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MARGINAL AMP CHAIN GRAPHS

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ABSTRACT. We present a new family of models that is based on graphs that may have undirected, directed and bidirected edges. We name these new models marginal AMP (MAMP) chain graphs because each of them is Markov equivalent to some AMP chain graph under marginalization of some of its nodes. However, MAMP chain graphs do not only subsume AMP chain graphs but also multivariate regression chain graphs. We describe global and pairwise Markov properties for MAMP chain graphs and prove their equivalence for compositional graphoids. We also characterize when two MAMP chain graphs are Markov equivalent.

For Gaussian probability distributions, we also show that every MAMP chain graph is Markov equivalent to some directed and acyclic graph with deterministic nodes under marginalization and conditioning on some of its nodes. This is important because it implies that the independence model represented by a MAMP chain graph can be accounted for by some data generating process that is partially observed and has selection bias. Finally, we modify MAMP chain graphs so that they are closed under marginalization for Gaussian probability distributions. This is a desirable feature because it guarantees parsimonious models under marginalization.

1. INTRODUCTION

Chain graphs (CGs) are graphs with possibly directed and undirected edges, and no semidirected cycle. They have been extensively studied as a formalism to represent independence models, because they can model symmetric and asymmetric relationships between the random variables of interest. However, there are four different interpretations of CGs as independence models (Cox and Wermuth, 1993, 1996; Drton, 2009; Sonntag and Peña, 2013). In this paper, we are interested in the AMP interpretation (Andersson et al., 2001; Levitz et al., 2001) and in the multivariate regression (MVR) interpretation (Cox and Wermuth, 1993, 1996). Although MVR CGs were originally represented using dashed directed and undirected edges, we prefer to represent them using solid directed and bidirected edges.

In this paper, we unify and generalize the AMP and MVR interpretations of CGs. We do so by introducing a new family of models that is based on graphs that may have undirected, directed and bidirected edges. We call this new family marginal AMP (MAMP) CGs.

The rest of the paper is organized as follows. We start with some preliminaries and notation in Section 2. We continue by proving in Section 3 that, for Gaussian probability distributions, every AMP CG is Markov equivalent to some directed and acyclic graph with deterministic nodes under marginalization and conditioning on some of its nodes. We extend this result to MAMP CGs in Section 4, which implies that the independence model represented by a MAMP chain graph can be accounted for by some data generating process that is partially observed and has selection bias. Therefore, the independence models represented by MAMP CGs are not arbitrary and, thus, MAMP CGs are worth studying. We also describe in Section 4 global and pairwise Markov properties for MAMP CGs and prove their equivalence for compositional graphoids. Moreover, we also characterize in that section when two MAMP CGs are Markov equivalent. We show in Section 5 that MAMP CGs are not closed under marginalization and modify them so that they become closed under marginalization for Gaussian probability distributions. This is important because it guarantees parsimonious

models under marginalization. Finally, we discuss in Section 6 how MAMP CGs relate to other existing models based on graphs such as regression CGs, maximal ancestral graphs, summary graphs and MC graphs.

2. PRELIMINARIES

In this section, we introduce some concepts of models based on graphs, i.e. graphical models. Most of these concepts have a unique definition in the literature. However, a few concepts have more than one definition in the literature and, thus, we opt for the most suitable in this work. All the graphs and probability distributions in this paper are defined over a finite set V . All the graphs in this paper are simple, i.e. they contain at most one edge between any pair of nodes. The elements of V are not distinguished from singletons. The operators set union and set difference are given equal precedence in the expressions. The term maximal is always wrt set inclusion.

If a graph G contains an undirected, directed or bidirected edge between two nodes V_1 and V_2 , then we write that $V_1 - V_2$, $V_1 \rightarrow V_2$ or $V_1 \leftrightarrow V_2$ is in G . We represent with a circle, such as in $\circ\rightarrow$ or $\circ\leftrightarrow$, that the end of an edge is unspecified, i.e. it may be an arrow tip or nothing. The parents of a set of nodes X of G is the set $pa_G(X) = \{V_1 | V_1 \rightarrow V_2 \text{ is in } G, V_1 \notin X \text{ and } V_2 \in X\}$. The children of X is the set $ch_G(X) = \{V_1 | V_1 \leftarrow V_2 \text{ is in } G, V_1 \notin X \text{ and } V_2 \in X\}$. The neighbors of X is the set $ne_G(X) = \{V_1 | V_1 - V_2 \text{ is in } G, V_1 \notin X \text{ and } V_2 \in X\}$. The spouses of X is the set $sp_G(X) = \{V_1 | V_1 \leftrightarrow V_2 \text{ is in } G, V_1 \notin X \text{ and } V_2 \in X\}$. The adjacents of X is the set $ad_G(X) = ne_G(X) \cup pa_G(X) \cup ch_G(X) \cup sp_G(X)$. A route between a node V_1 and a node V_n in G is a sequence of (not necessarily distinct) nodes V_1, \dots, V_n st $V_i \in ad_G(V_{i+1})$ for all $1 \leq i < n$. If the nodes in the route are all distinct, then the route is called a path. The length of a route is the number of (not necessarily distinct) edges in the route, e.g. the length of the route V_1, \dots, V_n is $n - 1$. A route is called undirected if $V_i - V_{i+1}$ is in G for all $1 \leq i < n$. A route is called descending if $V_i \rightarrow V_{i+1}$ or $V_i - V_{i+1}$ is in G for all $1 \leq i < n$. A route is called strictly descending if $V_i \rightarrow V_{i+1}$ is in G for all $1 \leq i < n$. The descendants of a set of nodes X of G is the set $de_G(X) = \{V_n | \text{there is a descending route from } V_1 \text{ to } V_n \text{ in } G, V_1 \in X \text{ and } V_n \notin X\}$. The non-descendants of X is the set $nde_G(X) = V \setminus X \setminus de_G(X)$. The strict ascendants of X is the set $san_G(X) = \{V_1 | \text{there is a strictly descending route from } V_1 \text{ to } V_n \text{ in } G, V_1 \notin X \text{ and } V_n \in X\}$. A route V_1, \dots, V_n in G is called a cycle if $V_n = V_1$. Moreover, it is called a semidirected cycle if $V_n = V_1$, $V_1 \rightarrow V_2$ is in G and $V_i \rightarrow V_{i+1}$, $V_i \leftrightarrow V_{i+1}$ or $V_i - V_{i+1}$ is in G for all $1 < i < n$. An AMP chain graph (AMP CG) is a graph whose every edge is directed or undirected st it has no semidirected cycles. A MVR chain graph (MVR CG) is a graph whose every edge is directed or bidirected st it has no semidirected cycles. A set of nodes of a graph is connected if there exists a path in the graph between every pair of nodes in the set st all the edges in the path are undirected or bidirected. A connectivity component of a graph is a maximal connected set. The subgraph of G induced by a set of its nodes X , denoted as G_X , is the graph over X that has all and only the edges in G whose both ends are in X .

Let X, Y, Z and W denote four disjoint subsets of V . An independence model M is a set of statements $X \perp_M Y | Z$. Moreover, M is called graphoid if it satisfies the following properties: Symmetry $X \perp_M Y | Z \Rightarrow Y \perp_M X | Z$, decomposition $X \perp_M Y \cup W | Z \Rightarrow X \perp_M Y | Z$, weak union $X \perp_M Y \cup W | Z \Rightarrow X \perp_M Y | Z \cup W$, contraction $X \perp_M Y | Z \cup W \wedge X \perp_M W | Z \Rightarrow X \perp_M Y \cup W | Z$, and intersection $X \perp_p Y | Z \cup W \wedge X \perp_p W | Z \cup Y \Rightarrow X \perp_p Y \cup W | Z$. Moreover, M is called compositional graphoid if it is a graphoid that also satisfies the composition property $X \perp_M Y | Z \wedge X \perp_M W | Z \Rightarrow X \perp_M Y \cup W | Z$.

