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Japhet Niyobuhungiro and Eric Setterqvist
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Japhet Niyobuhungiro∗
Division of Mathematics and Applied Mathematics, Department of Mathematics,
Linköping University, SE–581 83 Linköping, Sweden

Department of Mathematics, School of pure and applied sciences, College of Science and Technology,
University of Rwanda, P.O. Box 117 Butare, Rwanda

Eric Setterqvist
Division of Mathematics and Applied Mathematics, Department of Mathematics,
Linköping University, SE–581 83 Linköping, Sweden

Abstract

In this paper we consider an analogue of the well–known in image processing, Rudin–Osher–Fatemi (ROF) denoising model on a general finite directed and connected graph. We consider the space $BV$ on the graph and show that the unit ball of its dual space can be described as the image of the unit ball of the space $\ell^\infty$ on the graph by a divergence operator. Based on this result, we propose a new fast algorithm to find the exact minimizer for the ROF model. Finally we prove convergence of the algorithm and illustrate its performance on some image denoising test examples.

Keywords: ROF model, Directed graph, $L$–functional, Image processing, Dual BV, Regularization.

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1. Classical ROF model and ROF model on the graph

Let us suppose that we observed noisy image $f_{ob} \in L^2$ defined on a square domain $\Omega = [0, 1]^2$ in $\mathbb{R}^2$

$$f_{ob} = f_s + \eta,$$

where $f_s \in BV$ is the original image and $\eta \in L^2$ is noise. Denoising is one of the problems which appear in image processing: "How to recover the image $f_s$ from the noisy image $f_{ob}$?". Variational methods using the total variation minimization are often employed to solve this problem. The total variation regularization technique was introduced by Rudin, Osher and Fatemi in [1] and is called the ROF model. It suggests to take as an approximation to the original image $f_s$ the function $f_{opt,t} \in BV$, which is the exact minimizer for the $L_{2,1}$– functional for the couple $(L^2, BV)$:

$$L_{2,1} \left(t, f_{ob}; L^2, BV\right) = \inf_{g \in BV} \left( \frac{1}{2} \|f_{ob} - g\|_{L^2}^2 + t \|g\|_{BV}\right), \text{ for some } t > 0,$$

i.e., $f_{opt,t} \in BV$ is such that

$$L_{2,1} \left(t, f_{ob}; L^2, BV\right) = \frac{1}{2} \|f_{ob} - f_{opt,t}\|_{L^2}^2 + t \|f_{opt,t}\|_{BV}.$$

By $BV$ we denote the space of functions of bounded variation defined by the seminorm

$$\|f\|_{BV(\Omega)} = \int_0^1 var_x f(x,y) dy + \int_0^1 var_y f(x,y) dx,$$

∗Corresponding author

Email addresses: japhet.niyobuhungiro@liu.se, jniyobuhungiro@ur.ac.rw (Japhet Niyobuhungiro), eric.setterqvist@liu.se (Eric Setterqvist)

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where

\[ \text{var}_x f(x,y) = \sup_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} \sum_{j=1}^{n-1} |f(x_{j+1},y) - f(x_j,y)| \]

is a function of \( y \) equal to variation of \( f \) on the horizontal axis for \( y \) fixed, and

\[ \text{var}_y f(x,y) = \sup_{0 \leq y_1 \leq \ldots \leq y_n \leq 1} \sum_{i=1}^{n-1} |f(x,y_{i+1}) - f(x,y_i)| \]

is a function of \( x \) equal to variation of \( f \) on the vertical axis for \( x \) fixed.

However the problem of actual calculation of the function \( f_{opt} \) (see (1) and (2)) is non-trivial. Standard approach is connected with discretization of the functional (1), i.e. we divide \( \Omega \) into \( N \times N \) square cells and instead of the space \( L^2(\Omega) \) consider its finite dimensional subspace \( S_N \) which consists of functions that are constant on each cell.

\[ S_N = \left\{ f \in L^2(\Omega) : f = \sum_{i,j=1}^{N} f_{ij} \chi_{ij} \right\}, \tag{3} \]

where

\[ \chi_{ij}(x,y) = \begin{cases} 
1 & \text{if } \frac{i-1}{N} \leq x < \frac{i}{N} \text{ and } \frac{j-1}{N} \leq y < \frac{j}{N}; \\
0 & \text{otherwise}. \end{cases} \tag{4} \]

Throughout, we will consider our discretization grid as a 2D Cartesian coordinate in (computer) screen space, i.e., with the origin at the top left corner and +y axis pointing down, which is the way matrices are represented on the computer, as illustrated in Figure 1.

![Rectangular grid illustration](illustration.png)

Figure 1: Illustration of the rectangular grid. The grid contains \( N^2 \) cells referred to as vertices; \( N(N-1) \) horizontal edges directed left right and \( (N-1)N \) vertical edges directed up down. Total of \( 2N(N-1) \) edges.

As the BV seminorm of a function \( g \in S_N \) is equal to

\[ \| g \|_{BV}(S_N) = \frac{1}{N} \left( \sum_{i=1}^{N} \sum_{j=1}^{N-1} |g_{i,j+1} - g_{ij}| + \sum_{j=1}^{N} \sum_{i=1}^{N-1} |g_{i+1,j} - g_{ij}| \right), \]
the discrete analogue of the functional \( f \) can be written as

\[
L_{2,1} \left( t, f_{ob}; L^2, BV \right) = \inf_{g \in S_N} \left( \frac{1}{2N^2} \left( \sum_{i,j=1}^{N} (f_{obij} - g_{ij})^2 \right) + \frac{t}{N} \left( \sum_{i=1}^{N} \sum_{j=1}^{N-1} |g_{ij+1} - g_{ij}| + \sum_{i=1}^{N} \sum_{j=1}^{N-1} |g_{i+1,j} - g_{ij}| \right) \right).
\]

Now we can consider the graph \( G = (V, E) \), where the set of vertices \( V \) corresponds to cells and the set of edges \( E \) corresponds to set of pairs of cells which have common faces (see Figure 1), i.e.

\[
V = \{ v_{ij}; i, j = 1, 2, \ldots, N \},
\]

and \( E = E_H \cup E_V \), where

\[
E_H = \{ e^H_{ij} = (v_{ij}, v_{i,j+1}) ; i = 1, 2, \ldots, N \text{ and } j = 1, 2, \ldots, N - 1 \}
\]

are the \( N(N - 1) \) horizontal edges (that we choose to be directed left right) and

\[
E_V = \{ e^V_{ij} = (v_{ij}, v_{i+1,j}) ; i = 1, 2, \ldots, N - 1 \text{ and } j = 1, 2, \ldots, N \}
\]

are the \( N(N - 1) \) vertical edges (that we choose to be directed top down). So we have that \( \dim V = N^2 \) and \( \dim E = 2N (N - 1) \). Let us denote by \( S_V \) and \( S_E \) the set of real-valued functions on \( V \) and \( E \) respectively. Next we consider the analogue of the gradient operator on the graph, i.e., \( \text{grad} : S_V \rightarrow S_E \) which maps function \( f \in S_V \) to function \( \text{grad} f \in S_E \) defined as (see Figure 2)

\[
(\text{grad} f) \left( e^H_{ij} \right) = f \left( v_{i,j+1} \right) - f \left( v_{ij} \right), i = 1, 2, \ldots, N \text{ and } j = 1, 2, \ldots, N - 1;
\]

\[
(\text{grad} f) \left( e^V_{ij} \right) = f \left( v_{i+1,j} \right) - f \left( v_{ij} \right), i = 1, 2, \ldots, N - 1 \text{ and } j = 1, 2, \ldots, N.
\]

In this framework, the observed image \( f_{ob} \in S_N \) can be considered as an element of \( S_V \), and the functional \( f \) can be written as

\[
L_{2,1} \left( t, f_{ob}; L^2, BV \right) = \inf_{g \in S_V} \left( \frac{1}{2N^2} \| f_{ob} - g \|_{L^2(S_V)}^2 + \frac{t}{N} \| \text{grad} g \|_{L^1(S_E)} \right),
\]

(6)
where
\[ \|f\|_{L^2(S_V)} = \left( \sum_{v \in V} (f(v))^2 \right)^{\frac{1}{2}}, \|h\|_{L^1(S_E)} = \sum_{e \in E} |h(e)| = \sum_{e \in E_H} |h(e)| + \sum_{e \in E_V} |h(e)|. \]

Of course, exact minimizer of (6) coincides with exact minimizer of
\[ L_{2,1} \left( s, f_{ob}; L^2(S_V), BV(S_V) \right) = \inf_{g \in S_V} \left( \frac{1}{2} \|f_{ob} - g\|_{L^2(S_V)}^2 + s \|\text{grad } g\|_{L^1(S_E)} \right), \quad s = Nt. \]

This leads to the following analogue of the ROF model on a general finite directed and connected graph \( G = (V,E) \).

