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Sobolev-Type Spaces

Properties of Newtonian Functions Based on Quasi-Banach Function Lattices in Metric Spaces

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Cover picture: modified Koch snowflake as an illustration of a fractal metric space with rectifiable curves

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Abstract

This thesis consists of four papers and focuses on function spaces related to first-order analysis in abstract metric measure spaces. The classical (i.e., Sobolev) theory in Euclidean spaces makes use of summability of distributional gradients, whose definition depends on the linear structure of $\mathbb{R}^n$. In metric spaces, we can replace the distributional gradients by (weak) upper gradients that control the functions' behavior along (almost) all rectifiable curves, which gives rise to the so-called Newtonian spaces. The summability condition, considered in the thesis, is expressed using a general Banach function lattice quasi-norm and so an extensive framework is built. Sobolev-type spaces (mainly based on the L^p norm) on metric spaces, and Newtonian spaces in particular, have been under intensive study since the mid-1990s.

In Paper I, the elementary theory of Newtonian spaces based on quasi-Banach function lattices is built up. Standard tools such as moduli of curve families and the Sobolev capacity are developed and applied to study the basic properties of Newtonian functions. Summability of a (weak) upper gradient of a function is shown to guarantee the function's absolute continuity on almost all curves. Moreover, Newtonian spaces are proven complete in this general setting.

Paper II investigates the set of all weak upper gradients of a Newtonian function. In particular, existence of minimal weak upper gradients is established. Validity of Lebesgue's differentiation theorem for the underlying metric measure space ensures that a family of representation formulae for minimal weak upper gradients can be found. Furthermore, the connection between pointwise and norm convergence of a sequence of Newtonian functions is studied.

Smooth functions are frequently used as an approximation of Sobolev functions in analysis of partial differential equations. In fact, Lipschitz continuity, which is (unlike C^1-smoothness) well-defined even for functions on metric spaces, often suffices as a regularity condition. Thus, Paper III concentrates on the question when Lipschitz functions provide good approximations of Newtonian functions. As shown in the paper, it suffices that the function lattice quasi-norm is absolutely continuous and a fractional sharp maximal operator satisfies a weak norm estimate, which it does, e.g., in doubling Poincaré spaces if a non-centered maximal operator of Hardy–Littlewood type is locally weakly bounded. Therefore, such a local weak boundedness on rearrangement-invariant spaces is explored as well.

Finer qualitative properties of Newtonian functions and the Sobolev capacity get into focus in Paper IV. Under certain hypotheses, Newtonian functions are proven to be quasi-continuous, which yields that the capacity is an outer capacity. Various sufficient conditions for local boundedness and continuity of Newtonian functions are established. Finally, quasi-continuity is applied to discuss density of locally Lipschitz functions in Newtonian spaces on open subsets of doubling Poincaré spaces.

Populärvetenskaplig sammanfattning

Avhandlingen består av fyra artiklar som behandlar så kallade *newtonrum* baserade på vissa *kvasinormer*. Låt oss försöka belysa det som döljs bakom dessa begrepp.

Många företeelser i naturvetenskap kan beskrivas med hjälp av differentialekvationer. Till exempel kan temperaturfördelningen i en kropp uttryckas som en funktion beroende på tid och position och den löser den så kallade värmeledningsekvationen. Vanligtvis kan man visa att det finns en entydig lösning, men det är sällan möjligt att finna den explicit. Å andra sidan kan man med hjälp av datorer försöka beräkna en approximation av den sökta lösningen. Ett problem med detta angreppssätt är att man inte i förväg vet om det numeriska resultatet närmar sig den verkliga lösningen. I allmänhet kan lösningarna vara ganska ”vilda” funktioner. Emellertid går det ofta att visa flera trevliga egenskaper hos dem. Dessa egenskaper kan t.ex. vara uppskattningar av hur stora värden funktionen antar och hur snabbt värdena svänger. Med hjälp av ett lämpligt mätningssätt kan man avgöra om en approximation till differentialekvationens lösning representerar verkligheten tillräckligt väl. Det finns olika sätt att mäta storleken av funktioner samt deras variationer (alltså i princip derivator), och denna avhandling behandlar olika sådana sätt samt den rika teorin omkring dessa.

Teorin för mätning av icke-släta funktioner på våra vanliga (åskådliga) rum studerades noggrant under större delen av 1900-talet. Under de senaste 15–20 åren har man försökt utvidga detta till allmänna så kallade metriska rum, där man endast känner till avståndet mellan två godtyckliga punkter. Metriska rum kan vara väldigt oregelbundna (t.ex. fraktaler) så att det i själva verket inte behöver finnas några räta linjer mellan olika punkter, vilket förhindrar tillämpning av den klassiska teorin. Ändå kan man ofta förbinda punkterna med kurvor. Om man nöjer sig med att mäta funktionen längs kurvor, så förenklas problemet. Finns det tillräckligt många kurvor, så kan man ändå dra viktiga slutsatser om beteendet av en funktion med hjälp av mätningar längs alla möjliga kurvor. Detta är ungefär den princip som står bakom teorin om newtonrum.

Newtonrum har redan studerats för vissa typer av storleksbegrepp. Teorin i avhandlingen byggs, mer allmänt än tidigare, baserad på *kvasinormer i banachfunktionslattice*, vilket ger många generaliseringar och kvalitativa förfiningar av hittills kända resultat.

Låt oss nu betrakta ett flertal frågor som man ofta vill få besvarade och som besvaras i avhandlingen. Antag att man vill undersöka en funktion som beskriver någon fysikalisk storhet och att funktionens variation redan har mätts (eller snarare uppskattats) längs alla kurvor. Man kan fråga sig huruvida det verkligen var nödvändigt att kolla på funktionens beteende på alla kurvor. Det visar sig att man mycket väl kunde ha struntat i ett litet antal kurvor (i en väldigt precis mening) utan att förlora väsentliga data. Dessutom inser man att funktionen egentligen är ganska snäll (*absolutkontinuerlig*) längs alla kurvor med undantag av ett litet antal (där ”litet antal” har samma betydelse som ovan).

Man kan behöva ta reda på om det finns ställen (mängder) där funktionsvärdena inte styrs av mätningar längs kurvor och hur stora dessa mängder egentligen är. I avhandlingen visas att det exakt är de mängder vars så kallade *sobolevkapacitet* är lika med noll. *Sobolevkapaciteten* kan i princip tolkas som ett volymmått med en finare upplösning än det vanliga volymmåttet. Strängt taget innebär det att mängderna där man saknar kontroll över funktionsvärden är ännu mindre än när man använder sig av den klassiska *sobolevteorin* om icke-släta funktioner i det vanliga rummet $\mathbb{R}^n$.

Trots att man i de flesta fall endast överuppskattar funktionernas variationer, så känns det bara naturligt att förvänta sig att det finns en bästa (alltså minsta) uppskattning. Frågan om huruvida den existerar är i själva verket mer komplicerad med tanke på att funktionernas beteende kan vara oförutsebart längs ett visst litet antal kurvor. Dessutom beror den egentliga innebörden av frasen ”litet antal” på storleksbegreppet som bestäms av *funktionslatticets kvasinorm*. För vissa storleksbegrepp var den här existensfrågan ett öppet problem inom newtonteori tills den besvarades jakande i en av avhandlingens artiklar.