We now recall the semantics of AMP, MVR and LWF CGs. A node B in a path ρ in an AMP CG G is called a triplex node in ρ if $A \rightarrow B \leftarrow C$, $A \rightarrow B - C$, or $A - B \leftarrow C$ is a subpath of ρ . Moreover, ρ is said to be Z -open with $Z \subseteq V$ when

- every triplex node in ρ is in $Z \cup san_G(Z)$, and

- every non-triplex node B in ρ is outside Z , unless $A - B - C$ is a subpath of ρ and $pa_G(B) \setminus Z \neq \emptyset$.

A node B in a path ρ in a MVR CG G is called a triplex node in ρ if $A \leftrightarrow B \leftrightarrow C$ is a subpath of ρ . Moreover, ρ is said to be Z -open with $Z \subseteq V$ when

- every triplex node in ρ is in $Z \cup san_G(Z)$, and
- every non-triplex node B in ρ is outside Z .

A section of a route ρ in a CG is a maximal undirected subroute of ρ . A section $V_2 - \dots - V_{n-1}$ of ρ is a collider section of ρ if $V_1 \rightarrow V_2 - \dots - V_{n-1} \leftarrow V_n$ is a subroute of ρ . A route ρ in a CG is said to be Z -open when

- every collider section of ρ has a node in Z , and
- no non-collider section of ρ has a node in Z .

Let X, Y and Z denote three disjoint subsets of V . When there is no Z -open path/path/route in an AMP/MVR/LWF CG G between a node in X and a node in Y , we say that X is separated from Y given Z in G and denote it as $X \perp_G Y | Z$. The independence model represented by G is the set of separations $X \perp_G Y | Z$. We denote it as $I_{AMP}(G)$, $I_{MVR}(G)$ or $I_{LWF}(G)$. In general, these three independence models are different. However, if G is a directed and acyclic graph (DAG), then they are the same. Given an AMP, MVR or LWF CG G and two disjoint subsets L and S of V , we denote by $[I(G)]_L^S$ the independence model represented by G under marginalization of the nodes in L and conditioning on the nodes in S . Specifically, $X \perp_G Y | Z$ is in $[I(G)]_L^S$ iff $X \perp_G Y | Z \cup S$ is in $I(G)$ and $X, Y, Z \subseteq V \setminus L \setminus S$.

Finally, we denote by $X \perp_p Y | Z$ that X is independent of Y given Z in a probability distribution p . We say that p is Markovian wrt an AMP, MVR or LWF CG G when $X \perp_p Y | Z$ if $X \perp_G Y | Z$ for all X, Y and Z disjoint subsets of V . We say that p is faithful to G when $X \perp_p Y | Z$ iff $X \perp_G Y | Z$ for all X, Y and Z disjoint subsets of V .

3. ERROR AMP CGS

Any regular Gaussian probability distribution that can be represented by an AMP CG can be expressed as a system of linear equations with correlated errors whose structure depends on the CG (Andersson et al., 2001, Section 5). However, the CG represents the errors implicitly, as no nodes in the CG correspond to the errors. We propose in this section to add some deterministic nodes to the CG in order to represent the errors explicitly. We call the result an EAMP CG. We will show that, as desired, every AMP CG is Markov equivalent to its corresponding EAMP CG under marginalization of the error nodes, i.e. the independence model represented by the former coincides with the independence model represented by the latter. We will also show that every EAMP CG under marginalization of the error nodes is Markov equivalent to some LWF CG under marginalization of the error nodes, and that the latter is Markov equivalent to some DAG under marginalization of the error nodes and conditioning on some selection nodes. The relevance of this result can be best explained by extending to AMP CGs what Koster (2002, p. 838) stated for summary graphs and Richardson and Spirtes (2002, p. 981) stated for ancestral graphs: The fact that an AMP CG has a DAG as departure point implies that the independence model associated with the former can be accounted for by some data generating process that is partially observed (corresponding to marginalization) and has selection bias (corresponding to conditioning). We extend this result to MAMP CGs in the next section.

It is worth mentioning that Andersson et al. (2001, Theorem 6) have identified the conditions under which an AMP CG is Markov equivalent to some LWF CG.¹ It is clear from

¹To be exact, Andersson et al. (2001, Theorem 6) have identified the conditions under which all and only the probability distributions that can be represented by an AMP CG can also be represented by some LWF CG. However, for any AMP or LWF CG G , there are Gaussian probability distributions that have all and only the independencies in the independence model represented by G , as shown by Levitz et al. (2001, Theorem

these conditions that there are AMP CGs that are not Markov equivalent to any LWF CG. The results in this section differ from those by Andersson et al. (2001, Theorem 6), because we show that every AMP CG is Markov equivalent to some LWF CG with error nodes under marginalization of the error nodes.

It is also worth mentioning that Richardson and Spirtes (2002, p. 1025) show that there are AMP CGs that are not Markov equivalent to any DAG under marginalization and conditioning. However, the results in this section show that every AMP CG is Markov equivalent to some DAG with error and selection nodes under marginalization of the error nodes and conditioning of the selection nodes. Therefore, the independence model represented by any AMP CG has indeed some DAG as departure point and, thus, it can be accounted for by some data generating process. The results in this section do not contradict those by Richardson and Spirtes (2002, p. 1025), because they did not consider deterministic nodes while we do (recall that the error nodes are deterministic).

Finally, it is also worth mentioning that EAMP CGs are not the first graphical models to have DAGs as departure point. Specifically, summary graphs (Cox and Wermuth, 1996), MC graphs (Koster, 2002), ancestral graphs (Richardson and Spirtes, 2002), and ribbonless graphs (Sadeghi, 2013) predate EAMP CGs and have the mentioned property. However, none of these other classes of graphical models subsumes AMP CGs, i.e. there are independence models that can be represented by an AMP CG but not by any member of the other class (Sadeghi and Lauritzen, 2012, Section 4). Therefore, none of these other classes of graphical models subsumes EAMP CGs under marginalization of the error nodes.

3.1. AMP and LWF CGs with Deterministic Nodes. We say that a node A of an AMP or LWF CG is determined by some $Z \subseteq V$ when $A \in Z$ or A is a function of Z . In that case, we also say that A is a deterministic node. We use $D(Z)$ to denote all the nodes that are determined by Z . From the point of view of the separations in an AMP or LWF CG, that a node is determined by but is not in the conditioning set of a separation has the same effect as if the node were actually in the conditioning set. We extend the definitions of separation for AMP and LWF CGs to the case where deterministic nodes may exist.

Given an AMP CG G , a path ρ in G is said to be Z -open when

- every triplex node in ρ is in $D(Z) \cup \text{san}_G(D(Z))$, and
- no non-triplex node B in ρ is in $D(Z)$, unless $A - B - C$ is a subpath of ρ and $\text{pa}_G(B) \setminus D(Z) \neq \emptyset$.

Given an LWF CG G , a route ρ in G is said to be Z -open when

- every collider section of ρ has a node in $D(Z)$, and
- no non-collider section of ρ has a node in $D(Z)$.

It should be noted that we are not the first to consider models based on graphs with deterministic nodes. For instance, Geiger et al. (1990, Section 4) consider DAGs with deterministic nodes. However, our definition of deterministic node is more general than theirs.

