\tilde{C} = \begin{bmatrix} -1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & -1 \end{bmatrix}.$$

$\tilde{C}$ has the same eigenvalues $(-3, 0, 0)$ as the matrix C . This means that M_N and hence B_N is positive semi-definite with $\Sigma_N = X\hat{\Sigma}_N$. Finally we let $\hat{\Sigma}_N^+ = 0$ since it has no importance for stability.

We summarize the result in

Proposition 4. *The approximation (20) with the boundary operators (21) and penalty coefficients (31), (32) is strongly stable.*

Proof. By time-integration of the approximation (34) with the penalty coefficients (31), (32) we obtain,

$$\|u\|_{P \otimes I}^2 \leq \|f\|_{P \otimes I}^2 + \int_0^T g_0^T (\Lambda^+ + G_0) g_0 + g_1^T (|\Lambda^-| + G_1) g_1 dt \quad (38)$$

The estimate (38) includes both the initial and boundary data. $\square$

Remark 4. *Note that the conditions in proposition 3 and 4 are identical. If we can estimate the problem using non-zero boundary data it is strongly stable. If zero boundary data is necessary for obtaining an estimate, it is called stable. See [43, 44] for more details.*

Remark 5. *Notice the similarity between the continuous estimate (19) and the discrete estimate (38).*

4. The stochastic problem

In this section we focus on the stochastic properties of (1) which we formulate as,

$$\begin{aligned} u_t + Au_x &= F(x, t, \xi) = \mathbb{E}[F](x, t) + \delta F(x, t, \xi) \\ H_0 u(0, t, \xi) &= g_0(t, \xi) = \mathbb{E}[g_0](t) + \delta g_0(t, \xi) \\ H_1 u(1, t, \xi) &= g_1(t, \xi) = \mathbb{E}[g_1](t) + \delta g_1(t, \xi) \\ u(x, 0, \xi) &= f(x, \xi) = \mathbb{E}[f](x) + \delta f(x, \xi). \end{aligned} \quad (39)$$

In (39), $\mathbb{E}[\cdot]$ denotes the expected value or mean, and δ indicates the variation from the mean. $\mathbb{E}[\cdot]$ is defined as,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx,$$

where f_X is the probability density function of the stochastic variable X .

By taking the expected value of (39) and letting $E[u] = v$ we obtain,

$$\begin{aligned} v_t + Av_x &= \mathbb{E}[F](x, t) \\ H_0 v(0, t) &= \mathbb{E}[g_0](t) \\ H_1 v(1, t) &= \mathbb{E}[g_1](t) \\ v(x, 0) &= \mathbb{E}[f](x). \end{aligned} \quad (40)$$

The difference between (39) and (40) and the notation $e = u - \mathbb{E}[u]$ for the deviation from the mean leads to,

$$\begin{aligned} e_t + Ae_x &= \delta F(x, t, \xi) \\ H_0 e(0, t, \xi) &= \delta g_0(t, \xi) \\ H_1 e(1, t, \xi) &= \delta g_1(t, \xi) \\ e(x, 0, \xi) &= \delta f(x, \xi). \end{aligned} \quad (41)$$

Remark 6. Note that (41) is of the same form as (39). Therefore, since (39) has an energy estimate in terms of the data, so does (41). Note also that the data in (41) is the deviation from the mean.

4.1. The variance formulation

The energy method on (41) (multiply by e and integrate in space) leads similarly as in the general non-homogeneous case (9) to,

$$\begin{aligned} \|e\|_t^2 &= \begin{bmatrix} e_0^- \\ \delta g_0 \\ e_1^+ \\ \delta g_1 \end{bmatrix}^T \begin{bmatrix} R_0^T \Lambda^+ R_0 + \Lambda^- & R_0^T \Lambda^+ \\ \Lambda^+ R_0 & \Lambda^+ \\ R_1^T \Lambda^- R_1 + \Lambda^+ & R_1^T \Lambda^- \\ \Lambda^- R_1 & \Lambda^- \end{bmatrix} \begin{bmatrix} e_0^- \\ \delta g_0 \\ e_1^+ \\ \delta g_1 \end{bmatrix} \\ &- \begin{bmatrix} e_1^+ \\ \delta g_1 \end{bmatrix}^T \begin{bmatrix} R_1^T \Lambda^- R_1 + \Lambda^+ & R_1^T \Lambda^- \\ \Lambda^- R_1 & \Lambda^- \end{bmatrix} \begin{bmatrix} e_1^+ \\ \delta g_1 \end{bmatrix}, \end{aligned} \quad (42)$$

Unlike the derivation of (10), where our ambition was to *bound* the solution in terms of the data, our ambition is now to *evaluate* the right hand side of (42).