1.1. Formulation of the problem on general graph

Let \( G = (V,E) \) be a finite directed and connected graph with \( N \) vertices \( V = \{v_1, v_2, \ldots, v_N\} \) and \( M \) directed edges \( E = \{e_1, e_2, \ldots, e_M\} \), where each edge is determined by a pair of two vertices \( e = (v_i, v_j) \) for some \( i,j \in \{1,2,\ldots,N\} \). Let \( S_V = \{f : f : V \rightarrow \mathbb{R}\} \) denote the set of of real–valued functions defined on the vertices and \( S_E = \{g : g : E \rightarrow \mathbb{R}\} \) denote the set of of real–valued functions defined on the edges. Then the analogue of ROF model on the graph \( G \) can be formulated as follows:

**Problem 1.1.** Suppose that we know function \( f_{ob} \in S_V \). For given \( t > 0 \), find exact minimizer of the functional
\[ L_{2,1} \left( t, f_{ob}; L^2(S_V), BV(S_V) \right) = \inf_{g \in S_V} \left( \frac{1}{2} \|f_{ob} - g\|_{L^2(S_V)}^2 + t \|\text{grad } g\|_{BV(S_V)} \right), \]

where
\[ \|f\|_{L^2(S_V)} = \left( \sum_{v \in V} (f(v))^2 \right)^{\frac{1}{2}}, \|f\|_{BV(S_V)} = \|\text{grad } f\|_{L^1(S_E)}; \|h\|_{L^1(S_E)} = \sum_{e \in E} |h(e)|, \]
and operator \( \text{grad} : S_V \rightarrow S_E \) is defined by the formula
\[ (\text{grad } f)(e) = f(v_j) - f(v_i) \text{ if } e = (v_i, v_j). \]

In this paper we will suggest a fast reiterative algorithm that constructs exact minimizer of (8). We would like to note that the case of a general graph could be particularly useful when instead of usually considered rectangular domain we have some manifold, for example map of the Earth is an image on the sphere.

2. Description of the dual space to \( BV(S_V) \)

It was shown in [2] that exact minimizer for the \( L_{2,1} \)-functional
\[ L_{2,1} \left( t, f_{ob}; L^2, X \right) = \inf_{g \in X} \left( \frac{1}{2} \|f_{ob} - g\|_{L^2}^2 + t \|g\|_{X} \right) \]
for the couple \((L^2, X)\), where space \( L^2 \) is defined by the standard Euclidean norm \( \| \cdot \|_2 \), \( X \) is a Banach space on \( \mathbb{R}^n \) and \( t \) is a given positive parameter, is equal to the difference between \( f_{ob} \) and the nearest element to \( f_{ob} \) of the ball of radius \( t > 0 \) of the space \( X^* \). Note that proofs in [2] are also true when \( X \) is equiped with seminorm. So to construct exact minimizer for the \( L_{2,1} \)-functional for the couple \((L^2(S_V), BV(S_V))\) we first need to describe the ball of radius \( t > 0 \) of the space \( BV^*(S_V) \) with norm defined by
\[ \|h\|_{BV^*(S_V)} = \sup_{\|f\|_{BV(S_V)} \leq 1} \langle h, f \rangle_{S_V}, \text{ where } \langle h, f \rangle_{S_V} = \sum_{v \in V} h(v)f(v). \]

**Remark 1.** Note that \( \|f\|_{BV(S_V)} = 0 \) if and only if \( f = C \) for some constant \( C \in \mathbb{R} \). Therefore
\[ \|h\|_{BV^*(S_V)} = \left\{ \begin{array}{ll} \sup_{\|f\|_{BV(S_V)} \leq 1} \langle h, f \rangle_{S_V}, & \text{if } \sum_{v \in V} h(v) = 0; \\
+\infty, & \text{otherwise.} \end{array} \right. \]
In order to formulate the main result of this section, let us consider divergence operator on the graph, i.e. the operator \( \text{div} : S_E \to S_V \) defined by

\[
(\text{div} g)(v) = \sum_{i: (v, \nu) \in E} g((v, \nu)) - \sum_{j: (v, \nu) \in E} g((v, \nu)).
\] (11)

We can now formulate the result

**Theorem 2.1.** The unit ball of the space \( BV^*(S_V) \) is equal to the image of the unit ball of the space \( \ell^\infty(S_E) \) under the operator \( \text{div} \), i.e.,

\[
B_{BV^*(S_V)} = \text{div} \left( B_{\ell^\infty(S_E)} \right).
\]

The proof of Theorem 2.1 is based on two elementary lemmas. The first one is

**Lemma 1.** The operator \( \text{div} : S_E \to S_V \) is conjugate to the operator \( \text{grad} : S_V \to S_E \), i.e.

\[
\langle \text{div} g, f \rangle_{S_V} = \langle g, \text{grad} f \rangle_{S_E},
\]

where \( \langle f_1, f_2 \rangle_{S_V} = \sum_{v \in V} f_1(v) f_2(v) \) and \( \langle g_1, g_2 \rangle_{S_E} = \sum_{e \in E} g_1(e) g_2(e) \).

**Proof.** The expression \( \langle \text{grad} f, g \rangle_{S_E} \) is, by definition of the operator \( \text{grad} \), equal to

\[
\langle \text{grad} f, g \rangle_{S_E} = \sum_{e = (v, \nu) \in E} \left[ f(v) - f(\nu) \right] g(e) = \sum_{e = (v, \nu) \in E} f(v) g((v, \nu)) - \sum_{e = (v, \nu) \in E} f(\nu) g((v, \nu)).
\]

\[
= \sum_{v \in V} \left[ \sum_{i: (v, \nu) \in E} f(\nu) g((v, \nu)) - \sum_{j: (v, \nu) \in E} f(v) g((v, \nu)) \right].
\]

The expression \( \langle f, \text{div} g \rangle_{S_V} \) is, by definition of the operator \( \text{div} \), equal to

\[
\langle f, \text{div} g \rangle_{S_V} = \sum_{v \in V} f(v) (\text{div} g)(v) = \sum_{v \in V} f(v) \left[ \sum_{i: (v, \nu) \in E} g((v, \nu)) - \sum_{j: (v, \nu) \in E} g((v, \nu)) \right].
\]

We conclude that

\[
\langle \text{div} g, f \rangle_{S_V} = \langle g, \text{grad} f \rangle_{S_E}.
\]

For the statement of the second lemma, we will need the following definition

**Definition 2.1.** Let \( X \) be a Banach space and let \( V \) be a subspace of \( X \). The annihilator of \( V \) denoted \( \text{Ann} (V) \) is the set of bounded linear functionals that vanish on \( V \). That is the set defined by

\[
\text{Ann} (V) = \{ x^* \in X^* : \langle x^*, v \rangle = 0, \text{ for all } v \in V \},
\] (12)

where \( X^* \) is the dual space of \( X \).