När partiella differentialekvationer studeras, så kräver många resonemang att släta funktioner utgör bra approximationer till alla funktioner som kommer i fråga som svaga lösningar till ekvationen. Om området har goda geometriska egenskaper, så beror approximerbarheten på storleken av så kallade *maximalfunktioner*. Därför undersöks maximalfunktioner samt deras uppskattningar noggrant i avhandlingen. Följaktligen får man fram några enkla villkor som räcker för att kunna använda sig av släta approximationer. Därtill ges typexempel där sådana approximationer inte är möjliga.

Till sist studeras det om uppskattningar av storleken av en funktions variationer längs kurvor leder till att själva funktionen är begränsad eller till och med kontinuerlig. Så är fallet om det storleksbegrepp som används vid uppskattningar (d.v.s. *kvasinormen av banachfunktionslatticet*) är tillräckligt restriktivt i jämförelse med dimensionen av volymmåttet. Man inser nämligen att det då inte kan finnas några icke-tomma mängder som har *sobolevkapacitet* noll. Det innebär i sin tur att funktionen är kontrollerad överallt av sina variationer som enbart mätts eller uppskattats längs nästan alla kurvor.

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It has been 11 years since I set out on a journey through the wondrous lands ruled by the mighty Queen of the Sciences; and some journey it has been! Not always did I know where the path I decided to take was about to lead. I have had the chance to experience the marvelous beauty of mathematics, but I have also come across places I personally deem somewhat gloomy. Fortunately enough, I did not have to wade through the dark and murky recesses of unknown territories on my own. I have been surrounded by people, who would not let me remain stuck in peril, who would encourage me to go on with and even guide me through my voyage of exploration, and I am most thankful for that.

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Lukáš Malý
Linköping, April 2014

Contents

Introduction

Background

The aim of this thesis is to define and build up the theory around Newtonian spaces based on quasi-Banach function lattices, eventually proving some interesting properties in this setting. Newtonian spaces present perhaps the most fruitful generalization of first-order Sobolev spaces to abstract metric measure spaces. Such generalizations have been under intensive study since the mid-1990s. We refer the reader interested in analysis on metric spaces, for example, to Ambrosio and Tilli [3], Björn and Björn [6], Cheeger [12], Heinonen and Koskela [31], Heinonen [28], or Heinonen, Koskela, Shanmugalingam and Tyson [32]. This theory can be applied in diverse areas of analysis, such as linear and non-linear potential theory on Riemannian manifolds or Carnot groups, first-order analysis on fractal sets, the theory of degenerate elliptic equations and quasi-conformal mappings.

1 Sobolev spaces and variations thereupon

The theory of Sobolev spaces is an essential tool for analysis of various properties of partial differential equations, including calculus of variations. In the Euclidean setting, the Sobolev space $W^{1,p}(\Omega)$ for an open set $\Omega \subset \mathbb{R}^n$ consists of functions integrable in the pth power whose distributional gradient is integrable in the pth power as well. For more details on Sobolev spaces, see, for example, Evans [19], Maz′ya [38], Stein [50], or Ziemer [54]. The distributional derivatives are defined via integration by parts, for which the linear structure of $\mathbb{R}^n$ is crucial. On the other hand, the classical Sobolev norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p\right)^{1/p}$$

does not really depend on the vector of the weak gradient, but only on its length. One possible approach to generalizations of Sobolev spaces is to replace $|\nabla u|$ by a function that preserves certain fundamental properties of the modulus of the gradient. This is the idea that lies behind the definition of Newtonian spaces via (weak) upper gradients. Some other approaches make use of various characterizations of Sobolev functions as in Theorems 1 and 2 below.

First, let us introduce some notation. The open ball centered at $x \in \mathcal{P}$ with radius $r > 0$ will be denoted by $B(x,r)$. Given a ball $B = B(x,r)$ and a scalar $\sigma > 0$, we let $\sigma B = B(x, \sigma r)$. We define the *integral mean* of a measurable function u over a set E of finite positive measure as

$$u_E := \fint_E u\, d\mu = \frac{1}{\mu(E)} \int_E u\, d\mu$$

whenever the integral on the right-hand side exists, not necessarily finite though. The *(centered) Hardy–Littlewood maximal operator* is defined as

$$Mf(z) = \sup_{r>0} \fint_{B(z,r)} |f(x)|\, dx, \quad z \in \mathbb{R}^n, \tag{1}$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a measurable function (cf. Section 5 below).

Theorem 1 (cf. Hajłasz [22, Theorem 2.1]). *For $u \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$, the following conditions are equivalent:*

- *$u \in W^{1,p}(\mathbb{R}^n)$;*
- *there is $0 \le g \in L^p(\mathbb{R}^n)$ such that*

$$\fint_B |u - u_B|\, dx \le r \fint_B g\, dx \tag{2}$$

  for every ball B of radius $r > 0$;
- *there is $0 \le g \in L^p(\mathbb{R}^n)$ and $\sigma \ge 1$ such that*

$$\fint_B |u - u_B|\, dx \le r \biggl(\fint_{\sigma B} g^p\, dx \biggr)^{1/p} \tag{3}$$

  for every ball B of radius $r > 0$;
- *there is $0 \le g \in L^p(\mathbb{R}^n)$ and $\sigma \ge 1$ such that*

$$\frac{|u(z) - u(y)|}{|z - y|} \le \sup_{0<r<\sigma|z-y|} \biggl(\fint_{B(z,r)} g^p\, dx \biggr)^{1/p} + \sup_{0<r<\sigma|z-y|} \biggl(\fint_{B(y,r)} g^p\, dx \biggr)^{1/p}$$

  whenever $z, y \in \mathbb{R}^n \setminus E$, $z \ne y$, where $|E| = 0$.

Moreover, there is $c > 0$ such that $|\nabla u| \le cg$ a.e. whenever g satisfies either of the conditions above.

Theorem 2 (cf. Hajłasz [22, Theorem 2.2]). *Let $u \in L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$. Then, $u \in W^{1,p}(\mathbb{R}^n)$ if and only if there is $0 \le h \in L^p(\mathbb{R}^n)$ and $E \subset \mathbb{R}^n$ with $|E| = 0$ such that*

$$|u(z) - u(y)| \le |z - y|(h(z) + h(y)), \quad z, y \in \mathbb{R}^n \setminus E.$$

Furthermore, there is $c > 0$ such that $M|\nabla u| \le ch$.

1.1 Newtonian spaces – an approach based on (weak) upper gradients

The concept of Newtonian spaces is based on the Newton–Leibniz formula in the following way. Let $\Omega \subset \mathbb{R}^n$ be open. Suppose that $u \in \mathcal{C}^1(\Omega)$ and let $\gamma : [0, l_\gamma] \to \Omega$ be a $\mathcal{C}^1$-curve parametrized by arc length ds. Then,

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| = \left| \int_0^{l_\gamma} (u \circ \gamma)'(t)\, dt \right| \le \int_0^{l_\gamma} |\nabla u| \cdot |\gamma'(t)|\, dt = \int_\gamma |\nabla u|\, ds. \quad (4)$$

Thus, the modulus of the usual gradient can be used to estimate the difference of function values in two distinct points, which is the main idea lying behind the upper gradients. Namely, having an extended real-valued function u defined on a metric measure space $\mathcal{P}$, we call a Borel function $g : \mathcal{P} \to [0, \infty]$ an *upper gradient* of u if it satisfies

$$|u(\gamma(l_\gamma)) - u(\gamma(0))| \le \int_\gamma g\, ds \quad (5)$$

for every rectifiable curve $\gamma : [0, l_\gamma] \to \mathcal{P}$. Obviously, the upper gradient is not given uniquely as we can add a non-negative Borel function to an upper gradient of u obtaining another upper gradient of u. Upper gradients, under the name *very weak gradients*, were first defined and studied by Heinonen and Koskela in [30, 31].