To ease the notation we introduce $e_0^+ = (X_+^T e)_0$, $e_0^- = (X_-^T e)_0$, $e_1^+ = (X_+^T e)_1$ and $e_1^- = (X_-^T e)_1$. By replacing δg_0 and δg_1 from some parts of the boundary terms in (42) by using the boundary conditions we get,

$$\begin{aligned}
\|e\|_t^2 &= (R_0 e_0^-)^T \Lambda^+ (R_0 e_0^-) - (e_0^-)^T \Lambda^- (e_0^-) \\
&+ (R_1 e_1^+)^T \Lambda^- (R_1 e_1^+) + (e_1^+)^T \Lambda^+ (e_1^+) \\
&= \delta g_0^T \Lambda^+ \delta g_0 + (e_0^+ - R_0 e_0^-)^T \Lambda^+ (R_0 e_0^-) \\
&- \delta g_1^T \Lambda^- \delta g_1 - (e_1^- - R_1 e_1^+)^T \Lambda^- (R_1 e_1^+) \\
&+ (R_0 e_0^-)^T \Lambda^+ (e_0^+ - R_0 e_0^-) \\
&- (R_1 e_1^+)^T \Lambda^- (e_1^- - R_1 e_1^+).
\end{aligned} \tag{43}$$

Taking the average of (42) and (43) cancels out the $(R_0 e_0^-)^T \Lambda^+ (R_0 e_0^-)$ and $(R_1 e_1^+)^T \Lambda^- (R_1 e_1^+)$ terms and gives us

$$\begin{aligned}
\|e\|_t^2 &= (e_0^-)^T \Lambda^- (e_0^-) + (e_1^+)^T \Lambda^+ (e_1^+) \\
&= (\delta g_0)^T \Lambda^+ \delta g_0 + (\delta g_0 + e_0^+)^T \Lambda^+ (R_0 e_0^-) \\
&- (\delta g_1)^T \Lambda^- \delta g_1 - (\delta g_1 + e_1^-)^T \Lambda^- (R_1 e_1^+).
\end{aligned} \tag{44}$$

Note that

$$\begin{aligned}
\mathbb{E}[\|e\|^2] = \mathbb{E}[\int_0^1 e^2 dx] &= \int_0^1 \mathbb{E}[e^2] dx = \int_0^1 \mathbb{E}[(u - \mathbb{E}[u])^2] dx \\
&= \int_0^1 \text{Var}[u] dx = \|\text{Var}[u]\|_1.
\end{aligned} \tag{45}$$

Taking the expected value of (44) and using (45) gives us,

$$\begin{aligned}
\|\text{Var}[u]\|_t &= \mathbb{E}[(e_0^-)^T \Lambda^- (e_0^-)] + \mathbb{E}[(e_1^+)^T \Lambda^+ (e_1^+)] \\
&= \mathbb{E}[\delta g_0^T \Lambda^+ \delta g_0] + \mathbb{E}[(\delta g_0 + e_0^+)^T \Lambda^+ (R_0 e_0^-)] \\
&- \mathbb{E}[\delta g_1^T \Lambda^- \delta g_1] - \mathbb{E}[(\delta g_1 + e_1^-)^T \Lambda^- (R_1 e_1^+)].
\end{aligned} \tag{46}$$

Equation (46) gives us a clear description of when and how much the variance decays depending on i) the boundary operators, and ii) the correlation of the stochastic boundary data.

The estimate (46) has its semi-discrete counterpart, which is strongly stable as was shown in Proposition 4. We do not repeat that derivation for the deviation e , but use (46) in our discussion below on the effect on the variance.