3.2. From AMP CGs to DAGs Via EAMP CGs. Andersson et al. (2001, Section 5) show that any regular Gaussian probability distribution p that is Markovian wrt an AMP CG G can be expressed as a system of linear equations with correlated errors whose structure depends on G . Specifically, assume without loss of generality that p has mean 0. Let K_i denote any connectivity component of G . Let Ω_{K_i, K_i}^i and $\Omega_{K_i, \text{pa}_G(K_i)}^i$ denote submatrices of the precision matrix Ω^i of $p(K_i, \text{pa}_G(K_i))$. Then, as shown by Bishop (2006, Section 2.3.1),

$$K_i | \text{pa}_G(K_i) \sim \mathcal{N}(\beta^i \text{pa}_G(K_i), \Lambda^i)$$

6.1) and Peña (2011, Theorems 1 and 2). Then, our formulation is equivalent to the original formulation of the result by Andersson et al. (2001, Theorem 6).

where

$$\beta^i = -(\Omega_{K_i, K_i}^i)^{-1} \Omega_{K_i, pa_G(K_i)}^i$$

and

$$(\Lambda^i)^{-1} = \Omega_{K_i, K_i}^i.$$

Then, p can be expressed as a system of linear equations with normally distributed errors whose structure depends on G as follows:

$$K_i = \beta^i pa_G(K_i) + \epsilon^i$$

where

$$\epsilon^i \sim \mathcal{N}(0, \Lambda^i).$$

Note that for all $A, B \in K_i$ st $A - B$ is not in G , $A \perp_G B | pa_G(K_i) \cup K_i \setminus A \setminus B$ and thus $(\Lambda^i)_{A, B}^{-1} = 0$ (Lauritzen, 1996, Proposition 5.2). Note also that for all $A \in K_i$ and $B \in pa_G(K_i)$ st $A \leftarrow B$ is not in G , $A \perp_G B | pa_G(A)$ and thus $(\beta^i)_{A, B} = 0$. Let β_A contain the nonzero elements of the vector $(\beta^i)_{A, \bullet}$. Then, p can be expressed as a system of linear equations with correlated errors whose structure depends on G as follows. For any $A \in K_i$,

$$A = \beta_A pa_G(A) + \epsilon^A$$

and for any other $B \in K_i$,

$$\text{covariance}(\epsilon^A, \epsilon^B) = \Lambda_{A, B}^i.$$

It is worth mentioning that the mapping above between probability distributions and systems of linear equations is bijective (Andersson et al., 2001, Section 5). Note that no nodes in G correspond to the errors ϵ^A . Therefore, G represent the errors implicitly. We propose to represent them explicitly. This can easily be done by transforming G into what we call an EAMP CG G' as follows:

- 1 Let $G' = G$
- 2 For each node A in G
- 3 Add the node ϵ^A to G'
- 4 Add the edge $\epsilon^A \rightarrow A$ to G'
- 5 For each edge $A - B$ in G
- 6 Add the edge $\epsilon^A - \epsilon^B$ to G'
- 7 Remove the edge $A - B$ from G'

The transformation above basically consists in adding the error nodes ϵ^A to G and connect them appropriately. Figure 1 shows an example. Note that every node $A \in V$ is determined by $pa_{G'}(A)$ and, what will be more important, that ϵ^A is determined by $pa_{G'}(A) \setminus \epsilon^A \cup A$. Thus, the existence of deterministic nodes imposes independencies which do not correspond to separations in G . Note also that, given $Z \subseteq V$, a node $A \in V$ is determined by Z iff $A \in Z$. The if part is trivial. To see the only if part, note that $\epsilon^A \notin Z$ and thus A cannot be determined by Z unless $A \in Z$. Therefore, a node ϵ^A in G' is determined by Z iff $pa_{G'}(A) \setminus \epsilon^A \cup A \subseteq Z$ because, as shown, there is no other way for Z to determine $pa_{G'}(A) \setminus \epsilon^A \cup A$ which, in turn, determine ϵ^A . Let ϵ denote all the error nodes in G' . Note that we have not yet given a formal definition of EAMP CGs. We define them as all the graphs resulting from applying the pseudocode above to an AMP CG. It is easy to see that every EAMP CG is an AMP CG over $V \cup \epsilon$ and, thus, its semantics are defined. The following theorem confirms that these semantics are as desired. The formal proofs of our results appear in the appendix at the end of the paper.

Theorem 1. $I_{AMP}(G) = [I_{AMP}(G')]_{\epsilon}^{\emptyset}$.

Theorem 2. Assume that G' has the same deterministic relationships no matter whether it is interpreted as an AMP or LWF CG. Then, $I_{AMP}(G') = I_{LWF}(G')$.

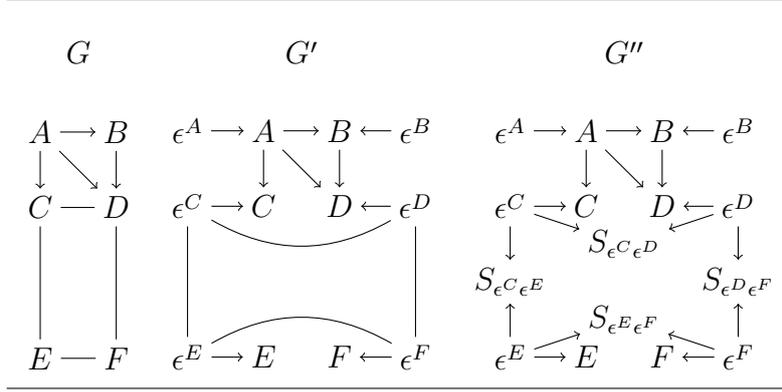


FIGURE 1. Example of the different transformations for AMP CGs.

The following corollary links the two most popular interpretations of CGs. Specifically, it shows that every AMP CG is Markov equivalent to some LWF CG with deterministic nodes under marginalization. The corollary follows from Theorems 1 and 2.

Corollary 1. $I_{AMP}(G) = [I_{LWF}(G')]_{\epsilon}^{\emptyset}$.

Now, let G'' denote the DAG obtained from G' by replacing every edge $\epsilon^A - \epsilon^B$ in G' with $\epsilon^A \rightarrow S_{\epsilon^A \epsilon^B} \leftarrow \epsilon^B$. Figure 1 shows an example. The nodes $S_{\epsilon^A \epsilon^B}$ are called selection nodes. Let S denote all the selection nodes in G'' . The following theorem relates the semantics of G' and G'' .

Theorem 3. *Assume that G' and G'' have the same deterministic relationships. Then, $I_{LWF}(G') = [I(G'')]_{\epsilon}^S$.*

The main result of this section is the following corollary, which shows that every AMP CG is Markov equivalent to some DAG with deterministic nodes under marginalization and conditioning. The corollary follows from Corollary 1 and Theorem 3.

Corollary 2. $I_{AMP}(G) = [I(G'')]_{\epsilon}^S$.

4. MARGINAL AMP CGS

In this section, we present the main contribution of this paper, namely a new family of graphical models that unify and generalize AMP and MVR CGs. Specifically, a graph G containing possibly directed, bidirected and undirected edges is a marginal AMP (MAMP) CG if

- C1.** G has no semidirected cycle,
- C2.** G has no cycle $V_1, \dots, V_n = V_1$ st $V_1 \leftrightarrow V_2$ is in G and $V_i - V_{i+1}$ is in G for all $1 < i < n$, and
- C3.** if $V_1 - V_2 - V_3$ is in G and $sp_G(V_2) \neq \emptyset$, then $V_1 - V_3$ is in G too.

A set of nodes of a MAMP CG G is undirectly connected if there exists a path in G between every pair of nodes in the set st all the edges in the path are undirected. An undirected connectivity component of G is a maximal undirectly connected set. We denote by $uc_G(A)$ the undirected connectivity component a node A of G belongs to.

The semantics of MAMP CGs is as follows. A node B in a path ρ in a MAMP CG G is called a triplex node in ρ if $A \leftrightarrow B \leftrightarrow C$, $A \leftrightarrow B - C$, or $A - B \leftrightarrow C$ is a subpath of ρ . Moreover, ρ is said to be Z -open with $Z \subseteq V$ when

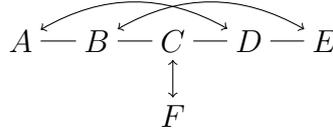
- every triplex node in ρ is in $Z \cup san_G(Z)$, and
- every non-triplex node B in ρ is outside Z , unless $A - B - C$ is a subpath of ρ and $sp_G(B) \neq \emptyset$ or $pa_G(B) \setminus Z \neq \emptyset$.

Let X , Y and Z denote three disjoint subsets of V . When there is no Z -open path in G between a node in X and a node in Y , we say that X is separated from Y given Z in G and denote it as $X \perp_G Y|Z$. We denote by $X \not\perp_G Y|Z$ that $X \perp_G Y|Z$ does not hold. Likewise, we denote by $X \perp_p Y|Z$ (respectively $X \not\perp_p Y|Z$) that X is independent (respectively dependent) of Y given Z in a probability distribution p . The independence model represented by G , denoted as $I(G)$, is the set of separation statements $X \perp_G Y|Z$. We say that p is Markovian wrt G when $X \perp_p Y|Z$ if $X \perp_G Y|Z$ for all X , Y and Z disjoint subsets of V . Moreover, we say that p is faithful to G when $X \perp_p Y|Z$ iff $X \perp_G Y|Z$ for all X , Y and Z disjoint subsets of V .

