Master Thesis

Local Volatility Calibration on the Foreign Currency Option Market

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Abstract

In this thesis we develop and test a new method for interpolating and extrapolating prices of European options. The theoretical base originates from the local variance gamma model developed by Carr (2008), in which the local volatility model by Dupire (1994) is combined with the variance gamma model by Madan and Seneta (1990). By solving a simplified version of the Dupire equation under the assumption of a continuous five parameter diffusion term, we derive a parameterization defined for strikes in an interval of arbitrary size. The parameterization produces positive option prices which satisfy both conditions for absence of arbitrage in a one maturity setting, i.e. all adjacent vertical spreads and butterfly spreads are priced non-negatively.

The method is implemented and tested in the FX-option market. We suggest two sub-models, one with three and one with five degrees of freedom. By using a least-square approach, we calibrate the two sub-models against 416 Reuters quoted volatility smiles. Both sub-models succeeds in generating prices within the bid-ask spread for all options in the sample. Compared to the three parameter model, the model with five parameters calibrates more exactly to market quoted mids but has a longer calibration time. The three parameter model calibrates remarkably quickly; in a MATLAB implementation using a Levenberg-Marquardt algorithm the average calibration time is approximately 1 ms. Both sub-models produce volatility smiles which are $C^2$ and well-behaving.

Further, we suggest a technique allowing for arbitrage-free interpolation of calibrated option price functions in the maturity dimension. The interpolation is performed in parameter space, where every set of parameters uniquely determines an option price function. Furthermore, we produce sufficient conditions to ensure absence of calendar spread arbitrage when calibrating the proposed model to several maturities. We use this technique to produce implied volatility surfaces which are sufficiently smooth, satisfy all conditions for absence of arbitrage and fit market quoted volatility surfaces within the bid-ask spread. In the final chapter we use the results for producing Dupire local volatility surfaces and for pricing variance swaps.

Keywords: FX-options, local volatility calibration, local variance gamma, volatility interpolation/extrapolation, variance swaps, option pricing

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Nomenclature

Most of the reoccurring abbreviations and symbols are described here.

Symbols

- $S_t$: Spot price/exchange rate at time $t$
- $f(t,T)$: Exchange rate of a zero-price forward contract entered at time $t$ and maturing at time $T$
- $K$: Strike/Moneyness
- $x$: Initial value of underlying asset
- $T$: Maturity time
- $\tau$: Time to maturity
- $\alpha(K)$: Local Variance Gamma (LVG) volatility function
- $\omega$: Parameter set which uniquely determines a LVG-volatility function $\alpha(K)$
- $\sigma(K,T)$: Dupire local volatility function at strike $K$ and maturity $T$
- $C(K,T)$: Value of a European call option contract with strike $K$ and maturity $T$
- $P(K,T)$: Value of a European put option contract with strike $K$ and maturity $T$

Abbreviations

- FX: Foreign Exchange
- OTC: Over The Counter
- ATM: At-The-Money
- OTM: Out-of-The-Money
- ITM: In-The-Money
- SDE: Stochastic Differential Equation
- PDE: Partial Differential Equation
- LVG: Local Variance Gamma
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Chapter 1

Introduction

1.1 Purpose

The problem formulation can, in its simplest form, be described in the following way: given market quoted prices for a sparse grid of European options with different strikes and maturities, provide prices for European options at an arbitrary strike and maturity. Hence, the task can be described as an interpolation and extrapolation problem in the option price surface\(^1\). Further, the generated option price surface should be as smooth as possible but not contain arbitrage opportunities.

1.2 Objective

Since the early 90’s, a large quantity of publications has explored ways to solve the problem specified in Section 1.1 in an efficient way; a task proven to be non-trivial. As concluded by Carr (2009), a good pricing model needs to be very robust; generating arbitrage-free option prices even with low quality input data. After calibration, a pricing model should price market quoted options within the bid-ask spread. Time is often a critical parameter in the financial industry, and therefore a good model needs to calibrate very quickly to new data. Additionally, a pricing model should ideally allow for a hedging strategy where uncertainty can theoretically be entirely avoided. In many markets it is common to trade instruments which are more complex than European vanilla options, such as American options, barrier options and variance swaps\(^2\). The usefulness of a pricing model is significantly improved if it is possible to generate arbitrage-free prices for such instruments as well.

\(^1\)The formulation is often, especially in the FX-option market, presented as an interpolation in implied volatility rather than in price.

\(^2\)Such instruments are generally not market quoted, which makes it impossible to calibrate pricing models directly against them. The goal is to generate prices for such instruments which are consistent with quoted prices for vanilla options.
Chapter 1. Introduction

1.3 Background

The log-normal model, usually referred to as the Black & Scholes model, was first introduced by Black and Scholes (1973). Merton (1973) further explored the implications and limitations of the model. Using a replicative approach based on stochastic calculus, these important publications gave the financial markets the necessary tools for pricing and hedging European type options. Consequently, the market for financial derivatives has expanded rapidly since the Black & Scholes model was published. As of today, options with a vast spectra of different underlying assets are traded liquidly on exchanges all over the world\(^3\).

However, the Black & Scholes model has some important flaws which have been discussed in many publications; see the article by Jackwerth and Rubinstein (1996) among others. One of the most problematic issues is the assumption that asset returns have a log-normal distribution. This assumption fails to explain effects such as volatility clustering and has been proven false in empirical studies. The most significant difference in distribution characteristics between a normal distribution and observed log-return time series is that the latter generally have larger tails, enhancing tail risks. Jackwerth and Rubinstein (1996) presents some numbers exemplifying how the log-normality hypothesis fails to explain some important historical events. For example, on October 19, 1987, the two month future on S&P500 fell with 29%. Observing the standard deviation at the time, this corresponds to a fall of -27 standard deviations, an event with a probability of approximately \(10^{-160}\) under the log-normality hypothesis. This means that the event can be thought of as virtually impossible. Two years later, the index fell by 6% again, an event which under log-normality should occur once every 14756 years. Today, market quoted option prices take these enhanced tail risks into account and it is therefore generally impossible to generate more than one market price using the Black & Scholes model without manipulating the volatility parameter.

There are several newer pricing models which deal with these imperfections of the Black & Scholes model. Jackwerth and Rubinstein (1996) extracts the risk-neutral probability distribution from market quoted option prices. However, this approach does not allow for interpolation in the maturity dimension. Hagan et al. (2002) and Heston (1993) assume the volatility to be a stochastic process. These models\(^4\) tend to calibrate more accurately than the Black & Scholes model, but will not necessarily match all market quoted option prices. Further, stochastic volatility models introduce uncertainty which cannot be hedged by trading in the underlying asset. Madan and Seneta (1990) present another approach to make the tail probabilities larger by introducing a Gamma process working as a subordinator to the Brownian motion used in the Black & Scholes model. The resulting model is referred to as the “Variance Gamma” model and is further developed by Milne and Madan (1991) and Madan et al. (1998). The return distribution in the variance gamma model captures tail-risk more

\(^3\)The vast majority of FX-option contracts are still entered OTC.

\(^4\)Stochastic volatility models have become extremely popular amongst banks and other financial institutions.
Dupire (1994) proposes a model where the volatility is assumed to be a deterministic function dependent on time and on the contemporaneous value of the underlying asset. This model is denoted the “Local Volatility” model and is studied further by Derman and Kani (1998) among others. A new important equation is introduced in these papers, usually called the Dupire equation or the Forward equation. This PDE describes the dynamics in a grid of European option prices. In order for the local volatility model to be usable, the volatility function must be calibrated to market prices on European options. In many implementations, problems arise when calibrating this local volatility function. Calibration methods are often both unstable and computationally costly. Further, many implementations such as the one proposed by Andersen and Bortherton-Ratcliffe (1997-1998), assume the existence of a continuum of market quoted strikes. Obviously, this is a condition which is never satisfied in practice.

1.4 Earlier Work

Some earlier publications are closely related to the approach taken in this thesis. Carr (2008) shows that one finite difference step in the Dupire equation can be seen as option prices coming from a model denoted the “Local Variance Gamma” model. This model is further examined by Carr and Nadtochiy (2013), where a calibration algorithm is presented while assuming the LVG-volatility to be a discontinuous piecewise constant function. Andreasen and Huge (2011) use a numerical technique for generating arbitrage-free call price surfaces by using the LVG-version of the Dupire equation. This method assumes piecewise constant LVG-volatility and generates call prices on different maturities by using a systematic approach where the prices on earlier maturities are used as boundary conditions. From the produced call price surface, a Dupire local volatility surface is derived. Due to the discontinuities in LVG-volatility, the Dupire local volatility surface suffers from problems with discontinuities as well. Another obstacle with discontinuous LVG-volatility lies in regularity problems of the generated implied volatility smiles, which is shown by Carr and Nadtochiy (2013, pp. 28).

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5See Definition 8.
6The LVG-volatility function is a deterministic function of strike used in the LVG model.
1.5 Topics Covered

This thesis consists of eight chapters and two appendixes. The main topics covered are:

Chapter 2: We present some preliminary results which are important for the derivation of a new model in Chapter 3. This includes the Black & Scholes model, the variance gamma model, and the local volatility model.

Chapter 3: The central equations in the LVG model are derived. Further, we present an analytical solution in the case of a continuous piecewise linear LVG-volatility function with five parameters.

Chapter 4: We explore terminology, quoting conventions, and other unique attributes of the FX-option market.

Chapter 5: We develop and test a calibration strategy used for fitting the model developed in Chapter 3 to implied volatility smiles quoted in the FX-option market.

Chapter 6: The newly developed model is extended to support maturity dimension interpolation. An algorithm for calibrating the model to an entire implied volatility surface is presented.

Chapter 7: The extended model from Chapter 6 is used for producing Dupire local volatility surfaces and for pricing variance swaps.

Chapter 8: Conclusion

Appendix A: We give proofs of some of the theoretical results in the thesis.

Appendix B: We present data from the calibration in Chapter 5.
Chapter 2

Preliminaries

In this chapter we present some preliminary results necessary for understanding the local variance gamma model presented in Chapter 3. We will be working in the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We will let \((W_t)_{t \in [0,T]}\) denote a standard Brownian motion defined on the probability space and \(\mathcal{F} = (\mathcal{F}_t, t \in [0,T])\) the right continuous filtration generated by \(W\). The following definition will be used throughout the thesis:

**Definition 1** Let \(X_t\) be a stochastic process which is adapted to the filtration \(\mathcal{F}_t\) and satisfies \(E[|X_t|] < \infty, \forall t \geq 0\). If \(X_t\) satisfies the martingale property:

\[
E^\mathbb{P}[X_{t_2} | \mathcal{F}_{t_1}] = X_{t_1}, \quad \forall 0 \leq t_1 \leq t_2 < \infty,
\]

we call \(X_t\) a martingale under \(\mathbb{P}\).

A more extensive definition is made by Applebaum (2009, pp.85) among others.

2.1 The Black & Scholes Model

In this section we present the Black & Scholes valuation model for contingent claims of European type. The model was introduced by Black and Scholes (1973) and further analysed by Merton (1973). This model is the foundation for all models considered in this thesis. The flaws of the Black & Scholes model define the purpose of the more complicated approaches considered in later chapters.

Assume that the price of some asset is described by a stochastic process \((S_t)_{t \in [0,T]}\) which follows the geometric Brownian motion

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad t \in [0,T],
\]

where \(\mu\) is the rate of return and \(\sigma\) is the volatility. Both of these parameters are so far assumed to be constant. The process \((W_t)_{t \in [0,T]}\) is a standard Brownian motion under the real world probability measure. The following theorem, originally formulated by Black and Scholes (1973), is a powerful tool for pricing contingent claims of European type.

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Theorem 1 (The Black & Scholes Equation) Let $V(S,t)$ be a European type contingent claim traded on an arbitrage-free market. Let $V(S,t)$ have a constant strike $K > 0$ and constant maturity $T > 0$. Assume that the asset underlying the claim follows the dynamics in (2.1), and that no dividends are paid. Assume further the existence of a risk-free asset with constant rate of return $r$, available for investments on all maturities. Transaction costs and other factors restricting the possibility for trading in the underlying or risk-free asset are assumed to be absent. The discounted value of the claim then satisfies:

$$\frac{\partial V}{\partial t}(S,t) + rS\frac{\partial V}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) = rV(S,t).$$

(2.2)

This parabolic PDE has the boundary condition:

$$\lim_{t \to T} V(S,t) = \phi(S_T,T),$$

where $\phi(S_T,T)$ is the payoff function of the claim. Further, the solution of the above PDE can be found by solving the conditional expectation:

$$V(S,t) = \mathbb{E}_Q\left[e^{-r(T-t)}V(S_T,T)|S_t = S\right],$$

(2.3)

where $Q$ is the risk-neutral probability measure\(^1\). The dynamics of the underlying asset under $Q$ follow:

$$\frac{dS_t}{S_t} = rdt + \sigma dW^Q_t, \quad t \in [0,T].$$

(2.4)

The proof of this famous theorem is rather straightforward. Consider an investment strategy consisting of a short position in a European contingent claim and a long position in the underlying asset. Let the position in the underlying asset follow a continuously updated $\Delta$-hedging strategy\(^2\). By using Itô’s lemma, it can be shown that such an investment strategy does not contain any uncertainty. Assuming that the market is free from arbitrage, all assets without uncertainty must have the same rate of return as the risk-free asset. Setting the drift of the $\Delta$-hedged portfolio equal to the risk-free return produces (2.2). A more detailed proof is shown by Hull (2008, pp.287). In order to reach the equations (2.3) and (2.4), one popular approach is to use the Feynman-Kac formula, in which a relationship between parabolic PDE’s and conditional expectations is formulated.

In the case of a European call option, the payoff is described by the function

$$\phi(S_T,T) = (S_T - K)^+ := (S_T - K)1_{\{S_T > K\}}.$$  

For this specific case, a unique analytical solution to (2.2) exists. The solution is often referred to as the Black & Scholes formula and can be seen in the theorem below.

---

\(^1\)The risk-neutral probability measure is discussed further in Section 2.1.1.

\(^2\)A continuously updated $\Delta$-hedge has the position $\Delta = \frac{\partial V}{\partial S}$ in the underlying asset at all times.
2.1. The Black & Scholes Model

**Theorem 2** (The Black & Scholes pricing formula for European call options)
Assume that all conditions in Theorem 1 are satisfied and that the payoff corresponds to that of a European call option, \( \phi(S,T) = (S - K)^+ \). Equation (2.2) will then have the unique solution

\[
C(S,t) = SN(d_+) - e^{-r(T-t)}KN(d_-), \quad t \in [0,T],
\]

where \( N \) is the cumulative distribution function of a standard normally distributed random variable.

This formula can be derived by using the fact that the SDE in (2.4) has the unique solution

\[
S_T = S_t \exp \left( (r - \frac{1}{2} \sigma^2)(T - t) + \sigma (W_T^Q - W_t^Q) \right), \quad 0 \leq t \leq T. \tag{2.6}
\]

This can be shown by defining a process \( dZ_t = \ln S_t \) and applying Itô’s lemma.

Theorem 2 can then be proven by substituting the payoff of a European call option into (2.3):

\[
C(S,t) = \mathbb{E}^Q \left[ e^{-r(T-t)}(S_T - K)^+ | S_t = S \right]. \tag{2.7}
\]

Substituting (2.6) into (2.7) yields an expression which is analytically solvable. The \( \Delta \) is found by differentiating (2.5) with respect to the contemporaneous value of the underlying asset:

\[
\frac{\partial C}{\partial S} := \Delta C(S,t) = N(d_+). \tag{2.8}
\]

Put options can be priced in a similar way or by using the theorem below; the put-call parity.

**Theorem 3** (The Put-Call Parity) Assume that the price of a European call option is given by \( C(S,t) \). The value of a put option with the same underlying, strike and maturity \( P(S,t) \) is then:

\[
P(S,t) = C(S,t) - S + Ke^{-r(T-t)} \tag{2.9}
\]

**Proof:** The price of the put option can be found by discounting its expected payoff under the risk-neutral probability measure \( Q \):

\[
P(S,t) = \mathbb{E}^Q[e^{-r(T-t)}(K - S_T)(1 - 1_{\{S_T < K\}})|S_t = S].
\]

Using some algebra and assuming the discounted price of the underlying asset to be a \( Q \)-martingale, we can rewrite this into:

\[
\mathbb{E}^Q[e^{-r(T-t)}(K - S_T)(1 - 1_{\{S_T < K\}})|S_t = S] = e^{-r(T-t)}K - \mathbb{E}^Q[e^{-r(T-t)}S_T|S_t = S] + C(S,t)
\]

\[
= C(S,t) - S + Ke^{-r(T-t)}.
\]

\(^3\)Notice that the Black & Scholes equation, but not its analytical solution, is valid also when the drift and volatility of the underlying asset are deterministic functions of time and the contemporaneous value of the underlying asset.
Notice that the above result does not require any assumptions concerning the probability distribution of \( S_T \). Hence, the Put-Call parity is applicable for all models where discounted asset prices are \( \mathbb{Q} \)-martingales\(^4\).

### 2.1.1 Risk-Neutral Valuation and Absence of Arbitrage

The strength of the Black & Scholes model is a consequence of the fact that (1) does not contain the asset-specific parameter \( \mu \). Hence, as Hull (2008, pp.289) points out, no variable in the equation is dependent on the risk preferences of the investor. This is the law of one price; the value of a financial instrument is the same for all investors. One important and related concept is that of risk-neutral valuation, which will be discussed briefly below. For a more comprehensive study in the subject the reader can refer to Björk (2004, pp.85) among others.

First, let us define the concept of an arbitrage opportunity.

**Definition 2** An arbitrage opportunity is an investment strategy which with probability \( P > 0 \) achieves a wealth \( W > 0 \) without any probability of generating a loss.

In order for an asset pricing model to be consistent with the effective market hypothesis, no arbitrage opportunities can exist. It is of great importance that models are arbitrage-free. This leads us to the first fundamental theorem of asset pricing, a theorem that will not be proven in this thesis.

**Theorem 4** (First Fundamental Theorem of Asset Pricing) A market is free from arbitrage opportunities if and only if there exists a probability measure \( \mathbb{Q} \) which is equivalent to the real world probability measure and under which all discounted, non-dividend paying asset prices are martingales.

To be rigorous, this theorem is only valid in a time-discrete setting with a discrete probability space. In order to define absence of arbitrage and the first fundamental theorem of asset pricing in a continuous time setting, additional theory must be introduced. Interested readers are referred to Delbaen and Schachermayer (1994). It is easily shown that if a discounted asset price is a \( \mathbb{Q} \)-martingale, its expected return under \( \mathbb{Q} \) must be equal to the return of the risk-free asset.