**Lemma 2.** Let \( X \) be a Banach space with dual space \( X^* \) and let \( V \) be a finite dimensional subspace of \( X \) and \( \text{Ann} (V) \) be the annihilator of \( V \). Let \( x_0 \in X \). Then

\[
\inf_{v \in V} \| x_0 + v \|_X = \sup_{x^* \in B_{X^*} \cap \text{Ann}(V)} \langle x^*, x_0 \rangle,
\] (13)

where \( B_{X^*} \) is the unit ball of \( X^* \).
Proof. The case \(x_0 \in V\) is obvious. From now on we suppose \(x_0 \notin V\). Let us take an arbitrary \(x^* \in B_{X^*} \cap \text{Ann}(V)\) and \(v \in V\). Since \(x^* \in \text{Ann}(V)\) (i.e., \(\langle x^*, v \rangle = 0, \forall v \in V\)) then we have
\[
\langle x^*, x_0 \rangle = \langle x^*, x_0 + v \rangle \leq \|x^*\|_{X^*} \|x_0 + v\|_X, \forall v \in V.
\]
Since \(x^* \in B_{X^*}\) (i.e., \(\|x^*\|_{X^*} \leq 1\)) it follows that
\[
\langle x^*, x_0 \rangle \leq \|x_0 + v\|_X, \forall v \in V.
\]
Therefore since \(x^* \in B_{X^*} \cap \text{Ann}(V)\) and \(v \in V\) were arbitrary, we have that
\[
\inf_{v \in V} \|x_0 + v\|_X \geq \sup_{x^* \in B_{X^*} \cap \text{Ann}(V)} \langle x^*, x_0 \rangle.
\]
(14)

In order to prove the reverse inequality let us consider the space \(W\), which is the algebraic sum between the span of \(x_0\) and the space \(V\):
\[
W = \{x_0\} + V = \{w \in X : w = \lambda x_0 + v, v \in V \text{ and } \lambda \in \mathbb{R}\},
\]
(15)
and take \(v_0 \in V\) such that \(\inf_{v \in V} \|x_0 + v\|_X = \|x_0 + v_0\|_X\). The existence of such \(v_0\) follows from the assumption that \(V\) is a finite dimensional subspace of \(X\). Without loss of generality we can assume that \(\|x_0 + v_0\|_X = 1\). Since \(W\) is a normed vector space by itself, it is possible to consider its dual space. Moreover \(V\) is a linear subspace of \(W\) and then by the Hahn-Banach Theorem, since \(x_0 + v_0 \in W\) and \(x_0 + v_0 \notin V\) there exists a bounded linear functional \(x^*_0 \in W^*\) such that
\[
\langle x^*_0, v \rangle = 0 \text{ for all } v \in V, \text{ and } \langle x^*_0, x_0 + v_0 \rangle = 1.
\]
(16)

It follows that
\[
x^*_0 \in \text{Ann}(V) \text{ and } \langle x^*_0, x_0 \rangle = 1.
\]
(17)

Let us now investigate the action of \(x^*_0\) on \(W\). Let \(\lambda \in \mathbb{R}, v \in V\) and let \(w = \lambda x_0 + v\) be an element of \(W\). Then we have
\[
\langle x^*_0, w \rangle = \langle x^*_0, \lambda x_0 + v \rangle = \langle x^*_0, \lambda x_0 + \lambda v_0 + v - \lambda v_0 \rangle = \lambda \langle x^*_0, x_0 + v_0 \rangle + \langle x^*_0, v - \lambda v_0 \rangle = \lambda,
\]
(18)
because \(\langle x^*_0, x_0 + v_0 \rangle = 1\) and \(\langle x^*_0, v - \lambda v_0 \rangle = 0\) since \(x^*_0 \in \text{Ann}(V)\) and \(v - \lambda v_0 \notin V\). Let us now describe the unit ball \(B_W\) of \(W\). Suppose that \(w = \lambda x_0 + v \in B_W\) where \(\lambda \neq 0\). We have that
\[
1 \geq \|w\|_X = \|\lambda x_0 + v\|_X = |\lambda| \|x_0 + \frac{v}{\lambda}\|_X \geq |\lambda| \|x_0 + v_0\|_X = |\lambda|.
\]
(19)

Therefore \(w = \lambda x_0 + v \in B_W\) implies that \(|\lambda| \leq 1\). The \(W^*\)- norm of the functional \(x^*_0\) is by definition
\[
\|x^*_0\|_{W^*} = \sup_{w \in B_W} \langle x^*_0, w \rangle.
\]
From (18) and (19), it follows that
\[
\|x^*_0\|_{W^*} = \sup_{w \in B_W} \lambda = \sup_{|\lambda| \leq 1} \lambda = 1.
\]
(20)

By invoking the Hahn-Banach theorem, we can extend the functional \(x^*_0\):
\[
\exists \tilde{x}^*_0 \in X^* \text{ such that } \tilde{x}^*_0|_W = x^*_0 \text{ and } \|\tilde{x}^*_0\|_{X^*} = \|x^*_0\|_{W^*} = 1.
\]
(21)

From (17) and (21) we conclude that
\[
\tilde{x}^*_0 \in B_{X^*} \cap \text{Ann}(V).
\]
It follows that
\[
\inf_{v \in V} \| x_0 + v \|_X = \| x_0 + v_0 \|_X = \langle x^*_0, x_0 + v_0 \rangle = \langle x^*_0, x_0 \rangle = \sum_{v \in B_X \cap \text{Ann}(V)} (x^*, x_0).
\] (22)

Putting (14) and (22) together, we obtain
\[
\inf_{v \in V} \| x_0 + v \|_X = \sup_{x^* \in B_X \cap \text{Ann}(V)} \langle x^*, x_0 \rangle.
\]

**Proof of Theorem 2.1** The operator grad has a kernel which is given by
\[
\ker(\text{grad}) = \{ f \in S_V : f = C, \text{ for some } C \in \mathbb{R} \},
\] (23)
and the annihilator (or orthogonal complement) of the ker(\text{grad}) is given by
\[
\text{Ann}(\ker(\text{grad})) = \left\{ F \in S_V : \sum_{v \in V} F(v) = 0 \right\}.
\] (24)

Since from Lemma 1, \text{div} = (\text{grad})^*, where (\text{grad})^* is the conjugate operator of grad, then it is a simple result from linear algebra that
\[
\text{Im}(\text{div}) = \text{Ann}(\ker(\text{grad})) \text{ and } \text{Im}(\text{grad}) = \text{Ann}(\ker(\text{div})).
\] (25)

From Remark 1 note that if \( \sum_{v \in V} h(v) \neq 0 \) then \( \| h \|_{BV^*(S_V)} = +\infty \), so to avoid this case we will restrict ourselves on functions \( h \in \text{Ann}(\ker(\text{grad})) \). Then from relations (24) and (25), we conclude that \( BV^*(S_V) = \text{Im}(\text{div}) \). Therefore for all \( h \in BV^*(S_V) \), there exists at least one \( g \in SE \) such that \( h = \text{div} g \).

So let \( h \in BV^*(S_V) \) and fix \( g_0 \in SE \) such that \( h = \text{div} g_0 \). Now let us consider the infimum of \( \| g \|_{\ell^\infty(SE)} \) over all functions \( g \) in the fiber of \( h \in BV^*(S_V) \) under the operator div.

Put \( U = \ker(\text{div}) \). Then we have \( \text{Ann}(U) = \text{Im}(\text{grad}) \). By applying Lemma 2 we have that
\[
\inf_{h = \text{div} g} \| g \|_{\ell^\infty(SE)} = \inf_{\{ u \in \ker(\text{div}) \}} \| g_0 + u \|_{\ell^\infty(SE)}.
\]

From Lemma 1, \( h = \text{div} g_0 \). (9) and (10) follows that
\[
\inf_{h = \text{div} g} \| g \|_{\ell^\infty(SE)} = \sup_{\| f \|_{\ell^1(SE)} \leq 1} \langle f, g_0 \rangle_{SE} = \sup_{\| f \|_{\ell^1(SE)} \leq 1} \langle \text{grad} f, g_0 \rangle_{SE}.
\]

It means that
\[
\| h \|_{BV^*(S_V)} \leq 1 \text{ if and only if } \inf_{h = \text{div} g} \| g \|_{\ell^\infty(SE)} \leq 1.
\] (26)

Note that the infimum is attained because ker(\text{div}) is a finite dimensional subspace of \( SE \). Therefore there exists an element
\[
g_h \in g_0 + \ker(\text{div}), \text{ such that } \| g_h \|_{\ell^\infty(SE)} \leq 1 \text{ and } \text{div} g_h = h.
\]

We conclude that
\[
B_{\ell^\infty(SE)} = \text{div} \left( B_{\ell^\infty(SE)} \right).
\]
2.1. Corollary for the discrete ROF model

As a corollary of Theorem 2.1 we will have the following result (for a reminder of space \(S_N\), see expression (3)).