Note that if a MAMP CG G has a path $V_1 - V_2 - \dots - V_n$ st $sp_G(V_i) \neq \emptyset$ for all $1 < i < n$, then $V_1 - V_n$ must be in G . Therefore, the independence model represented by a MAMP CG is the same whether we use the definition of Z -open path above or the following simpler one. A path ρ in a MAMP CG G is said to be Z -open when

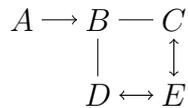
- every triplex node in ρ is in $Z \cup san_G(Z)$, and
- every non-triplex node B in ρ is outside Z , unless $A - B - C$ is a subpath of ρ and $pa_G(B) \setminus Z \neq \emptyset$.

The motivation behind the three constraints in the definition of MAMP CGs is as follows. The constraint C1 follows from the semidirected acyclicity constraint of AMP and MVR CGs. For the constraints C2 and C3, note that typically every missing edge in the graph of a graphical model corresponds to a separation. However, this may not be true for graphs that do not satisfy the constraints C2 and C3. For instance, the graph G below does not contain any edge between B and D but $B \not\perp_G D|Z$ for all $Z \subseteq V \setminus \{B, D\}$. Likewise, G does not contain any edge between A and E but $A \not\perp_G E|Z$ for all $Z \subseteq V \setminus \{A, E\}$.



Since the situation above is counterintuitive, we enforce the constraints C2 and C3. Theorem 5 below shows that every missing edge in a MAMP CG corresponds to a separation.

Note that AMP and MVR CGs are special cases of MAMP CGs. However, MAMP CGs are a proper generalization of AMP and MVR CGs, as there are independence models that can be represented by the former but not by the two latter. An example follows (we postpone the proof that it cannot be represented by any AMP or MVR CG until after Theorem 7).



Given a MAMP CG G , let \widehat{G} denote the AMP CG obtained by replacing every bidirected edge $A \leftrightarrow B$ in G with $A \leftarrow L_{AB} \rightarrow B$. Note that G and \widehat{G} represent the same separations over V . Therefore, every MAMP CG can be seen as the result of marginalizing out some nodes in an AMP CG, hence the name. Furthermore, Corollary 2 shows that every AMP CG can be seen as the result of marginalizing out and conditioning on some nodes in a DAG. Consequently, every MAMP CG can also be seen as the result of marginalizing out and conditioning on some nodes in a DAG. Therefore, the independence model represented by a MAMP CG can be accounted for by some data generating process that is partially observed and has selection bias. This implies that the independence models represented by MAMP CGs are not arbitrary and, thus, MAMP CGs are worth studying. The theorem below provides another way to see that the independence models represented by MAMP CGs

are not arbitrary. Specifically, it shows that each of them coincides with the independence model of some probability distribution.

Theorem 4. *For any MAMP CG G , there exists a regular Gaussian probability distribution p that is faithful to G .*

Corollary 3. *Any independence model represented by a MAMP CG is a compositional graphoid.*

Finally, we show below that the independence model represented by a MAMP CG coincides with certain closure of certain separations. This is interesting because it implies that a few separations and rules to combine them characterize all the separations represented by a MAMP CG. Moreover, it also implies that we have a simple graphical criterion to decide whether a given separation is or is not in the closure without having to find a derivation of it, which is usually a tedious task. Specifically, we define the pairwise separation base of a MAMP CG G as the separations

- $A \perp B | pa_G(A)$ for all non-adjacent nodes A and B of G st $B \notin de_G(A)$, and
- $A \perp B | ne_G(A) \cup pa_G(A \cup ne_G(A))$ for all non-adjacent nodes A and B of G st $A \in de_G(B)$ and $B \in de_G(A)$, i.e. $uc_G(A) = uc_G(B)$.

We define the compositional graphoid closure of the pairwise separation base of G , denoted as $cl(G)$, as the set of separations that are in the base plus those that can be derived from it by applying the compositional graphoid properties. We denote the separations in $cl(G)$ as $X \perp_{cl(G)} Y | Z$.

Theorem 5. *For any MAMP CG G , if $X \perp_{cl(G)} Y | Z$ then $X \perp_G Y | Z$.*

Theorem 6. *For any MAMP CG G , if $X \perp_G Y | Z$ then $X \perp_{cl(G)} Y | Z$.*

4.1. Markov Equivalence. We say that two MAMP CGs are Markov equivalent if they represent the same independence model. In a MAMP CG, a triplex $(\{A, C\}, B)$ is an induced subgraph of the form $A \leftrightarrow B \leftrightarrow C$, $A \leftrightarrow B - C$, or $A - B \leftrightarrow C$. We say that two MAMP CGs are triplex equivalent if they have the same adjacencies and the same triplexes.

Theorem 7. *Two MAMP CGs are Markov equivalent iff they are triplex equivalent.*

We mentioned in the previous section that MAMP CGs are a proper generalization of AMP and MVR CGs, as there are independence models that can be represented by the former but not by the two latter. Moreover, we gave the an example and postponed the proof. With the help of Theorem 7, we can now give the proof.

Example 1. *The independence model represented by the MAMP CG G below cannot be represented by any AMP or MVR CG.*

$$\begin{array}{ccc} A \longrightarrow B & \text{---} & C \\ & \downarrow & \uparrow \\ & D \longleftrightarrow & E \end{array}$$

To see it, assume to the contrary that it can be represented by an AMP CG H . Note that H is a MAMP CG too. Then, G and H must have the same triplexes by Theorem 7. Then, H must have triplexes $(\{A, D\}, B)$ and $(\{A, C\}, B)$ but no triplex $(\{C, D\}, B)$. So, $C - B - D$ must be in H . Moreover, H must have a triplex $(\{B, E\}, C)$. So, $C \leftarrow E$ must be in H . However, this implies that H does not have a triplex $(\{C, D\}, E)$, which is a contradiction because G has such a triplex. To see that no MVR CG can represent the independence model represented by G , simply note that no MVR CG can have triplexes $(\{A, D\}, B)$ and $(\{A, C\}, B)$ but no triplex $(\{C, D\}, B)$.

We end this section with two lemmas that identify some interesting distinguished members of a triplex equivalence class of MAMP CGs. We say that two nodes form a directed node pair if there is a directed edge between them.

Lemma 1. *For every triplex equivalence class of MAMP CGs, there is a unique maximal set of directed node pairs st some CG in the class has exactly those directed node pairs.*

A MAMP CG is a maximally directed CG (MDCG) if it has exactly the maximal set of directed node pairs corresponding to its triplex equivalence class. Note that there may be several MDCGs in the class. For instance, the triplex equivalence class that contains the MAMP CG $A \rightarrow B$ has two MDCGs (i.e. $A \rightarrow B$ and $A \leftarrow B$).

Lemma 2. *For every triplex equivalence class of MDCGs, there is a unique maximal set of bidirected edges st some MDCG in the class has exactly those bidirected edges.*

A MDCG is a maximally bidirected MDCG (MBMDCG) if it has exactly the maximal set of bidirected edges corresponding to its triplex equivalence class. Note that there may be several MBMDCGs in the class. For instance, the triplex equivalence class that contains the MAMP CG $A \rightarrow B$ has two MBMDCGs (i.e. $A \rightarrow B$ and $A \leftarrow B$). Note however that all the MBMDCGs in a triplex equivalence class have the same triplex edges, i.e. the edges in a triplex.

5. ERROR MAMP CGS

Unfortunately, MAMP CGs are not closed under marginalization, meaning that the independence model resulting from marginalizing out some nodes in a MAMP CG may not be representable by any MAMP CG. An example follows.