### 2.1.2 Static Arbitrage

In this section we discuss the concept of static arbitrage, and how absence of this type of arbitrage can be established in a grid of European options. Before we continue, let us define three different types of trading strategies which will be referenced in this thesis.

**Definition 3** Let \( K_1 < K_2 \) be two strikes for European call options with the same underlying asset and time to maturity. We denote a trading strategy vertical spread if the trader simultaneously buys one option with strike \( K_2 \) and sells one option with strike \( K_1 \).

\(^4\)According to Theorem 4, the first fundamental theorem of asset pricing, this condition is satisfied if and only if the model is free from arbitrage.
2.1. The Black & Scholes Model

Definition 4 Let $K_1 < K_2 < K_3$ be three strikes for European call options with the same underlying asset and time to maturity. We denote a trading strategy butterfly spread if the trader simultaneously sells two options with strike $K_2$, buys one option with strike $K_1$, and buys one option with strike $K_3$.

Definition 5 Let $T_1 < T_2$ be two maturities for European call options with the same underlying asset and strike. We denote a trading strategy calendar spread if the trader simultaneously sells one option with maturity $T_1$ and buys one option with maturity $T_2$.

These strategies are popular ways for traders to take positions in European options. Similar strategies are also available for put options, which can be proven by using the put-call parity. More information concerning these trading strategies can be found in the book by Hull (2008, pp.221-229) among others.

As mentioned earlier, establishing absence of arbitrage in a set of asset prices can be a difficult task in a continuous time setting. As concluded by Carr and Madan (2005), this is related to the construction of the information set available for building trading strategies. In order to establish absence of arbitrage in the classical sense, one would need to specify the structure of possible paths for the price process. The concept of “Static Arbitrage”, introduced by Carr et al. (2003), restricts the information set that may be used for taking positions. Hence, static arbitrage corresponds to a simpler meaning of what defines an arbitrage opportunity. While static arbitrage still follows Definition 2, we add the restriction that a position in the underlying asset may only be dependent on time and on the contemporaneous value of the underlying asset. In other words, the path of the price process cannot be part of the information set that a position is based upon. When referring to arbitrage later in this thesis, this will generally correspond to static arbitrage if nothing else is stated.

Carr and Madan (2005) derives how absence of static arbitrage in a rectangular grid of European call option prices can be established. This result will not be proven in this thesis.

Condition 1 Consider a grid of European call options $C(K,T)$, consisting of option prices defined for a countable set of strikes $K \in \kappa$ and a common, countable set of maturities $T \in \theta$. Assume that all interest rates are zero and that the underlying asset hence is a $Q$-martingale. The grid will not possess static arbitrage if the prices satisfy the following conditions:

1. Non-increasing in strike, $\frac{\partial C}{\partial K}(K,T) \leq 0$, $\forall K \in \kappa, T \in \theta$. This condition, which is equivalent to positive prices for all adjacent vertical spreads, is usually denoted absence of vertical spread arbitrage.

2. Convexity in strike, $\frac{\partial^2 C}{\partial K^2}(K,T) \geq 0$, $\forall K \in \kappa, T \in \theta$. This condition, which is equivalent to positive prices for all butterfly spreads, is usually denoted absence of butterfly spread arbitrage.

3. Non-decreasing in maturity, $\frac{\partial C}{\partial T}(K,T) \geq 0$, $\forall K \in \kappa, T \in \theta$. This condition, which is equivalent to positive prices for all calendar spreads, is usually denoted absence of calendar spread arbitrage.
2.1.3 Implied Volatility

The parameter $\sigma$ in (2.5) corresponds to the average volatility of the underlying asset during the lifetime of the contract. This is the only parameter in the Black & Scholes model that is not directly observable in the financial markets. In order to find the market price of a publicly traded option by using the Black & Scholes model, it is necessary to find a matching value for $\sigma$. Generally, the value of $\sigma$ which produces a market price is called implied volatility.

**Definition 6** The implied volatility is the volatility which makes (2.5) generate a price consistent with the price of a market quoted call option.

The implied volatility is found by inverting (2.5) with respect to the volatility parameter, using an option price quoted on the financial markets with known strike and maturity. Since (2.5) is not analytically invertible with respect to $\sigma$, the implied volatility has to be found using numerical techniques.

The implied volatility for a grid of market quoted options with different maturities and strikes is generally not constant. The strike and maturity dependency in implied volatility is caused by the financial markets attaching higher probabilities to extreme movements in log-returns compared to the normal distribution. As a function of strike, the implied volatility for FX-options usually assumes the shape of a smile. This curve is generally denoted the implied volatility smile. See Figure 5.2 for examples of implied volatility smiles.

It is possible to add maturity dependency in $\sigma$ while keeping a closed-form solution similar to (2.5). Assume that the volatility is a deterministic function of time, $\sigma(t)$. Assuming $\sigma(t)$ to be a step function allows for arbitrage-free calibration to market prices on options with any set of maturities. This simple approach is sadly not applicable in the strike dimension, which leads us to the next section.

2.2 The Local Volatility Model

The local volatility model is a generalization of the Black & Scholes model. The model was first proposed by Dupire (1994) and has been further developed by Derman and Kani (1998) among others. The model is based on the assumption that the volatility is a general deterministic function dependent on time and the contemporaneous value of the underlying asset. This generalization makes it possible to create a risk-neutral probability distribution of the underlying asset which is consistent with an entire market quoted implied volatility surface. When the model was first introduced, the underlying asset was assumed to be a Q-martingale. In later publications, the model has been generalized to allow for

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5 Or the corresponding formula for a put-options.

6 This phenomena first became observable in the markets after the financial breakdown in 1987. For more information, the reader is referred to Rubinstein (1994) and Jackwerth and Rubinstein (1996).

7 While implied volatility functions in the FX-market usually assume the shape of smiles, the corresponding shape for equity-options is often monotonically decreasing in strike. This type of shape is often called implied volatility skew. See Hull (2008, pp.386) for more information concerning implied volatilities for equity options.
interest rates, but since this will not be used for later results in this thesis we will here present the Dupire equation in its original format. To emphasize one of the strengths of the local volatility model we need the following definition.

**Definition 7** A model is complete if all contingent claims can be perfectly hedged.

Since the local volatility model does not introduce any further sources of risk, the model is a complete market model where assets theoretically can be perfectly hedged by using a continuously updated Δ-hedging strategy.

**Theorem 5** (The Second Fundamental Theorem of Asset Pricing) An arbitrage-free market is complete if and only if the equivalent risk-neutral probability measure \( Q \) is unique.

This theorem will not be proven in this thesis. The most important effect of Theorem 5 is that prices generated by the local volatility model are unique\(^8\), an attribute which is very useful.

Assume that the underlying asset follows a stochastic process with the dynamics

\[
dS_t = S_t \sigma(S_t, t)dW^Q_t, \quad t \in [0, T], \tag{2.10}
\]

where \((W^Q_t)_{t \in [0, T]}\) is a standard Brownian motion under the risk-neutral probability measure. The central equation in the local volatility model, the Dupire equation, is presented below together with a proof similar to the one by Dupire (1994).

**Theorem 6** (The Dupire Equation for European Call Options) Let \( C(K, T) \) denote European call options with strikes \( K \in \kappa \), maturities \( T \in \theta \), and underlying asset \( S \). Assume that the price of \( S \) follows the dynamics in (2.10), with initial condition \( S_t = S \). The prices of the call options at time \( t \) will then satisfy the equation:

\[
\frac{\partial C}{\partial t}(K, T) = \frac{1}{2} \sigma^2(K, T) K^2 \frac{\partial^2 C}{\partial K^2}(K, T), \tag{2.11}
\]

with boundary condition:

\[
\lim_{T \to t} C(K, T) = (S - K)^+.
\]

While this equation looks very similar to the Black & Scholes equation (2.2), the two equations have fundamentally different meaning. The Black & Scholes equation describes the evolution of the price of any European contingent claim over time, holding claim-specific parameters such as strike \( K \) and maturity \( T \) constant. The Dupire equation describes the dynamics in a grid of European call option prices, holding the contemporaneous value of the underlying asset \( S \) and time \( t \) constant. The two equations are often referred to as the forward and backward equations. This convention originates from the fact that the Black & Scholes equation has boundary conditions at the terminal date \( t = T \), and hence needs to be solved backwards in time, while the Dupire equation has boundary conditions at expiration.\(^8\)

\(^8\)Just as in the case of the first fundamental theorem of asset pricing, a more complicated approach is needed to make the definition rigorous in a continuous time setting, see Delbaen and Schachermayer (1994).
Chapter 2. Preliminaries

Consider a European call option \( C(S,t) \) with some underlying asset \( S \), following the dynamics in (2.10). Assume further that the call option has maturity \( T \) and strike \( K \). Denote the risk-neutral density of the underlying at time \( t \) as \( \phi(S,t) \).

The price of the option can then, by using (2.3), be calculated as the expected payoff under the risk-neutral probability measure:

\[
C(S,t) = \mathbb{E}^Q[(S_T - K)1_{\{S_T - K > 0\}}]\big|_{S_t = S}.
\]

Using the definition of expected value, this expression can be formulated as the integral

\[
C(S,t) = \int_0^\infty (x - K)1_{\{x > K\}} \varphi(x,T)dx = \int_K^\infty (x - K) \varphi(x,T)dx,
\]

(2.12)

where \( \varphi(x,T) \) is the probability density function of the underlying asset at maturity, conditioned on \( S_t = S \). In order to continue the derivation, the following theorem, usually called the Fokker-Planck theorem, is needed. No extensive discussion concerning this theorem will be presented here, interested readers are referred to Risken (1996).

**Theorem 7** (The Fokker-Planck Theorem) Let \( X_t \) be a \( N \)-dimensional stochastic process with uncertainty driven by a \( M \)-dimensional standard Brownian motion \( W_t \):

\[
dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t,
\]

where \( \mu(X_t,t) = (\mu_1(X_t,t), \ldots, \mu_N(X_t,t)) \) is a \( N \)-dimensional drift vector and \( \sigma(X_t,t) \) is a diffusion tensor. Then the joint probability function \( f(x,t) \) satisfies the Fokker-Planck equation

\[
\frac{\partial f(x,t)}{\partial t} = -N \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i} \left[ \mu_i(x,t)f(x,t) \right] + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left[ D_{i,j}(x,t)f(x,t) \right],
\]

(2.13)

where \( D_{i,j} = \frac{1}{2} \sum_{k=1}^{M} \sigma_{i,k}(x,1) \sigma_{j,k}(x,1), \quad 1 \leq i, j \leq N \).

Applying this theorem to the driftless one-factor framework which is of interest yields the Fokker-Planck equation

\[
\frac{\partial}{\partial t} \varphi(S,t) = \frac{1}{2} \frac{\partial^2}{\partial S^2} \left( S^2 \sigma^2(S,t) \varphi(S,t) \right).
\]

(2.14)

The solution of this equation will, in accordance with Theorem 5, be assumed to be unique provided that the probability distribution is restricted to the risk-neutral one\(^{10}\). In order to proceed, some differentials of the option price with respect to the strike \( K \) and maturity \( T \) are needed. We will from now on

---

9 Notice that the both Black & Scholes equation and the Dupire equation easily can be reformulated into instead having a dependency with respect to time to maturity \( \tau = T - t \). This type of formulation is used in Chapter 3.

10 Since no additional sources of risk have been introduced, one can assume that the market model is complete and that the risk-neutral probability measure is unique, see Björk (2004, pp.105) for a deeper discussion.
2.3. The Variance Gamma Model

let \( K \) and \( T \) be variables and \( S \) and \( t \) static parameters. The notation for a call option is also changed to \( C(K, T) \), but keep in mind that the price is still conditioned on \( S_t = S \). Using the Leibniz integral rule on (2.12) and assuming that \( \lim_{S \to \infty} \varphi(S,T) = 0 \) yields:

\[
\frac{\partial C}{\partial K}(K, T) = - \int_k^\infty \varphi(x, T)dx, \quad (2.15)
\]

\[
\frac{\partial^2 C}{\partial K^2}(K, T) = \varphi(K, T), \quad (2.16)
\]

\[
\frac{\partial C}{\partial T}(K, T) = \int_k^\infty (x-K) \frac{\partial}{\partial T}(\varphi(x, T))dx. \quad (2.17)
\]

Substituting (2.14) at time \( t = T \) into (2.17) yields:

\[
\frac{\partial C}{\partial T}(S, t) = \int_k^\infty (x-K) \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} (x^2 \sigma^2(x, T) \varphi(x, T)) \right)dx. \quad (2.18)
\]

Integrating this expression by parts two times and using the equations (2.15) and (2.16) yields:

\[
\int_k^\infty (x-K) \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} (x^2 \sigma^2(x, T) \varphi(x, T)) \right)dx = \frac{1}{2} \sigma^2(K, T)K^2 \frac{\partial^2 C}{\partial K^2}. \quad (2.19)
\]

Substituting (2.19) into (2.18) yields the Dupire PDE for European call options\(^{11}\). The boundary condition of the above equation simply states that an option with time to maturity \( \tau = T - t = 0 \) will have the value equal to the call option payoff function:

\[
\lim_{T \to t} C(K, T) = (S - K)^+. \quad (2.20)
\]

Generalizations of this equation are shown by Derman and Kani (1998), where deterministic interest rates are introduced, and by Deelstra and Rayée (2012), where a version allowing for stochastic interest rates is derived.

2.3 The Variance Gamma Model

The variance gamma model was first introduced by Madan and Seneta (1990) and can also be considered a generalization of the Black & Scholes model. The purpose of the model is similar to that of the local volatility model; to generalize the assumption of log-return normality. In the variance gamma model, this is accomplished by allowing the Brownian motion to be driven by a Gamma process instead of a predictable time parameter. Each calendar day is considered to be an independent random variable with positive variance. This can be interpreted as the introduction of the concept of financial time; the level of activity in the financial markets is given a possibility to manifest itself in the Gamma process. Below, we define a measure which will be used frequently in this section.

\(^{11}\)By using the put-call parity, it is easily shown that (2.11) is satisfied by put option prices as well.
Definition 8 Let $X$ be a random variable. We define the kurtosis of $X$ as the fourth moment divided by the second moment squared:

$$K_X = \frac{E[X^4]}{E[X^2]^2}$$

The kurtosis measures how fat the tails of the distribution of a random variable is.

Madan and Seneta (1990) show that a new parameter $t^*$, originating from the Gamma process, manipulates the kurtosis of the return distribution. This addresses the fact that empirical studies show that the kurtosis of log-return time series is higher than normal distribution kurtosis. These results are generalized by Milne and Madan (1991) and Madan et al. (1998), where yet another parameter is introduced making it possible to manipulate the skewness of the return distribution as well. This parameter is not used in the LVG model and will therefore not be discussed further in this thesis.

2.3.1 The Gamma process

The Gamma process is a non-decreasing one dimensional Lévy process, sometimes called a subordinator\(^{12}\). Let the Gamma process be defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with the right continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. The Gamma process is, unlike the Brownian motion, a discontinuous pure jump process. Using results and notations from Applebaum (2009) and Madan and Seneta (1990), the Gamma process $V(t)$ is defined through its density $g_{V(t)}$ with variables $t > 0$, $v > 0$ and parameters $c > 0$ and $t^* > 0$:

$$g_{V(t)}(v) = \frac{c^{t/t^*} v^{t/t^* - 1} e^{-cv}}{\Gamma(t/t^*)},$$

where $\Gamma(x)$ is the Gamma function. Notice that the Gamma process, being a Lévy process, has independent increments. The mean and variance of the Gamma process can be shown as

$$E[V(t)] = \frac{t}{ct^*} \text{ and } \text{var}[V(t)] = \frac{t}{c^2 t^*}.$$

In this thesis, the Gamma process will be assumed to follow the normal time axis in mean. This condition can be imposed by demanding the relationship $c = 1/t^*$ to be satisfied. This yields a process called the unbiased Gamma process which has the probability density

$$g_{V(t)}(v) = \frac{v^{t/t^* - 1} e^{-v/t^*}}{(t^*)^{t/t^*} \Gamma(t/t^*)}$$

and first two moments

$$E[V(t)] = t, \quad \text{var}[V(t)] = tt^*.$$

\(^{12}\)This term comes from the fact that the process is often used to describe the evolution of time in other processes, see Applebaum (2009, pp.52) for more information.
2.3. The Variance Gamma Model

2.3.2 Using the Gamma process

In this section, the Gamma process is used to modify the return distribution kurtosis. Define the process \( (B_t)_{t \in [0,T]} \) as

\[
B_t = W^Q_{V(t)}, \quad t \in [0,T],
\]

where \( (W^Q_t)_{t \in [0,T]} \) is a standard Brownian motion under the risk-neutral measure and \( V(t) \) is an unbiased Gamma process. Assume that the log-returns of some asset follow the process \( X_t; \)

\[
X_t = \sigma B_t, \quad t \in [0,T], \text{ and } X_0 = x.
\]

Conditioned on a specific trajectory of the Gamma process, \( X_t \) will be a normally distributed random variable. Hence, the unconditional distribution function of \( X_t \) can be found by integrating the normal density over the density of the Gamma process:

\[
f_{X_t}(x) = \int_0^\infty \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi \frac{\sigma^4}{t^4}}} e^{\frac{-v}{\sigma^4 t^4}} \, dv.
\]

Madan et al. (1998) derive the first four moments of the process \( X_t \) as

\[
E[X_t] = 0, \quad E[X_t^2] = \sigma^2 t, \quad E[X_t^3] = 0, \quad \text{and } E[X_t^4] = 3\sigma^4 t^4 + 3\sigma^4 t^2.
\]

The first three moments are the expected results from a normal distribution. Using the standard definition of kurtosis we can see that

\[
K_X := \frac{E[X_t^4]}{(E[X_t^2])^2} = 3 + 3t^*/t.
\]

For \( t = 1 \), this corresponds to a kurtosis \( K_x = 3(1 + t^*) \). Since the kurtosis for a normal distribution is equal to 3, the variable \( t^* \) can be seen as the percentage additional kurtosis. Recall that the kurtosis controls the tails of the distribution. By using the variance gamma model it is therefore possible to create return distributions with higher tail risks; something that corresponds well to empirical studies of return time series. Out of a valuation perspective this is very interesting since it can be used to create implied volatility smile effects.
Chapter 3

Local Variance Gamma

In this chapter, we explore the local variance gamma model first introduced by Carr (2008). This model combines the local volatility model with the variance gamma model. Many of the results in the chapter are shown by Carr and Nadtochiy (2013). Our approach differs from earlier publications by considering a new assumption concerning the structure of the LVG-volatility function. Initially, only mutually independent option price curves with constant time to maturity are considered. In Chapter 5, the proposed model is calibrated to market quotes from the FX-option market and in Chapter 6 we interpolate calibrated option price functions between maturities in order to produce arbitrarily fine option price grids.