4.2. Dependence on stochastically varying data

In this section we will study the effects of different boundary conditions for different stochastic properties of the initial and boundary data.

4.2.1. Zero variance on the boundary

If we have perfect knowledge of the boundary data, that is $\delta g_0 = 0$ and $\delta g_1 = 0$ and use this in (46) we obtain,

$$\begin{aligned} \|\text{Var}[u]\|_t &= \mathbb{E}[(R_0 e_0^-)^T \Lambda^+ (R_0 e_0^-)] + \mathbb{E}[(e_0^-)^T \Lambda^- (e_0^-)] \\ &- \mathbb{E}[(R_1 e_1^+)^T \Lambda^- (R_1 e_1^+)] - \mathbb{E}[(e_1^+)^T \Lambda^+ (e_1^+)]. \end{aligned} \quad (47)$$

From (47) we conclude that any non-zero value of R_0 and R_1 gives a positive contribution to the growth of the L_1 -norm of the variance of the solution. The optimal choices of R_0 and R_1 for zero variance on the boundary is $R_0 = R_1 = 0$. Note also that the sign of R_0 and R_1 cancels in the quadratic form.

4.2.2. Decaying variance on the boundary

Next, assume that the variance of the boundary data is non-zero but decays with time. The difference in the energy estimate (46) for the characteristic and non-characteristic case include the term,

$$\begin{aligned} \mathbb{E}[(\delta g_0 + e_0^+)^T \Lambda^+ (R_0 e_0^-)] &= 2\mathbb{E}[(e_0^+)^T \Lambda^+ R_0 e_0^-] \\ &- \mathbb{E}[(R_0 e_0^-)^T \Lambda^+ R_0 e_0^-], \end{aligned} \quad (48)$$

where we used $\delta g_0 = e_0^+ - R_0 e_0^-$. With a similar argument we conclude that,

$$\begin{aligned} \mathbb{E}[(\delta g_1 + e_1^-)^T \Lambda^- (R_1 e_1^+)] &= 2\mathbb{E}[(e_1^-)^T \Lambda^- R_1 e_1^+] \\ &- \mathbb{E}[(R_1 e_1^+)^T \Lambda^- R_1 e_1^+]. \end{aligned} \quad (49)$$

Furthermore, the correlation coefficient between e_0^+ and $R_0 e_0^-$ is,

$$\begin{aligned} \rho_{e_0^+, R_0 e_0^-} &= \frac{\mathbb{E}[(e_0^+ - \mathbb{E}[e_0^+])^T (R_0 e_0^- - \mathbb{E}[R_0 e_0^-])]}{\sqrt{\mathbb{E}[(e_0^+ - \mathbb{E}[e_0^+])^2]^T} \sqrt{\mathbb{E}[(R_0 e_0^- - \mathbb{E}[R_0 e_0^-])^2]}} \\ &= \frac{\mathbb{E}[(e_0^+)^T R_0 e_0^-]}{\sqrt{\mathbb{E}[(e_0^+)^2]^T \mathbb{E}[(R_0 e_0^-)^2]}}. \end{aligned} \quad (50)$$

Similarly,

$$\rho_{e_1^-, R_1 e_1^+} = \frac{\mathbb{E}[(e_1^-)^T R_1 e_1^+]}{\sqrt{\mathbb{E}[(e_1^-)^2]^T \mathbb{E}[(R_1 e_1^+)^2]}}. \quad (51)$$

It follows from (48), (49) and the correlation coefficients in (50) and (51) that the contribution to the decay of the variance depends on the correlation between the characteristic variables at the boundaries.

A negative correlation lowers the variance and a positive correlation increases the variance. By choosing the initial data to be sufficiently correlated we can possibly make $\mathbb{E}[(\delta g_0 + e_0^+)^T \Lambda^+ (R_0 e_0^-)] - \mathbb{E}[(\delta g_1 + e_1^-)^T \Lambda^- (R_1 e_1^+)]$ negative and hence get a greater variance decay than in the characteristic case (when $R_0 = R_1 = 0$). For long times however, the variance decay is well approximated by the zero boundary case (47) which means that the characteristic boundary conditions are optimal.