Example 2. *The independence model resulting from marginalizing out E and I in the MAMP CG G below cannot be represented by any MAMP CG.*

$$\begin{array}{ccccccccc} & & A & & & & B & & & & \\ & & \downarrow & & & & \downarrow & & & & \\ C & - & D & - & E & \rightarrow & F & \leftarrow & I & - & J & - & K \end{array}$$

To see it, assume to the contrary that it can be represented by a MAMP CG H . Note that C and D must be adjacent in H , because $C \perp_G D | Z$ for all $Z \subseteq \{A, B, F, J, K\}$. Similarly, D and F must be adjacent in H . However, H cannot have a triplex $(\{C, F\}, D)$ because $C \perp_G F | A \cup D$. Moreover, $C \leftarrow D$ cannot be in H because $A \perp_G C$, and $D \rightarrow F$ cannot be in H because $A \perp_G F$. Then, $C - D - F$ must be in H . Following an analogous reasoning, we can conclude that $F - J - K$ must be in H . However, this contradicts that $D \perp_G J$.

A solution to the problem above is to represent the marginal model by a MAMP CG with extra edges so as to avoid representing false independencies. This, of course, has two undesirable consequences: Some true independencies may not be represented, and the complexity of the CG increases. See (Richardson and Spirtes, 2002, p. 965) for a discussion on the importance of the class of models considered being closed under marginalization. In this section, we propose an alternative solution to this problem: Much like we did in Section 3 with AMP CGs, we modify MAMP CGs into what we call EMAMP CGs, and show that the latter are closed under marginalization.²

²The reader may think that parts of this section are repetition of Section 3 and, thus, that both sections should be unified. However, we think that this would harm readability.

5.1. MAMP CGs with Deterministic Nodes. We say that a node A of a MAMP CG is determined by some $Z \subseteq V$ when $A \in Z$ or A is a function of Z . In that case, we also say that A is a deterministic node. We use $D(Z)$ to denote all the nodes that are determined by Z . From the point of view of the separations in a MAMP CG, that a node is determined by but is not in the conditioning set of a separation has the same effect as if the node were actually in the conditioning set. We extend the definition of separation for MAMP CGs to the case where deterministic nodes may exist.

Given a MAMP CG G , a path ρ in G is said to be Z -open when

- every triplex node in ρ is in $D(Z) \cup \text{san}_G(D(Z))$, and
- no non-triplex node B in ρ is in $D(Z)$, unless $A - B - C$ is a subpath of ρ and $pa_G(B) \setminus D(Z) \neq \emptyset$.

5.2. From MAMP CGs to EMAMP CGs. Andersson et al. (2001, Section 5) and Kang and Tian (2009, Section 2) show that any regular Gaussian probability distribution that is Markovian wrt an AMP or MVR CG G can be expressed as a system of linear equations with correlated errors whose structure depends on G . As we show below, these two works can easily be combined to obtain a similar result for MAMP CGs.

Let p denote any regular Gaussian distributions that is Markovian wrt a MAMP CG G . Assume without loss of generality that p has mean 0. Let K_i denote any connectivity component of G . Let Ω_{K_i, K_i}^i and $\Omega_{K_i, pa_G(K_i)}^i$ denote submatrices of the precision matrix Ω^i of $p(K_i, pa_G(K_i))$. Then, as shown by Bishop (2006, Section 2.3.1),

$$K_i | pa_G(K_i) \sim \mathcal{N}(\beta^i pa_G(K_i), \Lambda^i)$$

where

$$\beta^i = -(\Omega_{K_i, K_i}^i)^{-1} \Omega_{K_i, pa_G(K_i)}^i$$

and

$$(\Lambda^i)^{-1} = \Omega_{K_i, K_i}^i.$$

Then, p can be expressed as a system of linear equations with normally distributed errors whose structure depends on G as follows:

$$K_i = \beta^i pa_G(K_i) + \epsilon^i$$

where

$$\epsilon^i \sim \mathcal{N}(0, \Lambda^i).$$

Note that for all $A, B \in K_i$ st $uc_G(A) = uc_G(B)$ and $A - B$ is not in G , $A \perp_G B | pa_G(K_i) \cup K_i \setminus A \setminus B$ and thus $(\Lambda_{uc_G(A), uc_G(A)}^i)^{-1}_{A, B} = 0$ (Lauritzen, 1996, Proposition 5.2). Note also that for all $A, B \in K_i$ st $uc_G(A) \neq uc_G(B)$ and $A \leftrightarrow B$ is not in G , $A \perp_G B | pa_G(K_i)$ and thus $\Lambda_{A, B}^i = 0$. Finally, note also that for all $A \in K_i$ and $B \in pa_G(K_i)$ st $A \leftarrow B$ is not in G , $A \perp_G B | pa_G(K_i)$ and thus $(\beta^i)_{A, B} = 0$. Let β_A contain the nonzero elements of the vector $(\beta^i)_{A, \bullet}$. Then, p can be expressed as a system of linear equations with correlated errors whose structure depends on G as follows. For any $A \in K_i$,

$$A = \beta_A pa_G(A) + \epsilon^A$$

and for any other $B \in K_i$,

$$\text{covariance}(\epsilon^A, \epsilon^B) = \Lambda_{A, B}^i.$$

It is worth mentioning that the mapping above between probability distributions and systems of linear equations is bijective. We omit the proof of this fact because it is unimportant in this work, but it can be proven much in the same way as Lemma 1 in Peña (2011). Note that each equation in the system of linear equations above is a univariate recursive regression, i.e. a random variable can be a regressor in an equation only if it has been the regressand in a previous equation. This has two main advantages, as Cox and Wermuth (1993, p. 207)

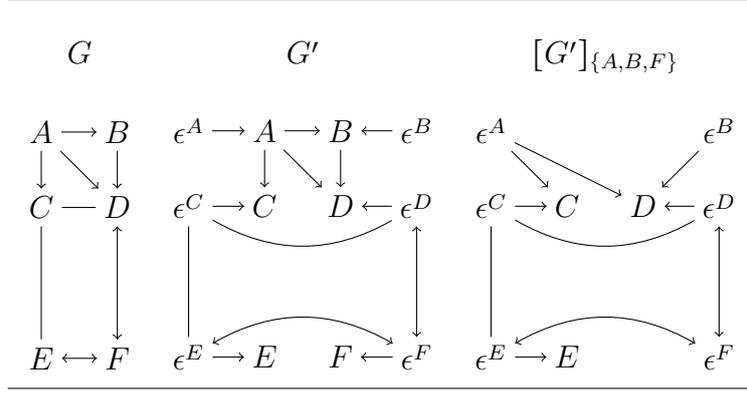


FIGURE 2. Example of the different transformations for MAMP CGs.

explain: "First, and most importantly, it describes a stepwise process by which the observations could have been generated and in this sense may prove the basis for developing potential causal explanations. Second, each parameter in the system [of linear equations] has a well-understood meaning since it is a regression coefficient: That is, it gives for unstandardized variables the amount by which the response is expected to change if the explanatory variable is increased by one unit and all other variables in the equation are kept constant." Therefore, a MAMP CG can be seen as a data generating process and, thus, it gives us insight into the system under study.

Note that no nodes in G correspond to the errors ϵ^A . Therefore, G represent the errors implicitly. We propose to represent them explicitly. This can easily be done by transforming G into what we call an EMAMP CG G' as follows, where $A \mapsto B$ means $A \leftrightarrow B$ or $A - B$:

- 1 Let $G' = G$
- 2 For each node A in G
- 3 Add the node ϵ^A to G'
- 4 Add the edge $\epsilon^A \rightarrow A$ to G'
- 5 For each edge $A \mapsto B$ in G
- 6 Add the edge $\epsilon^A \mapsto \epsilon^B$ to G'
- 7 Remove the edge $A \mapsto B$ from G'

The transformation above basically consists in adding the error nodes ϵ^A to G and connect them appropriately. Figure 2 shows an example. Note that every node $A \in V$ is determined by $pa_{G'}(A)$ and, what will be more important, that ϵ^A is determined by $pa_{G'}(A) \setminus \epsilon^A \cup A$. Thus, the existence of deterministic nodes imposes independencies which do not correspond to separations in G . Note also that, given $Z \subseteq V$, a node $A \in V$ is determined by Z iff $A \in Z$. The if part is trivial. To see the only if part, note that $\epsilon^A \notin Z$ and thus A cannot be determined by Z unless $A \in Z$. Therefore, a node ϵ^A in G' is determined by Z iff $pa_{G'}(A) \setminus \epsilon^A \cup A \subseteq Z$ because, as shown, there is no other way for Z to determine $pa_{G'}(A) \setminus \epsilon^A \cup A$ which, in turn, determine ϵ^A . Let ϵ denote all the error nodes in G' . It is easy to see that G' is a MAMP CG over $V \cup \epsilon$ and, thus, its semantics are defined. The following theorem confirms that these semantics are as desired.