**Corollary 1.** Let \(h \in S_N\). Then \(\|h\|_{BV^*(S_N)} \leq 1\) if and only if \(h\) can be decomposed as \(h = h_1 + h_2\), with \(h_1, h_2 \in S_N\) such that

\[
\sup_{0 \leq x \leq 1} \left| \int_0^x h_1(s,y) \, ds \right| \leq 1 \quad \text{and} \quad \int_0^1 h_1(s,y) \, ds = 0, \text{ for all } y \in [0,1];
\]

and

\[
\sup_{0 \leq y \leq 1} \left| \int_0^y h_2(x,t) \, dt \right| \leq 1 \quad \text{and} \quad \int_0^1 h_2(x,t) \, dt = 0, \text{ for all } x \in [0,1].
\]

**Proof.** Let us take \(h \in S_N\) such that \(\|h\|_{BV^*(S_N)} \leq 1\). By interpretation of the space \(S_N\) on the graph, we have from Theorem 2.1 that

\[
\|h\|_{BV^*(S_N)} \leq 1 \quad \text{if and only if} \quad h = \text{div} \, g \quad \text{for some} \quad g \in S_E \quad \text{such that} \quad \|g\|_{\ell^\infty(S_E)} \leq N,
\]

where

\[
h(v_{ij}) = (\text{div} \, g)(v_{ij}) = \left[ g(e_{i,j-1}^H) - g(e_{ij}^H) \right] + \left[ g(e_{i-1,j}^V) - g(e_{ij}^V) \right], \quad i,j = 1, \ldots, N,
\]

with the assumption \(g(e_{i,0}^H) = g(e_{0,j}^V) = 0\). It follows that

\[
h = h_1 + h_2,
\]

where functions \(h_1, h_2 \in S_V\) are defined as follows

\[
h_1(v_{ij}) = \left[ g(e_{i,j-1}^H) - g(e_{ij}^H) \right] \quad \text{and} \quad h_2(v_{ij}) = \left[ g(e_{i-1,j}^V) - g(e_{ij}^V) \right]. \quad (27)
\]

From (27) we deduce that

\[
\begin{align*}
g(e_{i,0}^H) &= -h_1(v_{i1}); \\
g(e_{i1}^H) &= - (h_1(v_{i1}) + h_1(v_{i2})); \\
& \vdots \\
g(e_{i,N-1}^H) &= - (h_1(v_{i1}) + h_1(v_{i2}) + \ldots + h_1(v_{i,N-1})); \\
g(e_{i,N}^H) &= - (h_1(v_{i1}) + h_1(v_{i2}) + \ldots + h_1(v_{i,N-1}) + h_1(v_{iN})).
\end{align*}
\]

From the first \(N - 1\) equations of (28), we have that

\[
\sup_{1 \leq k \leq N-1} \left| g(e_{ik}^H) \right| = \sup_{1 \leq k \leq N-1} \left| \sum_{ij=1}^k h_1(v_{ij}) \right|, \quad \forall i = 1, 2, \ldots, N. \quad (29)
\]

From the the last equations of (28), we deduce that

\[
0 = g(e_{iN}^H) = - \sum_{j=1}^N h_1(v_{ij}), \quad \forall i = 1, 2, \ldots, N. \quad (30)
\]

From the correspondance between \(S_V\) and space \(S_N\), the expression (29) implies that

\[
\sup_{1 \leq k \leq N-1} \left| g(e_{ik}^H) \right| = \sup_{1 \leq k \leq N-1} N \left| \int_0^k h_1(s,y) \, ds \right| = \sup_{0 \leq x \leq 1} N \left| \int_0^x h_1(s,y) \, ds \right|, \forall y \in [0,1]. \quad (31)
\]

Since \(\|g\|_{\ell^\infty(S_E)} \leq N\), it follows from (31) that

\[
\sup_{0 \leq x \leq 1} N \left| \int_0^x h_1(s,y) \, ds \right| \leq N, \quad \forall y \in [0,1].
\]
Therefore

\[
\sup_{0 \leq x \leq 1} \left| \int_0^x h_1(s, y) \, ds \right| \leq 1, \quad \forall y \in [0, 1].
\]

From (30), we have that

\[
\sum_{j=1}^{N} h_1(v_{ij}) = N \int_0^N h_1(s, y) \, ds = 0, \quad \forall y \in [0, 1].
\]

Therefore

\[
\int_0^1 h_1(s, y) \, ds = 0, \quad \forall y \in [0, 1].
\]

We conclude that

\[
\sup_{0 \leq x \leq 1} \left| \int_0^x h_1(s, y) \, ds \right| \leq 1 \text{ and } \int_0^1 h_1(s, y) \, ds = 0, \text{ for all } y \in [0, 1].
\]

Similar arguments for the function \(h_2\) and the function \(g\) acting on the edges \(e_{ij}\) will yield that

\[
\sup_{0 \leq y \leq 1} \left| \int_0^y h_2(x, t) \, dt \right| \leq 1 \text{ and } \int_0^1 h_2(x, t) \, dt = 0, \text{ for all } x \in [0, 1].
\]

### 3. Algorithm for construction of exact minimizer

Below we present a fast reiterative algorithm for the actual construction of the element \(f_{opt,t}\). This algorithm is a specialized version of an algorithm constructed in [3] for problems concerning a weighted divergence operator on a general finite directed graph.

As was mentioned in the previous section, we know from [2] that the exact minimizer \(f_{opt,t}\) for the \(L_{2,1}\)-functional

\[
L_{2,1} \left( t, f_{ob}; \ell^2(S_V), BV(S_V) \right) = \inf_{g \in BV(S_V)} \left( \frac{1}{2} \| f_{ob} - g \|_{\ell^2(S_V)}^2 + t \| g \|_{BV(S_V)} \right)
\]

is given by

\[
f_{opt,t} = f_{ob} - \tilde{h}, \quad (32)
\]

where \(\tilde{h}\) is the nearest element to \(f_{ob}\) in the metric of \(\ell^2(S_V)\), in the ball \(tBV(S_V)\), i.e.

\[
\inf_{h \in tBV(S_V)} \| f_{ob} - h \|_{\ell^2(S_V)} = \| f_{ob} - \tilde{h} \|_{\ell^2(S_V)}. \quad (33)
\]

Next, from Theorem [2.1] we know that \(tBV(S_V) = t \text{div} \left( B_{\ell^\infty(S_E)} \right)\) for \(t > 0\). The proposed algorithm constructs \(\tilde{h}\) through a sequence of elements \(g_n \in tB_{\ell^\infty(S_E)}\) such that

\[
\text{div}(g_n) \rightarrow \tilde{h} \text{ as } n \rightarrow +\infty \text{ in the metric of } \ell^2(S_V).
\]
3.1. Algorithm

The algorithm consists of several steps outlined in this section. Let \( f_{ob} \in S_V \) on \( G = (V = \{v_1, \ldots, v_N\}, E = \{e_1, \ldots, e_M\}) \), a regularization parameter \( t \) and a maximum number of iterations \( N_{iter} \) be given. Set

\[
e_k = (v_i, v_j) \in E, \quad k = 1, 2, \ldots, M; \quad \text{for some } i, j \in \{1, 2, \ldots, N\}.
\]

Define the operator \( T \) as follows:

\[
T = T_MT_{M-1}T_{M-2} \ldots T_2T_1,
\]

where for \( k = 1, 2, \ldots, M, \quad T_k : tB_{l^\infty(S_V)} \rightarrow tB_{l^\infty(S_V)} \) is defined as follows:

\[
(T_k g)(e) = \begin{cases} 
Kg(e_k) & \text{if } Kg(e_k) \in [-t, +t]; \\
-t & \text{if } Kg(e_k) < -t; \\
+t & \text{if } Kg(e_k) > +t.
\end{cases}
\]

where

\[
Kg(e_k) = \left[ f_{ob}(v_j) - (\nabla \chi_{e_k}) (v_j) \right] - \left[ f_{ob}(v_i) - (\nabla \chi_{e_k})(v_i) \right].
\]

and

\[
\begin{align*}
(\nabla \chi_{e_k})(v_i) &= (\nabla g)(v_i) + g(e_k); \\
(\nabla \chi_{e_k})(v_j) &= (\nabla g)(v_j) - g(e_k);
\end{align*}
\]

and

\[
(\nabla \chi_{e_k})(v_\ell) = (\nabla g)(v_\ell), \quad \forall \ell \neq i, j.
\]

With these notions at our disposal, we now outline the steps of the algorithm.