Notice also that as data decays the boundary terms can be approximated as,

$$\begin{aligned} e_0^+ &= R_0 e_0^- + \delta g_0 \approx R_0 e_0^- \\ e_1^- &= R_1 e_1^+ + \delta g_1 \approx R_1 e_1^+. \end{aligned} \quad (52)$$

A consequence of (52) is that the characteristic variables at the boundaries become more and more correlated as the boundary data decays. Using (52) in (50) and (51) ensures that the correlation coefficients are positive. With zero data we get a perfect correlation between the characteristic variables.

4.2.3. Large variance on the boundary

In the case where the variance of the boundary is large, no general conclusions can be drawn since the terms $\mathbb{E}[\delta g_0^T \Lambda^+ (R_0 e_0^-)]$, $-\mathbb{E}[(R_0 e_0^-)^T \Lambda^+ \delta g_0]$, $-\mathbb{E}[\delta g_1^T \Lambda^- (R_1 e_1^+)]$ and $\mathbb{E}[(R_1 e_1^+)^T \Lambda^- \delta g_1]$ will dominate.

5. A detailed analysis of a model problem

We consider the problem (1), where we for simplicity and clarity choose $A = X \Lambda X^T$ to be the 2×2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}. \quad (53)$$

The boundary conditions are of the type (5). In this simplified case R_0 and R_1 are scalars.

5.1. The spectrum

To quantify the effect of boundary conditions on the decay of the variance we compute the continuous and discrete spectrum for our model problem. For details on this technique, see [22], [23], [43], [49], [50], [51], [52], [53].

5.1.1. The spectrum for the continuous problem

Consider the problem (1) with the matrix given in (53). Laplace transforming (1) gives the following system of ordinary differential equations,

$$\begin{aligned} s\hat{u} + A\hat{u}_x &= 0 & 0 \leq x \leq 1, \\ H_0\hat{u} = (X_+^T - R_0X_-^T)\hat{u} &= 0 & x = 0, \\ H_1\hat{u} = (X_-^T - R_1X_+^T)\hat{u} &= 0 & x = 1, \end{aligned} \quad (54)$$

where we have used zero data since that does not influence the spectrum.

The set of ordinary differential equations with boundary conditions (54) form an eigenvalue problem. The ansatz $\hat{u} = e^{\kappa x}\psi$ leads to,

$$(sI + A\kappa)\psi = 0. \quad (55)$$

Equation (55) has a non-trivial solution only when $(sI + A\kappa)$ is singular. Hence, κ is given by $|sI + A\kappa| = 0$, which lead to $\kappa_1 = s$ and $\kappa_2 = -s/3$. Equation (55) provide us with the eigenvectors and the general solution

$$\hat{u} = \alpha_1 e^{sx} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_2 e^{-\frac{s}{3}x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (56)$$

We insert (56) into the boundary conditions, which gives,

$$E\tilde{\alpha} = \begin{bmatrix} -R_0 & 1 \\ e^s & -R_1 e^{-\frac{s}{3}} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0. \quad (57)$$

A non-trivial solution of (57) require s -values such that

$$|E| = R_0 R_1 e^{-\frac{s}{3}} - e^s = 0. \quad (58)$$

The singular values of $|E|$ which yields the spectrum of the continuous problem (1) are,

$$s = \begin{cases} \frac{3}{4} \ln(|R_0 R_1|) + \frac{3n\pi i}{2}, & n \in \mathbb{Z}, \text{ if } R_0 R_1 > 0, \\ \frac{3}{4} \ln(|R_0 R_1|) + \frac{3n\pi i}{2} + \frac{3\pi i}{4}, & n \in \mathbb{Z}, \text{ if } R_0 R_1 < 0. \end{cases} \quad (59)$$

Note that no spectrum is defined when R_0 and/or R_1 are equal to zero, which corresponds to the characteristic boundary conditions. Note also that the real part of s is negative for $|R_0 R_1| < 1$.

