

# Curvature and geometric modules of noncommutative spheres and tori

Joakim Arnlind

**Linköping University Post Print**



N.B.: When citing this work, cite the original article.

Original Publication:

Joakim Arnlind, Curvature and geometric modules of noncommutative spheres and tori, 2014, Journal of Mathematical Physics, (55), 4, 041705.

<http://dx.doi.org/10.1063/1.4871175>

Copyright: American Institute of Physics (AIP)

<http://www.aip.org/>

Postprint available at: Linköping University Electronic Press

<http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-107859>

## Curvature and geometric modules of noncommutative spheres and tori

Joakim Arnlind

Citation: [Journal of Mathematical Physics](#) **55**, 041705 (2014); doi: 10.1063/1.4871175

View online: <http://dx.doi.org/10.1063/1.4871175>

View Table of Contents: <http://scitation.aip.org/content/aip/journal/jmp/55/4?ver=pdfcov>

Published by the [AIP Publishing](#)

---

### Articles you may be interested in

[Coherent states in noncommutative quantum mechanics](#)

J. Math. Phys. **50**, 043505 (2009); 10.1063/1.3105926

[Invariant noncommutative connections](#)

J. Math. Phys. **46**, 123503 (2005); 10.1063/1.2131206

[Covariants, joint invariants and the problem of equivalence in the invariant theory of Killing tensors defined in pseudo-Riemannian spaces of constant curvature](#)

J. Math. Phys. **45**, 4141 (2004); 10.1063/1.1805728

[Algebraic classification of the curvature of three-dimensional manifolds with indefinite metric](#)

J. Math. Phys. **44**, 4374 (2003); 10.1063/1.1592611

[State vector reduction as a shadow of a noncommutative dynamics](#)

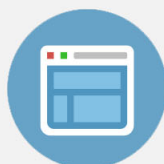
J. Math. Phys. **41**, 5168 (2000); 10.1063/1.533400

---



## Re-register for Table of Content Alerts

Create a profile.



Sign up today!



## Curvature and geometric modules of noncommutative spheres and tori

Joakim Arnlind<sup>a)</sup>

*Department of Mathematics, Linköping University, 581 83 Linköping, Sweden*

(Received 24 January 2014; accepted 28 March 2014; published online 18 April 2014)

When considered as submanifolds of Euclidean space, the Riemannian geometry of the round sphere and the Clifford torus may be formulated in terms of Poisson algebraic expressions involving the embedding coordinates, and a central object is the projection operator, projecting tangent vectors in the ambient space onto the tangent space of the submanifold. In this note, we point out that there exist noncommutative analogues of these projection operators, which implies a very natural definition of noncommutative tangent spaces as particular projective modules. These modules carry an induced connection from Euclidean space, and we compute its scalar curvature. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4871175>]

### I. INTRODUCTION

Linear connections on modules over noncommutative algebras, and associated differential calculi have been studied from many different points of view (see, e.g., Refs. 9, 11, and 15 for a derivation based approach). In most cases, the definition of the curvature operator is immediately given as the failure of the connection to be commutative, in analogy with classical differential geometry. However, the Ricci and scalar curvature does not come as easily. In commutative geometry, they arise as contractions over a basis of the tangent space, which does not always have an apparent noncommutative analogue (however, see Refs. 7, 14, 10, and 16). There are also more sophisticated definitions relying on the appearance of the scalar curvature in the expansion of the heat kernel (see, e.g., Ref. 8).

In a series of papers,<sup>5,3,4</sup> it was proven that one may formulate the metric geometry of embedded manifolds in terms of multi-linear algebraic expressions in the embedding coordinates. For surfaces, and, in general, almost Kähler manifolds, a Poisson algebraic formulation exists<sup>1,2</sup> (see also Ref. 6). These results were then used to construct noncommutative geometric concepts (such as curvature) by simply replacing Poisson brackets by commutators, and, in the context of matrix regularizations, these concepts were proven to be useful.<sup>5</sup> However, matrix regularizations rely on a sequence of algebras converging (in a certain sense) to the commutative algebra of smooth functions on the manifold, and therefore it was not clear how well adapted these concepts are to a single noncommutative algebra.

In this note, we will show that the projector of classical geometry, projecting tangent vectors from the ambient space to the tangent space of the embedded manifold, has a natural analogue in the noncommutative algebras of the sphere and the torus. This allows for the definition of a projective module which one may call the tangent bundle of the corresponding noncommutative geometry. Furthermore, an analogue of the Riemannian connection can be found and the corresponding scalar curvatures are computed. Note that our approach is in principle not limited to surfaces, and can be applied to noncommutative algebras corresponding to submanifolds of any dimension.

---

<sup>a)</sup>E-mail: [joakim.arnlind@liu.se](mailto:joakim.arnlind@liu.se)

## II. POISSON ALGEBRAIC FORMULATION OF SURFACE GEOMETRY

In Ref. 5, it was shown that the geometry of embedded Riemannian manifolds can be reformulated in terms of multi-linear brackets of the embedding coordinates; moreover, in the case of almost Kähler manifolds, a Poisson bracket formulation can be obtained.<sup>1</sup> Let us recall the basic facts of this reformulation, in the case of embedded surfaces.

Let  $(\Sigma, g)$  be a two-dimensional Riemannian manifold, and let  $\theta$  be a Poisson bivector defining the bracket

$$\{f, h\} = \theta^{ab}(\partial_a f)(\partial_b h),$$

for  $f, h \in C^\infty(\Sigma)$ . On a two-dimensional manifold, every Poisson bivector is of the form  $\theta^{ab} = \epsilon^{ab}/\rho$  for some density  $\rho$  (where  $\epsilon^{12} = -\epsilon^{21} = 1$ ). The cofactor expansion of the inverse of a matrix gives the following way of writing the inverse of the metric:

$$g^{ab} = \frac{1}{g} \epsilon^{ap} \epsilon^{bq} g_{ab} \implies g^{ab} = \frac{\rho^2}{g} \theta^{ap} \theta^{bq} g_{ab}, \quad (2.1)$$

which, upon setting  $\gamma = \sqrt{g}/\rho$ , becomes  $\gamma^2 g^{ab} = \theta^{ap} \theta^{bq} g_{ab}$ .