Step 1. Take \( g_0 = 0 \), or choose any \( g_0 \in tB_{l^\infty(S_V)} \).

Step 2. Calculate \( g = Tg_0 \). i.e., calculate \( (Tg_0)(e_k) \) for \( k = 1, 2, \ldots, M \). If \( g = g_0 \) then take \( h = \nabla g(0) \), otherwise go to Step 3.

Step 3. Put \( g_0 = g \) and go to Step 2.

We continue this process applying the operator \( T \) to the new element \( g \in tB_{l^\infty(S_V)} \) generating the sequence of elements \( g_0, g_1 = Tg_0, g_2 = Tg_1, \ldots, g_n = Tg_{n-1} \) with \( g_n \in tB_{l^\infty(S_V)}, n = 0, 1, 2, \ldots \) until the maximum number of iterations \( N_{iter} \) is reached. In the next Section we will show that

\[
\nabla (g_n) \rightarrow h \quad \text{as } n \rightarrow +\infty \text{ in the metric of } l^2(S_V).
\]

To prove the convergence of the proposed algorithm we need the following proposition

Proposition 3.1. Let \( \tilde{h} \) be the minimizer defined by \( \Box \). The operator \( T \) is continuous and satisfies the following two conditions

(a) For any \( g \in tB_{l^\infty(S_V)} \), \( \nabla g = \tilde{h} \) if and only if \( Tg = g \);

(b) For any \( g \in tB_{l^\infty(S_V)} \), if \( \nabla g \neq \tilde{h} \) and

\[
\|f_{ob} - \nabla (Tg)\|_{l^2(S_V)} < \|f_{ob} - \nabla g\|_{l^2(S_V)}.
\]

Proof. It is clear from definition that small changes of \( g \in tB_{l^\infty(S_V)} \) leads to small changes of any operator \( T_k \) and therefore the operator \( T \) is continuous as a product of continuous operators.

We now prove (a). Assume first that \( g \in tB_{l^\infty(S_V)} \) is such that \( \nabla g = \tilde{h} \). Let \( e_k = (v_i, v_j) \in E \). We note that \( g(e_k) \) appears only in the following two terms of \( \|f_{ob} - \nabla g\|_{l^2(S_V)}^2 \):

\[
\begin{align*}
(f_{ob}(v_j) - (\nabla g)(v_j))^2 + (f_{ob}(v_i) - (\nabla g)(v_i))^2 &= \\
\left[ f_{ob}(v_j) - (\nabla \chi_{e_k})(v_j) - g(e_k) \right]^2 + \left[ f_{ob}(v_i) - (\nabla \chi_{e_k})(v_i) + g(e_k) \right]^2.
\end{align*}
\]
Since \( \text{div} \ g = \tilde{h}, \ g(e_k) \) in particular must minimize \( \omega \) in the interval \([-t, t]\). We note that by the Jensen’s inequality
\[
\omega(g(e_k)) = \left[ f_{ob}(v_i) - (\text{div}_w g)(v_i) - g(e_k) \right]^2 + \left[ f_{ob}(v_i) - (\text{div}_w g)(v_i) + g(e_k) \right]^2 \geq 2 \left( \frac{f_{ob}(v_i) - (\text{div}_w g)(v_i)}{2} \right)^2 + \left( \frac{f_{ob}(v_i) - (\text{div}_w g)(v_i)}{2} \right)^2 = 2 \left( \frac{f_{ob}(v_i) - (\text{div}_w g)(v_i)}{2} \right)^2 + \left( \frac{f_{ob}(v_i) - (\text{div}_w g)(v_i)}{2} \right)^2.
\]
with equality if and only if
\[
f_{ob}(v_i) - (\text{div}_w g)(v_i) - g(e_k) = f_{ob}(v_i) - (\text{div}_w g)(v_i) + g(e_k),
\]
i.e. when
\[
g(e_k) = \frac{f_{ob}(v_i) - (\text{div}_w g)(v_i)}{2} - \frac{f_{ob}(v_i) - (\text{div}_w g)(v_i)}{2} = (K_g)(e_k).
\]
Moreover, \( \omega(x) \) is strictly convex and therefore strictly decreasing for \( x < (K_g)(e_k) \) and strictly increasing for \( x > (K_g)(e_k) \). So the minimal value of \( \omega(x) \) on the interval \([-t, t]\) is only attained at i) the point \((K_g)(e_k)\) if \((K_g)(e_k) \in [-t, t]\), ii) the point \(-t\) if \((K_g)(e_k) < -t\) and iii) the point \(t\) if \((K_g)(e_k) > t\). The assumption \( \text{div} \ g = \tilde{h} \) then implies that \( g(e_k) \) must be the nearest point in the interval \([-t, t]\) to \((K_g)(e_k)\), i.e. \( T_g(e_k) = g(e_k) \). Since \( e_k \in E \) was arbitrary it follows that \( T_{kg} (e_k) = g(e_k), k = 1, \ldots, M \) which implies \( T_{kg} g = g, k = 1, \ldots, M \) and therefore \( T_g = g \).

We now show the other direction of (a). Let us assume that \( g \in tB_{\ell^\infty(S_E)} \) and \( T_g = g \). Then for any edge \( e \in E, g(e) \) coincides with the point of the interval \([-t, t]\) which is nearest to \((K_g)(e)\). As \( ||f_{ob} - \text{div} g||_{\ell^2(S_V)} \) is a convex function of \( g \) on \( tB_{\ell^\infty(S_E)} \), to show this direction it is enough to show that \( g \) minimizes \( ||f_{ob} - \text{div} g||_{\ell^2(S_V)} \) locally, i.e. it is enough to show that for some small \( \varepsilon > 0 \) we have
\[
||f_{ob} - \text{div} g||_{\ell^2(S_V)} = \inf_{f \in R_\varepsilon} ||f_{ob} - \text{div} f||_{\ell^2(S_V)},
\]
where \( R_\varepsilon \) is the tubular set given by
\[
R_\varepsilon = \left\{ f \in tB_{\ell^\infty(S_E)} : ||g - f||_{\ell^\infty(S_E)} \leq \varepsilon \right\}.
\]
Note that for any \( f \in R_\varepsilon \) and \( e \in E \) we have
\[
f(e) \in [-t, t] \cap [g(e) - \varepsilon, g(e) + \varepsilon].
\]
The set \( R_\varepsilon \) is a closed convex subset of \( S_E \), therefore there exists a function \( f_\varepsilon \in R_\varepsilon \) such that
\[
||f_{ob} - \text{div} f_\varepsilon||_{\ell^2(S_V)} = \inf_{f \in R_\varepsilon} ||f_{ob} - \text{div} f||_{\ell^2(S_V)}
\]
so we need to prove that
\[
||f_{ob} - \text{div} g||_{\ell^2(S_V)} = ||f_{ob} - \text{div} f_\varepsilon||_{\ell^2(S_V)}.
\]
We first note that it follows from the necessity direction proved above that for any edge \( e \in E, f_\varepsilon(e) \) will coincide with the point of the interval \([-t, t] \cap [g(e) - \varepsilon, g(e) + \varepsilon] \) which is nearest to \((K_{f_\varepsilon})(e)\).