The continuous spectrum given by (59) tells us how the analytical solution grows or decays in time. That growth or decay is given by the specific boundary conditions used, as can be seen in (57)

5.1.2. The spectrum for the semi-discrete problem

Consider the semi-discrete problem (20) with homogeneous boundary conditions after rearrangement,

$$u_t + \underbrace{(D \otimes A + (P^{-1}E_0 \otimes \Sigma_0 H_0) + (P^{-1}E_N \otimes \Sigma_N H_1))}_M u = 0. \quad (60)$$

The eigenvalues of the matrix M is the discrete spectrum of (20). Figure 1 shows the discrete and continuous spectrum for different discretizations and combinations of R_0 and R_1 . Similar to the continuous case, the discrete spectrum tells us how the numerical solution grows or decays in time.

Figure 1 show that the discrete spectrum converges to the continuous one as the number of grid points increase. The relation (59) show that the real part of s decreases when $|R_0 R_1|$ decrease, which is also seen in figure 1.

Figure 2 show the discrete spectrum for the characteristic case (where no continuous spectrum exist) when both R_0 and R_1 are zero. Note that the eigenvalues move to the left in the complex plane as the number of grid points increases.

5.2. Accuracy of the model problem

The order of accuracy p is defined as,

$$p = \log_2 \left(\frac{\|e_h\|_P}{\|e_{\frac{h}{2}}\|_P} \right), \quad \|e_h\| = \|u_a - u_h\| \quad (61)$$

where e_h is the error, u_h the computed solution using grid spacing h and data from u_a which is the manufactured analytical solution. The manufactured analytical solution is used to compute the exact error e , see [47, 48].

The time-integration is done using the classical 4th order explicit Runge-Kutta scheme [27]. Table 1 shows the p -value for different grid sizes in space using SBP-SAT schemes of order two and three, see [46]. In all simulations below we use the SBP-SAT scheme of order three with a mesh containing 50 grid points in space and 1000 grid points in time.

5.3. Random distributions

We consider uniformly and normally distributed randomness denoted $\mathcal{U}(a, b)$ and $\mathcal{N}(\mu, \sigma^2)$. The uniform distribution denoted $\mathcal{U}(a, b)$ is defined by the parameters a and b which denotes the minimum and maximum value

Figure 1: The discrete and continuous spectrum for different grids and values of R_0, R_1 .

SBP operator	$N_x = 20$	$N_x = 40$	$N_x = 80$	$N_x = 160$
2nd order	2.293	2.078	2.021	2.006
3rd order	3.250	3.089	3.034	3.014

Table 1: The order of accuracy for the 2nd and 3rd order SBP-SAT schemes for different number of grid points in space.

Figure 2: The discrete spectrum for characteristic boundary conditions and different grids.

of the distribution respectively. The probability density function f for a uniformly distributed random variable is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{for } x < a, \text{ or } x > b. \end{cases}$$

The normal distribution denoted $\mathcal{N}(\mu, \sigma^2)$ is defined using its mean μ and variance σ^2 . The probability density function for the normal distribution is

$$f(x) = \frac{1}{2\pi\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The numerical simulations are carried out using the scheme in section 3 with an additional discretization of the random variable $\xi = [\xi_0, \xi_1, \dots, \xi_S]$ using 80 grid points, which give the variance with four significant digits. The variance is computed using a Gaussian quadrature rule where the weights are calculated using the stochastic variable's probability density function [34, 35].

5.4. Zero variance on the boundary

To investigate the zero variance case on the boundary, we prescribe,

$$\begin{aligned} g_0(t, \xi) &= 0 \\ g_1(t, \xi) &= 0 \\ f(x, \xi) &= [2 + 2 \sin(2\pi x) \xi^3, 1 - 3 \sin(2\pi x) \xi]^T, \end{aligned} \tag{62}$$

Figure 3: The L_1 -norm of the variance as a function of time for a normally distributed ξ for characteristic and non-characteristic boundary conditions. Zero variance on the boundary.

as the initial and boundary conditions. Figure 3 and 4 shows the variance decay with time comparing characteristic and non-characteristic boundary conditions when using a normally ($\xi \sim \mathcal{N}(0, 1)$) and uniformly distributed ($\xi \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$) ξ respectively. Note the similarity between studying variance decay in the case of zero variance on the boundary and analyzing convergence to steady state, see [49, 50, 51].