Given a space X of measurable functions endowed with a quasi-seminorm $\|\cdot\|_X$, we define the *Newtonian space* N^1X as the space of extended real-valued functions (not equivalence classes) that belong to X and have an upper gradient in X. The Newtonian space N^1X can also be characterized as the set of measurable functions $u : \mathcal{P} \to \overline{\mathbb{R}}$ with finite N^1X quasi-seminorm, i.e.,

$$\|u\|_{N^1X} := \|u\|_X + \inf_g \|g\|_X < \infty\,, \quad (6)$$

where the infimum is taken over all upper gradients g of u. Note that if there are no rectifiable curves in the metric space, then the constant zero function is an upper gradient of any function whence the Newtonian space is rendered trivial in the sense that $N^1X = X$.

The theory of Newtonian spaces becomes more flexible if we relax the definition of upper gradient by allowing (5) to fail for a family of curves with so-called modulus equal to zero. This relaxation gives rise to the *weak upper gradients*, which were introduced by Koskela and MacManus [35] and can be considered a standard tool in the area of Newtonian spaces.

We have already mentioned that upper gradients are not given uniquely. The same holds for weak upper gradients. Nevertheless, among all weak upper gradients, there are significant ones, namely minimal weak upper gradients. They are minimal both pointwise a.e., and normwise, hence the infimum in (6) is attained.

Existence of minimal weak upper gradients has been established for L^p, $p \in (1, \infty)$, by Shanmugalingam in [48], and for $p \in [1, \infty)$ by Hajłasz in [22]. Tuominen has shown the existence in the setting of reflexive Orlicz spaces in [51]. Using a similar approach as Shanmugalingam and Tuominen, Mocanu [39] has applied the James characteristic of reflexive spaces to prove that minimal weak upper gradients exist in any strictly convex Banach function space. Even though reflexivity of the underlying function space is crucial for the argument in Mocanu's paper, it is not mentioned there. Newtonian spaces based on quasi-Banach function lattices (see Section 2 below) were introduced in [I] and minimal weak upper gradients were proven to exist in this very general setting in [II].

1.2 Hajłasz spaces and Hajłasz gradients

Hajłasz [21] proposed a Sobolev-type space $M^{1,p}(\mathcal{P})$ using a "gradient" that satisfies another pointwise inequality, inspired by the characterization of Sobolev functions in Theorem 2. Namely, a function $u \in L^p(\mathcal{P})$ belongs to $M^{1,p}(\mathcal{P})$ if there is a non-negative function $g \in L^p(\mathcal{P})$ and a set $E \subset \mathcal{P}$ with $\mu(E) = 0$ such that

$$|u(x) - u(y)| \le \mathrm{d}(x, y)(g(x) + g(y))$$

holds for every $x, y \in \mathcal{P} \setminus E$. The symbol $\mathrm{d}(x, y)$ denotes the distance between the points x and y. Such a function g is called a *Hajłasz gradient* of u. Since Hajłasz gradients are defined via an inequality (and hence not uniquely), the norm on the *Hajłasz space* $M^{1,p}(\mathcal{P})$ needs to be established by a minimization process, i.e.,

$$\|u\|_{M^{1,p}} := \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all Hajłasz gradients g of u. In [22], Hajłasz showed that for every $u \in M^{1,p}(\mathcal{P})$, $1 \le p < \infty$, there is $\tilde{u} \in N^{1,p}(\mathcal{P}) := N^1L^p(\mathcal{P})$ such that $u = \tilde{u}$ a.e. Therefore, $M^{1,p}(\mathcal{P}) \subset N^{1,p}(\mathcal{P})$ as long as the Newtonian space is equipped with a.e.-equivalence classes. If there are no rectifiable curves, then $M^{1,p}(\mathcal{P})$ is usually a proper subset of $N^{1,p}(\mathcal{P}) = L^p(\mathcal{P})$ for every $p \in [1, \infty]$, apart from some pathological cases such as when $\mathcal{P}$ is a finite set.

We could see in (4) that the notion of upper gradient is inspired by the length of the vector of the ordinary gradient. On the other hand, the Hajłasz gradient in the Euclidean setting can be understood as the Hardy–Littlewood maximal function of the ordinary gradient as we have seen in Theorem 2. Therefore, when working with a.e.-equivalence classes, it can be shown that $M^{1,p}(\mathbb{R}^n) = N^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$ for $p > 1$, but $M^{1,1}(\mathbb{R}^n) \subsetneq N^{1,1}(\mathbb{R}^n) = W^{1,1}(\mathbb{R}^n)$. Aïssaoui extended Hajłasz's approach to define Orlicz–Sobolev spaces on metric measure spaces in [1].

1.3 Calderón–Sobolev spaces and fractional sharp maximal functions

Calderón's characterization of $u \in W^{1,p}(\mathbb{R}^n)$ in [11] motivated Shvartsman [49] to define Calderón–Sobolev spaces using the *fractional sharp maximal function*

$$u_1^\sharp(x) = \sup_{r>0} \frac{1}{r} \fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu, \quad x \in \mathcal{P}.$$

Provided that $1 < p < \infty$, a function $u \in L^p(\mathbb{R}^n)$ belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if $u_1^\sharp \in L^p(\mathbb{R}^n)$. It follows from the Poincaré inequality for Sobolev functions, i.e., (2) with $g = c|\nabla u|$, that $u_1^\sharp \le cM|\nabla u|$, where M is the Hardy–Littlewood maximal operator defined in (1).

The *Calderón–Sobolev space* is $CW^{1,p}(\mathcal{P}) = \{u \in L^p(\mathcal{P}) : u_1^\sharp \in L^p(\mathcal{P})\}$. Hajłasz and Kinnunen [23] showed that $cu_1^\sharp$ is a Hajłasz gradient of u for some $c \ge 1$, provided that the metric measure space is equipped with a doubling measure (see Section 3 below). On the other hand, direct calculation shows that we can estimate $u_1^\sharp \le cMg$, where g is a Hajłasz gradient of u. If $1 < p \le \infty$ and if the measure is doubling (whence $M : L^p \to L^p$ is bounded), then $M^{1,p}(\mathcal{P}) = CW^{1,p}(\mathcal{P})$.

1.4 Other approaches

Heikkinen, Koskela, and Tuominen [27] defined a Sobolev-type space by means of a generalized Poincaré inequality, extending the ideas of Franchi, Hajłasz, and Koskela in [20], and relate this space to the corresponding Newtonian space. Function spaces based on a weak p-Poincaré inequality analogous with (3) were also discussed by Hajłasz and Koskela in [24, 25]. Koskela and MacManus [35] studied the role of (weak) upper gradients in the Poincaré inequality.

Cheeger [12] defined Sobolev-type spaces consisting of such $u \in L^p(\mathcal{P})$ that can be approximated by a sequence $\{u_i\}_{i=1}^\infty$ of L^p functions with upper gradients $g_i \in L^p(\mathcal{P})$ in the sense that $u_i \to u$ in L^p as $i \to \infty$, while $\liminf_{i\to\infty} \|g_i\|_{L^p} < \infty$. Shanmugalingam showed in [47] that for $p > 1$ the Newtonian space $N^{1,p}$ with a.e. equivalence classes coincides with the corresponding Cheeger space. On the other hand, if $p = 1$, then the Cheeger approach yields a generalization of the space of functions of bounded variation.