Now, assume that  $\Sigma$  is isometrically embedded in a  $m$ -dimensional Riemannian manifold  $(M, \bar{g})$ , via the embedding functions  $x^1, \dots, x^m$ ; i.e.,

$$g_{ab} = \bar{g}_{ij}(\partial_a x^i)(\partial_b x^j),$$

where  $\partial_a = \frac{\partial}{\partial x^a}$ . (Indices  $i, j, k, \dots$  run from 1 to  $m$  and indices  $a, b, c, \dots$  run from 1 to 2.) Relation (2.1) allows one to rewrite geometric object in terms of Poisson brackets of the embedding functions  $x^1, \dots, x^m$ . For instance, one notes that by defining  $\mathcal{D} : T_p M \rightarrow T_p M$  as

$$\mathcal{D}^i_j = \frac{1}{\gamma^2} \{x^i, x^k\} \bar{g}_{kl} \{x^j, x^l\} \bar{g}_{jm},$$

$$\mathcal{D}(X) \equiv \mathcal{D}^i_j X^j \partial_i,$$

for  $X = X^i \partial_i \in T_p M$ , one computes

$$\begin{aligned} \mathcal{D}(X)^i &= \frac{1}{\gamma^2} \theta^{ab} (\partial_a x^i)(\partial_b x^k) \bar{g}_{kl} \theta^{pq} (\partial_p x^j)(\partial_q x^l) \bar{g}_{jm} X^m \\ &= \frac{1}{\gamma^2} \theta^{ab} \theta^{pq} g_{bq} (\partial_a x^i)(\partial_p x^j) \bar{g}_{jm} X^m = g^{ap} (\partial_a x^i)(\partial_p x^j) \bar{g}_{jm} X^m, \end{aligned}$$

by using (2.1). Hence, the map  $\mathcal{D}$  is identified as the orthogonal projection onto  $T_p \Sigma$ , seen as a subspace of  $T_p M$  and, for convenience, we also introduce the complementary projection as

$$\Pi = \mathbb{1} - \mathcal{D}.$$

Having the projection operator at hand, one may proceed to develop the theory of submanifolds. For instance, the Levi-Civita connection  $\nabla$  on  $\Sigma$  is given by

$$\nabla_X Y = \mathcal{D}(\bar{\nabla}_X Y),$$

where  $X, Y \in T_p \Sigma$  and  $\bar{\nabla}$  is the Levi-Civita connection on  $M$ . Let us now turn to the particular case we shall be interested in. Namely, we assume that  $(M, \bar{g}_{ij}) = (\mathbb{R}^m, \delta_{ij})$  (which one may always do) and choose  $\gamma = 1$  (i.e.,  $\theta^{ab} = \epsilon^{ab}/\sqrt{g}$ ). In this setting, the connection becomes

$$\nabla_X Y^i = \mathcal{D}^i_k X(Y^k),$$

where  $X(f)$  denotes the action of  $X \in T_p \Sigma$  on  $f \in C^\infty(\Sigma)$  as a derivation; as usual, one introduces the curvature operator as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

In the non-commutative setting, we shall be interested in a particular set of derivations; namely, let

$$\partial^i(\cdot) = \{x^i, \cdot\} = \{x^i, x^j\} \partial_j(\cdot)$$

and set  $\nabla^i Y^k = \nabla_{\partial^i} Y^k = \mathcal{D}^k_l \partial^i(Y^l)$ . With respect to this set of derivations, one introduces the operator

$$\tilde{R}^{ij}(Z) = \nabla^i \nabla^j Z - \nabla^j \nabla^i Z - \nabla_{[\partial^i, \partial^j]} Z,$$

$$\tilde{R}(X, Y)Z = X^i Y^j \tilde{R}_{ij}(Z),$$

and computes that

$$\begin{aligned} \tilde{R}^{ij}(Z)^k &= \partial^i (\mathcal{D}^k_m) \partial^j (\mathcal{D}^m_l) Z^l - \partial^j (\mathcal{D}^k_m) \partial^i (\mathcal{D}^m_l) Z^l \\ &\equiv \tilde{R}^{ijk}_l Z^l. \end{aligned}$$

The relation to the curvature operator  $R$  is given by

$$R(X, Y)Z = \tilde{R}(\mathcal{P}(X), \mathcal{P}(Y))Z,$$

where  $\mathcal{P}(X) = \mathcal{P}^i_j X^j \partial_i$  with  $\mathcal{P}^{ij} = \{x^i, x^j\}$ . To compute the scalar curvature  $S$ , one has to contract indices of  $R_{ijkl}$  with the projection operator  $\mathcal{D}^{ij}$ , since one is summing over a basis of  $T_p \Sigma$  (seen as a subspace of  $T_p M$ ); i.e.,  $S = \mathcal{D}^{jl} \mathcal{D}^{ik} R_{ijkl}$ . Subsequently, the scalar curvature is given in terms of  $\tilde{R}$  as

$$S = \mathcal{P}_{jl} \mathcal{P}_{ik} \tilde{R}^{ijkl}, \quad (2.2)$$

which is a formula we shall use to define scalar curvature in the non-commutative setting. Let us now recall how the differential geometry of the sphere and the torus can be described in terms of Poisson brackets.

### A. The sphere

One considers the sphere as isometrically embedded in  $\mathbb{R}^3$  via

$$\vec{x} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$

giving

$$(g_{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix} \quad \text{and} \quad \sqrt{g} = \sin \theta.$$

By defining

$$\{f, h\} = \frac{1}{\sqrt{g}} \varepsilon^{ab} (\partial_a f) (\partial_b h) = \frac{1}{\sin \theta} \varepsilon^{ab} (\partial_a f) (\partial_b h),$$

one obtains

$$\{x^i, x^j\} = \varepsilon^{ijk} x^k,$$

where  $\varepsilon^{ijk}$  is a totally antisymmetric tensor with  $\varepsilon^{123} = 1$ . It is then straightforward to show that

$$\mathcal{D}^{ij} = \{x^i, x^k\} \{x^j, x^k\} = \delta^{ij} - x^i x^j,$$

$$\Pi^{ij} = x^i x^j,$$

$$\tilde{R}^{ijkl} = (\varepsilon^{ikm} \varepsilon^{jln} - \varepsilon^{jkm} \varepsilon^{iln}) x^m x^n,$$

$$S = \mathcal{P}^{jl} \mathcal{P}^{ik} \tilde{R}^{ijkl} = 2.$$

## B. The torus

The Clifford torus is considered as embedded in  $\mathbb{R}^4$  via

$$\vec{x} = \frac{1}{\sqrt{2}}(\cos \varphi_1, \sin \varphi_1, \cos \varphi_2, \sin \varphi_2)$$

giving rise to the induced metric

$$(g_{ab}) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \sqrt{g} = \frac{1}{2}.$$