Theorem 8. $I(G) = [I(G')]_{\epsilon}^{\emptyset}$.

5.3. EMAMP CGs Are Closed under Marginalization. Finally, we show that EMAMP CGs are closed under marginalization, meaning that for any EMAMP CG G' and $L \subseteq V$ there is an EMAMP CG $[G']_L$ st $[I(G')]_{L \cup \epsilon} = [I([G']_L)]_{\epsilon}$. We actually show how to transform G' into $[G']_L$. Note that our definition of closed under marginalization is an adaptation of the

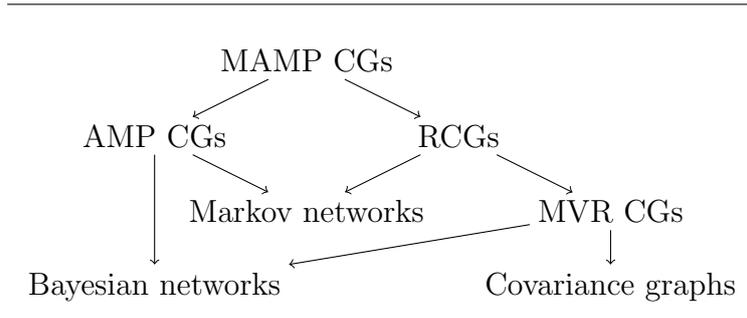


FIGURE 3. Subfamilies of MAMP CGs.

standard one to the fact that we only care about independence models under marginalization of the error nodes.

To gain some intuition into the problem and our solution to it, assume that L contains a single node B . Then, marginalizing out B from the system of linear equations associated with G implies the following: For every C st $B \in pa_G(C)$, modify the equation $C = \beta_C pa_G(C) + \epsilon^C$ by replacing B with the right-hand side of its corresponding equation, i.e. $\beta_B pa_G(B) + \epsilon^B$ and, then, remove the equation $B = \beta_B pa_G(B) + \epsilon^B$ from the system. In graphical terms, this corresponds to C inheriting the parents of B in G' and, then, removing B from G' . The following pseudocode formalizes this idea for any $L \subseteq V$.

- 1 Let $[G']_L = G'$
- 2 Repeat until all the nodes in L have been considered
- 3 Let B denote any node in L that has not been considered before
- 4 For each pair of edges $A \rightarrow B$ and $B \rightarrow C$ in $[G']_L$ with $A, C \in V \cup \epsilon$
- 5 Add the edge $A \rightarrow C$ to $[G']_L$
- 6 Remove B and all the edges it participates in from $[G']_L$

Note that the result of the pseudocode above is the same no matter the ordering in which the nodes in L are selected in line 3. Note also that we have not yet given a formal definition of EMAMP CGs. We define them recursively as all the graphs resulting from applying the first pseudocode in this section to a MAMP CG, plus all the graphs resulting from applying the second pseudocode in this section to an EMAMP CG. It is easy to see that every EMAMP CG is a MAMP CG over $W \cup \epsilon$ with $W \subseteq V$ and, thus, its semantics are defined. Theorem 8 together with the following theorem confirm that these semantics are as desired.

Theorem 9. $[I(G')]_{L \cup \epsilon} = [I([G']_L)]_\epsilon$.

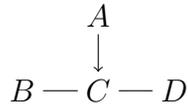
6. DISCUSSION

In this paper we have introduced MAMP CGs, a new family of graphical models that unify and generalize AMP and MVR CGs. We have described global and pairwise Markov properties for them and proved their equivalence for compositional graphoids. We have shown that every MAMP CG is Markov equivalent to some DAG with deterministic nodes under marginalization and conditioning on some of its nodes. Therefore, the independence model represented by a MAMP CG can be accounted for by some data generating process that is partially observed and has selection bias. We have also characterized when two MAMP CGs are Markov equivalent. We conjecture that every Markov equivalence class of MAMP CGs has a distinguished member. We are currently working on this question. It is worth mentioning that such a result has been proven for AMP CGs (Roverato and Studený, 2006). Finally, we have modified MAMP CGs so that they are closed under marginalization. This

is a desirable feature because it guarantees parsimonious models under marginalization. We are currently studying how to modify MAMP CGs so that they are closed under conditioning too. We are also working on a constraint based algorithm for learning a MAMP CG a given probability distribution is faithful to. The idea is to combine the learning algorithms that we have recently proposed for AMP CGs (Peña, 2012) and MVR CGs (Sonntag and Peña, 2012).

We believe that the most natural way to generalize AMP and MVR CGs is by allowing undirected, directed and bidirected edges. However, we are not the first to introduce a family of models that is based on graphs that may contain these three types of edges. In the rest of this section, we review some works that have done it before us, and explain how our work differs from them. Cox and Wermuth (1993, 1996) introduced regression CGs (RCGs) to generalize MVR CGs by allowing them to have also undirected edges. The separation criterion for RCGs is identical to that of MVR CGs. Then, there are independence models that can be represented by MAMP CGs but that cannot be represented by RCGs, because RCGs generalize MVR CGs but not AMP CGs. An example follows.

Example 3. *The independence model represented by the AMP CG G below cannot be represented by any RCG.*

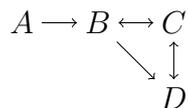


To see it, assume to the contrary that it can be represented by a RCG H . Note that H is a MAMP CG too. Then, G and H must have the same triplexes by Theorem 7. Then, H must have triplexes $(\{A, B\}, C)$ and $(\{A, D\}, C)$ but no triplex $(\{B, D\}, C)$. So, $B \circlearrowright C \rightarrow D$, $B \circlearrowleft C - D$, $B \leftarrow C \circlearrowright D$ or $B - C \circlearrowleft D$ must be in H . However, this implies that H does not have the triplex $(\{A, B\}, C)$ or $(\{A, D\}, C)$, which is a contradiction.

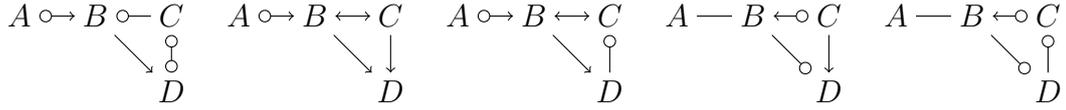
It is worth mentioning that, although RCGs can have undirected edges, they cannot have a subgraph of the form $A \circlearrowright B - C$. Therefore, RCGs are a subfamily of MAMP CGs. Figure 3 depicts this and other subfamilies of MAMP CGs.

Another family of models that is based on graphs that may contain undirected, directed and bidirected edges is maximal ancestral graphs (MAGs) (Richardson and Spirtes, 2002). Although MAGs can have undirected edges, they must comply with certain topological constraints. The separation criterion for MAGs is identical to that of MVR CGs. Therefore, the example above also serves to illustrate that MAGs generalize MVR CGs but not AMP CGs, as MAMP CGs do. See also (Richardson and Spirtes, 2002, p. 1025). Therefore, MAMP CGs are not a subfamily of MAGs. The following example shows that MAGs are not a subfamily of MAMP CGs either.

Example 4. *The independence model represented by the MAG G below cannot be represented by any MAMP CG.*



To see it, assume to the contrary that it can be represented by a MAMP CG H . Obviously, G and H must have the same adjacencies. Then, H must have a triplex $(\{A, C\}, B)$ because $A \perp_G C$, but it cannot have a triplex $(\{A, D\}, B)$ because $A \perp_G D | B$. This is possible only if the edge $A \leftarrow B$ is not in H . Then, H must have one of the following induced subgraphs:



However, the first and second cases are impossible because $A \perp_H D | B \cup C$ whereas $A \not\perp_G D | B \cup C$. The third case is impossible because it does not satisfy the constraint C1. In the fourth case, note that $C \leftrightarrow B - D$ cannot be in H because, otherwise, it does not satisfy the constraint C1. Then, the fourth case is impossible because $A \perp_H D | B \cup C$ whereas $A \not\perp_G D | B \cup C$. Finally, the fifth case is also impossible because it does not satisfy the constraint C1 or C2.