We clearly see from figures 3 and 4 that the analytical predictions were correct and that characteristic boundary conditions gives a smaller variance than non-characteristic boundary conditions. We also see that the theoretical predictions from the spectrum calculations indicating further decay for decreasing $|R_0 R_1|$ are confirmed.

The similarity of the results in figures 3 and 4 despite the difference in stochastic distribution is typical for all the calculations performed. In the rest of this section we only include results obtained using $\xi \sim \mathcal{N}(0, 1)$.

Figure 4: The L_1 -norm of the variance as a function of time for a uniformly distributed ξ for characteristic and non-characteristic boundary conditions. Zero variance on the boundary.

5.5. Decaying variance on the boundary

For the case of decaying variance on the boundary, we choose,

$$\begin{aligned} g_0(t, \xi) &= 3 + 3e^{-1.1t} \cos(2\pi t) \xi^3 - R_0(2 + 3e^{-1.1t} \cos(2\pi t) \xi) \\ g_1(t, \xi) &= 2 + 3e^{-1.1t} \cos(2\pi t) \xi - R_1(3 + 3e^{-1.1t} \cos(2\pi t) \xi^3) \\ f(x, \xi) &= [3 + 3\xi^3, 2 + 3\xi]^T, \end{aligned}$$

as initial and boundary data. In the figures 5 below we compare the L_1 -norm of the variance between the characteristic and non-characteristic boundary conditions for different combinations of R_0 and R_1 .

Figure 6 show the correlation between the characteristic variables at the boundaries for one of the cases. In order to study how the correlation between the characteristic variables at the boundary effect the variance we show the case $R_0 R_1 = 0.25$ from $t = 0$ to $t = 1$ in figures 7 and 8. Figure 5 show that the characteristic boundary conditions give us the smallest variance for long times. The enhanced decay of the variance with decreasing $R_0 R_1$ from the spectrum analysis is also confirmed.

By studying the expanded figures 7 and 8 we see a negative correlation in the approximate region of $t = 0.4$ to $t = 0.7$ and an indication of a faster

Figure 5: The L_1 -norm of the variance as a function of time for characteristic and non-characteristic boundary conditions. Decaying variance on the boundary.

Figure 6: The correlation between the characteristic variables at the boundaries for the non-characteristic boundary condition ($R_0R_1 = 0.25$). Decaying variance on the boundary.

Figure 7: A blow-up of the correlation between the characteristic variables at the boundaries for the non-characteristic boundary condition ($R_0 R_1 = 0.25$). Decaying variance on the boundary.

Figure 8: The L_1 -norm of the variance as a function of time for characteristic and non-characteristic boundary conditions ($R_0 R_1 = 0.25$). Decaying variance on the boundary.

$R_0 R_1 =$	0	0.25	0.5	0.75	1.0
$\int_0^T \ Var[u]\ dt$	62.6	82.7	193.4	581.8	2427.7

Table 2: The integral of the L_1 -norm of the variance for different values of R_0 and R_1 for $\xi \sim \mathcal{N}(0, 1)$ and $T = 10$.

decay of the variance than in the characteristic case. We also see in figure 6 that as the data decays, the characteristic variables at the boundaries become increasingly correlated.

Finally, we compare the different cases by integrating the L_1 -norm of the variance from zero to T , which can be interpreted as the total amount of uncertainty. Table 2 shows the total L_1 -norm of the variance of the solution for the five different cases. The characteristic boundary condition gives in total the lowest L_1 -norm of the variance and confirms the spectrum analysis.

5.6. Large variance on the boundary

For the case of non-decaying variance on the boundary, we choose,

$$\begin{aligned} g_0(t, \xi) &= 3 + 3 \cos(2\pi t) \xi^3 - R_0(2 + 3 \cos(2\pi t) \xi) \\ g_1(t, \xi) &= 2 + 3 \cos(2\pi t) \xi - R_1(3 + 3 \cos(2\pi t) \xi^3) \\ f(x, \xi) &= [3 + 3\xi^3, 2 + 3\xi]^T, \end{aligned}$$

as the initial and boundary data. In the figures below we compare the L_1 -norm of the variance between the characteristic and non-characteristic boundary conditions for different combinations of R_0 and R_1 .