3.1 The Forward and Backward Equations in the LVG model

In this section we present non-detailed proofs of the most important equations in the local variance gamma model. The reader is referred to the detailed proofs by Carr and Nadtochiy (2013) for the more technical aspects.

Assume that the value for some underlying asset follows the stochastic process $M_s$ with the dynamics

$$dM_s = \alpha(M_s)dW^Q_s, \quad s \in [0, \xi], \quad M_0 = x,$$

(3.1)

where $0 < L < x < U < \infty$, $x$ is the initial value of the process and $\xi$ is the stopping time when the process $M_s$ first exits the interval $[L, U]$. The roles of the two parameters $L$ and $U$ are further studied in Section 3.6. Further, assume $\alpha$ to be a positive piecewise continuous function with a finite number of discontinuities of the first order that is bounded from above and away from zero. Under these assumptions, $M_s$ is a true $Q$-martingale. Additionally, it is possible to show that the family of solutions to the SDE, $M^x = M_{[x \in [L, U]]}$, is unique and that the solutions have a strong Markov property. This result is proven by Karatzas and Shreve (1998, pp 322 & 335). Using a similar method as in (2.3), prices can be found for any European type contingent claim $V_t$ by discounting...
the expected payoff conditioned on the initial value of the underlying asset:

$$V_t = E^Q[\phi(M, T) | M_t = x].$$

Using the Markov property of $M^x$ and slightly different notations, the value function can be expressed as

$$V^M(x, \tau) = E^Q[\phi(M^x, \tau)], \quad (3.2)$$

where $\tau = T - t$ is the time to maturity of the claim. Further, European contingent claims are assumed to satisfy (2.2), the Black & Scholes equation. Below, this equation is expressed using a differential with respect to $\tau$:

$$\frac{\partial V^M}{\partial \tau}(x, \tau) = \frac{1}{2} \alpha^2(x) \frac{\partial^2 V^M}{\partial x^2}(x, \tau), \quad (3.3)$$

with boundary condition

$$V^M(x, 0) = \phi(x, 0), \quad x \in [L, U],$$

where $\phi(M^x, \tau)$ is the payoff function. Further, let $V^M(x, \tau)$ represent a European call option for which $\phi(x, 0) = (x - K)^+$ for $K \geq 0$. $V^M(x, \tau) = C^M(x, \tau)$ will then satisfy the Dupire equation, (2.11). Below this equation is expressed using a differential with respect to $\tau$:

$$\frac{\partial C^M}{\partial \tau}(K, \tau) = \frac{1}{2} \sigma^2(K) \frac{\partial^2 C^M}{\partial K^2}(K, \tau), \quad (3.4)$$

with boundary condition

$$C^M(K, 0) = (x - K)^+. \quad (3.5)$$

We will now introduce the Gamma process into the framework presented above. Assume that $G(t)$ is an unbiased Gamma process with a probability distribution from (2.22). Define the stochastic process:

$$X_t = M^G(t), \quad t \in [0, T]. \quad (3.6)$$

Notice that $X_t$ inherits the martingale and Markov property from $M^x$. Conditioned on a specific trajectory of the Gamma process, $X_t$ will have a dynamic similar to that of $M$. The proposition below concludes that the unconditional probability distribution can be found by integrating over the distribution function of the Gamma process\(^1\).

**Proposition 1** If $B \in \mathcal{B}(\mathbb{R})$, $\tau \geq 0$, and $0 \leq t_1 < \cdots < t_n < t$, the following equation is satisfied:

$$\mathbb{Q}(X_{t+s} \in B | X_t, \ldots, X_1) = \int_0^\infty \frac{u^{s/t^* - 1} e^{-u/t^*}}{(t^*)^{s/t^*} \Gamma(s/t^*)} \mathbb{Q}(M_u \in B) du, \quad (3.7)$$

where $\mathbb{Q}$ is the risk-neutral probability measure and $\mathcal{B}$ is the set of Borel Measurable functions.

\(^1\)A similar procedure was followed in Section 2.3.2.
A rigorous proof of this result is shown by Carr and Nadtochiy (2013, Prop. 4). Using Proposition 1, it is possible to price any European claim \( V(x, \tau) \) with underlying asset \( X_t \) by taking the expectation of the payoff;

\[
V(x, \tau) = \mathbb{E}[\phi(X_T) | \mathcal{F}_t] = \int_0^\infty \frac{u^{\tau/t^*} - 1}{(t^*)^{\tau/t^*}} \mathbb{E}[\phi(M_u^x, u)] du
\]

where (3.2) has been used in the last equality. Now, let us assume that the time to maturity \( \tau \) is equal to the characteristic time parameter of the Gamma process \( t^* \). The above equation then simplifies into:

\[
V(x, t^*) = \int_0^\infty \frac{e^{-u/t^*}}{t^*} V^M(x, u) du.
\]

Integrating (3.3) over this simplified gamma density yields:

\[
\int_0^\infty \frac{e^{-u/t^*}}{t^*} \frac{\partial V^M}{\partial u}(x, u) du = \int_0^\infty \frac{e^{-u/t^*}}{t^*} \frac{1}{2} \alpha^2(x) \frac{\partial^2 V^M}{\partial x^2}(x, u) du
\]

\[
\Rightarrow \left[ \frac{e^{-u/t^*}}{t^*} V^M(x, u) \right]_0^\infty + \frac{1}{t^*} \int_0^\infty \frac{e^{-u/t^*}}{t^*} V^M(x, u) du = \frac{1}{2} \alpha^2(x) \frac{\partial^2}{\partial x^2} \left( \int_0^\infty \frac{e^{-u/t^*}}{t^*} V^M(x, u) du \right).
\]

Simplifying this formula and substituting in (3.8) yields the LVG version of the Black & Scholes equation:

\[
\frac{1}{2} \alpha^2(x) \frac{\partial^2 V}{\partial x^2}(x, t^*) = \frac{V(x, t^*) - \phi(x, 0)}{t^*}.
\]

In order to make this derivation rigorous, some additional requirements concerning the integrability of \( V(x, t) \) and \( \alpha(x) \) need to be established; see Carr and Nadtochiy (2013, Theorem 8). If \( V(x, t) \) represents a European call option, we can also derive a LVG version of the Dupire equation. The derivation is practically identical to that of the LVG Black & Scholes equation;

\[
\frac{1}{2} \alpha^2(K) \frac{\partial^2 C}{\partial K^2}(K, t^*) = \frac{C(K, t^*) - (x - K)^+}{t^*}.
\]

The following theorem motivates us to find a solution to (3.10).

**Theorem 8** (Absence of Arbitrage in the LVG Version of the Dupire Equation)

Let \( C(K, t^*), K \in [L, U], t^* > 0 \) be a solution to (3.10). Then the set of call options \( C(K, t^*) \) does not contain butterfly or vertical spread arbitrage as defined in Condition 1.

A proof of Theorem 8 is provided in Appendix A. The proof is inspired by the more technical proof given by Carr and Nadtochiy (2013). As a consequence, calculated prices must also be positive and greater or equal to the intrinsic value of the option. Replacing \( (x - K)^+ \) with \( (K - x)^+ \) yields a similar formula for European put options.
3.2 Proposed LVG-volatility Function

The equations (3.9) and (3.10) lack a general solution. In order to find a closed-form solution, it is necessary to restrict the LVG-volatility function $\alpha$ to some specific family of functions. Earlier assumptions have been made by Carr and Nadtochiy (2013) and Andreasen and Huge (2011), where $\alpha$ is assumed to be piecewise constant. Using a high number of degrees of freedom, these models are usually able to calibrate well to any market quoted volatility smile but suffers from problems mentioned in Section 1.3. Most of these problems arise from the assumption of a discontinuous LVG-volatility function. In this thesis, we will instead assume $\alpha$ to be a continuous function, restricted to the set of piecewise linear four interval functions which are constant in the outer subintervals, see Figure 5.3 for an example.

Since this family of LVG-volatility functions have a very limited amount of degrees of freedom, the solution cannot be assumed to fit all market quoted prices perfectly. The objective will be focused on pricing European options on all available strikes within the bid-ask spread. Let us begin by partitioning the space $K \in [L,U]$ into four separate subintervals:

$$I_1 = \{ K \in \mathbb{R}^+ | L \leq K < \nu_1 \},$$
$$I_2 = \{ K \in \mathbb{R}^+ | \nu_1 \leq K \leq x \},$$
$$I_3 = \{ K \in \mathbb{R}^+ | x < K \leq \nu_2 \},$$
$$I_4 = \{ K \in \mathbb{R}^+ | \nu_2 < K \leq U \}.$$ (3.11)

Recall that $x$ is the initial value of the underlying asset. The constants $\nu_1, \nu_2$ are assumed to satisfy $L < \nu_1 < x$ and $x < \nu_2 < U$. Below, we present an assumption concerning the structure of the LVG-volatility function. This assumption separates the approach in this thesis from Carr and Nadtochiy (2013).

**Assumption 1** Let the LVG-volatility function $\alpha(K)$ be a continuous function defined as:

$$\alpha(K) = \sum_{i=1}^{4} \alpha_i(K) I_{\{K \in I_i\}},$$ (3.12)

where the local LVG-volatility functions are defined as:

$$\begin{align*}
\alpha_1(K) &= \gamma_1 & \forall K \in I_1, \\
\alpha_2(K) &= \gamma_2 K + b_2 & \forall K \in I_2, \\
\alpha_3(K) &= \gamma_3 K + b_3 & \forall K \in I_3, \\
\alpha_4(K) &= \gamma_4 & \forall K \in I_4.
\end{align*}$$

The constants $\gamma_1, \gamma_2, b_2, \gamma_3, b_3, \text{and} \gamma_4$ are determined by $x$ and the members of the set $\omega$, defined as:

$$\omega := \{ \sigma_1, \sigma_x, \sigma_2, \nu_1, \nu_2 \}.$$ (3.13)

All elements in $\omega$ are positive and bounded from above. The parameters in $\alpha(K)$
are related to the underlying set \( \omega \) by the conditions:

\[
\begin{align*}
\gamma_1 &= \sigma_1, & \gamma_2 &= \frac{\sigma_x - \sigma_1}{x - \nu_1}, & b_2 &= \sigma_1 - \nu_1 \frac{\sigma_x - \sigma_1}{x - \nu_1}, \\
\gamma_3 &= \frac{\sigma_2 - \sigma_x}{\nu_2 - x}, & \gamma_4 &= \sigma_2, & b_3 &= \sigma_x - \sigma_1 \frac{\sigma_2 - \sigma_x}{\nu_2 - x}.
\end{align*}
\]

This family of continuous LVG-volatility functions are constant on \( I_1 \) and \( I_4 \), and linear on \( I_2 \) and \( I_3 \). Notice that the constants \( a_2, a_3 \) correspond to the slope of the LVG-volatility function in the linear regions. Furthermore; \( \alpha \) is positive, integrable, locally differentiable and bounded from above by \( \max(\sigma_1, \sigma_x, \sigma_2) \). In this chapter, all of the elements in \( \omega \) are assumed to be known.

### 3.3 Analytic Solution

In this section we present an analytical solution to (3.10) given the four-interval piecewise linear LVG-volatility function proposed in Assumption 1.

**Proposition 2** There is a unique solution to (3.10) for \( K \in [L,U] \), given a LVG-volatility function following Assumption 1. The solution is \( C^2 \) for \( K \in [L,U] \) and satisfies the two boundary conditions:

\[
\begin{align*}
\lim_{K \to L} C(K) &= x - L, \\
\lim_{K \to U} C(K) &= 0.
\end{align*}
\]

Further, the unique closed-form solution is given by the expression

\[
C(K) = \Psi(K) + (X - K)^+, \quad \Psi(K) = \sum_{i=1}^{4} \psi_i(K) \mathbf{1}_{\{K \in I_i\}}
\]

where

\[
\begin{align*}
\psi_1(K) &= \frac{1}{\beta_x} \left( \frac{\sigma_1}{\sigma_x} \right)^{q_1} \frac{1 - \mu_1}{1 - \mu_1 \left( \frac{\sigma_1}{\sigma_x} \right)^{Q}} e^{p(K - \nu_1)} \left( \frac{1 - e^{2p(L - K)}}{1 - e^{2p(L - \nu_1)}} \right), \\
\psi_2(K) &= \frac{1}{\beta_x} \left( \frac{\alpha_2(K)}{\sigma_x} \right)^{q_1} \frac{1 - \mu_1}{1 - \mu_1 \left( \frac{\sigma_1}{\sigma_x} \right)^{Q}} e^{p(K - \nu_1)}, \\
\psi_3(K) &= \frac{1}{\beta_x} \left( \frac{\alpha_3(K)}{\sigma_x} \right)^{r_1} \frac{1 - \mu_2}{1 - \mu_2 \left( \frac{\sigma_1}{\sigma_x} \right)^{R}} e^{s(K - \nu_2)}, \\
\psi_4(K) &= \frac{1}{\beta_x} \left( \frac{\sigma_2}{\sigma_x} \right)^{r_1} \frac{1 - \mu_2}{1 - \mu_2 \left( \frac{\sigma_1}{\sigma_x} \right)^{R}} e^{s(K - \nu_2)} \left( \frac{1 - e^{2s(U - K)}}{1 - e^{2s(U - \nu_2)}} \right).
\end{align*}
\]
The coefficients in the above expression are given by:

\[ p = \sqrt{\frac{2}{\sigma_1^2 t^*}}, \quad s = \sqrt{\frac{2}{\sigma_2^2 t^*}}, \quad Q = \sqrt{1 + \frac{8}{\gamma_2^2 t^*}}, \quad R = \sqrt{1 + \frac{8}{\gamma_3^2 t^*}}, \]

\[ q_{1/2} = \frac{1}{2} \pm \frac{1}{2} Q, \quad r_{1/2} = \frac{1}{2} \pm \frac{1}{2} R, \]

\[ \mu_1 = \frac{\gamma_2 q_1 (1 - e^{2p(L - \nu_1)}) - \sigma_1 p(1 + e^{2p(L - \nu_1)})}{\gamma_2 q_2 (1 - e^{2p(L - \nu_1)}) - \sigma_1 p(1 + e^{2p(L - \nu_1)})}, \]

\[ \mu_2 = \frac{\gamma_3 r_1 (1 - e^{2s(U - \nu_2)}) - \sigma_2 s(1 + e^{2s(U - \nu_2)})}{\gamma_3 r_2 (1 - e^{2s(U - \nu_2)}) - \sigma_2 s(1 + e^{2s(U - \nu_2)})}, \]

\[ \beta_x = \frac{\gamma_2 q_1}{\sigma_x} \left( 1 - \mu_1 \frac{\sigma_1}{\sigma_x} \right)^Q - \frac{\gamma_3 r_1}{\sigma_x} \left( 1 - \mu_2 \frac{\sigma_2}{\sigma_x} \right)^R. \]

The corresponding formula for put options is

\[ P(K) = \Psi(K) + (K - x)^+. \]

Recall that \( \gamma_2 \) and \( \gamma_3 \) are the slopes of the LVG-volatility function on the linear regions, which can be calculated from the set \( \omega \) as in Assumption 3.12. From the construction of the proposed LVG-volatility function, we can see that parameterizations of the type in Proposition 2 generally have five unknown parameters corresponding to the elements of the earlier defined set \( \omega \). Recall that Methods concerning how to find these parameters will be presented in Chapter 5. The remaining part of this section will be dedicated to proving and explaining Proposition 2.

### 3.4 Derivation of the Analytical Solution

The approach will be to solve the homogeneous version of (3.10) locally on each of the subintervals from (3.11). Boundary conditions will be imposed on each subinterval, such that the resulting option price function is unique and \( C^1 \).

Notice that since \( \alpha \) is assumed to be continuous, solutions to (3.10) which are \( C^1 \) will, in fact, also be \( C^2 \). Before starting, let us define a new function which will prove useful.

**Definition 9** Let \( \Psi(K) \) be the value of a continuum of European call options \( C(K), K \in [L, U] \) with constant time to maturity \( t^* \), after the intrinsic value has been removed:

\[ \Psi(K) := C(K) - (x - K)^+, \quad \forall K \in [L, U]. \]

Notice that \( \Psi(K) \) will be the homogeneous solution to (3.10), which means that \( \Psi(K) \) will solve the homogeneous equation

\[ \frac{1}{2} t^* \alpha^2(K) \frac{\partial^2 \Psi(K)}{\partial K^2} - \Psi(K) = 0, \quad \forall K \in [L, U]. \tag{3.14} \]

This can be shown by simple substitution. The following proposition provides the necessary conditions to proceed with the calculations.
Proposition 3  The homogeneous solution vanishes in the two endpoints $L$ and $U$:

\[
\lim_{K \to L} \Psi(K) = \lim_{K \to U} \Psi(K) = 0.
\]

One of our objectives is to find a function $C(K)$ that satisfies the two boundary conditions:

\[
C(K) = \begin{cases} 
0, & \text{For } K \to U, \\
(x - L), & \text{For } K \to L.
\end{cases}
\]

Demanding Proposition 3 to be satisfied ensures that these two conditions are met.

Considering the partitioning of $\alpha(K)$, $\Psi(K)$ will also be partitioned on the four subintervals:

\[
\Psi(K) = \sum_{i=1}^{4} \psi_i(K) 1_{\{K \in I_i\}}.
\]

(3.15)

In the subsections below, local solutions are found and merged using constants.