Let us now decompose the set of edges \( E \) into two parts, the first part which we denote by \( \Omega_g \) consists of the edges for which \((K_g)(e)\) does not belong to the interval \([-t, t]\), i.e.
\[
\Omega_g = \left\{ e \in E : (K_g)(e) \notin [-t, t] \right\}.
\]
As \( g(e) \) is the nearest point in the interval \([-t, t] \) to \((K_g)(e)\) we have
\[
g(e) = \begin{cases} -t, & \text{if } K_g(e) < -t; \\ +t, & \text{if } K_g(e) > +t. \end{cases} \quad \text{for edges } e \in \Omega_g.
\]
If the number \( \varepsilon > 0 \) is small enough, it follows from \( \| g - f \|_{L^2(\mathcal{S}_V)} \leq \varepsilon \) that on \( e \in \Omega_S \) where we have \( (Kg)(e) < -t \) we will also have \( (Kf_\varepsilon)(e) < -t \) and therefore \( f_\varepsilon(e) = -t = g(e) \). Analogously, if \( \varepsilon > 0 \) is small enough, on \( e \in \Omega_S \) where \( (Kg)(e) > t \) we will have \( (Kf_\varepsilon)(e) > t \) and therefore \( f_\varepsilon(e) = t = g(e) \). So we have proved that

\[
f_\varepsilon(e) = g(e) \quad \text{for all } e \in \Omega_S.
\]

The next step is to consider the graph \( G' = (V, E \setminus \Omega_S) \), i.e. the graph \( G \) with edges in \( \Omega_S \) removed. \( G' \) is the union of several connected components \( (V_k, E_k) \), \( k = 1, \ldots, \ell \) (we have \( V_1 \cup \cdots \cup V_\ell = V \) and \( E_1 \cup \cdots \cup E_\ell = E \setminus \Omega_S \)). Note that some of the graphs \((V_k, E_k)\) can consist just of one single vertex. For these graphs there is nothing to prove as \( E_k = \emptyset \).

Let us now consider a subgraph \( (V_k, E_k) \) where \( E_k \neq \emptyset \). On each \( e \in E_k \) we have \( (Kg)(e) \in [-t, t] \) and therefore \( g(e) = (Kg)(e) \), i.e. if \( e = (v_i, v_j) \) then

\[
g(e) = (Kg)(e) = \frac{f_{ob}(v_i) - (\text{div}_V g)(v_i)}{2} - \frac{f_{ob}(v_j) - (\text{div}_V g)(v_j)}{2},
\]
or equivalently

\[
f_{ob}(v_i) - (\text{div}_V g)(v_i) = f_{ob}(v_j) - (\text{div}_V g)(v_j).
\]

(Here it is important to note that operators \( K, \text{div} \) and \( \text{div}_V \) are considered in the original setting of \( G = (V, E) \).) Therefore, for all \( v \in V_k \) the values of

\[
f_{ob}(v) - (\text{div}_V g)(v)
\]

are equal. So we have

\[
\sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V g)(v)]^2 = |V_k| \left( \frac{\sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V g)(v)]}{|V_k|} \right)^2.
\]

(37)

For \( f_\varepsilon \) we can with Jensen’s inequality derive the corresponding inequality

\[
\sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V f_\varepsilon)(v)]^2 \geq |V_k| \left( \frac{\sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V f_\varepsilon)(v)]}{|V_k|} \right)^2.
\]

(38)

Now, note that flow on edges in \( E_k \) are cancelled in the sums

\[
\sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V g)(v)] \text{ and } \sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V f_\varepsilon)(v)].
\]

(39)

Therefore, only flows on edges in \( \Omega_S \) remain in these sums. It then follows from (36) that

\[
\sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V f_\varepsilon)(v)] = \sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V g)(v)].
\]

Therefore, taking into account (37) and (38), we obtain

\[
\sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V f_\varepsilon)(v)]^2 \geq \sum_{v \in V_k} [f_{ob}(v) - (\text{div}_V g)(v)]^2.
\]

Summing over all \( V_k \) then gives

\[
\| f_{ob} - \text{div}_V f_\varepsilon \|_{L^2(\mathcal{S}_V)}^2 \geq \| f_{ob} - \text{div}_V g \|_{L^2(\mathcal{S}_V)}^2.
\]

Hence, from the definition of \( f_\varepsilon \) we must have

\[
\| f_{ob} - \text{div}_V f_\varepsilon \|_{L^2(\mathcal{S}_V)}^2 = \| f_{ob} - \text{div}_V g \|_{L^2(\mathcal{S}_V)}^2.
\]

and (a) is therefore established.
To prove (b) note that for \( \forall g \in tB_{\ell^\infty(S_E)} \) the operators \( T_k, k = 1, \ldots, M, \) by definition (see also necessity part of (a)) satisfy
\[
\|f_{ob} - \text{div} (T_kg)\|_{\ell^2(S_V)} \leq \|f_{ob} - \text{div} \ g\|_{\ell^2(S_V)}
\]
with equality if and only if \( T_kg(\epsilon_k) = g(\epsilon_k) \). This implies for the operator \( T = TM_{M-1} \ldots T_1 \) that
\[
\|f_{ob} - \text{div} (Tg)\|_{\ell^2(S_V)} \leq \|f_{ob} - \text{div} \ g\|_{\ell^2(S_V)}
\]
with equality if and only if \( Tg = g \) which in turn by condition (a) is equivalent to \( \text{div} g = \tilde{h} \). Hence for any \( g \in tB_{\ell^\infty(S_E)} \), if \( \text{div} g \neq \tilde{h} \) then
\[
\|f_{ob} - \text{div} (Tg)\|_{\ell^2(S_V)} < \|f_{ob} - \text{div} \ g\|_{\ell^2(S_V)},
\]
which establishes (b) and completes the proof. 

3.2. Convergence of the algorithm
With Proposition 3.1 at hand, we can now prove convergence of the Algorithm.

**Theorem 3.1.** Let \( \tilde{h} \) be the minimizer defined by (33), \( g \in tB_{\ell^\infty(S_E)} \) and let \( T \) be the operator constructed in Section 3.1. Then
\[
\text{div} (T^n g) \longrightarrow \tilde{h} \text{ as } n \rightarrow +\infty \text{ in the metric of } \ell^2(S_V).
\]

**Proof.** Let \( T \) be the operator constructed in Section 3.1. From Proposition 3.1, \( T : tB_{\ell^\infty(S_E)} \longrightarrow tB_{\ell^\infty(S_E)} \) is continuous and satisfies the conditions (a) and (b). Then
\[
T^n g \in tB_{\ell^\infty(S_E)}, \ \forall g \in tB_{\ell^\infty(S_E)}; \ n = 0, 1, 2, \ldots
\]
Since \( T \) satisfies conditions (a) and (b), the sequence of numbers \( \|f_{ob} - \text{div} (T^n g)\|_{\ell^2(S_V)} \) is monotonically decreasing and is bounded below by \( \inf_{g \in tB_{\ell^\infty(S_E)}} \|f_{ob} - \text{div} \ g\|_{\ell^2(S_V)} \). Therefore it converges. Let us now consider the sequence \( (T^n g)_{n \in \mathbb{N}} \subset tB_{\ell^\infty(S_E)} \). Since \( tB_{\ell^\infty(S_E)} \) is a compact set, then \( (T^n g)_{n \in \mathbb{N}} \) contains a convergent subsequence in \( tB_{\ell^\infty(S_E)} \), say \( (T^n g)_{k \in \mathbb{N}} \):
\[
\lim_{k \rightarrow \infty} (T^n g) = g_h \in tB_{\ell^\infty(S_E)}.
\]
Since \( T, \text{div} \) and \( \| \cdot \|_{\ell^2(S_V)} \) are continuous operators, we have
\[
\|f_{ob} - \text{div} (Tg_h)\|_{\ell^2(S_V)} = \lim_{k \rightarrow \infty} \|f_{ob} - \text{div} (T (T^n g))\|_{\ell^2(S_V)} = \lim_{k \rightarrow \infty} \|f_{ob} - \text{div} (T^n g)\|_{\ell^2(S_V)}
\]
From Proposition 3.1 and continuity of \( \text{div} \) and \( \| \cdot \|_{\ell^2(S_V)} \) it follows that
\[
\|f_{ob} - \text{div} (Tg_h)\|_{\ell^2(S_V)} = \lim_{k \rightarrow \infty} \|f_{ob} - \text{div} (T^n g_{k+1})\|_{\ell^2(S_V)} \geq \lim_{k \rightarrow \infty} \|f_{ob} - \text{div} (T^n g_{k+1})\|_{\ell^2(S_V)}
\]
Therefore by Proposition 3.1 we conclude that \( \text{div} g_h = \tilde{h} \). By continuity of \( \text{div} \) and \( \| \cdot \|_{\ell^2(S_V)} \) we have that
\[
\lim_{k \rightarrow \infty} \|f_{ob} - \text{div} (T^n g)\|_{\ell^2(S_V)} = \|f_{ob} - \text{div} \ g_h\|_{\ell^2(S_V)}.
\]
Now, since the subsequence \( \left( \|f_{ob} - \text{div} (T^n g)\|_{\ell^2(S_V)} \right)_{k \in \mathbb{N}} \) must converge to the same limit as \( \left( \|f_{ob} - \text{div} (T^n g)\|_{\ell^2(S_V)} \right)_{n \in \mathbb{N}} \), we conclude that
\[
\lim_{n \rightarrow \infty} \|f_{ob} - \text{div} (T^n g)\|_{\ell^2(S_V)} = \|f_{ob} - \text{div} \ g_h\|_{\ell^2(S_V)} = \|f_{ob} - \tilde{h}\|_{\ell^2(S_V)}.
\]
Therefore, as $\tilde{h}$ is the unique nearest element to $f_{\text{ob}}$ in $tB_{BV^1(S_V)}$ we conclude that