Although there is no clear theoretical support for what happens when we have a large variance on the boundary, the trend from the previous cases prevail. With the exception of the case when $R_0 R_1 = 0.25$, figure 9 show that the characteristic boundary conditions give us the smallest variance for long times. In this case there is no convergence of the correlation between the characteristic variables on the boundaries.

Finally, we compare the different cases by integrating the L_1 -norm of the variance from zero to T which can be interpreted as the total amount of uncertainty. Table 3 shows the total L_1 -norm of the variance of the solution for the five different cases. A decreasing $R_0 R_1$ lead to a decreasing L_1 -norm of the variance.

Figure 9: The L_1 -norm of the variance as a function of time for characteristic and non-characteristic boundary conditions. Large variance on the boundary.

Figure 10: The correlation between the characteristic variables at the boundaries for the non-characteristic boundary condition ($R_0R_1 = 0.25$). Large variance on the boundary.

$R_0 R_1 =$	0	0.25	0.5	0.75	1.0
$\int_0^T \ Var[u]\ dt$	747.3	602.6	1157.5	2280.5	4738.9

Table 3: The integral of the L_1 -norm of the variance for different values of R_0 and R_1 for $\xi \sim \mathcal{N}(0, 1)$ and $T = 10$.

6. An application in fluid mechanics

In this section we study a common problem in fluid dynamics, namely the effects of uncertainty in subsonic outflow boundary conditions. We study different subsonic outflow boundary conditions for the Euler equations with random boundary data.

The linearized one dimensional symmetrized form of the Euler equations with frozen coefficients is, see [42],

$$U_t + \bar{A}U_x = 0. \quad (63)$$

In (63),

$$U = \left[\frac{\bar{c}}{\sqrt{\gamma\bar{\rho}}}\rho, u, \frac{1}{\bar{c}\sqrt{\gamma(\gamma-1)}}T \right]^T, \quad \bar{A} = \begin{bmatrix} \bar{u} & \frac{\bar{c}}{\sqrt{\gamma}} & 0 \\ \frac{\bar{c}}{\sqrt{\gamma}} & \bar{u} & \sqrt{\frac{\gamma-1}{\gamma}}\bar{c} \\ 0 & \sqrt{\frac{\gamma-1}{\gamma}}\bar{c} & \bar{u} \end{bmatrix} \quad (64)$$

and $\bar{A} = X\Lambda X^T$ where

$$X = \begin{bmatrix} -\sqrt{\frac{\gamma-1}{\gamma}} & \frac{1}{\sqrt{2\gamma}} & \frac{1}{\sqrt{2\gamma}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{\gamma}} & \sqrt{\frac{\gamma-1}{2\gamma}} & \sqrt{\frac{\gamma-1}{2\gamma}} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \bar{u} & 0 & 0 \\ 0 & \bar{u} + \bar{c} & 0 \\ 0 & 0 & \bar{u} - \bar{c} \end{bmatrix}. \quad (65)$$

The perturbation variables ρ , u , p , T , c and γ represent the normalized density, velocity, pressure, temperature, speed of sound and the ratio of specific heat. The overbar denotes variables at the constant state where $\bar{A}$ is calculated. We have used $\bar{u} = 1$, $\bar{c} = 2$, $\bar{\rho} = 1$ and $\gamma = 1.4$.