3.4.1 Solution on $I_1$

For $K \in I_1$, the local homogeneous solution is denoted $\psi_1(K)$. In accordance with Proposition 3, this local solution will be assumed to have the boundary condition $\psi_1(L) = 0$. Further, according to Assumption 3.12 the LVG-volatility function is constant on this subinterval; $\alpha_1(K) = \gamma_1 > 0$, $\forall K \in I_1$. The differential equation (3.14) will hence, locally in $I_1$, behave as the following linear equation with constant coefficients:

\[
\frac{1}{2} t^{\ast} \gamma_1 \frac{\partial^2 \psi_1(K)}{\partial K^2} - \psi_1(K) = 0, \quad \forall K \in I_1.
\]

This differential equation has solutions

\[
\psi_1(K) = \lambda_1 e^{pK} + B_1 e^{-pK},
\]

where

\[
p = \sqrt{\frac{2}{\gamma_1 t^{\ast}}},
\]

and $\lambda_1, B_1$ are some constants. Using one degree of freedom to impose the boundary condition $\psi_1(L) = 0$ yields the family of solutions

\[
\psi_1(K) = \lambda_1 \left( e^{pK} - e^{p(2L - K)} \right).
\]

(3.16)

Differentiating this solution with respect to $K$ yields an expression which will be used later:

\[
\frac{\partial \psi_1}{\partial K} = p \lambda_1 \left( e^{pK} + e^{p(2L - K)} \right).
\]

(3.17)
3.4.2 Solution on $I_2$

On this subinterval, the LVG-volatility is assumed to be a linear function; $\alpha_2(K) = \gamma_2 K + b_2 > 0, \ \forall K \in I_2$. In our partitioning, $\psi_2(K)$ is the local homogeneous solution on the subinterval. The differential equation (3.14) will locally behave as the quadratic coefficient differential equation

$$\frac{1}{2} t^* (\gamma_2 K + b_2) \frac{\partial^2 \psi_2(K)}{\partial K^2} - \psi_2(K) = 0, \ \forall K \in I_2.$$ 

Notice that $\psi_2(K)$ can be regarded as a function of $\alpha_2(K)$, rather than of $K$ itself. Since $t^*$ is assumed to be constant, the differential of $\psi_2(K)$ with respect to $\alpha_2(K)$ is well-defined. Hence, we may use the chain rule to change into differentials with respect to $\alpha_2(K)$:

$$\frac{\partial \psi_2(K)}{\partial K} = \frac{\partial \psi_2(K)}{\partial \alpha_2(K)} \frac{\partial \alpha_2(K)}{\partial K} = \gamma_2 \frac{\partial \psi_2(K)}{\partial \alpha_2(K)},$$

$$\frac{\partial^2 \psi_2(K)}{\partial K^2} = \frac{\partial^2 \alpha_2(K)}{\partial K^2} \frac{\partial \psi_2(K)}{\partial \alpha_2(K)} + \left( \frac{\partial \alpha_2(K)}{\partial K} \right)^2 \frac{\partial^2 \psi_2(K)}{\partial \alpha_2^2(K)} = \gamma_2^2 \frac{\partial^2 \psi_2(K)}{\partial \alpha_2^2(K)},$$

where we have used the identity

$$\frac{\partial \alpha_2(K)}{\partial K} = 0$$

in the last equality. Substituting (3.18) into (3.10) yields:

$$\frac{1}{2} \gamma_2^2 t^* \frac{\partial^2 \psi_2(K)}{\partial \alpha_2^2(K)} - \psi_2(K) = 0.$$ 

We recognize this as a Cauchy-Euler equation. Making the substitution $\psi_2(K) = (\alpha_2(K))^q$ reduces the equation into:

$$\frac{1}{2} \gamma_2^2 t^* \alpha_2^2(K)(q - 1)(\alpha_2(K))^{q - 2} - (\alpha_2(K))^q = 0.$$ 

This equation can be rewritten;

$$(q^2 - q - \frac{2}{\gamma_2^2 t^*})(\alpha_2(K))^q = 0.$$ 

Since $\alpha_2(K) > 0, \ \forall K \in I_2$ according to Assumption 1, we conclude that all solutions must be of the form $\psi_2(K) = (\alpha_2(K))^q$, with an exponent $q$ satisfying:

$$q = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{8}{\gamma_2^2 t^*}} \right).$$

From this we can conclude that the family of solutions to the local homogeneous equation is the linear combination of the two solutions:

$$\psi_2(K) = \lambda_2 \alpha_2^{q_1}(K) + B_2 \alpha_2^{q_2}(K),$$

where

$$q_{1/2} = \frac{1}{2} \pm \frac{1}{2} Q,$$

$$Q = \sqrt{1 + \frac{8}{\gamma_2^2 t^*}}.$$
and \(\lambda_2, B_2\) are some constants. In order to demand the homogeneous solution to be smooth at the point \(\nu_2\), corresponding to the border between \(I_1\) and \(I_2\), the two conditions

\[
\lim_{K \to \nu_1^+} \psi_1(K) = \lim_{K \to \nu_1^-} \psi_2(K), \\
\lim_{K \to \nu_1^+} \frac{\partial \psi_1}{\partial K}(K) = \lim_{K \to \nu_1^-} \frac{\partial \psi_2}{\partial K}(K)
\]

will be imposed. Substituting (3.16) and (3.17) into these conditions yields the system:

\[
\begin{aligned}
\lambda_2 q_1^{\nu_1}(\nu_1) + B_2 q_2^{\nu_2} = & \lambda_1 \left( e^{p\nu_1} - e^{p(2L-\nu_1)} \right), \\
\lambda_2 q_1^{\gamma_2} q_2^{\gamma_2-1}(\nu_1) + B_2 q_1^{\gamma_2} q_2^{\gamma_2-1} = & p \lambda_1 \left( e^{p\nu_1} + e^{p(2L-\nu_1)} \right).
\end{aligned}
\]

Solving this system for \(\lambda_1\) and \(B_2\) yields:

\[
B_2 = -\lambda_2 \gamma_1^Q \mu_1, \\
\lambda_1 = \lambda_2 \gamma_1^Q (1 - \mu_1)
\]

where

\[
\mu_1 = \frac{\gamma_1 q_1(1 - e^{2p(\nu_1)}) - \gamma_1 p(1 + e^{2p(\nu_1)})}{\gamma_1 q_2(1 - e^{2p(\nu_1)}) - \gamma_1 p(1 + e^{2p(\nu_1)})}.
\]

Substituting the above expression for \(B_2\) into (3.19) yields the local homogeneous solution on \(I_2\) as:

\[
\psi_2(K) = \lambda_2 \left( \alpha_2^{\nu_1}(K) - \gamma_1^Q \mu_1 \alpha_2^{\nu_1}(K) \right),
\]

which has the differential

\[
\frac{\partial \psi_2}{\partial K}(K) = \lambda_2 \left( q_1^{\gamma_2} q_2^{\gamma_2-1}(K) - \gamma_1^Q \mu_1 \gamma_2 q_2^{\gamma_2-1}(K) \right).
\]

The homogeneous solution on \(I_1\), denoted \(\psi_1(K)\), can in terms of \(\lambda_2\) be expressed as:

\[
\psi_1(K) = \lambda_2 \gamma_1^Q (1 - \mu_1) e^{p(K-\nu_1)} \left( 1 - \frac{e^{2p(L-K)}}{1 + e^{2p(L-\nu_1)}} \right).
\]

Before the remaining subintervals are dealt with, let us conclude that the sequence of the homogeneous solutions \(\psi_1(K)\) and \(\psi_2(K)\) is now guaranteed to be \(C^1\) on the interval \(I_1 \cup I_2\).

### 3.4.3 Solution on \(I_4\)

Let us now consider the subinterval \(I_4\). On this subinterval, the LVG-volatility function is assumed to be constant: \(\alpha_4(K) = \gamma_4 > 0, \quad \forall K \in I_4\). Notice that \(I_4\) borders the absorbing point \(U\) where the boundary condition \(\psi_4(U) = 0\) will

\footnote{Notice that since \(\alpha(K)\) is assumed to be a continuous function we may simplify the expression by using the notation \(\alpha_2(\nu_1) = \gamma_1\).}
be imposed. Using an identical method as on $I_1$, we find the family of local homogeneous solutions as:

$$
\psi_4(K) = \lambda_4 \left( e^{sK} - e^{s(2U-K)} \right),
$$

(3.23)

where

$$
s = \sqrt{\frac{2}{\gamma_4 t^*}}.
$$

For later use, we will also calculate the differential of (3.23);

$$
\frac{\partial \psi_4}{\partial K} = s \lambda_4 \left( e^{sK} + e^{s(2U-K)} \right).
$$

(3.24)

3.4.4 Solution on $I_3$

The subinterval $K \in I_3$ is similar to the subinterval $I_2$ in the sense that the LVG-volatility function is assumed to be linear. On $I_3$, the LVG-volatility function is assumed to follow:

$$
\alpha_3(K) = \gamma_3 K + b_3, \quad \forall K \in I_3.
$$

Using the identical procedure as on $I_2$, the local homogeneous solution can be shown as:

$$
\psi_3(K) = \lambda_3 \alpha_3^{r_1^3}(K) + B_3 \alpha_3^{r_2^3}(K),
$$

(3.25)

where

$$
r_{1/2} = \frac{1}{2} \pm \frac{1}{2} R,
$$

$$
R = \sqrt{1 + \frac{8}{\gamma_3^2 t^*}},
$$

and $\lambda_3, B_3$ are some constants. Imposing smoothness conditions in the border point between $I_3$ and $I_4$, called $\nu_2$, yields:

$$
\psi_3(K) = \lambda_3 \left( \alpha_3^{r_1^3}(K) - \gamma_4^R \mu_2 \alpha_3^{r_2^3}(K) \right)
$$

(3.26)

and

$$
\frac{\partial \psi_3}{\partial K}(K) = \lambda_3 \left( r_1 \gamma_3 \alpha_3^{r_1^3-1}(K) - \gamma_4^R \mu_2 r_2 \alpha_3^{r_2^3-1}(K) \right),
$$

(3.27)

where

$$
\mu_2 = \frac{\gamma_3 r_1 (1 - e^{2s(U-\nu_2)}) - \gamma_4 s (1 + e^{2s(U-\nu_2)})}{\gamma_3 r_2 (1 - e^{2s(U-\nu_2)}) - \gamma_4 s (1 + e^{2s(U-\nu_2)})}.
$$

Expressing $\psi_4(K)$ in terms of $\lambda_3$ yields:

$$
\psi_4(K) = \lambda_3 \gamma_4^{r_1^4} (1 - \mu_2) e^{s(K-\nu_2)} \left( \frac{1 - e^{2s(U-K)}}{1 - e^{2s(U-\nu_2)}} \right).
$$

(3.28)

Notice that the sequence of $\psi_3(K)$ and $\psi_4(K)$ is now $C^1$ on the interval $I_3 \cup I_4$. 

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3.4. Derivation of the Analytical Solution

3.4.5 Merging the Solutions

The local solutions to the homogeneous equation are now expressed in terms of the two constants \( \lambda_2 \) and \( \lambda_3 \). The final step towards reaching Proposition 2 is to impose conditions in the border point between \( I_2 \) and \( I_3 \). This point corresponds to the value of the underlying asset, \( x \). Recall that the purpose of these calculations is to find an option price function \( C(K) \in C^2 \). From \( \Psi(K) \), \( C(K) \) can be found by adding the intrinsic value of the call option; \( C(K) = \Psi(K) + (x - K)^+ \). Notice that \((x - K)^+\) is continuous, but has a discontinuous differential;

\[
\frac{\partial}{\partial K}(x - K)^+ = \begin{cases} 
0, & \text{if } K > x, \\
-1, & \text{if } K < x.
\end{cases}
\]

In order to compensate for this discontinuity, the following proposition is made concerning the behavior of the homogeneous solution \( \Psi(K) \) at the point \( K = x \).

**Proposition 4** The homogeneous solution is continuous at the point \( K = x \), but its differential with respect to \( K \) has a discontinuity corresponding to a fall of magnitude \( 1 \):

\[
\lim_{K \to x^-} \psi_2(K) = \lim_{K \to x^+} \psi_3(K),
\]
\[
\lim_{K \to x^-} \frac{\partial \psi_2}{\partial K}(K) - 1 = \lim_{K \to x^+} \frac{\partial \psi_3}{\partial K}(K).
\]

Plugging in the expressions from the equations (3.20), (3.26), (3.21) and (3.27) into the two conditions in Proposition 4 yields the system:

\[
\lambda_2 \left( \alpha_2^q(x) - \gamma_1^Q \mu_1 \alpha_2^q(x) \right) = \lambda_3 \left( \alpha_3^r(x) - \gamma_4^R \mu_2 \alpha_3^r(x) \right),
\]
\[
\lambda_2 \left( q_1 \gamma_2 \alpha_2^{q_1-1}(x) - \gamma_1^Q \mu_1 \gamma_2 \alpha_2^{q_2-1}(x) \right) = 1 + \lambda_3 \left( r_1 \gamma_3 \alpha_3^{r_1-1}(K) - \gamma_4^R \mu_2 \gamma_3 \alpha_3^{r_2-1}(x) \right).
\]

(3.29)

In order to simplify the solution to this system, notice that since the LVG-volatility function \( \alpha(K) \) is assumed to be continuous we can use the notations:

- \( \alpha_2(x) = \alpha_3(x) = \sigma_x \in \omega \),
- \( \alpha_1(\nu_1) = \alpha_2(\nu_1) = \gamma_1 = \sigma_1 \in \omega \),
- \( \alpha_3(\nu_2) = \alpha_4(\nu_2) = \gamma_4 = \sigma_2 \in \omega \).

Solving the system (3.29) with respect to \( \lambda_2 \) and \( \lambda_3 \) yields:

\[
\lambda_2 = \frac{1}{\beta_x \left( \sigma_2^{q_1} - \sigma_1^Q \mu_1 \sigma_2^{q_2} \right)},
\]
\[
\lambda_3 = \frac{1}{\beta_x \left( \sigma_2^{r_1} - \sigma_2^R \mu_2 \sigma_2^{r_2} \right)}.
\]
where

\[ \beta_x = \frac{\gamma_2 q_1}{\sigma_x} \left( \frac{1 - \mu_1 \frac{\sigma_x}{\sigma_z}}{1 - \mu_1 \left( \frac{\sigma_x}{\sigma_z} \right)^Q} \right) = \frac{\gamma_3 r_1}{\sigma_x} \left( \frac{1 - \mu_2 \frac{\sigma_x}{\sigma_z}}{1 - \mu_2 \left( \frac{\sigma_x}{\sigma_z} \right)^R} \right). \]

When put together, the results shown in the last few pages prove Proposition 2. Notice that in the derivation we have used all degrees of freedom and hence the solution is unique. The corresponding formula for put options can easily be found by using the put-call parity.

### 3.5 Special Cases

In this section, a few interesting special cases are considered. Let us first consider the case when the boundary points \( L \) and \( U \) go to 0 and \( \infty \) respectively.

Letting \( L \) decrease towards zero does not require any limits to be solved and it will therefore not be discussed further. Letting \( U \) increase towards infinity requires us to solve some simple limits which are presented below. First, let us investigate the limit for the local homogeneous solution on \( I_4 \):

\[
\lim_{U \to \infty} \psi_4(K) = \lim_{U \to \infty} \frac{1}{\beta_x} \left( \frac{\sigma_2}{\sigma_z} \right)^{r_1} \frac{1 - \mu_2}{1 - \mu_2 \left( \frac{\sigma_2}{\sigma_z} \right)^R} e^{s(K - \nu_2)} \frac{1 - e^{2s(U - \nu_2)}}{1 - e^{2s(U - \nu_2)}}
\]

which can be seen by multiplying both the nominator and denominator in the last fraction with \( e^{-2s(U - \nu_2)} \). In the derivation above it is assumed that the limit for \( \mu_2 \) exists. This limit can be shown as:

\[
\lim_{U \to \infty} \mu_2 = \lim_{U \to \infty} \frac{\gamma_3 r_1 - \sigma_2 s - e^{2s(U - \nu_2)}(\gamma_3 r_1 + \sigma_2 s)}{\gamma_3 r_2 - \sigma_2 s - e^{2s(U - \nu_2)}(\gamma_3 r_2 + \sigma_2 s)} = \frac{\gamma_3 r_1 + \sigma_2 s}{\gamma_3 r_2 + \sigma_2 s}
\]

where the last equality can be shown by using the formula of l’Hôpital or by multiplying both nominator and denominator with \( e^{-2s(U - \nu_2)} \).

The next modification of the solution in Proposition 2 we will consider is the case when the subintervals \( I_2 \) and \( I_3 \) vanish. Since this causes the LVG-volatility function to turn into a discontinuous piecewise constant function, the solution in Proposition 2 is no longer valid\(^3\).

Let us go back to the point where two solutions of the homogeneous equation (3.14) were found using constant coefficients. Denote these two solutions \( \psi^Y_1(K) \) and \( \psi^Y_2(K) \). In accordance results from this chapter, these two local solutions can be expressed as:

\[
\begin{align*}
\psi^Y_1(K) &= \lambda_1 \left( e^{sK} - e^{s(2L - K)} \right), & \forall K \in I_1, \\
\psi^Y_2(K) &= \lambda_2 \left( e^{sK} - e^{s(2U - K)} \right), & \forall K \in I_4,
\end{align*}
\]

\(^3\)When constructing the closed-form solution, \( \alpha(K) \) was assumed to be continuous.
where
\[ p = \sqrt{\frac{2}{\lambda_1 \sigma_1^2}}, \quad s = \sqrt{\frac{2}{\lambda_2 \sigma_2^2}}. \]

This expression can be reformulated as:
\[
\psi_1^V(K) = c_1^{1,1} e^{-Kz/\Sigma_1} + c_1^{2,1} e^{Kz/\Sigma_1},
\]
\[
\psi_2^V(K) = c_1^{1,2} e^{-Kz/\Sigma_2} + c_1^{2,2} e^{Kz/\Sigma_2},
\]
where
\[ c_1^{1,1} = -\lambda_1 e^{zL/\Sigma_1}, \quad c_1^{2,1} = \lambda_1 e^{-zL/\Sigma_1}, \quad c_1^{1,2} = \lambda_2 e^{zU/\Sigma_2}, \quad c_1^{2,2} = -\lambda_2 e^{-zU/\Sigma_2}. \]

This formula is identical to the solution given by Carr and Nadtochiy (2013, pp.23) for \( R = 1 \). The variables are translated from the notations used in Section 3.3 as:
\[ z = \frac{1}{\sqrt{\lambda_1}}, \quad \Sigma_1 = \frac{\sigma_1}{\sqrt{2}}, \quad \Sigma_2 = \frac{\sigma_2}{\sqrt{2}}, \quad \lambda_1 = \lambda_1 e^{pL}, \quad \lambda_2 = -\lambda_4 e^{sU}. \]

Let us now return to the old notations. The next step is to glue the option price curve together by imposing the smoothness conditions:
\[
\lim_{K \to x^-} \psi_1^V(K) = \lim_{K \to x^+} \psi_2^V(K),
\]
\[
\lim_{K \to x^-} \frac{\partial}{\partial K} \psi_1^V(K) - 1 = \lim_{K \to x^+} \frac{\partial}{\partial K} \psi_2^V(K).
\]

Substituting the expressions from (3.31) into the above conditions yields the system:
\[
\begin{align*}
\lambda_1 (e^{px} - e^{p(2L-x)}) &= \lambda_4 (e^{sx} - e^{s(2U-x)}), \\
\beta_1 (e^{px} + e^{p(2L-x)}) &= 1 + s \lambda_4 (e^{sx} + e^{s(2U-x)}).
\end{align*}
\]

Solving this system for \( \lambda_1 \) and \( \lambda_4 \) yields:
\[
\lambda_1 = \frac{1}{\beta_V (e^{px} - e^{p(2L-x)})},
\]
\[
\lambda_4 = \frac{1}{\beta_V (e^{sx} - e^{s(2U-x)})},
\]
where
\[ \beta_V = p \left( \frac{e^{px} + e^{p(2L-x)}}{e^{px} - e^{p(2L-x)}} \right) - s \left( \frac{e^{sx} + e^{s(2U-x)}}{e^{sx} - e^{s(2U-x)}} \right). \]

The call price function \( C^V(K) = \psi^V(K) + (x - K)^+ \) can therefore be expressed as:
\[
C^V(K) = (x - K)^+ + \frac{1}{\beta_V} \left[ \left( \frac{e^{pK} - e^{p(2L-K)}}{e^{px} - e^{p(2L-x)}} \right) 1\{K \leq x\} + \left( \frac{e^{sK} - e^{s(2U-K)}}{e^{sx} - e^{s(2U-x)}} \right) 1\{K > x\} \right]. \tag{3.32}
\]

It is now a trivial exercise to achieve the pure constant version by setting \( \sigma_1 = \sigma_2 \Rightarrow s = p \).
3.6 Possible Generalization

The parametrization presented in this chapter will suffer problems due to the restrictions on the LVG-volatility functions from Assumption 1. The main issue is the assumption that the LVG-volatility is constant for $K \in I_1$ and $K \in I_2$. Ideally, we would like diffusion functions in asset pricing models to start in the origin and go to infinity in the limit. These two desirable conditions are satisfied by the Black & Scholes model, where the diffusion function increases linearly from zero. Using a constant LVG-volatility in the tails will cause the tail probabilities to become larger for $K << X$ and smaller for $K >> X$ compared to the Black & Scholes model. In Figure 5.4, an example is given of an unwanted effect originating from the assumption of constant LVG-volatility in outer regions. Evidently, implied volatility smiles generated by the proposed model will start decreasing for very large strikes. A related, but much more powerful, effect can be seen in the Normal model for asset returns\(^4\), for which implied volatility function will decrease substantially for high strikes. The effect is decreased significantly by introducing the Gamma process into the model, which causes the tail probabilities in the return distribution to increase. In Section 5.5, we empirically study the impact of this effect on the model proposed in this chapter.