For comparison of various approaches to generalization of Sobolev spaces, see e.g. Hajłasz [22], Björn and Björn [6, Appendix B], or Heinonen, Koskela, Shanmugalingam and Tyson [32, Chapter 9]. The presented thesis however focuses solely on the theory of Newtonian spaces.

2 Quasi-Banach Function Lattices

Newtonian spaces have been studied for various underlying function norms in the past two decades. First, Shanmugalingam [47] studied the Newtonian spaces built upon the L^p norm for $p \in [1, \infty)$. Since then, a rich theory based on L^p has been developed. Thorough treatises on the Newtonian space theory in the L^p setting with $p \in [1, \infty)$ have been given by Björn and Björn [6], and Heinonen, Koskela, Shanmugalingam and Tyson [32]. Durand-Cartagena [16], together with Jaramillo [17], discussed the case $p = \infty$. Tuominen [51] and Aïssaoui [2] investigated Orlicz–Newtonian spaces. Harjulehto, Hästö, and Pere [26] examined Newtonian spaces based on Orlicz–Musielak variable exponent spaces. Newtonian theory built upon the Lorentz $L^{p,q}$ spaces was studied by Podbrdský [41] and Costea and Miranda [14]. Mocanu [39, 40] tried to generalize the concepts to the setting of Banach function spaces. In some of her results, uniform convexity or reflexivity of the underlying function space is however necessary. The theory that is developed in this thesis encompasses all these results and goes even further.

In the thesis, we assume that the function space X is a *quasi-Banach function lattice*, i.e., a complete quasi-seminormed linear space of measurable functions that has the lattice property. It can be described by the axioms its quasi-seminorm $\|\cdot\|_X$ needs to satisfy:

- $\|\cdot\|_X$ determines the set X, i.e., $X = \{u : \|u\|_X < \infty\}$;
- $\|\cdot\|_X$ is a *quasi-seminorm* giving rise to a.e. equivalence classes, i.e.,
  - ◇ $\|u\|_X = 0$ if and only if $u = 0$ a.e.,
  - ◇ $\|au\|_X = |a|\,\|u\|_X$ for every $a \in \mathbb{R}$ and every function u,
  - ◇ there is a constant $c_\vartriangle \ge 1$, the so-called *modulus of concavity*, such that $\|u + v\|_X \le c_\vartriangle(\|u\|_X + \|v\|_X)$ for all functions u, v;
- $\|\cdot\|_X$ satisfies the *lattice property*, i.e., if $|u| \le |v|$ a.e., then $\|u\|_X \le \|v\|_X$;
- $\|\cdot\|_X$ satisfies the *Riesz–Fischer property*, i.e., if $u_n \ge 0$ a.e. for all $n \in \mathbb{N}$, then $\left\|\sum_{n=1}^\infty u_n\right\|_X \le \sum_{n=1}^\infty c_\vartriangle^n \|u_n\|_X$, where $c_\vartriangle \ge 1$ is the modulus of concavity. Note that the function $\sum_{n=1}^\infty u_n$ needs be understood as a pointwise (a.e.) sum.
- $\|\cdot\|_X$ is continuous, i.e., if $\|u_n - u\|_X \to 0$ as $n \to \infty$, then $\|u_n\|_X \to \|u\|_X$.

The Riesz–Fischer property is for continuous quasi-seminorms equivalent to completeness of the function space (see Zaanen [53, Lemma 101.1] or Maligranda [36, Theorem 1.1]). Moreover, the assumption of continuity of $\|\cdot\|_X$ is not really restrictive since every discontinuous quasi-seminorm has an equivalent continuous quasi-seminorm (while the lattice property is preserved) such that

$$\|u + v\|^r \le \|u\|^r + \|v\|^r$$

with $r = 1/(1 + \log_2 c_\vartriangle) \in (0, 1]$, due to the Aoki–Rolewicz theorem, cf. Benyamini and Lindenstrauss [5, Proposition H.2]. If $c_\vartriangle = 1$, then the functional $\|\cdot\|_X$ is a seminorm. We then drop the prefix *quasi* and hence call X a *Banach function lattice*.

If $X \subset Y$ are quasi-Banach function lattices over the same measure space, then it follows from the Riesz–Fischer property that X is continuously embedded in Y, cf. Bennett and Sharpley [4, Theorem I.1.8].

2.1 Quasi-Banach function spaces

Quasi-Banach function spaces, and *Banach function spaces* in particular, form an important (and still very general) subclass of (quasi)Banach function lattices. In addition to the aforementioned axioms, (quasi)Banach function spaces satisfy the following ones as well:

- $\|\cdot\|_X$ has the *Fatou property*, i.e., if $0 \le u_n \nearrow u$ a.e., then $\|u_n\|_X \nearrow \|u\|_X$;
- if a measurable set $E \subset \mathcal{P}$ has finite measure, then $\|\chi_E\|_X < \infty$;
- for every measurable set $E \subset \mathcal{P}$ with $\mu(E) < \infty$ there is $C_E > 0$ such that $\int_E |u|\, d\mu \le C_E \|u\|_X$ for every measurable function u.

The Fatou property implies the Riesz–Fischer property, and it can be interpreted as the monotone convergence theorem. The other two conditions describe "local" continuous embeddings $L^\infty \hookrightarrow X$ and $X \hookrightarrow L^1$, where "local" actually means on sets of finite measure. A thorough treatise on Banach function spaces is given in Bennett and Sharpley [4, Chapter 1].

Example 3. (a) The Lebesgue $L^p(\mathcal{P}, \mu)$ spaces with $p \in [1, \infty]$ are Banach function spaces.

(b) The intersection of the Lebesgue spaces $(L^p \cap L^q)(\mathcal{P}, \mu)$ with (quasi)norm given as $\|\cdot\|_{L^p} + \|\cdot\|_{L^q}$, where $p \in [1, \infty]$ and $q \in (0, p)$, is a (quasi)Banach function space. Roughly speaking, the functions lying in this space have peaks controlled by the L^p norm, whereas their rate of decay "at infinity" (provided that $\mu(\mathcal{P}) = \infty$) is controlled by the L^q norm. If $q < 1$ and $\mu(\mathcal{P}) = \infty$, then these spaces are not normable. On the other hand, if $\mu(\mathcal{P}) < \infty$, then $L^p \cap L^q = L^p$ with equivalent (quasi)norms due to the Hölder inequality.

(c) The Lebesgue $L^p(\mathcal{P}, \mu)$ spaces with $p \in (0, 1)$ are **not** quasi-Banach function spaces as they fail the embedding $L^p \hookrightarrow L^1$ on sets of finite measure. However, they are quasi-Banach function lattices.

Example 4 (cf. Rao and Ren [42]). Let $\Phi : [0, \infty) \to [0, \infty]$ be a *Young function*, i.e., a convex function such that

$$\lim_{t \to 0} \Phi(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Phi(t) = \infty.$$

Then, the *Orlicz space* $L^\Phi(\mathcal{P},\mu)$ consists of measurable functions whose *Luxemburg norm*

$$\|u\|_{L^\Phi} := \inf\left\{\lambda > 0 : \int_{\mathcal{P}} \Phi\left(\frac{|u(x)|}{\lambda}\right) d\mu(x) \le 1\right\}$$

is finite. Orlicz spaces are Banach function spaces, providing a generalization of the Lebesgue spaces. Namely, if $\Phi(t) = t^p$ with $p \in [1,\infty)$ and $\Psi = \infty\chi_{[1,\infty)}$, then $L^\Phi = L^p$ and $L^\Psi = L^\infty$. The notion of a Young function however allows us to make finer distinction of function spaces.