By defining

$$\{f, h\} = \frac{1}{\sqrt{g}} \varepsilon^{ab} (\partial_a f) (\partial_b h) = 2 \varepsilon^{ab} (\partial_a f) (\partial_b h),$$

one obtains

$$(\{x^i, x^j\}) = 2 \begin{pmatrix} 0 & 0 & x^2 x^4 & -x^2 x^3 \\ 0 & 0 & -x^1 x^4 & x^1 x^3 \\ -x^2 x^4 & x^1 x^4 & 0 & 0 \\ x^2 x^3 & -x^1 x^3 & 0 & 0 \end{pmatrix},$$

from which it follows that

$$\mathcal{D} = 2 \begin{pmatrix} (x^2)^2 & -x^1 x^2 & 0 & 0 \\ -x^1 x^2 & (x^1)^2 & 0 & 0 \\ 0 & 0 & (x^4)^2 & -x^3 x^4 \\ 0 & 0 & -x^3 x^4 & (x^3)^2 \end{pmatrix},$$

$$\Pi = 2 \begin{pmatrix} (x^1)^2 & x^1 x^2 & 0 & 0 \\ x^1 x^2 & (x^2)^2 & 0 & 0 \\ 0 & 0 & (x^3)^2 & x^3 x^4 \\ 0 & 0 & x^3 x^4 & (x^4)^2 \end{pmatrix}.$$

Furthermore, a straightforward computation yields  $\tilde{R}^{ijkl} = 0$ .

## III. CONNECTIONS AND CURVATURE

Let  $\mathcal{A}$  be an associative  $*$ -algebra. A  $*$ -derivation is a derivation  $\partial$  such that  $\partial(a)^* = \partial(a^*)$  for all  $a \in \mathcal{A}$ ; by  $\text{Der}(\mathcal{A})$  we shall denote the vector space (over  $\mathbb{R}$ ) of  $*$ -derivations of  $\mathcal{A}$ . Moreover, assume that there exists a projector  $\mathcal{D}$ , acting on the (right) free module  $\mathcal{A}^m$ , i.e.,  $\mathcal{D} \in \text{End}(\mathcal{A}^m)$  and  $\mathcal{D}^2 = \mathcal{D}$ , and by  $T\mathcal{A}$  we denote the corresponding (finitely generated) projective module  $\mathcal{D}(\mathcal{A}^m)$ . Letting  $\{e_i\}_{i=1}^m$  denote the canonical basis of  $\mathcal{A}^m$ , one can write the action of  $\mathcal{D}$  as

$$\mathcal{D}(U) = e_i \mathcal{D}^i_j U^j,$$

for  $U = e_i U^i$  (note that there is no difference between lower and upper indices, but let us keep the notation that is familiar from differential geometry for now). We also introduce the complementary projection  $\Pi = \mathbb{1} - \mathcal{D}$ . Moreover, for every  $\partial \in \text{Der}(\mathcal{A})$  one defines

$$\tilde{\nabla}_\partial U = e_k \partial(U^k)$$

corresponding (in the commutative case) to the connection in the ‘‘ambient’’ space. Note that the two arguments of the connection are not on equal footing; one is a derivation and the other one belongs

to a free module. The map  $\bar{\nabla}_\partial$  is an affine connection on  $\mathcal{A}^m$  in the sense that

$$\begin{aligned} \bar{\nabla}_\partial(U + V) &= \bar{\nabla}_\partial U + \bar{\nabla}_\partial V, \\ \bar{\nabla}_{c\partial} U &= c\bar{\nabla}_\partial U, \\ \bar{\nabla}_{\partial+\partial'} U &= \bar{\nabla}_\partial U + \bar{\nabla}_{\partial'} U, \\ \bar{\nabla}_\partial(Ua) &= (\bar{\nabla}_\partial U)a + U\partial(a), \end{aligned} \tag{3.1}$$

for  $a \in \mathcal{A}$ ,  $c \in \mathbb{R}$ ,  $\partial, \partial' \in \text{Der}(\mathcal{A})$  and  $U, V \in \mathcal{A}^m$ . Furthermore, by introducing a metric on  $\mathcal{A}^m$  via

$$\langle U, V \rangle = (U_i)^* V^i, \tag{3.2}$$

for  $U = e_i U^i \in \mathcal{A}^m$  and  $V = e_i V^i \in \mathcal{A}^m$ , it is straightforward to show that  $\bar{\nabla}$  is a metric connection; i.e.,

$$\partial \langle U, V \rangle - \langle \bar{\nabla}_\partial U, V \rangle - \langle U, \bar{\nabla}_\partial V \rangle = 0,$$

for all  $\partial \in \text{Der}(\mathcal{A})$  and  $U, V \in \mathcal{A}^m$ . As for ordinary manifolds, one proceeds to define a connection on  $T\mathcal{A} = \mathcal{D}(\mathcal{A}^m)$  by setting

$$\nabla_\partial U = \mathcal{D}(\bar{\nabla}_\partial U) = e_i \mathcal{D}^j \partial(U^j)$$

for  $\partial \in \text{Der}(\mathcal{A})$  and  $U = e_i U^i \in T\mathcal{A}$ ; it follows that  $\nabla$  satisfies the requirements (3.1) of an affine connection. We shall assume that  $\mathcal{D}$  is symmetric with respect to the metric introduced in (3.2); i.e.,  $\langle \mathcal{D}(U), V \rangle = \langle U, \mathcal{D}(V) \rangle$  for all  $U, V \in \mathcal{A}^m$ . In this case,  $\nabla$  will also be a metric connection.

Now, let us choose a set of elements  $X^1, \dots, X^m \in \mathcal{A}$  together with their associated inner \*-derivations

$$\partial^i(a) = \frac{1}{i\hbar} [X^i, a]$$

for an arbitrary parameter  $\hbar \in \mathbb{R}$  (in the current setting, one might as well put  $\hbar = 1$ , but it will be convenient later on). In analogy with classical geometry, one should think of the  $X^i$ 's as embedding coordinates of a manifold into  $\mathbb{R}^m$ . A different choice of embedding does in general lead to a different induced metric on the submanifold. Therefore, the choice of  $X^i$ 's amount to a choice of the metric structure on the algebra.