$$\text{div } (T^n g) \rightarrow \text{div } g_h = \tilde{h} \text{ as } n \rightarrow +\infty \text{ in the norm of } \ell^2(S_V).$$

\[\blacksquare\]

4. Tikhonov functional

In this section, we consider the Tikhonov functional in order to use it for comparison purposes in the next section. Let us consider the Tikhonov functional for the couple of Sobolev spaces $(L^2(\Omega), \dot{W}^{1,2}(\Omega))$ defined on a square domain $\Omega = [0,1]^2$ in $\mathbb{R}^2$. It is given by:

$$L_{2,2} \left(t, f_{\text{ob}}; L^2(\Omega), \dot{W}^{1,2}(\Omega)\right) = \inf_{g \in \dot{W}^{1,2}(\Omega)} \left( \frac{1}{2} \| f_{\text{ob}} - g \|^2_{L^2(\Omega)} + t \| g \|^2_{\dot{W}^{1,2}(\Omega)} \right),$$

where

$$\| f \|^2_{\dot{W}^{1,2}(\Omega)} = \left( \left\| \frac{\partial}{\partial x_1} f(x) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial}{\partial x_2} f(x) \right\|_{L^2(\Omega)}^2 \right)^{1/2},$$

and

$$\| f \|^2_{L^2(\Omega)} = \left( \int_0^1 \int_0^1 (f(x_1, x_2))^2 \, dx_1 dx_2 \right)^{1/2}.$$  \hfill(45)

We need to find the element $f_{\text{opt},t} \in \dot{W}^{1,2}(\Omega)$, exact minimizer for (43). That is $f_{\text{opt},t} \in \dot{W}^{1,2}(\Omega)$ such that

$$L_{2,2} \left(t, f_{\text{ob}}; L^2(\Omega), \dot{W}^{1,2}(\Omega)\right) = \frac{1}{2} \| f_{\text{ob}} - f_{\text{opt},t} \|^2_{L^2(\Omega)} + t \| f_{\text{opt},t} \|^2_{\dot{W}^{1,2}(\Omega)}.$$  \hfill(46)

The exact minimizer $f_{\text{opt},t} \in \dot{W}^{1,2}$ for (43) can be found directly by solving the corresponding optimality condition. However functions from $S_N$ don’t have derivative (44). So we need to look for an approximate smooth function for which (44) exists. We consider a finite dimensional subspace of $\dot{W}^{1,2}(\Omega)$ which consists of functions that can be represented as trigonometric polynomials of degree $\leq N$. To this end, we will carry the processing in the Fourier transform domain where the image is represented as a weighted sum of the complex exponentials. Afterwards, we apply the inverse Fourier transform to return to the spatial domain. We will consider

$$f_{\text{ob}}(n_1, n_2) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \hat{f}_{\text{ob}}(k, \ell) e^{\frac{2\pi i}{N} (kn_1 + \ell n_2)}, \quad n_1, n_2 = 0, 1, \ldots, N - 1,$$  \hfill(47)

and

$$g(n_1, n_2) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \hat{g}(k, \ell) e^{\frac{2\pi i}{N} (kn_1 + \ell n_2)}, \quad n_1, n_2 = 0, 1, \ldots, N - 1,$$  \hfill(48)

where

$$\hat{f}_{\text{ob}}(k, \ell) = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{\text{ob}}(n_1, n_2) e^{-\frac{2\pi i}{N} (kn_1 + \ell n_2)}, \quad k, \ell = 0, 1, \ldots, N - 1,$$  \hfill(49)

and

$$\hat{g}(k, \ell) = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} g(n_1, n_2) e^{-\frac{2\pi i}{N} (kn_1 + \ell n_2)}, \quad k, \ell = 0, 1, \ldots, N - 1.$$  \hfill(50)
Equations (49)-(50) correspond to the discrete Fourier transforms of the images defined over the discrete finite 2D grid of size $N \times N$ and equations (47)-(48) correspond to their inverses. We use $N$ samples in both directions and in both spatial and frequency domains. Note that the image is periodized along both dimensions with period $N$. In this formulation, the functional (43) becomes

$$\inf_{\hat{g}(k,\ell)} \left( \frac{1}{2} \sum_{k,\ell=0}^{N-1} \left| \hat{f}_{\text{ob}}(k,\ell) - \hat{g}(k,\ell) \right|^2 + t \left( \sum_{k,\ell=0}^{N-1} \frac{2\pi}{N} k \right)^2 \left| \hat{g}(k,\ell) \right|^2 + \sum_{k,\ell=0}^{N-1} \frac{2\pi}{N} \ell \left| \hat{g}(k,\ell) \right|^2 \right).$$

So we need to minimize each term inside the sum, i.e.,

$$\inf_{\hat{g}(k,\ell)} \left( \frac{1}{2} \left| \hat{f}_{\text{ob}}(k,\ell) - \hat{g}(k,\ell) \right|^2 + t \left( \frac{2\pi}{N} k \right)^2 \left| \hat{g}(k,\ell) \right|^2 + \frac{2\pi}{N} \ell \left| \hat{g}(k,\ell) \right|^2 \right), \quad k,\ell = 0,1,\ldots,N-1. \quad (51)$$

We will represent the complex numbers $\hat{f}_{\text{ob}}(k,\ell)$ and $\hat{g}(k,\ell)$ in their polar form, i.e.,

$$\hat{f}_{\text{ob}}(k,\ell) = r_{k,\ell} e^{i\phi_{k,\ell}} \quad \text{and} \quad \hat{g}(k,\ell) = \tilde{r}_{k,\ell} e^{i\phi_{k,\ell}}. \quad (52)$$

Note that $r_{k,\ell}$ and $\phi_{k,\ell}$, $k,\ell = 0,1,\ldots,N-1$ are known because $f_{\text{ob}}$ is given. We only need to solve for $\tilde{r}_{k,\ell}$’s. Putting (52) in (51), we need to solve

$$\inf_{\tilde{r}_{k,\ell}} \left( \frac{1}{2} \left( r_{k,\ell} - \tilde{r}_{k,\ell} \right)^2 + \frac{2\pi}{N} k^2 \left( k^2 + \ell^2 \right) \tilde{r}_{k,\ell}^2 \right), \quad k,\ell = 0,1,\ldots,N-1. \quad (53)$$

Taking derivative with respect to $\tilde{r}_{k,\ell}$, we see that the infimum is attained at $\tilde{r}_{k,\ell}$ which satisfies

$$- (r_{k,\ell} - \tilde{r}_{k,\ell}) + \frac{2\pi}{N} k^2 \left( k^2 + \ell^2 \right) \cdot 2\tilde{r}_{k,\ell} = 0, \quad k,\ell = 0,1,\ldots,N-1,$$

we have

$$\tilde{r}_{k,\ell} = \frac{r_{k,\ell}}{1 + 2 \left( \frac{2\pi}{N} k^2 \right) \left( k^2 + \ell^2 \right) / 2}, \quad k,\ell = 0,1,\ldots,N-1. \quad (54)$$

Therefore the exact minimizer $f_{\text{opt},t}$ is given by the formula

$$f_{\text{opt},t}(n_1,n_2) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \hat{f}_{\text{opt},t}(k,\ell) e^{\frac{2\pi i}{N}(kn_1+\ell n_2)}, \quad n_1, n_2 = 0,1,\ldots,N-1, \quad (55)$$

where

$$\hat{f}_{\text{opt},t}(k,\ell) = \frac{r_{k,\ell}}{1 + 2 \left( \frac{2\pi}{N} k^2 \right) \left( k^2 + \ell^2 \right)} e^{i\phi_{k,\ell}}.$$

The matlab routines fft2 and ifft2 are employed for a fast computation of (53).