The *Zygmund spaces* $L\log L$ and $L_{\exp}$ can be defined as Orlicz spaces L^Φ and L^Ψ, respectively, where $\Phi(t) = t\max\{0,\log t\}$ and $\Psi(t) = \min\{t, e^{t-1}\}$.

Example 5 (cf. Diening, Harjulehto, Hästö, Růžička [15]). Let $p : \mathcal{P} \to [1,\infty]$ be a measurable function. The *variable exponent space* $L^{p(\cdot)}$ consists of measurable functions whose Luxemburg norm

$$\|u\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_{\{p<\infty\}} \left(\frac{|u(x)|}{\lambda}\right)^{p(x)} d\mu(x) \le 1 \quad \& \quad \|u\|_{L^\infty(\{p=\infty\})} < \lambda\right\}$$

is finite. If p is constant, then the variable exponent space coincides with the usual Lebesgue L^p space. The $L^{p(\cdot)}$ spaces can be understood as a special case of Orlicz–Musielak spaces, which are Banach function spaces.

Apparently, the Luxemburg norm of an Orlicz space (unlike a variable exponent space) does not depend on the exact distribution of function values in $\mathcal{P}$. Thus, Orlicz spaces belong to a wide class of spaces, the so-called *rearrangement-invariant spaces* (see [4, Chapter 2]), where the (quasi)norm of a function is invariant under measure-preserving transformations. For a measurable function $f : \mathcal{P} \to \overline{\mathbb{R}}$, we define its *distribution function* μ_f and its *decreasing rearrangement* f^* by

$$\mu_f(t) = \mu\{x \in \mathcal{P} : |f(x)| > t\} \quad \text{and} \quad f^*(t) = \inf\{s \ge 0 : \mu_f(s) \le t\}, \quad t \ge 0.$$

The Cavalieri principle implies that $\|f\|_{L^1(\mathcal{P},\mu)} = \|\mu_f\|_{L^1(\mathbb{R}^+,\lambda^1)} = \|f^*\|_{L^1(\mathbb{R}^+,\lambda^1)}$.

Example 6 (cf. [4, Chapters 4 and 5]). The real interpolation method applied to the Lebesgue spaces gives rise to the so-called Lorentz spaces. The *Lorentz space* $L^{p,q}(\mathcal{P},\mu)$ with $p,q \in [1,\infty)$ is defined by

$$\|u\|_{L^{p,q}} := \left(\frac{q}{p}\int_0^\infty \left(u^*(t)t^{1/p}\right)^q \frac{dt}{t}\right)^{1/q} < \infty.$$

The *Lorentz space* $L^{p,\infty}$, also called the *weak-L^p* space, is for $p \in [1,\infty)$ given via

$$\|u\|_{L^{p,\infty}} := \sup_{t>0} u^*(t)t^{1/p} < \infty.$$

The $L^{p,q}$ spaces are Banach function spaces whenever $1 \le q \le p$, but only (normable) quasi-Banach function spaces if $1 < p < q$. The $L^{1,q}$ spaces are **not** quasi-Banach function spaces if $q \in (1, \infty]$ since they are not locally embedded in L^1. They are mere quasi-Banach function lattices.

The Lorentz spaces $L^{p,q}$ form a finer scale of spaces that in some sense lie close to L^p. Namely, it holds that $L^{p,p} = L^p$ and $L^{p,q_1} \hookrightarrow L^{p,q_2}$ whenever $1 \le q_1 \le q_2 \le \infty$. Moreover, if $\mu(\mathcal{P}) < \infty$, then also $L^{p_1,q_1} \hookrightarrow L^{p_2,q_2}$ whenever $1 \le p_2 < p_1 < \infty$ regardless of the exact values of $q_1, q_2 \in [1, \infty]$.

Some rudimentary properties of a rearrangement-invariant space $X = X(\mathcal{P}, \mu)$ (where μ is non-atomic) can be described by its *fundamental function*, which is defined for $t \in [0, \mu(\mathcal{P}))$ as $\phi_X(t) = \|\chi_{E_t}\|_X$, where $E_t \subset \mathcal{P}$ is an arbitrary measurable set with $\mu(E_t) = t$. In general, the fundamental function of a rearrangement-invariant Banach function space (*r.i. space*) X is quasi-concave, but X may be equivalently renormed so that ϕ_X is concave. Moreover, ϕ_X is absolutely continuous except perhaps at the origin, where there may be a jump discontinuity.

Example 7. Given an r.i. space X with concave fundamental function ϕ_X, we define the *Lorentz space* $\Lambda(X)$ and the *Marcinkiewicz space* $M(X)$ by their norms

$$\|u\|_{\Lambda(X)} := \int_0^\infty u^*(t)\, d\phi_X(t) = \|u\|_{L^\infty}\phi(0+) + \int_0^\infty u^*(t)\phi_X'(t)\, dt,$$
$$\|u\|_{M(X)} := \sup_{t>0} M^*u^*(t)\phi_X(t) = \sup_{t>0} \phi_X(t) \fint_0^t u^*(s)\, ds,$$

where M^* is the non-centered Hardy–Littlewood maximal function (cf. (1); in fact, $M^* = M_1$ defined in (10) below). The Lorentz and the Marcinkiewicz spaces are the smallest and the largest spaces, respectively, among all r.i. spaces of the same fundamental function ϕ_X, i.e., $\Lambda(X) \hookrightarrow X \hookrightarrow M(X)$ whenever X is an r.i. space.

2.2 Other quasi-Banach function lattices

We have already seen in Examples 3 and 6 that the $L^p(\mathcal{P}, \mu)$ spaces with $0 < p < 1$ and the $L^{1,q}(\mathcal{P}, \mu)$ spaces with $q \in (1, \infty]$ are mere quasi-Banach function lattices as they fail to be locally embedded in L^1.

Example 8. Let X be an arbitrary (quasi)Banach function space. We can define a subspace Y of X by posing an additional condition that forces the function value to be zero at some point $x_0 \in \mathcal{P}$, e.g.,

$$\|u\|_Y = \|u\|_X + \sum_{k=1}^\infty \fint_{B(x_0,2^{-k})} |u|\, d\mu.$$

Then, Y is not a (quasi)Banach function space since L^∞ is not locally embedded in Y. It is however a (quasi)Banach function lattice.

Example 9. Spaces of continuous, C^k-smooth, or Sobolev functions are **not** quasi-Banach function lattices as they fail to comply with the lattice property.

2.3 Absolute continuity of the quasi-norm

The quasi-norm $\|\cdot\|_X$ in a quasi-Banach function lattice X is *absolutely continuous* if $\|u\chi_{E_n}\|_X \to 0$ as $n \to \infty$ whenever $u \in X$ and $\{E_n\}_{n=1}^\infty$ is a decreasing sequence of measurable sets with $\mu\bigl(\bigcap_{n=1}^\infty E_n\bigr) = 0$. The notion of absolute continuity of the quasi-norm turns out crucial for the results on approximability of Newtonian functions by Lipschitz continuous functions obtained in [III]. It is possible to find examples of function spaces $X = X(\mathbb{R}^n)$ lacking this property such that (locally) Lipschitz functions are not dense in the corresponding Newtonian space $N^1X(\mathbb{R}^n)$.