With the help of the above derivations we introduce, for  $U \in T\mathcal{A}$ ,

$$\tilde{R}^{ij}(U) = \nabla^i \nabla^j U - \nabla^j \nabla^i U - \nabla_{[\partial^i, \partial^j]} U,$$

where  $\nabla^i U = \nabla_{\partial^i} U$ . That  $\tilde{R}^{ij}$  is a module homomorphism becomes clear from the following result:

*Proposition 3.1.* For  $U = e_i U^i \in T\mathcal{A}$  it holds that

$$\tilde{R}^{ij}(U) = e_k \left( \partial^i (\mathcal{D}^k_m) \partial^j (\mathcal{D}^m_l) - \partial^j (\mathcal{D}^k_m) \partial^i (\mathcal{D}^m_l) \right) U^l.$$

*Proof.* Let  $U \in T\mathcal{A}$  with  $U = e_i U^i$ . Using that  $\mathcal{D}(U) = U$  and Leibnitz rule, one obtains

$$\nabla^i \nabla^j (U) = e_k \mathcal{D}^k_l \partial^i (\mathcal{D}^l_m \partial^j (U^m)) = e_k \mathcal{D}^k_l \partial^i (\mathcal{D}^l_m) \partial^j (U^m) + e_k \mathcal{D}^k_m \partial^i \partial^j (U^m),$$

and one may rewrite the first term as

$$\begin{aligned} e_k \mathcal{D}^k_l \partial^i (\mathcal{D}^l_m) \partial^j (U^m) &= e_k \partial^i (\mathcal{D}^k_m) \partial^j (U^m) - e_k \partial^i (\mathcal{D}^k_l) \mathcal{D}^l_m \partial^j (U^m) \\ &= e_k \partial^i (\mathcal{D}^k_l) \partial^j (\mathcal{D}^l_m) U^m. \end{aligned}$$

Hence, it holds that

$$\nabla^i \nabla^j (U) = e_k \partial^i (\mathcal{D}^k_l) \partial^j (\mathcal{D}^l_m) U^m + e_k \mathcal{D}^k_m \partial^i \partial^j (U^m),$$

from which the desired formula follows for  $\tilde{R}^{ij}$ . □

Consequently, one introduces

$$\tilde{R}^{ijk}{}_l = \partial^i(\mathcal{D}^k{}_m)\partial^j(\mathcal{D}^m{}_l) - \partial^j(\mathcal{D}^k{}_l)\partial^i(\mathcal{D}^l{}_m)$$

giving  $\tilde{R}^{ij}(U) = e_k \tilde{R}^{ijk}{}_l U^l$ . In analogy with formula (2.2), we define the scalar curvature of  $\nabla$  as

$$S = \mathcal{P}_{jl} \mathcal{P}_{ik} \tilde{R}^{ijkl}, \quad (3.3)$$

where  $\mathcal{P}^{ij} = \frac{1}{i\hbar}[X^i, X^j]$ .

Furthermore, let us introduce the divergence of an element  $U \in T\mathcal{A}$  as

$$\text{div}(U) = \nabla_i U^i = \mathcal{D}_{ik} \partial^i(U^k) \in \mathcal{A}.$$

Let  $\phi : \mathcal{A} \rightarrow \mathbb{C}$  be a  $\mathbb{C}$ -linear functional such that  $\phi(ab) = \phi(ba)$  for all  $a, b \in \mathcal{A}$ ; we shall refer to such a linear functional as a *trace*. Moreover, a trace  $\phi$  is said to be *closed* if it holds that

$$\phi(\text{div}(U)) = 0$$

for all  $U \in T\mathcal{A}$ .

Let us, for later convenience, slightly rewrite the condition that  $\phi$  is a trace.

*Lemma 3.2.* A trace  $\phi$  is closed if and only if it holds that

$$\phi([X^i, \Pi_{ik}]U^k) = 0 \quad (3.4)$$

for all  $U = e_i U^i \in T\mathcal{A}$ .

*Proof.* Using that  $\phi$  is a trace, one computes that

$$\begin{aligned} \phi(\text{div}(U)) &= \phi(\mathcal{D}_{ik} \partial^i(U^k)) = \phi(\partial^i(\mathcal{D}_{ik} U^k) - \partial^i(\mathcal{D}_{ik})U^k) \\ &= \phi(\partial^i(\mathcal{D}_{ik})U^k) = \phi(\partial^i(\Pi_{ik})U^k) = \frac{1}{i\hbar} \phi([X^i, \Pi_{ik}]U^k), \end{aligned}$$

from which the statement follows.  $\square$

#### IV. THE FUZZY SPHERE

For our purposes, we shall define the fuzzy sphere<sup>12,13</sup> as a (unital associative)  $*$ -algebra  $S_\hbar^2$  on three hermitian generators  $X^1, X^2, X^3$  satisfying the following relations:

$$[X^i, X^j] = i\hbar \varepsilon^{ijk} X^k,$$

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = \mathbb{1}.$$

It is easy to see that, by setting  $\Pi^{ij} = X^i X^j$  as a non-commutative analogue of the classical projection operator, it holds that

$$(\Pi^2)^{ij} = \Pi^{ik} \Pi^{kj} = X^i X^k X^k X^j = X^i \mathbb{1} X^j = \Pi^{ij},$$

which shows that  $\Pi$  is a projection operator when considered as an endomorphism of the free module  $(S_\hbar^2)^3$ ; moreover,  $\Pi$  is symmetric since  $(\Pi^{ij})^* = X^j X^i = \Pi^{ji}$ . Let us note that the similarity with the commutative formulas is even stronger; namely, one easily checks that

$$\mathcal{D}^{ij} = \delta^{ij} \mathbb{1} - X^i X^j = \frac{1}{(i\hbar)^2} [X^j, X^k][X^i, X^k].$$

One may proceed and define a connection  $\nabla$  on  $TS_\hbar^2 = \mathcal{D}((S_\hbar^2)^3)$  as in Sec. III, and since the projection operator is symmetric, this is a metric connection. As it will be helpful in computations, let us remind ourselves of a few identities involving  $\varepsilon^{ijk}$

$$\varepsilon^{ijk} \varepsilon^{imn} = \delta^{jm} \delta^{kn} - \delta^{jn} \delta^{km}, \quad \varepsilon^{ikl} \varepsilon^{jkl} = 2\delta^{ij},$$

$$\varepsilon^{ijk} X^j X^k = i\hbar X^i, \quad \varepsilon^{ijk} X^i X^j X^k = i\hbar \mathbb{1}.$$



Let us now compute the curvature of  $\nabla$ .