One could consider a generalization of the proposed model by allowing the LVG-volatility function on $I_1$ and $I_4$ to be linear instead of constant functions. This would let us use a LVG-volatility function which starts in the origin and goes to infinity in the limit. It should be possible to find a unique closed-form solution to (3.10) under this Assumption as well, but the analytical expression would be even more complicated. A slightly different approach would be to define the problem as a matrix equation. Sadly, the matrix will be very ill-conditioned, making the numerical errors substantial. Further, such an approach could make it problematic to develop a technique for performing arbitrage-free interpolation between calibrated option price functions.

The parametrization of European call option prices in Proposition 2 uses lower and upper bounds $L$ and $U$. In the applications in this thesis, these parameters will be set to 0 and $\infty$. This choice of parameters can be motivated in situations when modeling assets that theoretically can take any value on the positive real axis, such as exchange rates or stocks. In practice, other choices of boundary points could be used for forcing the option prices to follow some requested behavior. It is an interesting question whether the boundary points can efficiently be used to find an optimal balance between the effects of the Gamma process and the constant tail volatility. In implementations using other values than 0 and $\infty$ for the boundary points, it must be considered that the analytical formula in Proposition 2 will run into computational difficulties using a large upper boundary $U$, without simplifying it to infinity. One must also consider that option prices will not be well-defined for values $K \notin [L, U]$. If the analytical formula is used for such values, negative prices will appear. Further, such boundary points will cause $\Psi(K)$, and thus also the implied volatility, to drop to zero when approaching $L$ and $U$. As a consequence, produced volatility smiles will have an appearance which can be problematic in many aspects.

\(^{4}\)The Normal model assumes the diffusion function to be constant. The model has a closed form solution and is discussed by Hull (2008, pp. 692) among others.
Chapter 4

The FX-Option Market

As this thesis focuses on applications in the FX-option market\footnote{Much of the theory can possibly be extended to other underlying assets after some small adjustments.}, the following chapter is dedicated to briefly explaining FX-related notations used in this document and some of the terminology unique to the FX-option market. The interested reader can find more information concerning the terminology and quoting conventions in FX-markets in works by Reiswich and Wystup (2010) or Clark (2011).

4.1 Market Quotes

In the FX-option market, both call and put options give the owner the right, but not the obligation, to use a predefined exchange rate at some future point in time. The difference between the two contracts is which currency is used as strike and therefore has a predefined amount of units in the agreed-upon exchange rate. Hence, a call option in one currency can be interpreted as a put option in the other currency. This confusion is handled by instead quoting prices in implied volatility. In the theorem below it is shown that put and call options have the same implied volatility. The theorem, with an almost identical proof, is provided by Hull (2008, pp.381). A consequence of this theorem is that both put and call options, independent of the numéraire currency of the investor, can be found from an implied volatility quote.

**Theorem 9** Let $C_M$ be the price of a market traded call option with underlying asset $S$, strike $K$ and time to maturity $\tau$. Further let $P_M$ be a put option with the same underlying asset, strike, and time to maturity. Then $C_M$ and $P_M$ have the same implied volatility $\sigma_{imp}$.

**Proof.** First, notice that according to Theorem 2.9 the two market traded options are related as:

$$C_M - P_M = S - Ke^{-r(T-t)}.$$ 

Let $C_{BS}$ and $P_{BS}$ be a call and a put option found by using (2.5). Let both $C_{BS}$ and $P_{BS}$ have the same underlying asset, strike, and time to maturity as...
the market traded options. These prices will also satisfy Theorem 2.9:

\[ C_{BS} - P_{BS} = S - Ke^{-(T-t)}. \]

Hence, the pricing error for the put option and call option is the same:

\[ C_M - C_{BS} = P_M - P_{BS}. \]

The conclusion follows that when \( \sigma = \sigma_{imp} \), which means that \( C_{BS} = C_M \), the equality \( P_{BS} = P_M \) must also be satisfied. This implies that \( \sigma_{imp} \) must be the implied volatility of \( P_M \) as well.

Because of its flexibility, implied volatility is the unit generally found in FX-option market quotes.

### 4.2 Variable Space

When performing maturity dimension interpolation in Chapter 6, it will be important to exclude calendar spread arbitrage from a grid of European option prices. As concluded in Condition 1, this is accomplished by ensuring that the inequality \( \frac{\partial C}{\partial T} \geq 0 \) is satisfied for all option prices. Further, the underlying asset needs to be a \( \mathbb{Q} \)-martingale since the model developed in this thesis will be valid only for driftless diffusions. As concluded by Gatheral and Jacquir (2013), it is important to pick an appropriate variable space in order to achieve these results.

**Definition 10** Let \( C(S, t) \) be the value of a call option at time \( t \) with some underlying spot exchange rate \( S_t \). Assume that the option expires at time \( T \) and has strike \( K \). The moneyness of the option is defined as:

\[ K_M = \frac{K}{f(0, T)} \]

and the moneyness-forward as:

\[ X_t = \frac{f(t, T)}{f(0, T)}, \]

where \( f(t, T) \) is the forward exchange rate at time \( t \) with the same maturity as the call option.

Since the moneyness-forward has initial value \( X_0 = 1 \), it can be considered a forward contract which has been normalized at inception. The theorem below describes an important property of the moneyness-forward.

**Theorem 10** Assume that the spot exchange rate \( S_t \) is a diffusion process following the dynamics:

\[ dS_t = S_t(r_d(t) - r_f(t)) + S_t\sigma(S_t, t) dW^{Q_d}_t, \quad \text{(4.1)} \]

where \( r_d(t) \), \( r_f(t) \) and \( \sigma(S_t, t) \) are deterministic functions bounded from above and away from zero. Then \( X_t \), as in Definition 10, is a martingale under the domestic\(^2 \) risk-neutral probability measure \( Q_d \).

\(^2\)The phrases “Domestic” and “Foreign” currencies, investors et cetera are very common in financial literature covering foreign exchange related topics. The domestic currency simply corresponds to the numéraire currency of the investor, while the foreign currency can be considered some risky asset.
Proof. The martingale property of forward contracts was observed by Black (1976). Since then, the result has been used in a large quantity of papers. Assume that $S_t$ follows the dynamics in (4.1). Differentiating $X_t$ using the standard formula for forward contracts\(^3\) yields:

$$dX_t = \frac{1}{f(0,T)} d f(t,T) = \frac{1}{f(0,t)} d(S_t e^{\int_t^T (r_d(s) - r_f(s)) ds})$$

$$= \frac{1}{f(0,T)} \left( e^{\int_t^T (r_d(s) - r_f(s)) ds} dS_t - (r_d(t) - r_f(t)) S_t e^{\int_t^T (r_d(s) - r_f(s)) ds} dt \right)$$

$$= \frac{e^{\int_t^T (r_d(s) - r_f(s)) ds}}{f(0,T)} S_t \sigma(S_t, t) dW^Q_t.$$

Notice that since the second differential of the forward contract with respect to the spot rate is zero, we get no contribution from quadratic variation. This means that $X_t$ is a driftless diffusion. Since the diffusion function is assumed to be bounded from above and away from zero, this completes the proof. \(\square\)

In Lemma 1, we use Theorem 10 to formulate a very convenient way to use the closed form solution for European call options in the Black & Scholes model, (2.7). A very similar approach is taken by Gatheral and Jacquir (2013) among others.

**Lemma 1** The expression

$$E[(S_T - K)^+|S_0 = S]/f(0,T)$$

can be regarded as the price of a call option, $C(K, \tau)$, at time $t = 0$ with underlying asset $X_t$, strike $K_M$ and time to maturity $\tau = T - t$. Further, assuming the volatility to be constant, the undiscounted value of this call option is:

$$C(K_M, \tau) = N(d_1) - K_M N(d_2),$$

where

$$d_{1/2} = -\frac{\ln(K_M) \pm \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}}.$$

**Proof.** First notice that:

$$E[(S_T - K)^+|S_0 = S]/f(0,T) = E[(X_T - K_M)^+|X_0 = 1],$$

where $X_t$ and $K_M$ comes from Definition 10. Now, lets assume $X_t$ to have a constant volatility. Using (2.5) and the fact that $X_t$ is a $Q_d$-martingale it can be concluded that for $t = 0$:

$$C(K_M, \tau) = X_0 N(d_1) - K_M N(d_2) = N(d_1) - K_M N(d_2),$$

where $d_1$ and $d_2$ originates from the no-interest rate version of (2.5). \(\square\)

\(^3\)With standard formula we refer to the expression: $f(t,T) = S_t e^{\int_t^T (r_d(s) - r_f(s)) ds}$. For a derivation of this formula, see Hull (2008, pp.113).
This choice of variables makes it possible to formulate the proposition below.

**Proposition 5** Let $X_t$ be a $\mathbb{Q}_d$-martingale and $\tau_2 \geq \tau_1 > 0$ two time to maturities. Assuming the quotes to be arbitrage-free, the following inequality is true:

$$C(K_M, \tau_2) = E[(X_{\tau_2} - K_M)^+] \geq E[(X_{\tau_1} - K_M)^+] = C(K_M, \tau_1).$$

(4.2)

This result, which is shown by Gatheral and Jacquir (2013) among others, follows from the martingale property of $X_t$ and the fact that the payoff from the call option is bounded from below but not from above.

In (2.11), interest rates have been disregarded. Hence, model implementations using this version of the Dupire equation should be performed in $C(K_M, \tau)$ space, where the underlying asset is assumed to be a $\mathbb{Q}_d$-martingale. One could criticize that the prices coming from such implementations do not correspond to the “real” prices. Notice, however, that the input to option pricing models in the FX-market is in implied volatility, and the output should be in implied volatility as well. Therefore, it is important that the conversion between implied volatility and price is performed in a consistent way.

The notation $K_M$, for moneyness, will not be used in the later chapters in this thesis. Since the implementations will be valid in all models where the underlying asset is a $\mathbb{Q}_d$-martingale, only the notation $K$ will be used for strike.

### 4.3 Extracting Market Data

In this section we discuss how to convert FX-option market quotes into data we can use for calibrating the proposed model. Most of the information in this section is gathered from Reiswich and Wystup (2010) and Clark (2011). The applications in Chapter 5 and 6 will use the currency pairs EURUSD and EURSEK as examples. Since quoting conventions in the FX-option market differ between currency pairs, it will be necessary to develop two slightly different methodologies in order to transform market quotes into calibration data. The original market quoted data consists of implied volatility surfaces over time to maturity and $\Delta$. Since the model developed in Chapter 3 is in strike space, the market quotes must be transformed before calibration.

#### 4.3.1 $\Delta$-Conventions

Depending on the currency pair, the convention for what the quoted $\Delta$’s represents differ. Clark (2011, pp.43) mentions four different types of $\Delta$’s. All of these will be described in this section.

**Alternative 1: Spot or Forward $\Delta$**

The spot $\Delta$, $\Delta_S$, corresponds to the first differential of the option price with respect to the value of the underlying in the standard Black & Scholes model.

---

1By covering the two examples EURUSD and EURSEK we will in fact cover most examples that do not include currencies from Latin America
4.3. Extracting Market Data

The value can, using standard notations, be calculated as:

\[ \Delta S = \frac{\partial C}{\partial S} = \omega e^{-r_d(T-t)}N(\omega d_+), \]

where

\[ d_+ = \frac{\ln \left( \frac{S_t}{K} \right) + (r_d - r_f + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \]

\[ \omega = \begin{cases} 
1, & \text{if call option} \\
-1, & \text{if put option.} 
\end{cases} \]

While it is popular to use \( \Delta S \) for educational purposes, it is seldom used in practice in the FX-option market. This is because the formula includes discount factors, something which has proven to be unpractical\(^5\).

The forward \( \Delta \), denoted \( \Delta_F \), instead considers option prices as if they are coming from the Black model. This model was initially introduced by Black (1976) for applications on the commodity option market and is sometimes referred to as Black76. In this model, prices for European call options under constant volatility can be found through the expression

\[ C_t = e^{-r_d(T-t)}[f(t,T)N(\omega d_+ - K N(\omega d_+)), \]

where

\[ d_\pm = \frac{\ln \left( \frac{f(t,T)}{K} \right) \pm \frac{\sigma^2}{2}(T - t)}{\sigma \sqrt{T - t}}, \]

\[ f(t,T) = S_t e^{(r_d-r_f)(T-t)}. \]

We can find \( \Delta_F \) by differentiating the expected price of the future value of the option with respect to the price of a forward contract \( f(t,T) \):

\[ \Delta_F = \frac{\partial E[C_T]}{\partial f(t,T)} = \frac{\partial (e^{r_d(T-t)}C_t)}{\partial f(t,T)} = \omega N(\omega d_+). \]

As one can see, the above expression for \( \Delta_F \) does not explicitly use any discount factors. Generally, forward contracts are available for a large variety of maturities.

4.3.2 Application

In this thesis, the normalized price formula from Lemma 1 will be used. This formulation is consistent with \( \Delta_F \) in the sense that the underlying asset is assumed to be a martingale. The difference is that the initial value of the forward contract is normalized at inception.

\[ \Delta_X = \frac{\partial (E[C_T]/f(0,T))}{\partial (f(t,T)/f(0,T))} = \omega N(\omega d_+) \]

\(^5\)Clark (2011, pp.48) argues that the convention for \( \Delta \)-quotations changed after the financial crisis in 2008. During the crisis, liquidity in short term interest contracts supposedly went down dramatically. This created disagreements between banks over values for \( \Delta_S \).
where
\[ d_+ = \frac{-\ln(K) + \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}} \]

**Alternative 2: Premium Adjustment**

For a premium-adjusted \( \Delta \), the value\(^6\) of the option is subtracted from the \( \Delta \). As an example, consider a short position in a European call option on EURSEK. The EURSEK exchange rate is assumed to be equal to \( S \) when the contract is initiated. The price according to the Black & Scholes model is calculated and denoted \( C \). The option will, at time \( T \), allow the buyer to pay \( K \) SEK and receive 1 EUR. In order for the seller to \( \Delta \)-hedge the short position, \( \Delta_S \) EUR is needed. However, since the buyer has paid the premium of \( C \) SEK, corresponding to \( C \) EUR, only an additional amount of \( \Delta_S - \frac{C}{S} \) must be added in order to complete the hedge. This is the premium-adjusted spot \( \Delta \), defined by:

\[ \Delta_{S,pa} = \Delta_S - \frac{C}{S} \]

It is easily shown that:

\[ \Delta_{S,pa} = \omega e^{-r_f(T-t)}N(\omega d_+) - \omega \left[ \frac{e^{-r_f(T-t)}S_tN(\omega d_+)}{S_t} - e^{-r_f(T-t)}KN(\omega d_-) \right], \]

which can be simplified into

\[ \Delta_{S,pa} = \omega e^{-r_f(T-t)} \frac{K}{S_t} N(\omega d_-). \]

In a similar way, the value for a premium-adjusted forward \( \Delta \), defined by \( \Delta_{F,pa} = \Delta_F - \frac{F(V_T)}{f(t, T)} \), can be calculated as:

\[ \Delta_{F,pa} = \omega \frac{K}{f(t, T)} N(\omega d_-). \]

Normalizing the forward contract at inception results in the premium-adjusted \( \Delta \) in our setting:

\[ \Delta_{X,pa} = \omega KN(\omega d_-), \]

where

\[ d_- = \frac{-\ln(K) - \frac{1}{2}\sigma^2(T-t)}{\sigma \sqrt{T-t}}. \]

**4.3.3 Choosing a \( \Delta \)-Convention**

According to Reiswich and Wystup (2010), the two currencies considered in this thesis are both quoted as forward \( \Delta \)'s. Further, the premium adjustment is performed for EURSEK options but not for EURUSD options. The necessary conventions which will be used in Chapter 5 and 6 are summarized below, using the notation \( \tau = T - t \) for time to maturity:

\[ \Delta_X = \begin{cases} \omega N(\omega d_+), & \text{if EURUSD option} \\ \omega KN(\omega d_-), & \text{if EURSEK option} \end{cases} \]

where

\[ \omega = \begin{cases} 1, & \text{if Call option} \\ -1, & \text{if Put option} \end{cases} \quad \text{and} \quad d_\pm = \frac{-\ln(K) \pm \frac{1}{2}\sigma^2 \tau}{\sigma \sqrt{\tau}}. \]

\(^6\)For purposes such as this, the term risk premium or just premium is often used instead of value or price.
4.3. Extracting Market Data

4.3.4 At-The-Money Conventions

Out of the five options which are found for every maturity in FX-market quotes, the ATM option is usually the most liquid. There are two different definitions of ATM which are used in practice. In equity markets, an option is considered to be ATM if it has a strike which is equal to the contemporaneous value of the underlying asset. One of the ways of defining the meaning of ATM in the FX-option market is an option with a strike equal to the forward exchange rate at inception $f(0, T)$. This convention will be denoted ATMF:

$$K_{ATMF} = f(0, T).$$  \hspace{1cm} (4.3)

According to Clark (2011, pp.51), this convention is only used when one of the currencies involved are from a South American country.