The dominated convergence theorem yields that the L^p (quasi)norm is absolutely continuous for $p \in (0,\infty)$. On the other hand, the weak-L^p (i.e., the Lorentz $L^{p,\infty}$ spaces) and L^∞ spaces lack this property apart from a few pathological cases. In quasi-Banach function spaces, the dominated convergence theorem is in fact equivalent to the absolute continuity of the quasi-norm.

It is worth noting (even though it will not be used in the thesis) that the absolute continuity of the norm plays a vital role for establishing a connection between functional-analytic and measure-theoretic approaches to dual spaces of Banach function spaces (see [4, Sections I.3 and I.4]). Namely, the Banach space dual X^* that consists of bounded linear functionals on a Banach function space X is isometrically isomorphic to the *associate space*

$$X' = \left\{ v \in \mathcal{M}(\mathcal{P},\mu) : \|v\|_{X'} := \sup_{\|u\|_X = 1} \int_{\mathcal{P}} uv\, d\mu < \infty \right\}$$

if and only if the norm of X is absolutely continuous. The relation between $T \in X^*$ and $v \in X'$ is then given by the representation $Tu = \int_{\mathcal{P}} uv\, d\mu$, where $u \in X$.

3 Doubling measures

Some of the results on regularity and regularization of Newtonian functions in [III, IV] make use of the assumption that the measure is *doubling*, i.e., there is $c_{\mathrm{dbl}} \ge 1$ such that $\mu(2B) \le c_{\mathrm{dbl}}\mu(B)$ for every ball $B \subset \mathcal{P}$, where $2B$ denotes the ball of the same center as B but twice the radius. Roughly speaking, the doubling condition of the measure guarantees that the metric measure space has certain properties of finite-dimensional spaces. In particular, the Hardy–Littlewood maximal operator is a bounded mapping from $L^1(\mathcal{P})$ to $L^{1,\infty}(\mathcal{P})$, whence Lebesgue's differentiation theorem holds true.

In general, it is however not possible to define the dimension of the space (related to the measure) unambiguously. On the other hand, if μ is doubling, then there are $c_s > 0$ and $s > 0$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge c_s \left(\frac{r}{R}\right)^s \tag{7}$$

for every $x \in \mathcal{P}$, $0 < r < R < \infty$, and $y \in B(x,R)$. If (7) holds with some s, then it holds with all $s' \ge s$. By an iteration argument, it can be shown that (7) holds with $s = \log_2 c_{\mathrm{dbl}}$, which however need not be the optimal value. The set of admissible exponents may be open (see Björn, Björn, and Lehrbäck [7]), whence there need not exist an optimal value of s. The exponent s plays a fundamental role in [IV] when establishing sufficient conditions for Newtonian functions to be locally bounded and (Hölder) continuous. Roughly speaking, s is as significant as the dimension n is for the embeddings $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{0,\alpha}(\mathbb{R}^n)$.

If $\mathcal{P}$ is connected (which in particular holds if $\mathcal{P}$ supports a Poincaré inequality, see Section 4 below) and μ is doubling, then there are $c_\sigma > 0$ and $0 < \sigma \le s$ such that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \le c_\sigma \left(\frac{r}{R}\right)^\sigma \tag{8}$$

for every $x \in \mathcal{P}$, $0 < r < R < \infty$, and $y \in B(x,R)$. Similarly as above, if (8) holds with some σ, then it holds with all $\sigma' \le \sigma$. Note however that it may happen that $\sigma < s$ even if both σ and s are the best possible exponents (provided that these exist). Inequalities (7) and (8) may be used to estimate the Hausdorff dimension of the metric space as $\dim_H \mathcal{P} \in [\sigma, s]$. The metric measure space is called *Ahlfors Q-regular* if both (7) and (8) are satisfied with $\sigma = s =: Q$. Ahlfors regularity is a very restrictive condition that fails even in weighted $\mathbb{R}^n$, unless the weight is bounded away both from zero and from infinity.

If a metric space is endowed with a doubling measure, then the metric space is *doubling*, i.e., there is $N < \infty$ such that every ball of radius r can be covered by at most N balls of radius $r/2$. Consequently, bounded sets in such a metric space are totally bounded. Conversely, a doubling metric space may be equipped with a measure that is not doubling. However, if the doubling metric space is complete, then one can construct a doubling measure thereon (see Volberg and Konyagin [52]). Roughly speaking, doubling metric spaces possess certain properties of finite-dimensional linear spaces since they are bi-Lipschitz equivalent to a subset of $\mathbb{R}^n$ equipped with a metric that however need not be equivalent with the Euclidean one, see Semmes [45].

4 Poincaré inequalities

A metric space $\mathcal{P}$ supports a p-Poincaré inequality with $p \in [1, \infty)$ if there exist constants $c_{\mathrm{PI}} > 0$ and $\lambda \ge 1$ such that for all balls $B \subset \mathcal{P}$, all functions $u \in L^1_{\mathrm{loc}}(\mathcal{P})$ and all (p-weak) upper gradients g of u, we have

$$\fint_B |u - u_B| \, d\mu \le c_{\mathrm{PI}} \operatorname{diam}(B) \biggl(\fint_{\lambda B} g^p \, d\mu \biggr)^{1/p}. \tag{9}$$

Roughly speaking, if the metric space supports a Poincaré inequality, then there are plentiful curves forcing the upper gradient to be sufficiently large so that the volume integral of an upper gradient can be used to control the mean oscillation of a function. The Poincaré inequality allows us to do an advanced first-order analysis in metric spaces.

Regularity of Newtonian functions (such as continuity, Hölder continuity, approximability by Lipschitz functions) as well as Sobolev-type embeddings can be established if the metric space is endowed with a doubling measure and supports a Poincaré inequality.

Furthermore, a p-Poincaré inequality together with the doubling condition of the measure enables us to establish a relation between (weak) upper gradients, Hajłasz gradients and fractional sharp maximal functions that are used to define Newtonian, Hajłasz and Calderón–Sobolev spaces, respectively. Namely, it is easy to see that the p-Poincaré inequality implies that $u^\sharp_1 \le c(M_1 g^p)^{1/p}$, where M_1 is the Hardy–Littlewood maximal operator (see (10) below) and g is an upper gradient of u. Hajłasz and Kinnunen [23] showed that there is $c > 0$ such that $cu^\sharp_1$ is a Hajłasz gradient of every $u \in L^1_{\mathrm{loc}}(\mathcal{P})$, provided that μ is doubling (even if $\mathcal{P}$ does not support any Poincaré inequality).

Obviously, if (9) holds for some $p \in [1, \infty)$, then it holds for every $p' \in [p, \infty)$ by the Hölder inequality. On the other hand, Keith and Zhong [34] showed that the Poincaré inequality is a self-improving property if the metric space $\mathcal{P}$ is complete and μ is doubling. Namely, if (9) holds for some $p \in (1, \infty)$, then there is $\varepsilon > 0$ such that it holds for every $p' \in (p - \varepsilon, \infty)$.

It is often difficult to check whether a metric measure space supports a Poincaré inequality. The list of known examples includes Euclidean spaces with the Lebesgue measure, weighted Euclidean spaces with p-admissible weights (e.g., weights of the Muckenhoupt class A_p), complete Riemannian manifolds with non-negative Ricci curvature, Heisenberg groups, Carnot–Carathéodory spaces, and Loewner spaces, see [9, 25, 28, 29, 31, 37, 43]. Poincaré inequalities are preserved under bi-Lipschitz mappings and they survive the Gromov–Hausdorff limits. Semmes [44] established that n-Ahlfors regular complete metric spaces (where $n \in \mathbb{N}$) support a 1-Poincaré inequality if there are sufficiently many rectifiable curves.