*Proposition 4.1.* For the fuzzy sphere, it holds that

$$\begin{aligned} \tilde{R}^{ijkl} &= (\varepsilon^{ikp}\varepsilon^{jlq} - \varepsilon^{jkp}\varepsilon^{ilq})X^pX^q - i\hbar\varepsilon^{jlq}X^kX^iX^q - i\hbar\varepsilon^{jkp}X^pX^iX^l \\ &\quad + i\hbar\varepsilon^{ikp}X^pX^jX^l + i\hbar\varepsilon^{ilq}X^kX^jX^q + i\hbar\varepsilon^{ijp}X^kX^pX^l, \\ S &= (2 - 3\hbar^2 + h^4)\mathbb{1}. \end{aligned}$$

*Proof.* The proof consists of a straightforward computation. Starting from

$$\begin{aligned} \tilde{R}^{ijkl} &= \partial^i(\mathcal{D}^{km})\partial^j(\mathcal{D}^{ml}) - \partial^j(\mathcal{D}^{km})\partial^i(\mathcal{D}^{ml}) \\ &= \partial^i(\Pi^{km})\partial^j(\Pi^{ml}) - \partial^j(\Pi^{km})\partial^i(\Pi^{ml}) \\ &= \frac{1}{(i\hbar)^2}[X^i, X^kX^m][X^j, X^mX^l] - \frac{1}{(i\hbar)^2}[X^j, X^kX^m][X^i, X^mX^l], \end{aligned}$$

one expands the expression, using  $[X^i, X^j] = i\hbar\varepsilon^{ijk}X^k$  and the  $\varepsilon$ -identities we previously recalled, to obtain

$$\begin{aligned} \tilde{R}^{ijkl} &= (\varepsilon^{ikp}\varepsilon^{jlq} - \varepsilon^{jkp}\varepsilon^{ilq})X^pX^q - i\hbar\varepsilon^{jlq}X^kX^iX^q - i\hbar\varepsilon^{jkp}X^pX^iX^l \\ &\quad + i\hbar\varepsilon^{ikp}X^pX^jX^l + i\hbar\varepsilon^{ilq}X^kX^jX^q + i\hbar\varepsilon^{ijp}X^kX^pX^l. \end{aligned}$$

From this expression, one derives

$$\mathcal{P}^{ik}\tilde{R}^{ijkl} = (1 - \hbar^2 - \hbar^4)\varepsilon^{ilm}X^m + i\hbar(1 - 3\hbar^2)X^jX^l + i\hbar^3\delta^{jl}$$

again by using the appropriate identities. Finally, the scalar curvature is computed

$$\begin{aligned} S &= \mathcal{P}^{jl}\mathcal{P}^{ik}\tilde{R}^{ijkl} \\ &= (1 - \hbar^2 - \hbar^4)\varepsilon^{jlk}X^k\varepsilon^{ilm}X^m + i\hbar(1 - 3\hbar^2)\varepsilon^{jlk}X^kX^jX^l + i\hbar^3\varepsilon^{jlk}X^k\delta^{jl} \\ &= 2(1 - \hbar^2 - \hbar^4)\mathbb{1} + i\hbar(1 - 3\hbar^2)i\hbar\mathbb{1} = (2 - 3\hbar^2 + h^4)\mathbb{1}, \end{aligned}$$

which proves the statement. □

Let us now show that every trace on the fuzzy sphere is closed.

*Proposition 4.2.* Let  $\phi$  be a trace on  $S_h^2$ . Then  $\phi$  is closed.

*Proof.* Starting from the formula in Lemma 3.2, one computes

$$\begin{aligned} \frac{1}{i\hbar}\phi([X^i, \Pi^{ik}]U^k) &= \frac{1}{i\hbar}\phi([X^i, X^iX^k]U^k) = \frac{1}{i\hbar}\phi(X^i[X^i, X^k]U^k) \\ &= \phi(\varepsilon^{ikl}X^iX^lU^k) = -i\hbar\phi(X^kU^k). \end{aligned}$$

Now, since  $U \in T\mathcal{A}$  it holds that  $\Pi(U) = 0$ , which is equivalent to

$$X^iX^kU^k = 0$$

for  $i = 1, 2, 3$ . Multiplying the above equation by  $X^i$  from the left, and summing over  $i$  yields

$$0 = X^iX^iX^kU^k = X^kU^k.$$

Thus,  $X^kU^k = 0$ , which proves that  $\phi$  is closed. □

Note that one may easily compute the rank of the module  $TS_{\hbar}^2$  and its complementary module  $\mathcal{N} = \Pi((S_{\hbar}^2)^3)$  as the trace of the corresponding projections; i.e.,

$$\text{rank}(TS_{\hbar}^2) = \sum_{i=1}^3 \mathcal{D}^{ii} = \sum_{i=1}^3 (\delta^{ii} \mathbb{1} - X^i X^i) = 2\mathbb{1},$$

$$\text{rank}(\mathcal{N}) = \sum_{i=1}^3 \Pi^{ii} = \sum_{i=1}^3 X^i X^i = \mathbb{1},$$

corresponding to the geometric dimensions in the commutative setting. Moreover, the module  $\mathcal{N}$  turns out to be a free module.