The second way of defining ATM is when the $\Delta$ of a portfolio consisting of one call option and one put option with the same strike and time to maturity is equal to zero. This convention will be denoted DNS:

$$\Delta_C(K_{DNS}, \tau) + \Delta_P(K_{DNS}, \tau) = 0.$$  \hspace{1cm} (4.4)

This expression is analytically solvable, but since the definition of $\Delta$ varies between currencies, so will the solution of (4.4). The expressions for $K_{DNS}$ for a normalized forward $\Delta$ depending on premium adjustment are:

$$K_{DNS} = \begin{cases} f(0, T)e^{\frac{1}{2}\sigma_{ATM}^{2}\tau}, & \text{if no premium adjustment (EURUSD)}, \\ f(0, T)e^{-\frac{1}{2}\sigma_{ATM}^{2}\tau}, & \text{if premium adjustment (EURSEK)}. \end{cases}$$  \hspace{1cm} (4.5)

4.3.5 Changing Coordinates

Below, the conventions necessary to derive price and strike from implied volatility and $\Delta$ are presented. The ATM option strike, $K_{ATM}$ can be calculated analytically by using (4.5). For the other four $\Delta$’s, the case is slightly more complicated. Quotes in the FX-option market consist of implied volatilities for the ATM option, Butterfly spreads with $\Delta_{BF} = 0.25$ and $\Delta_{BF} = 0.1$, and Risk reversal spreads with $\Delta_{RR} = 0.25$ and $\Delta_{RR} = 0.1$. First, we must transform these quotes into call and put option implied volatilities. In accordance with Clark (2011), this is achieved through the linear identities

$$\begin{align*}
\sigma_{10\Delta PUT} &= \sigma_{ATM} + \sigma_{10\Delta BF} - 0.5\sigma_{10\Delta RR}, \\
\sigma_{25\Delta PUT} &= \sigma_{ATM} + \sigma_{25\Delta BF} - 0.5\sigma_{25\Delta RR}, \\
\sigma_{25\Delta CALL} &= \sigma_{ATM} + \sigma_{25\Delta BF} + 0.5\sigma_{25\Delta RR}, \\
\sigma_{10\Delta CALL} &= \sigma_{ATM} + \sigma_{10\Delta BF} + 0.5\sigma_{10\Delta RR},
\end{align*}$$  \hspace{1cm} (4.6)

where $\sigma_{ATM}$ is found in the market quote. The next step is to find the strikes corresponding to these volatilities. By solving the equations

$$\begin{align*}
\Delta_X(\omega = -1, \sigma_{10\Delta PUT}, K_{10\Delta PUT}, T) &= -0.1, \\
\Delta_X(\omega = -1, \sigma_{25\Delta PUT}, K_{25\Delta PUT}, T) &= -0.25, \\
\Delta_X(\omega = 1, \sigma_{25\Delta CALL}, K_{25\Delta CALL}, T) &= 0.25, \\
\Delta_X(\omega = 1, \sigma_{10\Delta CALL}, K_{10\Delta CALL}, T) &= 0.1,
\end{align*}$$  \hspace{1cm} (4.7)
with respect to strike, we can generate four strikes corresponding to the implied volatilities from (4.6). In the EURUSD case, the expressions are invertible:

\[
\Delta_X(\omega, \sigma, K, T) = \omega N(\omega d_+)
\]

\[
\Rightarrow \omega N^{-1}(\omega \Delta_X) = \frac{-\ln(K) + \frac{\sigma^2}{2} \tau}{\sigma \sqrt{\tau}}
\]

\[
\Rightarrow K = \exp\left(\frac{\sigma^2}{2} \tau - \omega \sigma \sqrt{\tau} N^{-1}(\omega \Delta_X)\right).
\]

For the EURSEK, where the \(\Delta\) is premium adjustment, the expressions instead follow:

\[
\Delta_X(\omega, \sigma, K, T) = \omega KN(\omega d_-).
\]

This formula is not invertible. In order to find the corresponding strikes, the equations from (4.7) must be solved by using some numerical root searcher. In this implementation, the MATLAB function fzero is used.

### 4.4 Summary and Algorithm

In this section we summarize the information presented in this chapter into an algorithm. The following procedure transforms market quoted implied volatility surfaces into price-strike pairs for use in the calibration in Chapter 5.

1. Calculate the ATM strike by using (4.5).
2. Calculate implied volatilities for put and call options using (4.6).
3. Choose one of the conventions from Section 4.3.3 and use it to generate strikes by solving (4.7).
4. Calculate prices by using Lemma 1, or the corresponding formula for put options.

The strike-price pairs produced by this algorithm correspond to options with a normalized forward contract as underlying asset. Since such a contract is a \(Q_\mu\)-martingale, this assumption is consistent with the fact that interest rates were disregarded when the proposed model was developed in Chapter 3. Since a normalized forward contract always has initial value 1, we do not need market quotes with forward exchange rates.
Chapter 5
Calibration

In this chapter, the closed-form solution from Proposition 2 is calibrated to implied volatility smiles from the FX-option market. It is essential that we find parameter values such that the generated implied volatility smiles are within market quoted bid-ask spreads. The calibration scope is restricted to independent volatility smiles with constant time to maturity. We propose a calibration algorithm which uses a least-square optimization solver. In Chapter 6, the proposed model is generalized to allow for interpolation between calibrated option price functions. The data used in these calibrations corresponds to daily market data for 208 days between 2013-05-20 and 2014-03-12. The considered currency pairs are EURUSD and EURSEK. All data is collected from Reuters Eikon, provided by FENICS. All calculations are performed in MATLAB on a PC with a 3.4 GHz quad-core processor.

5.1 Two Sub-Models

As mentioned in the text after Proposition 2, the price function can be considered as having either three or five unknown parameters. The parameter set $\omega$ uniquely determines a LVG-volatility function, and therefore also determines a continuum of European option prices through the parameterization from Proposition 2. The set $\omega$ is defined as $\omega := \{\sigma_1, \sigma_x, \sigma_2, \nu_1, \nu_2\}$, where the parameters $\sigma_1, \sigma_2,$ and $\sigma_x$ are the LVG-volatilities in the two subintervals $K \in I_1, K \in I_4,$ and at $K = x$. The parameters $\nu_1$ and $\nu_2$ correspond to the two border points where the LVG-volatility function changes between constant and linear behavior. These two positive parameters are assumed to satisfy the criteria $L < \nu_1 < x$ and $x < \nu_2 < U$. Logically, these values should be within the range of calibration points since this is the region where the degrees of freedom are needed to fit the market quoted option prices. In Section 5.3, we evaluate the difference between the following two approaches:

1. Using the parameters $\nu_1, \nu_2$ as degrees of freedom in a residual minimization problem.

2. Setting the parameters $\nu_1, \nu_2$ to static values dependent on the structure of the market quote.
The quotes from the FX-option market are, as mentioned in Chapter 4, positioned at strikes corresponding to five ∆’s:

\[
\begin{align*}
K_{10∆PUT}, \\
K_{25∆PUT}, \\
K_{DNS}, \\
K_{25∆CALL}, \\
K_{10∆CALL}.
\end{align*}
\]

This list is in order of magnitude. The first model explored in this chapter is a five parameter model denoted model 5P. For this model, the calibration algorithm is allowed to use the entire parameter set \( \omega \), including \( \nu_1 \) and \( \nu_2 \), as degrees of freedom in order to minimize the residuals. The second model, denoted Model 3P, uses the market quote dependent values

\[
\begin{align*}
\nu_1 &= \frac{K_{10∆PUT} + K_{25∆PUT}}{2}, \\
\nu_2 &= \frac{K_{10∆CALL} + K_{25∆CALL}}{2}.
\end{align*}
\]  

(5.1)

Hence, the calibration for model 3P uses three degrees of freedom. Although other static values for \( \nu_1 \) and \( \nu_2 \) could be considered, a simple visual evaluation shows that the values from (5.1) produce smooth and well-fitted implied volatility smiles (see Figure 5.2). A visualization of the parameters \( \nu_1 \) and \( \nu_2 \) in the two models is shown in Figure 5.1.

### 5.2 Optimization Problem

At the core of our proposed calibration procedure is a non-linear minimization problem. In this implementation, the minimization is performed using a least-square approach. In order to achieve an algorithm which is as numerically stable as possible, put options are used for \( K < x \) and call options for \( K \geq x \).

This corresponds to solving the optimization problem:

\[
\min_{\omega \in \mathbb{R}^5} \frac{1}{2} \sum_{i=1}^{2} |P_{\omega}^A(K_i) - P_M(K_i)|^2 + \sum_{i=3}^{5} |C_{\omega}^A(K_i) - C_M(K_i)|^2,
\]

such that

\[
0 < \sigma_1, \quad 0 < \sigma_x, \quad 0 < \sigma_2, \quad L < \nu_1 < x, \quad x < \nu_2 < U,
\]

where the options marked \( A \) are prices coming from a price function using the parameter set \( \omega \), and the prices marked \( M \) come from market quotes. Notice that we have one residual for every market quoted implied volatility. The problem formulation corresponds to minimizing the square of the residual norm, provided that the set \( \omega \) is used to generate option prices. Notice that the intrinsic value for both put options with \( K \leq x \) and call options with \( K \geq x \) is zero. Connecting this with results from Chapter 3 yields:

\[
P(K) = \Psi(K), \quad \text{if } K \leq x,
\]

\[
C(K) = \Psi(K), \quad \text{if } K > x,
\]
5.2. Optimization Problem

Figure 5.1: Comparison between the parameters \(\nu_1\) and \(\nu_2\) for model 3P and model 5P. The parameters correspond to border points of the LVG-volatility function. The parameters are calibrated to a EURUSD implied volatility surface.

with \(\Psi(K)\) as in Proposition 2. Therefore, the optimization problem can be redefined using only \(\Psi(K)\):

\[
\min_{0 < \omega \in \mathbb{R}^5} \frac{1}{2} \sum_{i=1}^{5} |\Psi(\omega_i) - O^M(K_i)|^2, \tag{5.2}
\]

such that

\[
0 < \sigma_1, \quad 0 < \sigma_x, \quad 0 < \sigma_2, \quad L < \nu_1 < x, \quad x < \nu_2 < U, \tag{5.3}
\]

where

\[
O^M(K) = \begin{cases} 
-xN(-d_1) + KN(-d_2), & \text{if } K \leq x \\
xN(d_1) - KN(d_2), & \text{if } K > x.
\end{cases} \tag{5.4}
\]

In the above expressions, \(d_{1/2}\) correspond to the factors from Lemma 1. The formulation of the optimization problem is equivalent in the five and three parameter case. Ideally, the calibration technique should be able to quickly generate implied volatility smiles which are smooth, well-behaving, and within the quoted bid-ask spreads. The calibrations in this thesis uses a MATLAB implementation of the Levenberg-Marquardt algorithm\(^1\), which is popular algorithm

\(^1\)The utilized optimization function is a Levenberg-Marquardt algorithm in a shortened version, developed by Fletcher. The MATLAB implementation is created by Miroslav Balda and can be be found on http://www.mathworks.com/matlabcentral/fileexchange/
for solving non-linear curve fitting problems. Recall that we need to impose\textsuperscript{2} the conditions from (5.3) on the elements of \( \omega \). Since the Levenberg-Marquardt algorithm does not allow for parameter bounds, it is necessary to develop some other strategy. In the case when

\[ L = 0, \quad U = \infty, \quad x = 1, \]

the conditions can be imposed by using the function mapping

\[ \tilde{\omega} = \{|\sigma_1|, |\sigma_2|, |\sigma_3|, \sin^2(\nu_1), |\nu_2| + 1\}. \]

This mapping ensures that the conditions are met independently of the values suggested by the optimization algorithm. One could criticize that the mapping does not restrict the parameters from being zero. However, this has not caused any obstacles in the implementation. By optimizing over \( \tilde{\omega} \) instead of over \( \omega \), we can make sure that generated solutions to (5.2) meet all conditions from (5.3).

\section*{5.3 Results}

In this section, some calibration results are presented. The market quoted implied volatility smiles for EURUSD and EURSEK differ in some ways and therefore it is interesting to compare the calibration results for the two currency pairs. While EURUSD options are very liquid with narrow bid-ask spreads, EURSEK options will generally have large spreads. Further, the volatility smiles for the two currency pairs have different types of skew.

In Table 5.1, average absolute errors from market quoted mids in decimal implied volatility are shown. This example considers options maturing in three months. Calibration error data for eight different maturities can be seen in Appendix B. Model 5P has lower errors than model 3P and calibrates almost exactly to market quoted mids. This is usually considered to be an advantage, but can also cause some problems. In the case of a currency such as the EURSEK, the data is often of poor quality. Sometimes, the quoted mids do not follow a pattern to which it is possible to fit a well-behaving curve. In these cases, there is a risk of model 5P generating irregular implied volatility smiles with local maximums. This effect has not been observed when calibrating model 3P.

In Table 5.2, some additional calibration statistics are presented. The residual norm and calibration time are averages over all maturities. Evidently, model 3P calibrates faster than model 5P while no conclusion can be made regarding the residual norm. None of the two sub-models miss the bid-ask spread for any option in the sample. Considering the total test coverage of 3328 different volatility smiles, each consisting of 5 options, it can be concluded that both models tend to deliver implied volatility smiles that fit market quotes within the bid-ask spread.

A graphical example of calibrated implied volatility smiles can be seen in Figure 5.2. The calibrations behind these smiles considers eight different maturities\textsuperscript{2} Violating this condition would result in negative LVG volatilities. The conditions could hypothetically be relaxed without introducing arbitrage in the price function, but this will not be investigated in this thesis.
5.4. Model Comparison

Table 5.1: Average absolute errors in decimal implied volatility from calibration to 416 market quoted volatility smiles on EURUSD and EURSEK. The calibration considers options maturing in six months. More calibration results are shown in Appendix B.

<table>
<thead>
<tr>
<th>Currency/Model</th>
<th>10DPUT</th>
<th>25DPUT</th>
<th>ATM</th>
<th>25DCALL</th>
<th>10DCALL</th>
</tr>
</thead>
<tbody>
<tr>
<td>EURUSD/5P</td>
<td>0.000162</td>
<td>0.000148</td>
<td>0.000052</td>
<td>0.000041</td>
<td>0.000059</td>
</tr>
<tr>
<td>EURUSD/3P</td>
<td>0.000492</td>
<td>0.000310</td>
<td>0.000063</td>
<td>0.000266</td>
<td>0.000510</td>
</tr>
<tr>
<td>EURSEK/5P</td>
<td>0.000099</td>
<td>0.000029</td>
<td>0.000082</td>
<td>0.000013</td>
<td>0.000023</td>
</tr>
<tr>
<td>EURSEK/3P</td>
<td>0.000146</td>
<td>0.000101</td>
<td>0.000298</td>
<td>0.000806</td>
<td>0.001080</td>
</tr>
</tbody>
</table>

Table 5.2: Calibration data from calibration to totally 416 market quoted volatility smiles for EURUSD and EURSEK options from one week to one year. The price functions have been extrapolated into a rectangular grid of option prices.

<table>
<thead>
<tr>
<th>Currency/Model</th>
<th>B/A Misses</th>
<th>Residual Norm:</th>
<th>Calibration Time (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EURUSD/5P</td>
<td>0</td>
<td>6.29E-10</td>
<td>2.17</td>
</tr>
<tr>
<td>EURUSD/3P</td>
<td>0</td>
<td>2.21E-08</td>
<td>0.995</td>
</tr>
<tr>
<td>EURSEK/5P</td>
<td>0</td>
<td>1.07E-07</td>
<td>5.22</td>
</tr>
<tr>
<td>EURSEK/3P</td>
<td>0</td>
<td>1.18E-07</td>
<td>1.04</td>
</tr>
</tbody>
</table>

The calibrated LVG-volatility functions generally tend to look similar to the example in Figure 5.3, in this example on the EURUSD with model 3P for the maturity of 1 month.

5.4 Model Comparison

One remaining question is which model best suits the next step in the calibration procedure, where an algorithm will be developed for interpolation between maturities. Evidently, model 5P has significantly smaller deviations from the quoted implied volatility mids. However, model 3P does not miss the bid-ask spread on any occasion in the test sample, which was the objective defined in Section 1.1. Further, model 3P calibrates significantly faster.

In Figure 5.1, the main difference between the two models is shown; the two parameters $\nu_1$ and $\nu_2$. In both cases, $\nu_1$ is decreasing and $\nu_2$ is increasing in maturity. Notice though that this cannot be seen as a general result; it cannot be guaranteed that $\nu_1$ is decreasing and $\nu_2$ increasing in model 5P. In model 3P, these two conditions are satisfied by the construction of the parameters from quoting conventions in the FX-option market. This attribute will be helpful in the next chapter and therefore this attribute favors model 3P. Another aspect, which has been mentioned earlier, is that model 5P has proven to generate implied volatility smiles with local extreme points if the input data is of poor quality. This effect has not been found for model 3P. The higher calibration time in model 5P could also become problematic. One of the main objectives of this thesis is to develop an extremely fast model, and therefore the calibration
Chapter 5. Calibration

speed should be regarded as one of the most important factors.

Considering all of these aspects, the remaining part of this thesis will only consider the three parameter model, model 3P.

5.5 Extreme Situations

In this section we investigate how the option price function behaves for some extreme situations that could become problematic out of a theoretical perspective. The first situation we will consider concerns the behavior of implied volatility smiles for very large $K$’s. Ideally, the implied volatility should be constant when $K$ goes to infinity. As concluded in Section 3.6, there are theoretical reasons to believe that this will not be the case for the model proposed in this thesis. The effect is caused by the fact that the constant LVG-volatility used for high strikes will be unnaturally low for call options with very large strikes. The effect will become increasingly significant with increasing $K$. In Figure 5.4 the effect is observable and starts at $K \approx 5$. In this example, EURUSD options maturing in 6 months are considered. Between $K = 5$ and $K = 10$, the implied volatility decreases with a value around 0.007 which corresponds to a drop with roughly 5%. Since $K = 5$ is already an extremely large moneyness value, this effect can be considered a minor obstacle in this example.
A second interesting extreme case concern the values of the Greek $\Delta$. In the Black & Scholes model for put options, $\Delta_P = -1$ when the value of the underlying asset vanishes, $x = 0$. The corresponding relation for call options is that $\Delta_C = 1$ when the strike vanishes, $K = 0$. It is easily shown that these two attributes are satisfied by the intrinsic values as well. Ideally, the developed model should hence inherit these attributes.