5. Experimental results

In this section we will illustrate on some test images the performance of ROF model using our algorithm and we will compare with result obtained by using the Tikhonov functional described in the previous section. All the test observed images used have width 512 and height 512 pixels and the corresponding original image is provided for comparison purposes. In view of our analogue of ROF model on the graph (see [7]), note that the regularization parameter we use for ROF model is $s = Nt$, where $N = 512$ and $t$ is the regularization parameter appearing in the original model (see [1]). The original image used in Figures [3]-[4] is an 8–bit gray–scale image with intensity values ranging from 0 (black) to 255 (white). The original image used in Figures [5] is an artificial image with 4 parts of intensity values 180, 120, 90 and 50. The original image used in Figures [7]-[8] is an artificial binary image of intensity values 150 and 100. In all the cases the noisy image is the corrupted version of the original image where a Gaussian additive noise of
standard deviation $\varepsilon$ with different values of $\varepsilon$ is added to the original image. i.e, if we denote the original image by $f_*$, then the observed noisy image $f_{ob}$ is defined by

$$f_{ob} = f_* + \varepsilon \times \text{randn}(512,512),$$

where $\text{randn}(N,N)$ is a function returning a N–by–N matrix containing pseudorandom values drawn from the standard normal distribution as built in Matlab.

In Figures 4, 6 and 8 we have highlighted in white the parts of the reconstructed image $f_{opt,t}$ where $|f_{*ij} - f_{opt,t,ij}| \geq \alpha$, $i,j = 1,2,\ldots,N$, for different values of $\alpha$ in both reconstructions.

Figure 3: Lenna, the original is a $512 \times 512$ image with intensity values ranging from 0 to 255. Top left: Original image $f_*$. Top right: Noisy image $f_{ob} = f_* + \varepsilon \times \text{randn}(512,512)$, i.e image with Gaussian additive noise of standard deviation $\varepsilon = 23.6$. Bottom left: Tikhonov reconstruction using $t = 0.45$. Bottom right: ROF reconstruction using $s = 15.5$. 
Figure 4: Top: Residual image, i.e., the difference between the observed image and the reconstructed image $(f_{ob} - f_{opt})$. Top left: Residual from Tikhonov reconstruction. Top right: Residual from ROF reconstruction. Bottom: The white shows, matrix componentwise, parts of the reconstructed image $f_{opt}$ where $|f_{uj} - f_{opt,uj}| \geq 35$, $i,j = 1,2,\ldots,N$. Left: in Tikhonov reconstruction. Right: in ROF reconstruction.
Figure 5: Geometric features with stairs, the original is a $512 \times 512$ image with 4 regions of intensity values 180, 120, 90 and 50. Top left: Original image $f$; Top right: Noisy image $f_{ob} = f + \epsilon \times \text{randn}(512,512)$, i.e., image with Gaussian additive noise of standard deviation $\epsilon = 32.25$. Bottom left: Tikhonov reconstruction using $t = 1.59$; Bottom right: ROF reconstruction using $s = 46.6$. 
Figure 6: Geometric features with stairs. Top: Residual image, i.e., the difference between the observed image and the reconstructed image \( f_{ob} - f_{opt,t} \). Top left: Residual from Tikhonov reconstruction. Top right: Residual from ROF reconstruction. Bottom: The white shows, matrix componentwise, parts of the reconstructed image \( f_{opt,t} \) where \( |f_{opt,t}| \geq 20, i, j = 1,2,\ldots, N \). Left: in Tikhonov reconstruction. Right: in ROF reconstruction.
Figure 7: Geometric features with a cusp, the original is a $512 \times 512$ image with regions of intensity values 150 and 100. Top left: Original image $f$; Top right: Noisy image $f_{\delta} = f + \epsilon \times \text{randn}(512,512)$, i.e. image with Gaussian additive noise of standard deviation $\epsilon = 30.85$. Bottom left: Tikhonov reconstruction using $t = 4.37$. Bottom right: ROF reconstruction using $s = 36.5$. 
Figure 8: Geometric features with a cusp. Top: Residual image, i.e., the difference between the observed image and the reconstructed image \((f_{ob} - f_{opt,t})\). Top left: Residual from Tikhonov reconstruction. Top right: Residual from ROF reconstruction. Bottom: The white shows, matrix componentwise, parts of the reconstructed image \(f_{opt,t}\) where \(|f_{ij} - f_{opt,i,j}| \geq 20\), \(i, j = 1, 2, \ldots, N\). Left: in Tikhonov reconstruction. Right: in ROF reconstruction.
6. Final remarks and discussions

Since its appearance in 1992, the ROF model has received a large amount of popularity for its efficiency in regularizing images without smoothing the boundaries, and it has since been applied to a multitude of other imaging problems (see for example the book [4]). Some of the earlier works for minimizing the total variation based on dual formulation include [5], where the methods presented is based on removal of some of the singularity caused by the non-differentiability of the quantity \( \text{grad} f \) appearing in the regularization term. In 2004, Chambolle (see [6]) provided an algorithm related to [5] for minimizing the total variation of an image and proved its convergence. In the last few years image decomposition models into a piecewise-smooth and oscillating components that usually people refer to as cartoon and textures (or textures + noise) respectively, have received interest in the image processing community. For example \( f_{\text{opt},t} \in BV \) satisfying (2) is such that

\[
f_{\text{ob}} = (f_{\text{ob}} - f_{\text{opt},t}) + f_{\text{opt},t}.
\]

This is the decomposition of \( f_{\text{ob}} \) into the piecewise-smooth component \( f_{\text{opt},t} \in BV \) and the component \((f_{\text{ob}} - f_{\text{opt},t}) \in L^2\) which contains textures and noise. The original theoretical model for such an image decomposition was introduced in 2002 by Yves Meyer in [7] by using the total variation to model the piecewise-smooth component and an appropriate dual space named \( G \) which is the Banach space composed of the distributions \( f \) which can be written \( f = \partial_1 g_1 + \partial_2 g_2 = \text{div}(g) \), with \( g_1 \) and \( g_2 \in L^\infty(\Omega) \) with the following norm

\[
\| f \|_G = \inf \left\{ \| g \|_{L^\infty(\Omega;\mathbb{R}^2)} : f = \text{div}(g), \| g \|_{L^\infty(\Omega;\mathbb{R}^2)} = \text{ess sup}_{x \in \Omega} \sqrt{|g_1(x)|^2 + |g_2(x)|^2} \right\},
\]

to model the oscillating component. Some of the most known works proposed in the literature for numerically solving the Meyer’s model or its variants include for instance the works of [8] who proposed a model that splits the image into three components, a geometrical components modeled by the total variation, a texture component modeled by a negative Sobolev norm and a noise component modeled by a negative Besov norm; the works of [6, 5, 9, 10] that proposed an efficient projection algorithm to minimize the total variation. In 2006 [11] and [12] adapted Chambolle’s algorithm in cases of presence of an operator like a convolution kernel for example. Recently, [13] showed that the Bregman iteration is a very efficient and fast way to solve TV problems among other \( L^1 \)-regularized optimization problems and proposed a split Bregman method which they applied to the Rudin–Osher–Fatemi functional for image denoising and [14] designed an algorithm by using the Split Bregman iterations and the duality used by Chambolle to find the minimizer of a functional based on Meyer G–norm. Other works based on the Meyer’s G–norm include for example [15] and [16].

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References