It is worth mentioning that the models represented by AMP and MVR CGs are smooth, i.e. they are curved exponential families, for Gaussian probability distributions. However, only the models represented by MVR CGs are smooth for discrete probability distributions. The models represented by MAGs are smooth in the Gaussian and discrete cases. See Drton (2009) and Evans and Richardson (2013).

Finally, three other families of models that are based on graphs that may contain undirected, directed and bidirected edges are summary graphs after replacing the dashed undirected edges with bidirected edges (Cox and Wermuth, 1996), MC graphs (Koster, 2002), and loopless mixed graphs (Sadeghi and Lauritzen, 2012). As shown in (Sadeghi and Lauritzen, 2012, Sections 4.2 and 4.3), every independence model that can be represented by summary graphs and MC graphs can also be represented by loopless mixed graphs. The separation criterion for loopless mixed graphs is identical to that of MVR CGs. Therefore, the example above also serves to illustrate that loopless mixed graphs generalize MVR CGs but not AMP CGs, as MAMP CGs do. See also (Sadeghi and Lauritzen, 2012, Section 4.1). Moreover, summary graphs and MC graphs have a rather counterintuitive and undesirable feature: Not every missing edge corresponds to a separation (Richardson and Spirtes, 2002, p. 1023). MAMP CGs, on the other hand, do not have this disadvantage (recall Theorem 5).

In summary, MAMP CGs are the only graphical models we are aware of that generalize both AMP and MVR CGs.

ACKNOWLEDGMENTS

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APPENDIX: PROOFS

Proof of Theorem 1. It suffices to show that every Z -open path between α and β in G can be transformed into a Z -open path between α and β in G' and vice versa, with $\alpha, \beta \in V$ and $Z \subseteq V \setminus \alpha \setminus \beta$.

Let ρ denote a Z -open path between α and β in G . We can easily transform ρ into a path ρ' between α and β in G' : Simply, replace every maximal subpath of ρ of the form $V_1 - V_2 - \dots - V_{n-1} - V_n$ ($n \geq 2$) with $V_1 \leftarrow \epsilon^{V_1} - \epsilon^{V_2} - \dots - \epsilon^{V_{n-1}} - \epsilon^{V_n} \rightarrow V_n$. We now show that ρ' is Z -open.

First, if $B \in V$ is a triplex node in ρ' , then ρ' must have one of the following subpaths:

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \leftarrow \epsilon^B - \epsilon^C \quad \epsilon^A - \epsilon^B \rightarrow B \leftarrow C$$

with $A, C \in V$. Therefore, ρ must have one of the following subpaths (specifically, if ρ' has the i -th subpath above, then ρ has the i -th subpath below):

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \text{ --- } C \quad A \text{ --- } B \leftarrow C$$

In either case, B is a triplex node in ρ and, thus, $B \in Z \cup \text{san}_G(Z)$ for ρ to be Z -open. Then, $B \in Z \cup \text{san}_{G'}(Z)$ by construction of G' and, thus, $B \in D(Z) \cup \text{san}_{G'}(D(Z))$.

Second, if $B \in V$ is a non-triplex node in ρ' , then ρ' must have one of the following subpaths:

$$A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B \leftarrow \epsilon^B \text{ --- } \epsilon^C \quad \epsilon^A \text{ --- } \epsilon^B \rightarrow B \rightarrow C$$

with $A, C \in V$. Therefore, ρ must have one of the following subpaths (specifically, if ρ' has the i -th subpath above, then ρ has the i -th subpath below):

$$A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B \text{ --- } C \quad A \text{ --- } B \rightarrow C$$

In either case, B is a non-triplex node in ρ and, thus, $B \notin Z$ for ρ to be Z -open. Since Z contains no error node, Z cannot determine any node in V that is not already in Z . Then, $B \notin D(Z)$.

Third, if ϵ^B is a non-triplex node in ρ' (note that ϵ^B cannot be a triplex node in ρ'), then ρ' must have one of the following subpaths:

$$\begin{aligned} A \rightarrow B \leftarrow \epsilon^B \text{ --- } \epsilon^C \quad \epsilon^A \text{ --- } \epsilon^B \rightarrow B \leftarrow C \quad \alpha = B \leftarrow \epsilon^B \text{ --- } \epsilon^C \quad \epsilon^A \text{ --- } \epsilon^B \rightarrow B = \beta \\ A \leftarrow B \leftarrow \epsilon^B \text{ --- } \epsilon^C \quad \epsilon^A \text{ --- } \epsilon^B \rightarrow B \rightarrow C \quad \epsilon^A \text{ --- } \epsilon^B \text{ --- } \epsilon^C \end{aligned}$$

with $A, C \in V$. Recall that $\epsilon^B \notin Z$ because $Z \subseteq V \setminus \alpha \setminus \beta$. In the first case, if $\alpha = A$ then $A \notin Z$, else $A \notin Z$ for ρ to be Z -open. Then, $\epsilon^B \notin D(Z)$. In the second case, if $\beta = C$ then $C \notin Z$, else $C \notin Z$ for ρ to be Z -open. Then, $\epsilon^B \notin D(Z)$. In the third and fourth cases, $B \notin Z$ because $\alpha = B$ or $\beta = B$. Then, $\epsilon^B \notin D(Z)$. In the fifth and sixth cases, $B \notin Z$ for ρ to be Z -open. Then, $\epsilon^B \notin D(Z)$. The last case implies that ρ has the following subpath:

$$A \text{ --- } B \text{ --- } C$$

Thus, B is a non-triplex node in ρ , which implies that $B \notin Z$ or $pa_G(B) \setminus Z \neq \emptyset$ for ρ to be Z -open. In either case, $\epsilon^B \notin D(Z)$ (recall that $pa_{G'}(B) = pa_G(B) \cup \epsilon^B$ by construction of G').

Finally, let ρ' denote a Z -open path between α and β in G' . We can easily transform ρ' into a path ρ between α and β in G : Simply, replace every maximal subpath of ρ' of the form $V_1 \leftarrow \epsilon^{V_1} \text{ --- } \epsilon^{V_2} \text{ --- } \dots \text{ --- } \epsilon^{V_{n-1}} \text{ --- } \epsilon^{V_n} \rightarrow V_n$ ($n \geq 2$) with $V_1 \text{ --- } V_2 \text{ --- } \dots \text{ --- } V_{n-1} \text{ --- } V_n$. We now show that ρ is Z -open.

First, note that all the nodes in ρ are in V . Moreover, if B is a triplex node in ρ , then ρ must have one of the following subpaths:

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \text{ --- } C \quad A \text{ --- } B \leftarrow C$$

with $A, C \in V$. Therefore, ρ' must have one of the following subpaths (specifically, if ρ has the i -th subpath above, then ρ' has the i -th subpath below):

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \leftarrow \epsilon^B \text{ --- } \epsilon^C \quad \epsilon^A \text{ --- } \epsilon^B \rightarrow B \leftarrow C$$

In either case, B is a triplex node in ρ' and, thus, $B \in D(Z) \cup \text{san}_{G'}(D(Z))$ for ρ' to be Z -open. Since Z contains no error node, Z cannot determine any node in V that is not already in Z . Then, $B \in D(Z)$ iff $B \in Z$. Since there is no strictly descending route from B

to any error node, then any strictly descending route from B to a node $D \in D(Z)$ implies that $D \in V$ which, as seen, implies that $D \in Z$. Then, $B \in \text{san}_{G'}(D(Z))$ iff $B \in \text{san}_{G'}(Z)$. Moreover, $B \in \text{san}_{G'}(Z)$ iff $B \in \text{san}_G(Z)$ by construction of G' . These results together imply that $B \in Z \cup \text{san}_G(Z)$.