In the construction of the closed-form solution in Proposition 2, the boundary conditions imposed correspond to the requirement that the homogeneous solution to (3.10) has the value zero at the points $L$ and $U$. If additional boundary conditions are imposed, forcing the $\Delta$’s to be zero as well, the non-trivial solution space for (3.10) vanishes. Hence, these two conditions will theoretically not be perfectly satisfied. However, numerical testing shows that the error is small. For call options with $K = 0$, the error is not observable using computer arithmetic. For put options with $x = 0$, the error is slightly larger, usually around $\Delta_P \approx 10^{-11}$. 
Figure 5.4: Example of a heavily extrapolated EURUSD implied volatility smile for a maturity of six months
Chapter 6

Maturity Interpolation

In this chapter, we explore ways of extending the model proposed in Chapter 3 to allow for arbitrage-free option price interpolation between maturities. The goal is to generate arbitrage-free implied volatility surfaces with an arbitrarily fine grid and with an arbitrary level of extrapolation in the strike dimension. The main focus will concern possibilities to perform interpolation between the closed-form solutions from Chapter 3 in such a way that absence of calendar spread arbitrage is ensured. The proposition below implies that arbitrage will appear if and only if the function $\Psi(K)$ from Proposition 2 is decreasing in maturity:

**Proposition 6** Let $\Psi_i(K)$, $i = 1 \ldots N$ be the homogeneous solution to (3.10), as parameterized in Proposition 2, at the time to maturity $t^* = \tau_i$. Further, let $0 < \tau_1 \leq \cdots \leq \tau_N$ be a sequence of increasing maturities. Then the condition

$$\Psi_{i+1}(K) \geq \Psi_i(K), \quad i = 1 \ldots N, \quad \forall K \in [L, U].$$

(6.1)

is a sufficient and necessary condition for absence of calendar spread arbitrage between the two maturities $\tau_i$ and $\tau_{i+1}$ in the interval $K \in [L, U]$.

**Proof.** According to Condition 1, calendar spread arbitrage can be found if and only if $\frac{\partial C}{\partial \tau} < 0$. Using a finite difference approximation, this condition corresponds to:

$$\frac{C(K, \tau_{i+1}) - C(K, \tau_i)}{\tau_{i+1} - \tau_i} < 0.$$ 

Since the intrinsic value is independent of the time to maturity, the condition

$$C(K, \tau_{i+1}) - C(K, \tau_i) = \Psi_{i+1}(K) - \Psi_i(K)$$

is satisfied. This condition is also true for put options, which follows from the put-call parity. This proposition can easily be proven in a setting with continuous maturities as well. Holding $K$ constant and differentiating the solution in Proposition 2 with respect to maturity $\tau$ yields:

$$\frac{\partial C}{\partial \tau}(K, \tau) = \frac{\partial \Psi}{\partial \tau}(K, \tau), \quad \forall K \in [L, U].$$

However, in this thesis only the maturity-discrete case will be considered. □
6.1 Local Variance

Andreasen and Huge (2011) generates option prices in a systematic way using a numerical algorithm. Prices at an earlier maturity are used as boundary condition when prices at a later maturity are calculated from a finite difference discretization of (3.10). This interpolation technique ensures absence of calendar spread arbitrage by construction. In this thesis, this will be accomplished in a different way by utilizing the closed-form solution from Proposition 2. Since an option price function exists for every parameter set \( \omega > 0 \), prices on intermediate maturities will be found by generating new parameter sets. However, this approach requires some technique to ensure absence of arbitrage between option price functions. When developing such a technique, it will prove helpful to make use of a new variable:

Definition 11 The local variance \( V(K, \tau) \) is the product of the time to maturity and the square of the corresponding LVG-volatility function;

\[
V(K, \tau_i) := \tau_i \alpha_i^2(K), \quad \forall K \in [L, U], \quad \tau_i > 0 \text{ and } i = 0, 1 \ldots N, \quad (6.2)
\]

where \( \alpha_i(K) \) is the LVG-volatility function calibrated to market quoted option prices at maturity \( \tau_i \).

Comparing functions of the type presented in Proposition 2 analytically is not an easy task. However, if only a finite amount of strikes are considered it is trivial to perform a numerical comparison. Since the implementation in this thesis is restricted to an arbitrarily fine sample containing a finite number of strikes, it is easy to test the criteria in Proposition 6. However, our aim is to produce conditions on the relation between parameter sets and the corresponding maturities which ensure Proposition 6 to be satisfied. Finding such conditions is not as easy and we will therefore simplify the problem by using the theorem below.

Theorem 11 Let \( V(K, \tau_1) \) and \( V(K, \tau_2) \) be two local variance functions as in Definition 6.2. Assume that these two functions are related to two functions \( \Psi_1(K), \Psi_2(K) \) by the equations

\[
\frac{1}{2} V(K, \tau_1) \frac{\partial^2 \Psi_1(K)}{\partial K^2} = \Psi_1(K), \quad \frac{1}{2} V(K, \tau_2) \frac{\partial^2 \Psi_2(K)}{\partial K^2} = \Psi_2(K),
\]

with the common boundary conditions

\[
\lim_{K \to L} \Psi_1(K) = \lim_{K \to L} \Psi_2(K) = 0, \\
\lim_{K \to U} \Psi_1(K) = \lim_{K \to U} \Psi_2(K) = 0.
\]

Assume further that the functions \( \Psi_1(K) \) and \( \Psi_2(K) \) are continuously differentiable for \( K \in [L, U] \) except for in a point \( x \) where:

\[
\lim_{K \to x^-} \Psi_1(K) = 1 + \lim_{K \to x^+} \Psi_1(K), \\
\lim_{K \to x^-} \Psi_2(K) = 1 + \lim_{K \to x^+} \Psi_2(K).
\]
If the two local variance functions differ in size according to:

\[ V(K, t_1) \leq V(K, t_2), \quad \forall K \in [L, U], \]

the two solutions \( \Psi_1(K) \) and \( \Psi_2(K) \) will satisfy the inequality:

\[ \Psi_1(K) \leq \Psi_2(K), \quad \forall K \in [L, U]. \]

A proof of the above theorem can be found in Appendix A. A direct effect of this theorem is that increasing local variance is a sufficient condition for Proposition 6, and therefore also for absence of calendar spread arbitrage. Studying the evolution of local variance in maturity is significantly easier than studying the evolution of option prices. Notice that the theorem is independent of the assumptions regarding the structure of LVG-volatility functions, which makes the result valuable on its own.

6.2 Conditions for Absence of Calendar Spread Arbitrage

It is important to choose an appropriate interpolation technique. Interpolating linearly in local variance would satisfy the condition for absence of arbitrage. However, this type of interpolation will not produce LVG-volatility functions for intermediate maturities which follow Assumption 1. Hence, this would render it impossible to use the parametrization in Proposition 2. We will instead generate new option price functions at intermediate maturities by producing new parameter sets \( \omega \). Let \( \tau_i > 0, \quad i = 1 \ldots M \) be an increasing sequence of market quoted maturities. Denote a calibrated parameter set which alone determines an option price function at maturity \( \tau_i \) as:

\[ \omega_i = \{ \sigma_{1,i}, \sigma_{x,i}, \sigma_{2,i}, \nu_{1,i}, \nu_{2,i} \}, \quad i = 1 \ldots M. \]

Let us first investigate which conditions need to be satisfied for calibrated option price functions to be free from calendar spread arbitrage. It is sufficient to look at this problem in a two-maturity setting. It follows directly from Theorem 11 that the condition in (6.4) is sufficient to ensure absence of arbitrage between the two maturities \( \tau_i \) and \( \tau_{i+1} \):

\[ \frac{\tau_{i+1}\alpha_{i+1}^2(K)}{\tau_i\alpha_i^2(K)} \geq \frac{\alpha_{i+1}(K)}{\alpha_i(K)}, \quad \forall K \in [L, U] \]

\[ \Rightarrow \sqrt{\frac{\tau_i}{\tau_{i+1}}} \leq \frac{\alpha_{i+1}(K)}{\alpha_i(K)}. \]

Let us define the function \( \Phi_{i+1}(K) := \frac{\alpha_{i+1}(K)}{\alpha_i(K)} \). Ensuring that

\[ \min_{K \in [L, U]} \Phi_{i+1}(K) \geq \sqrt{\frac{\tau_i}{\tau_{i+1}}} \]

is then equivalent to ensuring that the condition in (6.4) is satisfied. We will now make some simplifying assumptions concerning the structure of the LVG-volatility functions. Assume that the size of the subintervals \( I_2 \) and \( I_3 \) are
increasing in maturity. This assumption is always satisfied for model 3P in Chapter 5. Using the notation

\[ I_j^i, \quad i = 1 \ldots M, \quad j = 1 \ldots 4 \]

as the subinterval \( j \) at time to maturity \( \tau_i \), it is possible to split up the problem on six different subintervals. Assuming \( \nu_1 \) and \( \nu_2 \) to be increasing, the relationship \( I_1^{i+1} \subset I_1^i \) and \( I_4^{i+1} \subset I_4^i \) will be satisfied. The subintervals that will be considered are:

\[
K \in \begin{cases}
I_1^{i+1}, \\
I_1^i \cap I_1^{i+1}, \\
I_2^i, \\
I_4^i \cap I_4^{i+1}, \\
I_4^{i+1}.
\end{cases}
\] (6.5)

In Figure 6.1 a graphical presentation of the subintervals can be seen. The union of these six subintervals is the entire range \([L, U]\). In accordance with Assumption 1, \( \Phi_{i+1}(K) \) will be constant for \( K \in I_1^{i+1} \), \( K \in I_1^i \), and linear for \( K \in I_1^i \cap I_1^{i+1} \), \( K \in I_2^i \), \( K \in I_4^i \cap I_4^{i+1} \), and the fraction between two linear functions for \( K \in I_2^i \), \( K \in I_3^i \). Since \( \Phi_{i+1}(K) \) will be a positive continuous function with this structure, we can conclude that its minimum value can be found in one of the points: \( K = \nu_{(i+1)} \), \( K = \nu_i \), \( K = x \), \( K = \nu_{(2,i)} \) or \( K = \nu_{(2,i+1)} \). Relating to the parameter set \( \omega \), this can be expressed as:

\[
\min_{K \in [L, U]} \Phi_{i+1}(K) = \min \left( \frac{\sigma_{(1,i)}}{\sigma_{(2,i)}}, \frac{\sigma_{(x,i)}}{\sigma_{(2,i)}}, \frac{\alpha_{i+1}(\nu_{(1,i)})}{\sigma_{(1,i)}}, \frac{\alpha_{i+1}(\nu_{(2,i)})}{\sigma_{(2,i)}} \right). \] (6.6)

For the first three elements in the vector from (6.6), the sufficient conditions can be formulated as in Condition 2. The remaining two elements of \( \min_{K \in [L, U]} \Phi_{i+1}(K) \) are:

\[
\frac{\alpha_{i+1}(\nu_{(1,i)})}{\sigma_{(1,i)}} \quad \text{and} \quad \frac{\alpha_{i+1}(\nu_{(2,i)})}{\sigma_{(2,i)}}.
\]

Since \( \alpha_{i+1}(\nu_{(1,i)}) \) and \( \alpha_{i+1}(\nu_{(2,i)}) \) are elements of neither \( \omega_i \) or \( \omega_{i+1} \), finding sufficient conditions for them is more complicated. However, empirical testing shows that the calibrated LVG-volatility functions usually have a structure similar to the example in Figure 6.1, satisfying the conditions \( \sigma_x > \sigma_{1} \) and \( \sigma_x > \sigma_{2} \). Assuming that these two conditions are satisfied for all maturities, the minimum of \( \Phi_{i+1}(K) \) cannot be in these two points since \( \alpha_{i+1}(\nu_{(1,i)}) > \sigma_{(1,i+1)} \) and \( \alpha_{i+1}(\nu_{(2,i)}) > \sigma_{(2,i+1)} \).

**Condition 2** Let \( \omega_i \) and \( \omega_{i+1} \) be parameter sets found through calibration to market quoted volatility smiles at maturity \( \tau_{i+1} > \tau_i \). Assume that the parameters satisfy:

\[
\sigma_{(x,i)} > \sigma_{(1,i)}, \quad \sigma_{(x,i)} > \sigma_{(2,i)}
\]

\[
\sigma_{(x,i+1)} > \sigma_{(1,i+1)}, \quad \sigma_{(x,i+1)} > \sigma_{(2,i+1)}.
\]
and

\[
\nu_{(1,i+1)} < \nu_{(1,i)}, \quad \nu_{(2,i+1)} > \nu_{(2,i)}.
\]

Then the following three conditions are sufficient to achieve absence of calendar spread arbitrage between the two maturities \(\tau_i\) and \(\tau_{i+1}\):

\[
\frac{\sigma_{(1,i+1)}}{\sigma_{(1,i)}} \geq \sqrt{\frac{\tau_i}{\tau_{i+1}}},
\]

\[
\frac{\sigma_{(x,i+1)}}{\sigma_{(x,i)}} \geq \sqrt{\frac{\tau_x}{\tau_{x+1}}},
\]

\[
\frac{\sigma_{(2,i+1)}}{\sigma_{(2,i)}} \geq \sqrt{\frac{\tau_2}{\tau_{2+1}}}.
\] (6.7)

Figure 6.1: Example of two EURUSD LVG-volatility functions, corresponding to two different time to maturities. The partitioning from (6.5) is here presented graphically.

Using this condition, we can introduce linear parameter bounds into the optimization problem from (5.2) which ensures that the local variance is increasing.
in maturity:

\[
\omega_{i+1} = \arg\min_{0 < \omega \in \mathbb{R}^5} \frac{1}{2} \sum_{j=1}^{5} |\Psi_\omega(K_j) - O^M(K_j)|^2,
\]

st.

\[
\begin{align*}
\sigma_{(1,i+1)} & \geq \sigma_{(1,i)} \sqrt{\frac{\tau_i}{\tau_{i+1}}} , \\
\sigma_{(x,i+1)} & \geq \sigma_{(x,i)} \sqrt{\frac{\tau_i}{\tau_{i+1}}} , \\
\sigma_{(2,i+1)} & \geq \sigma_{(2,i)} \sqrt{\frac{\tau_i}{\tau_{i+1}}} , \\
\nu_{(1,i+1)} & \leq \nu_{(1,i)} , \\
\nu_{(2,i+1)} & \geq \nu_{(2,i)} ,
\end{align*}
\]

Since parameter bounds were required in the original optimization problem as well, these conditions do not make the calibration significantly slower. However, since these conditions add restrictions to the calibration, they could hypothetically make the results worse. The negative effect is magnified by the fact that increasing local variance is not necessary for absence of arbitrage. This means that there is a risk that we prohibit parameter sets which generates arbitrage free option prices. However, empirical studies shows that the optimal parameter sets rarely violate Condition 2. Using the same calibration set as in Chapter 5, the conditions were only breached on a few occasions of which none corresponded to real calendar spread arbitrage. The two additional restrictions

\[
\nu_{(1,i+1)} \leq \nu_{(1,i)} \quad \text{and} \quad \nu_{(2,i+1)} \geq \nu_{(2,i)}
\]

are important in order to guarantee that Condition 2 is satisfied. Notice that in this chapter we are implementing the three parameter model from Chapter 5, for which these restrictions are satisfied by the construction of FX-option quotes, see Section 5.1.

### 6.3 Interpolation

Let \(0 < \tau_1 < \tau_2\) be two market quoted maturities with known parameter sets \(\omega_1\) and \(\omega_2\). The next step towards generating intermediate option price functions is to add a new parameter set \(\omega\) for a maturity \(\tau\), satisfying \(\tau_1 < \tau < \tau_2\). In this section we will interpolate linearly between parameter sets. In practice, however, a cubic spline interpolation would be more appropriate; especially if the objective is to generate Dupire local volatility surfaces, as is done in Chapter 7. As shown in Figure 7.1 and 7.2, the generated Dupire local volatility surfaces are discontinuous in maturity if parameters are interpolated linearly, but continuous if the interpolation is performed using cubic splines. The theorem below tells us that linear interpolation in parameter space ensures increasing local variance for intermediate maturities.

**Theorem 12** Let \(\omega_1\) and \(\omega_2\) be parameter sets found by solving (6.8) for the maturities \(0 < \tau_1 < \tau_2\). When performing a linear interpolation, the set \(\omega\) for
6.3. Interpolation

maturity $\tau_1 < \tau < \tau_2$ can be found as:

$$\omega = \omega_1 (1 - \lambda(\tau)) + \omega_2 \lambda(\tau), \quad \tau \in [\tau_1, \tau_2],$$

where $\lambda(\tau) = \frac{\tau_2 - \tau_1}{\tau_2 - \tau_1}$. Further, linear parameter interpolation ensures that the local variance is increasing for intermediate maturities as well.

**Proof.** Let us consider the point $K = x$, where the local variance can be found as $V(x) = \tau \sigma_x^2$. In this point, Condition 2 can be translated into:

$$\left(\sigma_{(x,1)}(1 - \lambda(\tau)) + \sigma_{(x,2)} \lambda(\tau)\right)^2 \tau \leq \left(\sigma_{(x,2)} \sqrt{\frac{\tau_2}{\tau_1}} (1 - \lambda(\tau)) + \sigma_{(x,2)} \lambda(\tau)\right)^2 \tau$$

$$= \sigma_{(x,2)}^2 \left(\sqrt{\frac{\tau_2}{\tau_1}} - \lambda(\tau) \left(\sqrt{\frac{\tau_2}{\tau_1}} - 1\right)\right) \tau \leq \sigma_{(x,2)}^2 \tau_2, \quad \forall \tau \in [\tau_1, \tau_2].$$

The identical inequalities are also satisfied for $\sigma_1, \sigma_2$, and similar conditions can be established for the lower limit. The theorem is proven by combining the above results with Condition 2.

6.3.1 Algorithm

In this section, an algorithm which can be used for generating new option price functions on intermediate maturities is presented. Let $\tau_1$ and $\tau_N$, where $0 < \tau_1 < \tau_N$, denote two market quoted maturities with corresponding calibrated parameter sets $\omega_1$ and $\omega_N$.

The algorithm below interpolates between the two parameter sets, but is trivially expandable to cover an entire surface. $N$ corresponds to the number of requested maturities. Assume that the grid of option prices is rectangular. This means that there exist factors $K_{min}$ and $K_{max}$ which are common for all maturities. Since option prices are represented by continuous functions which are well-defined in the arbitrary large region $[L, U]$, the existence of such factors can be ensured by using the same partitioning for all maturities$^1$.

1. Find parameter sets $\omega_1$ and $\omega_N$ for two consecutive market quoted volatility smiles with maturities $\tau_1 < \tau_N$ by solving the optimization problem in (6.8).