There are some geometric constraints imposed on a metric space $\mathcal{P}$ if it supports a Poincaré inequality. First of all, it is necessarily connected. Roughly speaking, the following example shows that $\mathcal{P}$ cannot have any slits.

Example 10. Suppose that there exists an open ball $B = B(z, r) \subset \mathcal{P}$ such that $3B = U_3 \cup V_3$, where U_3 and V_3 are disjoint and open, while both $U_1 := U_3 \cap B$ and $V_1 := V_3 \cap B$ have positive measure, and $\operatorname{dist}(U_1, V_1) = 0$. Let

$$u(x) = \begin{cases} \max\left\{0, \dfrac{\mathrm{d}(x,z)}{2r} - 1\right\} & \text{for } x \in U_3, \\ \min\left\{1, 2 - \dfrac{\mathrm{d}(x,z)}{2r}\right\} & \text{for } x \in V_3, \\ \dfrac{1}{2} & \text{for } x \in \mathcal{P} \setminus (U_3 \cup V_3). \end{cases}$$

Then, $u = 0$ in $U_2 := U_3 \cap 2B$ while $u = 1$ in $V_2 := V_3 \cap 2B$. Moreover, $g = \chi_{3B \setminus 2B}/2r$ is an upper gradient of u. We will now show by contradiction that u and g do not fulfill (9) and hence $\mathcal{P}$ does not support any p-Poincaré inequality.

Suppose on the contrary that there are $p \in [1, \infty)$, $\lambda \geq 1$, and $c_{\mathrm{PI}} > 0$ such that (9) holds. Since $\operatorname{dist}(U_1, V_1) = 0$, we can find a ball $\widetilde{B} \subset 2B$ such that $\lambda\widetilde{B} \subset 2B$, $\mu(U_2 \cap \widetilde{B}) > 0$, and $\mu(V_2 \cap \widetilde{B}) > 0$. For such a ball $\widetilde{B}$, the right-hand side of (9) is equal to zero. However, the left-hand side is equal to $\mu(U_2 \cap \widetilde{B})\mu(V_2 \cap \widetilde{B})/\mu(\widetilde{B})^2 > 0$.

Similarly, narrow passages or cusps may destroy the Poincaré inequality. If $\mathcal{P}$ is complete, endowed with a doubling measure, and supports a Poincaré inequality, then it is *quasi-convex*, i.e., every two points can be connected by a curve whose length is comparable with the distance of these two points.

5 Maximal operators of Hardy–Littlewood type

The classical Hardy–Littlewood maximal operator is a central tool in harmonic analysis. It can be also used in studying partial differential equations and Sobolev functions, see e.g. Bojarski and Hajłasz [10]. Given $1 \leq p < \infty$, we define the *non-centered maximal operator* by

$$M_p f(x) = \sup_{B \ni x} \left(\fint_B |f|^p \, d\mu \right)^{1/p}, \quad x \in \mathcal{P}, \tag{10}$$

where $f : \mathcal{P} \to \overline{\mathbb{R}}$ is a measurable function. Coifman and Weiss [13] showed that $M_1 : L^1(\mathcal{P}, \mu) \to L^{1,\infty}(\mathcal{P}, \mu)$ is bounded, given that μ is a doubling measure. As a direct consequence, we see that $M_p : L^p(\mathcal{P}, \mu) \to L^{p,\infty}(\mathcal{P}, \mu)$ is bounded for all $p \in [1, \infty)$. Apparently, $M_p : L^\infty(\mathcal{P}, \mu) \to L^\infty(\mathcal{P}, \mu)$ is bounded as well. The Marcinkiewicz interpolation theorem then yields that $M_p : L^q(\mathcal{P}, \mu) \to L^q(\mathcal{P}, \mu)$ is bounded whenever $1 \le p < q \le \infty$.

It has been mentioned earlier that the maximal operator M_1 conveys the relation between fractional sharp maximal functions and Hajłasz gradients in metric spaces endowed with a doubling measure. If the metric space in addition supports a p-Poincaré inequality, then the maximal operator M_p creates a link between (weak) upper gradients on one side, and fractional sharp maximal functions and Hajłasz gradients on the other side.

Hence in doubling measure spaces, the Hajłasz space M^1X (that consists of functions $u \in X$ that have a Hajłasz gradient $g \in X$) coincides with the Calderón–Sobolev space CW^1X (that consists of functions $u \in X$ with $u_1^\sharp \in X$) whenever $M_1 : X \to X$ is bounded. In doubling p-Poincaré spaces, the Newtonian space N^1X (with a.e.-equivalence classes) coincides with M^1X and CW^1X whenever $M_p : X \to X$ is bounded.

When applying the method of Lipschitz truncations in [III], weak boundedness of maximal operators of Hardy–Littlewood type is used in conjunction with the Poincaré inequality to prove that Lipschitz functions are dense in the Newtonian space both in the Newtonian norm and in the Luzin sense, i.e., restriction of a Newtonian function to the complement of a set of arbitrarily small measure is Lipschitz continuous. Such a result goes back to Shanmugalingam [46] in the L^p setting. Similar results were also obtained by Tuominen [51] in the Orlicz setting, by Harjulehto, Hästö and Pere [26] in the setting of variable exponent spaces, and by Costea and Miranda [14] for the Lorentz–Newtonian spaces $N^1L^{p,q}(\mathcal{P})$ with $1 \le q \le p < \infty$.

6 Quasi-continuity

In the theory of first-order Sobolev spaces (and spaces of Sobolev type), quasi-continuity plays an analogous role as Luzin's theorem in zeroth-order analysis. A function is called *quasi-continuous* if an open set of arbitrarily small Sobolev capacity can be found so that the restriction of the function to the complement of that set is continuous. The notion of quasi-continuity is closely related with the *Sobolev capacity* C_X whose definition depends on the used Sobolev-type function norm. In the Newtonian setting, it is customary to define $C_X(E) = \inf\{\|u\|_{N^1X} : u \ge 1 \text{ on } E\}$ for $E \subset \mathcal{P}$, which turns out to correspond well with the natural equivalence classes in N^1X. Unfortunately, when the capacity is defined this way, it is not a priori outer regular. Roughly speaking, the capacity is an outer capacity if and only if all Newtonian functions are quasi-continuous, cf. Björn, Björn and Shanmugalingam [8] and [IV]. Note also that $C_X(E) = 0$ implies that $\mu(E) = 0$, but not vice versa.

In the Euclidean setting, Sobolev functions in $W^{1,p}(\mathbb{R}^n, w(x)\,dx)$, where w is a p-admissible weight, have quasi-continuous representatives, cf. Heinonen, Kilpeläinen, Martio [29]. For a thorough treatise of the unweighted case, see e.g. Malý and Ziemer [37]. In metric spaces, it suffices that continuous functions are dense in the Newtonian space N^1X in order to obtain existence of quasi-continuous representatives of N^1X functions, cf. [8, 47, IV]. In fact, there is a qualitative difference between $W^{1,p}(\mathbb{R}^n)$ and $N^{1,p}(\mathbb{R}^n)$. Namely, the space $N^{1,p}(\mathbb{R}^n)$ corresponds to the *refined Sobolev space* (see [29]) since all the representatives are quasi-continuous in this case.