*Proposition 4.3.* *The module  $\mathcal{N} = \Pi((S_{\hbar}^2)^3)$  is a free module of rank 1, and it is generated by  $X = e_i X^i$ .*

*Proof.* An element  $N = e_i N^i \in \mathcal{N}$  satisfies

$$X^i X^j N^j = N^i$$

for  $i = 1, 2, 3$ , which implies that there exists an element  $a = X^j N^j \in \mathcal{A}$  such that  $N = e_i X^i \cdot a$ . This proves that  $e_i X^i$  generates  $\mathcal{N}$ . Furthermore, one computes that

$$0 = X^i a \Rightarrow 0 = X^i X^i a \Rightarrow 0 = a,$$

which shows that  $\mathcal{N}$  is indeed a free module. □

### V. THE NON-COMMUTATIVE TORUS

The non-commutative torus  $\mathcal{A}_{\theta}$  (for  $\theta \in \mathbb{R}$ )<sup>9</sup> is defined as the unital associative \*-algebra on two unitary generators  $U, V$  satisfying the following relation:

$$VU = qUV$$

with  $q = e^{2i\theta}$ . Defining hermitian elements

$$X^1 = \frac{1}{2\sqrt{2}}(U^* + U), \quad X^2 = \frac{i}{2\sqrt{2}}(U^* - U),$$

$$X^3 = \frac{1}{2\sqrt{2}}(V^* + V), \quad X^4 = \frac{i}{2\sqrt{2}}(V^* - V),$$

it follows that

$$[X^1, X^2] = [X^3, X^4] = 0, \tag{5.1}$$

$$[X^1, X^3] = i\hbar(X^2 X^4 + X^4 X^2), \tag{5.2}$$

$$[X^2, X^4] = i\hbar(X^1 X^3 + X^3 X^1), \tag{5.3}$$

$$[X^1, X^4] = -i\hbar(X^2 X^3 + X^3 X^2), \tag{5.4}$$

$$[X^2, X^3] = -i\hbar(X^1 X^4 + X^4 X^1), \tag{5.5}$$

$$(X^1)^2 + (X^2)^2 = (X^3)^2 + (X^4)^2 = \frac{1}{2}\mathbb{1}, \tag{5.6}$$

with  $\hbar = \tan \theta$ . Conversely, one can show that the above relations imply that

$$U = \sqrt{2}(X^1 + iX^2), \quad V = \sqrt{2}(X^3 + iX^4)$$

are unitary elements satisfying  $VU = qUV$ . Namely, since  $[X^1, X^2] = [X^3, X^4] = 0$  it follows immediately that  $[U, U^*] = [V, V^*] = 0$ , and from  $(X^1)^2 + (X^2)^2 = \mathbb{1}/2$  and  $(X^3)^2 + (X^4)^2 = \mathbb{1}/2$  it follows that

$$UU^* + U^*U = VV^* + V^*V = 2\mathbb{1},$$

which, together with  $[U, U^*] = [V, V^*] = 0$ , implies that  $U$  and  $V$  are unitary. Furthermore, noting that (5.2)–(5.5) imply that

$$X^3X^1 = \cos(2\theta)X^1X^3 - i\sin(2\theta)X^2X^4,$$

$$X^4X^2 = \cos(2\theta)X^2X^4 - i\sin(2\theta)X^1X^3,$$

$$X^4X^1 = \cos(2\theta)X^1X^4 + i\sin(2\theta)X^2X^3,$$

$$X^3X^2 = \cos(2\theta)X^2X^3 + i\sin(2\theta)X^1X^4,$$

one readily shows that  $VU = qUV$ .

Since there is a natural split of the  $X^i$ 's into two groups, let us develop some notation reflecting this fact. Greek indices  $\alpha, \beta, \dots$  will take values in  $\{1, 2\}$  and “barred” indices  $\bar{\alpha}, \bar{\beta}, \dots$  take values in  $\{3, 4\}$ . With this notation, the projector  $\Pi$  may be defined as (in analogy with the classical formula)

$$\Pi^{\alpha\bar{\alpha}} = \Pi^{\bar{\alpha}\alpha} = 0,$$

$$\Pi^{\alpha\beta} = 2X^\alpha X^\beta, \quad \Pi^{\bar{\alpha}\bar{\beta}} = 2X^{\bar{\alpha}} X^{\bar{\beta}},$$

and one checks that  $\Pi^2 = \Pi$ , as well as  $(\Pi^{ij})^* = \Pi^{ji}$ . For the forthcoming computations, one notes that

$$[X^\alpha, X^\beta] = [X^{\bar{\alpha}}, X^{\bar{\beta}}] = 0, \tag{5.7}$$

$$[X^\alpha, \Pi^{\beta\gamma}] = [X^{\bar{\alpha}}, \Pi^{\bar{\beta}\bar{\gamma}}] = 0. \tag{5.8}$$

*Proposition 5.1.* The curvature  $\tilde{R}$  of  $\mathcal{A}_\theta$  vanishes; i.e.,

$$\tilde{R}^{ijkl} = 0$$

for  $i, j, k, l \in \{1, 2, 3, 4\}$ .

*Proof.* Using (5.7) and (5.8), it is easy to see that

$$\tilde{R}^{\alpha\bar{\alpha}kl} = \tilde{R}^{\alpha\beta\gamma k} = \tilde{R}^{\alpha\beta k\gamma} = \tilde{R}^{\bar{\alpha}\bar{\beta}\bar{\gamma}k} = \tilde{R}^{\bar{\alpha}\bar{\beta}k\bar{\gamma}} = 0.$$

Thus, it remains to show that  $\tilde{R}^{\alpha\beta\bar{\alpha}\bar{\beta}} = \tilde{R}^{\bar{\alpha}\bar{\beta}\alpha\beta} = 0$ ; let us outline the calculation for  $\tilde{R}^{\alpha\beta\bar{\alpha}\bar{\beta}}$ . It turns out to be slightly easier to perform the computation using variables  $U$  and  $V$  instead of  $X^i$ , and one writes

$$X^\alpha = \frac{i^{\alpha-1}}{2\sqrt{2}}(U^* + (-1)^{\alpha-1}U) \quad (\alpha = 1, 2),$$

$$X^{\bar{\alpha}} = \frac{-i^{\bar{\alpha}-1}}{2\sqrt{2}}(V^* + (-1)^{\bar{\alpha}-1}V) \quad (\bar{\alpha} = 3, 4).$$