2. Create a maturity grid $\tau_1 < \tau_2 < \cdots < \tau_N$

3. For $1 \leq i < N - 1$

(a) Generate new intermediate parameter sets by linear interpolation:

$$\omega(\tau_{i+1}) = \omega_N (1 - \lambda(\tau_{i+1})) + \omega_i \lambda(\tau_{i+1}), \quad \text{where} \quad \lambda(\tau) = \frac{\tau_N - \tau_{i+1}}{\tau_N - \tau_i}.$$

$^1$This assumption can be relaxed into demanding that no strikes for shorter maturities are outside the range for longer maturities.
6.4 Least-Square Formulation

An alternative approach could be to instead use a least-square method. As mentioned earlier, detecting arbitrage between two discretized functions of option prices is an easy task. The problematic part is finding out how the underlying parameter set can be adjusted in order to remove the arbitrage. In (6.8), a formulation is presented which ensures that Condition 2 is satisfied. The main problem with this approach is that increasing local variance is sufficient, but not necessary, for absence of arbitrage. Using a least-square approach would instead involve solving a minimization problem with non-linear constraints. The optimization problem can be formulated as:

$$\omega_{i+1} = \arg\min_{\omega \in \mathbb{R}^{5}} \frac{1}{2} |\omega_i - \omega|^2$$

st.

$$\Psi_i(K) < \Psi_{i+1}(K) < \Psi_N(K), \quad \forall K \in [K_{min}, K_{max}].$$

This optimization problem needs to be solved every time calendar spread arbitrage is found. Notice that solving the problem in (6.9) takes significantly more time than the original calibration. However, empirical testing shows that it is very uncommon that calendar spread arbitrage appears between calibrated option function. Using the same notation as in Section 6.3.1, an algorithm could be formulated as:

1. Generate independent option price functions for maturities $\tau_1$ and $\tau_N$ by using (5.2).

2. if $\Psi_1(K) > \Psi_N(K)$ for any $K \in [K_{min}, K_{max}]$  
   Generate a new parameter set for maturity $\tau_N$ by using (6.9)

3. for $1 \leq i < N$
   Generate a new intermediate parameter set by linear interpolation.
   $$\omega(\tau_{i+1}) = \omega_N(1 - \lambda(\tau)) + \omega_i \lambda(\tau),$$
   where $\lambda(\tau) = \frac{\tau_N - \tau_{i+1}}{\tau_N - \tau_i}$.

Compared to the local variance approach, the strength with the least-square approach is that no unnecessary constraints are imposed on the LVG-volatility functions. One could imagine that the local variance approach would be more effective for non-liquid currency pairs where bad model fits are common, while the least-square approach is better suited for liquid, well-behaving currency pairs.

6.5 Results

An interpolated LVG-volatility surface is shown in Figure 6.2. This surface is based on calibrations to eight different EURUSD volatility smiles on maturities ranging from one week to one year. New parameter sets have been generated by interpolating linearly. The result is a surface consisting of piecewise linear LVG-volatility functions, as defined in Assumption 1, for 50 equally distributed maturities.

Given an interpolated LVG-volatility surface, it is possible to use Proposition 2
Figure 6.2: Example of a linearly interpolated LVG-volatility surface calibrated to a market quoted EURUSD implied volatility surface.

to generate the corresponding option price functions for every maturity. Additionally, an implied volatility surface can be produced by inverting (5.4) with respect to $\sigma$. The implied volatility surface will be $C^2$ in the strike dimension, while the smoothness in the maturity dimension is highly dependent on the choice of parameter-interpolation technique. If the interpolation is performed linearly, the surface will only be continuous in the maturity dimension. However, if the interpolation is performed using cubic splines, the resulting implied volatility surface will be $C^2$ in the maturity dimension as well. As a consequence, the model can be used to produce continuous representations of the greeks $^2 \Delta$ and $\Gamma$. This attribute is very useful from a hedging perspective$^3$. The implied volatility surface corresponding to the LVG-volatility surface in Figure 6.2 is shown in Figure 6.3. This surface allows for arbitrage-free pricing of European options for strikes in the extrapolated$^4$ region $K \in [0.8, 1.2]$.

$^2$The $\Gamma$ is the second order differential of the option with respect to the contemporaneous value of the underlying asset.

$^3$The implementations by Carr and Nadtochiy (2013) and Andreasen and Huge (2011) do not result in continuous $\Delta$’s and $\Gamma$’s.

$^4$Further extrapolation is possible without any additional calibrations. However, recall the discussion in Section 5.5 regarding the behavior of the model for extremely large moneynesses.
Figure 6.3: Example of a calibrated EURUSD implied volatility surface. The crosses on the surface correspond to market quoted mids.
Chapter 7

Additional Applications

In addition to pricing European options, there exist a variety of applications for arbitrage free option price surfaces which are smooth and allow for significant extrapolation. In this chapter, the option price parameterizations are used for producing Dupire local volatility surfaces and pricing variance swaps.

7.1 Dupire Local Volatility

In this section a Dupire local volatility surface is generated, which corresponds to the diffusion term from (2.11). In order to derive this surface from our prices of European options, we will use a finite difference approximation of the Dupire equation. The method is very similar to the one presented by Andreasen and Huge (2011). Recall the Dupire equation in an interest rate-free environment:

\[ \frac{1}{2} K^2 \sigma^2(K, \tau) \frac{\partial^2 C}{\partial K^2}(K, \tau) = \frac{\partial C}{\partial \tau}(K, \tau). \]

Let us define a grid \( j = 1, 2 \ldots M \) and \( i = 1, 2 \ldots N \). \( C(K_j, \tau_i) \) will then correspond to a European call option with time to maturity \( \tau_i > 0 \) and strike \( K_j > 0 \). Standard finite difference approximations for the differentials in the Dupire equation can be expressed as:

\[
\frac{\partial^2 C}{\partial K^2}(K_j, \tau_i) \approx \frac{C(K_{j-1}, \tau_i) + C(K_{j+1}, \tau_i) - 2C(K_j, \tau_i)}{(\Delta K)^2},
\]

\[
\frac{\partial C}{\partial \tau}(K_j, \tau_i) \approx \frac{C(K_j, \tau_{i+1}) - C(K_j, \tau_i)}{\Delta \tau}.
\]

Using these approximations, the Dupire equation can be discretized as:

\[
\frac{1}{2} K_j^2 \sigma^2(K_j, \tau_i) \frac{C(K_{j-1}, \tau_i) + C(K_{j+1}, \tau_i) - 2C(K_j, \tau_i)}{(\Delta K)^2} = \frac{C(K_j, \tau_{i+1}) - C(K_j, \tau_i)}{\Delta \tau},
\]

which can be rearranged into:

\[
\sigma(K_j, \tau_i) = \frac{1}{K_j} \sqrt{\frac{2(\Delta K)^2}{\Delta \tau} \frac{C(K_j, \tau_{i+1}) - C(K_j, \tau_i)}{C(K_{j-1}, \tau_i) + C(K_{j+1}, \tau_i) - 2C(K_j, \tau_i)}}.
\]
According to earlier theoretical results, $\sigma(K, \tau_i)$ will be real since neither butterfly or calendar spread arbitrage should exist in the option price surface. In Figure 7.1 and 7.2, two Dupire local volatility surfaces are presented. In Figure 7.1, the parameters are interpolated linearly in maturity. Evidently, the Dupire local volatility is therefore discontinuous in the maturity dimension. In Figure 7.2, the parameter sets were interpolated with cubic splines, ensuring that the surface is $C^1$ in both dimensions. Notice that the surfaces are real at all points and has no singularities or holes.

Figure 7.1: Example of a magnified Dupire local volatility surface calibrated against a market quoted EURUSD implied volatility surface. The parameter interpolation is here performed linearly.
7.2. Valuation of Variance Swaps

In this section, another useful application of the developed model is presented; valuation of variance swaps. After extrapolation, arbitrage-free analytical solutions for prices on European options enable for fast valuation of this type of instrument. The variance swap is an OTC traded derivative which exposes the buyer directly to market volatility. As concluded by Carr and Lee (2009), variance swaps are popular instruments in the FX-market. The payoff of the variance swap is the difference of the realized variance during the lifetime of the contract, $\sigma^2_R(t,T)$, and some set swap rate, $\sigma^2_S(T)$. Generally, the contract uses a fair-value swap rate, where the swap rate is picked in such a way that the contract has no value at inception. Let $VS(t,T), t \in [0,T]$ denote the risk premium of a variance swap with maturity $T$ at time $t$. The risk premium can then be calculated as

$$VS(t,T) = E[\sigma^2_R(t,T)|\mathcal{F}_t] - \sigma^2_S(T),$$

where

$$VS(0,T) = 0 \implies E[\sigma^2_R(0,T)|\mathcal{F}_0] = \sigma^2_S(T).$$

(7.2)

This calls for a technique to determine the fair swap rate. Carr and Lee (2009), among other examples, use a Taylor expansion to show that the fair swap rate
can be found by solving the integral
\[ \sigma_S^2(T) = \frac{2}{T} \int_0^\infty \frac{O(K,T)}{K^2} dK, \]
where \( O(K,T) \) is the price of European OTM-options with strike \( K \) and maturity \( T \). Using the same notations as previously in this thesis, we conclude that:
\[ \sigma_S^2(\tau_i) = \frac{2}{\tau_i} \int_0^\infty \frac{\Psi_i(K)}{K^2} dK. \quad (7.3) \]

Using the analytical expression for \( \Psi(K) \), one could consider trying to solve the above integral analytically. However, the analytical solution of the integral will have to be expressed in terms of the hyper-geometric function. Simplifying such a solution using standard integrals may enhance the calibration speed. However, this will not be investigated in this thesis.

The parametrization of OTM-option prices, \( \Psi(K) \), goes to zero at the point \( K = 0 \). However, it is easily shown that
\[ \lim_{K \to 0} \frac{\Psi(K)}{K^2} = \infty \]
by using the rule of l'Hôpital on the first subinterval of \( \Psi(K) \). The nature of the divergence of \( \Psi(K)/K^2 \) at the point \( K = 0 \) will not be analysed thoroughly. Empirical testing shows that a computer implementation yields:
\[ \Psi(K)/K^2 \approx 0 \text{ for } M < 0.222E - 16. \]
Hence, the divergence only becomes problematic in a single point, \( K = 0 \). Therefore, this obstacle will be solved by defining the function \( \Omega(K) \) as:
\[ \Omega_i(K) = \begin{cases} 0, & \text{if } K = 0, \\ \Psi_i(K)/K^2, & \text{if } K > 0, \end{cases} \]
and compute the fair swap rate of a Variance swap at maturity \( \tau_i \) as
\[ \sigma^2_S(\tau_i) = \frac{2}{\tau_i} \int_0^\infty \Omega_i(K) dK. \quad (7.4) \]

In Figure 7.3, the fair swap rate of a variance swap for maturities between 1 week and 1 year is presented. The numerical integration needed to produce Figure 7.3 was done using a small tolerance and took less than 10ms. In order to get a smooth curve, the maturity interpolation from Chapter 6 was done using cubic splines.
Figure 7.3: Example of fair swap rates computed by numerical integration over an option price surface calibrated against a market quoted EURUSD implied volatility surface.
Chapter 8
Conclusions

In this thesis we propose a fast and arbitrage-free way of interpolating and extrapolating prices for European options. The main result of Chapter 3 is a closed-form solution of the local variance gamma version of the Dupire equation. The method guarantees absence of butterfly and vertical spread arbitrage. The main difference between the approach in this thesis and those in earlier publications is the assumption regarding the underlying LVG-volatility function. We propose a continuous piecewise linear function which is constant for very low and very high strikes. Further, we propose two different sub-models; one with five and one with three degrees of freedom. In Chapter 5, we show that both sub-models calibrate accurately to Reuters quoted FX-volatility smiles for EURUSD and EURSEK options. Further, the model calibrates at a remarkable speed. In a MATLAB implementation using a Levenberg-Marquardt algorithm, the three parameter model has an average calibration time around 1 ms for both currency pairs in the calibration sample. The average calibration time for the five parameter model is approximately 2 ms for EURUSD options and 5 ms for EURSEK options.

In Chapter 6, we propose a technique for generating option prices for intermediate maturities while ensuring absence of calendar spread arbitrage. In this technique, a parameter set which uniquely determines an option price function is interpolated in maturity. If the interpolation is performed using cubic splines, the produced option price surface is arbitrage-free, $C^2$ in both maturity and strike, and fits market quotes within the bid-ask spread. Further, the model supports strike-extrapolation at an arbitrary level. In Chapter 7 we use the calibrated option price surfaces for producing Dupire local volatility surfaces and for valuating variance swaps. A possible extension of the proposed model would be to change the assumptions on the LVG-volatility function by allowing the tail sections to tend towards zero and infinity. This would solve one of the observed obstacles; implied volatilities start decreasing for extremely high strikes. However, such an extension would most likely have longer calibration time and could render it problematic to perform arbitrage-free maturity interpolation. Further, it would be interesting to investigate how the model performs on other markets with a greater amount of market quoted options.

\[^{1}\text{If the parameter interpolation is performed linearly, only continuity in maturity can be guaranteed}\]
Bibliography


Peter Carr. Local variance gamma option pricing model, April 2009. Lecture from Bloomberg/Courant Institute.


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Appendix A

Proof of Theorem 8. According to (3.8), the following condition is satisfied by the call prices coming from the LVG model:

\[
C(K, t^*) = \int_0^\infty \frac{e^{-u/t^*}}{t^*} C^M(K, u) du, \tag{1}
\]

where \(C^M(K, u)\) are prices coming from (3.2). Let us start with the condition to ensure absence of butterfly spread arbitrage:

\[
\frac{\partial^2 C}{\partial K^2}(K, t^*) \geq 0, \quad \forall K \in (L, U)
\]

Differentiating both sides of (1) two times with respect to \(K\) yields:

\[
\frac{\partial^2 C}{\partial K^2}(K, t^*) = \int_0^\infty \frac{e^{-u/t^*}}{t^*} \frac{\partial^2 C^M}{\partial K^2}(K, u) du
\]

Since \(t^*\) and the exponential function will always be positive, we have the following implication:

\[
\frac{\partial^2 C^M}{\partial K^2}(K, u) \geq 0, \quad \forall u \geq 0
\]

\[
\Rightarrow \quad \frac{\partial^2 C}{\partial K^2}(K, t^*) \geq 0
\]

Further, according to (2.15) we have:

\[
\frac{\partial^2 C^M}{\partial K^2}(K, t^*) = \phi(K, t^*)
\]

where \(\phi(K, t^*)\) is the risk-neutral probability density of the underlying asset. Since this density is by definition positive, this completes the proof for butterfly spread arbitrage.

Differentiating both sides of (1) one time with respect to \(K\) yields:

\[
\frac{\partial C}{\partial K}(K, t^*) = \int_0^\infty \frac{e^{-u/t^*}}{t^*} \frac{\partial C^M}{\partial K}(K, u) du
\]

Extending the above reasoning, the following implication can be concluded:

\[
\frac{\partial C^M}{\partial K}(K, u) \leq 0, \quad \forall u \geq 0
\]

\[
\Rightarrow \quad \frac{\partial C}{\partial K}(K, t^*) \leq 0
\]

According to (2.16) we have:

\[
\frac{\partial C^M}{\partial K}(K, u) = - \int_{-\infty}^{\infty} \phi(x, u) dx, \quad u \geq 0
\]

Since \(\phi(x, u)\) is positive by definition, this completes the proof for vertical spread arbitrage.
Proof of Theorem 11. Consider two functions $\Psi_1(K)$ and $\Psi_2(K)$ from the LVG model. The functions will be solutions to the boundary value problem:

$$
\begin{align*}
\Psi_1''(K) &= \chi_1(K)\Psi_1(K), \\
\Psi_2''(K) &= \chi_2(K)\Psi_2(K), \\
\end{align*}
$$

(2)

with the boundary conditions:

$$
\Psi_1(L) = \Psi_2(L) = \Psi_1(U) = \Psi_2(U) = 0
$$

The functions $\chi_1$ and $\chi_2$ are related to the local variance $V_i(K)$ by:

$$
\chi_1(K) = \frac{2}{t^2\alpha_2^2(K)} = \frac{2}{V_1(K)}
$$

$$
\chi_2(K) = \frac{2}{t^2\alpha_2^2(K)} = \frac{2}{V_2(K)}
$$

Further, $\Psi_1(K)$ and $\Psi_2(K)$ are continuous non-negative functions which are $C^2$ in all points except of $x$, where the following conditions are satisfied:

$$
\begin{align*}
\lim_{K \to x^+} \Psi_1'(K) + 1 &= \lim_{K \to x^-} \Psi_1'(K) \\
\lim_{K \to x^+} \Psi_2'(K) + 1 &= \lim_{K \to x^-} \Psi_2'(K)
\end{align*}
$$

(3)

Define the function

$$
G(K) = \Psi_2(K) - \Psi_1(K)
$$

Let us now assume that $\chi_1(K) > \chi_2(K), \forall K \in [L, U]$ and prove that this implies $G(K) > 0, \forall K \in [L, U]$. Further, Theorem 6 implies that $G(K) = C_2(K) - C_1(K)$, where the call price function $C_1(K)$ and $C_2(K)$ are $C^2$ for $[L, U]$. Hence, $G(K) = \Psi_2(K) - \Psi_1(K)$ will be $C^2$ on $[L, U]$ as well. Additionally, $G(K)$ will satisfy the boundary conditions:

$$
G(L) = G(U) = 0
$$

(4)

Let us now study the continuous second differential of $G(K)$:

$$
G''(K) = \Psi_2''(K) - \Psi_1''(K) = \chi_2(K)\Psi_2(K) - \chi_1(K)\Psi_1(K)
$$

From the assumption $\chi_1(K) > \chi_2(K), \forall K \in [L, U]$, we can then conclude the implication

$$
\Psi_1'(K) > \Psi_2'(K) \implies G''(K) < 0
$$

to be satisfied. Hence, $G(K)$ is a $C^2$ function which has to be concave in all points in $[L, U]$ where it is negative. A consequence of these properties is that if $G(K)$ is negative in any point $K \in [L, U]$, it can impossibly satisfy both of the boundary conditions from (4). We conclude that $G(K) < 0$ for any $K \in [L, U]$ contradicts our assumptions. Hence, the implication

$$
V_1(K) \leq V_2(K) \forall K \in [L, U] \\
\implies \Psi_1(K) \leq \Psi_2(K) \forall K \in [L, U],
$$

is satisfied, which finishes the proof of the theorem.
Appendix B

Here we display calibration data for all maturities in the calibration set. All numbers are the mean of calibrations to 208 daily Reuters quoted implied volatility surfaces from dates ranging between 2013-12-13 and 2014-03-12.

Currency Pair: EURUSD

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