Summary of papers

Paper I: Newtonian spaces based on quasi-Banach function lattices

In the first paper, we define Newtonian spaces based on quasi-Banach function lattices using the notion of upper gradient. We investigate generalizations of standard tools in the theory of Newtonian functions. Namely, we define and study the Sobolev capacity based on the quasi-norm of the function lattice. It serves as a finer (σ-quasi-additive) outer measure for sets of zero measure.

As in the L^p case, we see that a function has Newtonian quasi-seminorm equal to zero if and only if the function is equal to zero *quasi-everywhere*, i.e., with exception of a set of capacity zero. The natural equivalence classes are thus given by equality up to sets of capacity zero and it is exactly these sets that do not carry any information about a Newtonian function.

We also define the modulus of a family of curves and prove that Newtonian spaces can be equivalently defined using weak upper gradients. We show that all these objects retain their properties, well-known in the L^p and the Orlicz cases, even in our general setting. Most importantly, the Newtonian space with equivalence classes given by equality quasi-everywhere is a complete quasi-normed space. Moreover, all Newtonian functions are absolutely continuous along almost every rectifiable curve, where “almost every curve” means that the exceptional family of curves has zero modulus.

Paper II: Minimal weak upper gradients in Newtonian spaces based on quasi-Banach function lattices

Given an extended real-valued function, there is a corresponding (weak) upper gradient. It is, however, not unique. The definition of the Newtonian quasi-norm uses minimization of an energy functional over all (weak) upper gradients. In this paper, we use a method similar to that of Hajłasz [22] to show that in the fully general setting of quasi-Banach function lattices, there is a minimal weak upper gradient and thus the infimum in the Newtonian quasi-norm is indeed attained. This result applies, in particular, to the $N^{1,\infty} := N^1L^\infty$ spaces, where the question of existence of a minimal weak upper gradient was still open. Afterwards, we find a family of representation formulae for the minimal weak upper gradient under the assumption that Lebesgue's differentiation theorem holds in the underlying metric measure space $\mathcal{P}$. For example, it suffices that the measure satisfies the doubling condition.

Next, we study the sets of upper gradients and weak upper gradients. We show that the latter is the closure of the former in the convex cone of non-negative functions in the corresponding quasi-Banach function lattice. Furthermore, we investigate convergence properties of sequences of Newtonian functions and their weak upper gradients.

Paper III: Regularization of Newtonian functions on metric spaces via weak boundedness of maximal operators

Analysis of partial differential equations frequently makes use of approximability of Sobolev functions by smooth functions. As usual partial derivatives (and hence the notion of $\mathcal{C}^k$-smoothness with $k \geq 1$) are unavailable in metric spaces, we consider Lipschitz continuity as the regularity concept of interest.

First, we look into the question when bounded functions are dense in the Newtonian space. It turns out that absolute continuity of the quasi-norm of the base function space X is crucial for the density. We observe that bounded functions are not dense in weak Marcinkiewicz (weak Lorentz) spaces. As a particular consequence, (locally) Lipschitz functions are shown not to be dense in $N^1X(\mathbb{R}^n)$ whenever X is a weak Marcinkiewicz space whose fundamental function, roughly speaking, is not concave enough relative to the dimension n.

The regularizations in N^1X that are investigated in the paper are constructed via Lipschitz truncations, provided that Hajłasz gradients of Newtonian functions satisfy certain weak norm estimates. Interplay between the Hajłasz, the Calderón and the Newtonian approaches is employed to study the problem in metric spaces endowed with a doubling measure supporting a p-Poincaré inequality. Then, the regularization problem reduces to establishing sufficient conditions for a maximal operator of Hardy–Littlewood type to be weakly bounded on sets of finite measure.

Therefore, such a weak boundedness of the maximal operators is explored as well. Particular focus is given to the situation when X is a rearrangement-invariant space. Then, the desired boundedness can be obtained, e.g., if X is locally embedded in certain classical Lorentz spaces or if the reciprocal for the fundamental function of X is comparable with its L^p-means.

Paper IV: Fine properties of Newtonian functions and the Sobolev capacity on metric measure spaces

Regularity of Newtonian functions based on quasi-Banach function lattices is the object of interest of this paper. Two interlinked concepts of regularity are studied, namely, quasi-continuity and continuity. The former can be understood as a Luzin-type condition, where a set of arbitrarily small capacity can be found for each Newtonian function so that its restriction to the complement of that set is continuous.

Furthermore, the quasi-continuity bears a close relation to outer regularity of the Sobolev capacity. In the Euclidean setting, the Sobolev capacity is defined in such a way that it is automatically an outer capacity (see [29, Definition 2.35]). This is however not the case when the Sobolev capacity is built via the Newtonian function (quasi)norm so that it corresponds well to natural equivalence classes in the respective Newtonian space. If continuous functions are dense in the Newtonian space, then we prove that all Newtonian functions are quasi-continuous if and only if the Sobolev capacity is an outer capacity.

Applying the results on quasi-continuity and on outer regularity of the capacity, locally Lipschitz functions are shown to be dense in a Newtonian space on an open subset of a proper metric space, provided that (locally) Lipschitz functions are dense in the Newtonian space on the whole metric space. The noteworthy part of this claim is that no hypotheses are put on this open subset. In particular, the open subset as a metric space need not support any Poincaré inequality, nor does the restriction of the measure need to be doubling.

Given that the metric space is endowed with a doubling measure and supports a Poincaré inequality, Newtonian functions can be proven to be essentially bounded if the summability of (weak) upper gradients, in terms of the function norm, exceeds the "dimension" s of the measure defined in (7). This is a well-known result for $N^{1,p} := N^1L^p$, where $p > s$. On the other hand, there are unbounded Newtonian functions in $N^{1,p}$ if $p = s$. Since the theory in the paper is built considering general quasi-Banach function lattices, it is possible to analyze the borderline case using a finer scale of function spaces. It suffices that the gradient lies in the Zygmund space $L^s(\log L)^\alpha$ with $\alpha > 1 - 1/s$ or in the Lorentz space $L^{s,1}$ to see that the Newtonian functions are locally essentially bounded. In the unweighted Euclidean spaces $\mathbb{R}^n$ (where $s = n$), it is known (cf. [18]) that the Lorentz space $L^{n,1}$ is the optimal rearrangement-invariant Banach function space to obtain the embedding $W^1L^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $|\Omega| < \infty$. The methods applied in the current paper differ significantly from those used in the Euclidean case. Here, all the estimates are based on Poincaré inequalities, whereas pointwise estimates by the Riesz potential and symmetrization of the Riesz kernel (cf. [33]), or the Pólya–Szegő principle (cf. [18]) can be used in $\mathbb{R}^n$.

The estimates for local essential boundedness are uniform in a certain way so that one can conclude that Newtonian functions have continuous representatives if the gradient is summable in a sufficiently high degree. If the metric space is in addition complete, then all representatives are continuous, which illustrates yet again the qualitative difference between the usual Sobolev and the Newtonian spaces.

Furthermore, several technical tools are established in the paper. In the area of calculus of weak upper gradients, the product and the chain rule are proven and the minimal weak upper gradient is shown to depend only on the local behavior of a function. The Vitali–Carathéodory theorem (i.e., approximability of the norm of a function by the norms of its lower semicontinuous majorants) for general quasi-Banach function lattices is also looked into.

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