Since

$$\begin{aligned}\tilde{R}^{\alpha\beta\bar{\alpha}\bar{\beta}} &= \partial^\alpha(\Pi^{\bar{\alpha}\bar{\gamma}})\partial^\beta(\Pi^{\bar{\gamma}\bar{\beta}}) - \partial^\beta(\Pi^{\bar{\alpha}\bar{\gamma}})\partial^\alpha(\Pi^{\bar{\gamma}\bar{\beta}}) \\ &= \frac{4}{(i\hbar)^2}[X^\alpha, X^{\bar{\alpha}}X^{\bar{\gamma}}][X^\beta, X^{\bar{\gamma}}X^{\bar{\beta}}] - \frac{4}{(i\hbar)^2}[X^\beta, X^{\bar{\alpha}}X^{\bar{\gamma}}][X^\alpha, X^{\bar{\gamma}}X^{\bar{\beta}}],\end{aligned}\quad (5.9)$$

let us start by computing  $2[X^\alpha, X^{\bar{\alpha}}X^{\bar{\gamma}}]$

$$\begin{aligned}2[X^\alpha, X^{\bar{\alpha}}X^{\bar{\gamma}}] &= \frac{i^{\alpha+\bar{\alpha}+\bar{\gamma}+1}}{8\sqrt{2}}[U^* + (-1)^{\alpha-1}U, (V^* + (-1)^{\bar{\alpha}-1}V)(V^* + (-1)^{\bar{\gamma}-1}V)] \\ &= \frac{i^{\alpha+\bar{\alpha}+\bar{\gamma}+1}}{8\sqrt{2}}\left((1-q^2)U^*(V^*)^2 + (-1)^{\alpha+\bar{\alpha}+\bar{\gamma}-1}(1-q^2)UV^2\right. \\ &\quad \left.+ (-1)^{\alpha-1}(1-\bar{q}^2)U(V^*)^2 + (-1)^{\bar{\alpha}+\bar{\gamma}}(1-\bar{q}^2)U^*V^2\right)\end{aligned}$$

by using  $VU = qUV$  and  $V^*U = \bar{q}UV^*$ . Subsequently, using this result, one computes (sum over  $\bar{\gamma}$  implied)

$$\begin{aligned}[X^\alpha, X^{\bar{\alpha}}X^{\bar{\gamma}}][X^\beta, X^{\bar{\gamma}}X^{\bar{\beta}}] &= \frac{-i^{\alpha+\beta+\bar{\alpha}+\bar{\beta}}}{64}\left(\bar{q}^2(1-q^2)^2((-1)^{\beta+\bar{\beta}-1} + (-1)^{\alpha+\bar{\alpha}-1})\mathbb{1}\right. \\ &\quad \left.+ q^2(1-\bar{q}^2)((-1)^{\alpha+\bar{\beta}-1} + (-1)^{\bar{\alpha}+\beta-1})\mathbb{1}\right. \\ &\quad \left.+ (1-q^2)(1-\bar{q}^2)((-1)^{\bar{\alpha}}\bar{q}^2 + (-1)^{\bar{\beta}}q^2)(U^*)^2\right. \\ &\quad \left.+ (1-q^2)(1-\bar{q}^2)((-1)^{\alpha+\beta+\bar{\alpha}}q^2 + (-1)^{\alpha+\beta+\bar{\beta}}\bar{q}^2)U^2\right),\end{aligned}$$

where many terms vanish due to the fact that anything proportional to  $(-1)^{\bar{\gamma}}$  cancel when summing over  $\bar{\gamma}$ . Since

$$q^2(1-\bar{q}^2)^2 = q^2 + \bar{q}^2 - 2 = \bar{q}^2(1-q^2)^2$$

one notes that the previous expression is *symmetric* with respect to interchanging  $\alpha$  and  $\beta$ , which implies, via (5.9), that  $\tilde{R}^{\alpha\beta\bar{\alpha}\bar{\beta}} = 0$ .  $\square$

Let us now show that, as for the fuzzy sphere, every trace on  $\mathcal{A}_\theta$  is closed.

*Proposition 5.2.* *Let  $\phi$  be a trace on  $\mathcal{A}_\theta$ . Then  $\phi$  is closed.*

*Proof.* Let us prove that  $[X^i, \Pi^{ik}] = 0$  for  $k = 1, 2, 3, 4$ . Lemma 3.2 then implies that  $\phi$  is closed. First, assume that  $k = \beta$

$$[X^i, \Pi^{i\beta}] = [X^\alpha, \Pi^{\alpha\beta}] + [X^{\bar{\alpha}}, \Pi^{\bar{\alpha}\beta}] = 0,$$

since  $\Pi^{\bar{\alpha}\beta} = 0$  and  $[X^\alpha, \Pi^{\alpha\beta}] = 0$ . Similarly, when  $k = \bar{\beta}$  one obtains

$$[X^i, \Pi^{i\bar{\beta}}] = [X^\alpha, \Pi^{\alpha\bar{\beta}}] + [X^{\bar{\alpha}}, \Pi^{\bar{\alpha}\bar{\beta}}] = 0,$$

which proves that  $[X^i, \Pi^{ik}] = 0$ .  $\square$

The rank of  $T\mathcal{A}_\theta = \mathcal{D}(\mathcal{A}_\theta^4)$  and  $\mathcal{N}\mathcal{A}_\theta = \Pi(\mathcal{A}_\theta^4)$  can again be computed via the trace of the corresponding projection operators

$$\text{rank}(T\mathcal{A}_\theta) = \sum_{i=1}^4 (\delta^{ii} \mathbb{1} - 2X^i X^i) = 2\mathbb{1},$$

$$\text{rank}(\mathcal{N}\mathcal{A}_\theta) = \sum_{i=1}^4 (2X^i X^i) = 2\mathbb{1}.$$

Now, let us show that, in fact, both  $T\mathcal{A}_\theta$  and  $\mathcal{N}\mathcal{A}_\theta$  are free modules.

*Proposition 5.3.* The module  $T\mathcal{A}_\theta = \mathcal{D}(\mathcal{A}_\theta^4)$  is a free module of rank 2, with a basis given by  $E_1 = -e_1X^2 + e_2X^1$  and  $E_2 = -e_3X^4 + e_4X^3$ .