The boundary conditions are of the type (5), where in this case R_0 and R_1 are matrices of size 2×1 and 1×2 respectively. Together with the characteristic case we study the following two settings of R_0 and R_1 in the non-characteristic case,

	Case	Characteristic	Pressure	Velocity
$\int_0^T \ Var[u]\ dt$	Normal	2579.0	3264.0	3143.0
$\int_0^T \ Var[u]\ dt$	Uniform	663.9	840.3	809.1

Table 4: The integral of the L_1 -norm of the variance for different values of the matrices R_0 and R_1 for $\xi \sim \mathcal{N}(0, 1)$ and $\xi \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$.

$$\text{Characteristic} \quad R_0 = [0, 0]^T, \quad R_1 = [0, 0], \quad (66)$$

$$\text{Pressure} \quad R_0 = [0, 0]^T, \quad R_1 = [0, -1], \quad (67)$$

$$\text{Velocity} \quad R_0 = [0, 0]^T, \quad R_1 = [0, +1]. \quad (68)$$

In (67) and (68), R_0 and R_1 are chosen such that the conditions for an energy estimate in (7) are satisfied. Note that the non-characteristic boundary condition (67) corresponds to specifying the *pressure* p and (68) corresponds to specifying the *velocity* u .

For completeness, we introduce the characteristic variables,

$$X^T U = \left[\frac{1}{\sqrt{\gamma-1}\bar{\rho}\bar{c}}(p - \bar{c}^2\rho), \quad \frac{1}{\sqrt{2}\bar{\rho}\bar{c}}(p + \bar{c}\bar{\rho}u), \quad \frac{1}{\sqrt{2}\bar{\rho}\bar{c}}(p - \bar{c}\bar{\rho}u) \right]^T. \quad (69)$$

The non-decaying randomness in the boundary data is given by $\rho = u = p = 0.1 + 3\cos(2\pi t)\xi^3$. In all calculations below we use the characteristic boundary conditions on the inflow boundary. On the outflow boundary, three different boundary conditions are considered. We either specify the ingoing characteristic variable, or the pressure or the velocity. We restrict ourselves to the study of large variance on the boundary.

We show results for $\xi \sim \mathcal{N}(0, 1)$ and $\xi \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$ and compare the L_1 -norm of the variance for the characteristic, pressure and velocity boundary condition respectively. Figures 11 and 12, show that even without a decaying variance on the boundary, the characteristic boundary conditions give us the smallest variance. We also compare the different boundary conditions by integrating the L_1 -norm of the variance from zero to T . Table 4 shows the total L_1 -norm of the variance of the solution for the different cases. As seen from Table 4 the characteristic boundary condition gives in total the lowest L_1 -norm of the variance.

Figure 11: The L_1 -norm of the variance as a function of time for $\xi \in \mathcal{N}(0, 1)$ and characteristic, pressure and velocity boundary condition.

Figure 12: The L_1 -norm of the variance as a function of time for $\xi \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$ and characteristic, pressure and velocity boundary condition.

7. Summary and conclusions

We have studied how the choice of boundary conditions for randomly varying initial and boundary data influence the variance of the solution for a hyperbolic system of equations. Initially, the stochastic nature of the problem was ignored and the general theory for initial boundary value problems and the related stability theory was used. Energy estimates for the continuous problem and stability estimates for the numerical scheme were derived for both homogeneous and non-homogeneous boundary conditions.

Next, the stochastic nature of the problem were considered and it was shown how the variance of the solution depend on the variance of the boundary data and the correlation between the characteristic variables at the boundaries. We also confirmed the close relation between the sharpness of the energy estimate and the size of the variance in the solution.

The continuous and discrete spectrum of the problem was employed to draw conclusions about the quantitative variance decay of the solution. Numerical results for a hyperbolic model problem supporting our theoretical derivations were presented for three different cases.

With perfect knowledge of the boundary data, the optimal choice in terms of variance reduction is the characteristic boundary conditions. With decaying variance on the boundary, the correlation between initial and boundary data decide which type of boundary conditions gives the lowest variance for short times. For long times, the characteristic boundary conditions are superior. With large variance on the boundary, the conclusions drawn above continue to hold, although no clear theoretical support exist for that case.

In a fluid mechanics application, we study different subsonic outflow boundary conditions with large non-decaying randomness in the data for the Euler equations. The numerical studies showed that the characteristic boundary condition is more variance reducing than two common non-characteristic boundary conditions such as specifying the pressure or the normal velocity.

The general conclusions from this investigation are *i)* with the same knowledge of data, we can get a more or less accurate description of the uncertainty in the solution, *ii)* the size of the variance depend to a large extent on the boundary conditions and *iii)* the characteristic boundary conditions are generally a good choice.

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