*Proof.* First of all, it is easy to check that  $\mathcal{D}(E_1) = E_1$  and  $\mathcal{D}(E_2) = E_2$ , which implies that  $E_1, E_2 \in T\mathcal{A}_\theta$ . Moreover,  $E_1$  and  $E_2$  are linearly independent, since

$$E_1a + E_2b = 0 \Rightarrow (-X^2a, X^1a, -X^4b, X^3b) = (0, 0, 0, 0) \Rightarrow$$

$$\begin{cases} ((X^1)^2 + (X^2)^2)a = 0 \\ ((X^3)^2 + (X^4)^2)b = 0 \end{cases} \Rightarrow a = b = 0.$$

Let us now show that  $E_1$  and  $E_2$  span  $T\mathcal{A}_\theta$ . By definition of  $T\mathcal{A}_\theta$  there exists, for every  $Y \in T\mathcal{A}_\theta$ , and element  $U \in \mathcal{A}_\theta^4$  such that  $Y = \mathcal{D}(U)$ . One readily computes that

$$\begin{aligned} \mathcal{D}(U)^1 &= -X^2(2X^1U^2 - 2X^2U^1), \\ \mathcal{D}(U)^2 &= X^1(2X^1U^2 - 2X^2U^1), \\ \mathcal{D}(U)^3 &= -X^4(2X^3U^4 - 2X^4U^3), \\ \mathcal{D}(U)^4 &= X^3(2X^3U^4 - 2X^4U^3); \end{aligned}$$

that is, for every  $U \in \mathcal{A}_\theta^4$ , there exist  $a, b \in \mathcal{A}_\theta$  such that  $\mathcal{D}(U) = E_1a + E_2b$ , which implies that  $E_1$  and  $E_2$  span  $T\mathcal{A}_\theta$ . □

*Proposition 5.4.* The module  $\mathcal{N}\mathcal{A}_\theta = \Pi(\mathcal{A}_\theta^4)$  is a free module of rank 2, with a basis given by  $N_\pm = e_1X^1 + e_2X^2 \pm e_3X^3 \pm e_4X^4$ .

*Proof.* It is easy to check that  $\Pi(N_+) = N_+$  and  $\Pi(N_-) = N_-$  which shows that they are indeed elements of  $\mathcal{N}\mathcal{A}_\theta$ . Thus, every element of the form

$$N = e_1X^1a + e_2X^2a + e_3X^3b + e_4X^4b \tag{5.10}$$

is an element of  $\mathcal{N}\mathcal{A}_\theta$ . Now, let  $N = e_iN^i \in \mathcal{A}_\theta^4$  such that  $\Pi(N) = N$ , which is equivalent to

$$\begin{aligned} \Pi^{\alpha\beta} N^\beta &= N^\alpha \Leftrightarrow 2X^\alpha X^\beta N^\beta = N^\alpha, \\ \Pi^{\bar{\alpha}\bar{\beta}} N^{\bar{\beta}} &= N^{\bar{\alpha}} \Leftrightarrow 2X^{\bar{\alpha}} X^{\bar{\beta}} N^{\bar{\beta}} = N^{\bar{\alpha}}. \end{aligned}$$

This immediately implies that  $N$  can be written in the form (5.10). Thus, the elements  $N_+$  and  $N_-$  generate  $\mathcal{N}\mathcal{A}_\theta$ . Next, assume that

$$N = e_1X^1a + e_2X^2a + e_3X^3b + e_4X^4b = 0,$$

which is equivalent to  $X^\alpha a = 0$  and  $X^{\bar{\alpha}} b = 0$ . Multiplying by  $X^\alpha$  and  $X^{\bar{\alpha}}$ , respectively, and summing over the index yields  $a = b = 0$ . Hence,  $N_+$  and  $N_-$  is a basis for the module  $\mathcal{N}\mathcal{A}_\theta$ . □

Finally, we note that the set  $\{E_1, E_2, N_+, N_-\}$  is a set of mutually orthogonal elements with respect to the metric  $\langle \cdot, \cdot \rangle$ .

## ACKNOWLEDGMENTS

I would like to thank J. Choe and J. Hoppe for discussions, and the Korea Institute for Advanced Study and Sogang University for hospitality and financial support.

- <sup>1</sup>J. Arnlind and G. Huisken, “On the geometry of Kähler-Poisson structures,” e-print [arXiv:1103.5862](#).
- <sup>2</sup>J. Arnlind and G. Huisken, “Pseudo-Riemannian geometry in terms of multi-linear brackets,” e-print [arXiv:1312.5454](#).
- <sup>3</sup>J. Arnlind, J. Hoppe, and G. Huisken, “On the classical geometry of embedded manifolds in terms of Nambu brackets,” e-print [arXiv:1003.5981](#).
- <sup>4</sup>J. Arnlind, J. Hoppe, and G. Huisken, “On the classical geometry of embedded surfaces in terms of Poisson brackets,” e-print [arXiv:1001.1604](#).
- <sup>5</sup>J. Arnlind, J. Hoppe, and G. Huisken, “Multi-linear formulation of differential geometry and matrix regularizations,” *J. Differ. Geom.* **91**(1), 1–39 (2012).
- <sup>6</sup>D. Blaschke and H. Steinacker, “Curvature and gravity actions for matrix models,” *Class. Quant. Grav.* **27**, 165010 (2010).
- <sup>7</sup>A. H. Chamseddine, G. Felder, and J. Fröhlich, “Gravity in noncommutative geometry,” *Commun. Math. Phys.* **155**(1), 205–217 (1993).
- <sup>8</sup>A. Connes and H. Moscovici, “Modular curvature for noncommutative two-tori,” e-print [arXiv:1110.3500](#).
- <sup>9</sup>A. Connes, “ $C^*$  algèbres et géométrie différentielle,” *C. R. Acad. Sci. Paris Sér. A-B* **290**(13), A599–A604 (1980).
- <sup>10</sup>M. Chaichian, A. Tureanu, R. B. Zhang, and X. Zhang, “Riemannian geometry of noncommutative surfaces,” *J. Math. Phys.* **49**(7), 073511 (2008).
- <sup>11</sup>M. Dubois-Violette, “Dérivations et calcul différentiel non commutatif,” *C. R. Acad. Sci. Paris Sér. I Math.* **307**(8), 403–408 (1988).
- <sup>12</sup>J. Hoppe, “Quantum theory of a massless relativistic surface and a two-dimensional bound state problem,” Ph.D. thesis, Massachusetts Institute of Technology, 1982.
- <sup>13</sup>J. Madore, “The fuzzy sphere,” *Class. Quant. Grav.* **9**(1), 69–87 (1992).
- <sup>14</sup>J. Madore, T. Masson, and J. Mourad, “Linear connections on matrix geometries,” *Class. Quant. Grav.* **12**(6), 1429–1440 (1995).
- <sup>15</sup>J. Mourad, “Linear connections in non-commutative geometry,” *Class. Quant. Grav.* **12**(4), 965–974 (1995).
- <sup>16</sup>J. Rosenberg, “Levi-Civita’s theorem for noncommutative tori,” *SIGMA* **9** (2013), 071; e-print [arXiv:1307.3775](#).