Second, if B is a non-triplex node in ρ , then ρ must have one of the following subpaths:

$$A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B - C \quad A - B \rightarrow C \quad A - B - C$$

with $A, C \in V$. Therefore, ρ' must have one of the following subpaths (specifically, if ρ has the i -th subpath above, then ρ' has the i -th subpath below):

$$A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B \leftarrow \epsilon^B - \epsilon^C \quad \epsilon^A - \epsilon^B \rightarrow B \rightarrow C \\ \epsilon^A - \epsilon^B - \epsilon^C$$

In the first five cases, B is a non-triplex node in ρ' and, thus, $B \notin D(Z)$ for ρ' to be Z -open. Since Z contains no error node, Z cannot determine any node in V that is not already in Z . Then, $B \notin Z$. In the last case, ϵ^B is a non-triplex node in ρ' and, thus, $\epsilon^B \notin D(Z)$ for ρ' to be Z -open. Then, $B \notin Z$ or $pa_{G'}(B) \setminus \epsilon^B \setminus Z \neq \emptyset$. Then, $B \notin Z$ or $pa_G(B) \setminus Z \neq \emptyset$ (recall that $pa_{G'}(B) = pa_G(B) \cup \epsilon^B$ by construction of G'). \square

Proof of Theorem 2. Assume for a moment that G' has no deterministic node. Note that G' has no induced subgraph of the form $A \rightarrow B - C$ with $A, B, C \in V \cup \epsilon$. Such an induced subgraph is called a flag by Andersson et al. (2001, pp. 40-41). They also introduce the term biflag, whose definition is irrelevant here. What is relevant here is the observation that a CG cannot have a biflag unless it has some flag. Therefore, G' has no biflags. Consequently, every probability distribution that is Markovian wrt G' when interpreted as an AMP CG is also Markovian wrt G' when interpreted as a LWF CG and vice versa (Andersson et al., 2001, Corollary 1). Now, note that there are Gaussian probability distributions that are faithful to G' when interpreted as an AMP CG (Levitz et al., 2001, Theorem 6.1) as well as when interpreted as a LWF CG (Peña, 2011, Theorems 1 and 2). Therefore, $I_{AMP}(G') = I_{LWF}(G')$. We denote this independence model by $I_{NDN}(G')$.

Now, forget the momentary assumption made above that G' has no deterministic node. Recall that we assumed that $D(Z)$ is the same under the AMP and the LWF interpretations of G' for all $Z \subseteq V \cup \epsilon$. Recall also that, from the point of view of the separations in an AMP or LWF CG, that a node is determined by the conditioning set has the same effect as if the node were in the conditioning set. Then, $X \perp_{G'} Y | Z$ is in $I_{AMP}(G')$ iff $X \perp_{G'} Y | D(Z)$ is in $I_{NDN}(G')$ iff $X \perp_{G'} Y | Z$ is in $I_{LWF}(G')$. Then, $I_{AMP}(G') = I_{LWF}(G')$. \square

Proof of Theorem 3. Assume for a moment that G' has no deterministic node. Then, G'' has no deterministic node either. We show below that every Z -open route between α and β in G' can be transformed into a $(Z \cup S)$ -open route between α and β in G'' and vice versa, with $\alpha, \beta \in V \cup \epsilon$. This implies that $I_{LWF}(G') = [I(G'')]_{\emptyset}^S$. We denote this independence model by $I_{NDN}(G')$.

First, let ρ' denote a Z -open route between α and β in G' . Then, we can easily transform ρ' into a $(Z \cup S)$ -open route ρ'' between α and β in G'' : Simply, replace every edge $\epsilon^A - \epsilon^B$ in ρ' with $\epsilon^A \rightarrow S_{\epsilon^A \epsilon^B} \leftarrow \epsilon^B$. To see that ρ'' is actually $(Z \cup S)$ -open, note that every collider section in ρ' is due to a subroute of the form $A \rightarrow B \leftarrow C$ with $A, B \in V$ and $C \in V \cup \epsilon$. Then, any node that is in a collider (respectively non-collider) section of ρ' is also in a collider (respectively non-collider) section of ρ'' .

Second, let ρ'' denote a $(Z \cup S)$ -open route between α and β in G'' . Then, we can easily transform ρ'' into a Z -open route ρ' between α and β in G' : First, replace every subroute $\epsilon^A \rightarrow S_{\epsilon^A \epsilon^B} \leftarrow \epsilon^B$ of ρ'' with ϵ^A and, then, replace every subroute $\epsilon^A \rightarrow S_{\epsilon^A \epsilon^B} \leftarrow \epsilon^B$ of ρ'' with

$\epsilon^A - \epsilon^B$. To see that ρ' is actually Z -open, note that every undirected edge in ρ' is between two noise nodes and recall that no noise node has incoming directed edges in G' . Then, again every collider section in ρ' is due to a subroute of the form $A \rightarrow B \leftarrow C$ with $A, B \in V$ and $C \in V \cup \epsilon$. Then, again any node that is in a collider (respectively non-collider) section of ρ' is also in a collider (respectively non-collider) section of ρ'' .

Now, forget the momentary assumption made above that G' has no deterministic node. Recall that we assumed that $D(Z)$ is the same no matter whether we are considering G' or G'' for all $Z \subseteq V \cup \epsilon$. Recall also that, from the point of view of the separations in a LWF CG, that a node is determined by the conditioning set has the same effect as if the node were in the conditioning set. Then, $X \perp_{G''} Y | Z$ is in $[I(G'')]_{\emptyset}^S$ iff $X \perp_{G'} Y | D(Z)$ is in $I_{NDN}(G')$ iff $X \perp_{G'} Y | Z$ is in $I_{LWF}(G')$. Then, $I_{LWF}(G') = [I(G'')]_{\emptyset}^S$. \square

Proof of Theorem 4. It suffices to replace every bidirected edge $A \leftrightarrow B$ in G with $A \leftarrow L_{AB} \rightarrow B$ to create an AMP CG \widehat{G} , apply Theorem 6.1 by Levitz et al. (2001) to conclude that there exists a regular Gaussian probability distribution q that is faithful to \widehat{G} , and then let p be the marginal probability distribution of q over V . \square

Proof of Corollary 3. It follows from Theorem 4 by just noting that the set of independencies in any regular Gaussian probability distribution satisfies the compositional graphoid properties (Studený, 2005, Sections 2.2.2, 2.3.5 and 2.3.6). \square

Proof of Theorem 5. Since the independence model represented by G is a compositional graphoid by Corollary 3, it suffices to prove that the pairwise separation base of G is a subset of the independence model represented by G . We prove this next. Let A and B be two non-adjacent nodes of G . Consider the following two cases.

Case 1: $B \notin de_G(A)$. Then, every path between A and B in G falls within one of the following cases.

Case 1.1: $A = V_1 \leftarrow V_2 \dots V_n = B$. Then, this path is not $pa_G(A)$ -open.

Case 1.2: $A = V_1 \leftrightarrow V_2 \dots V_n = B$. Note that $V_2 \neq V_n$ because, by assumption, A and B are non-adjacent in G . Note also that $V_2 \notin pa_G(A)$ due to the constraint C1. Then, $V_2 \rightarrow V_3$ must be in G for the path to be $pa_G(A)$ -open. By repeating this reasoning, we can conclude that $A = V_1 \leftrightarrow V_2 \rightarrow V_3 \rightarrow \dots \rightarrow V_n = B$ is in G . However, this contradicts the assumption that $B \notin de_G(A)$.

Case 1.3: $A = V_1 - V_2 - \dots - V_m \leftarrow V_{m+1} \dots V_n = B$. Note that $V_m \notin pa_G(A)$ due to the constraint C1. Then, this path is not $pa_G(A)$ -open.

Case 1.4: $A = V_1 - V_2 - \dots - V_m \rightarrow V_{m+1} \dots V_n = B$. Note that $V_{m+1} \neq V_n$ because, by assumption, $B \notin de_G(A)$. Note also that $V_{m+1} \notin pa_G(A)$ due to the constraint C1. Then, $V_{m+1} \rightarrow V_{m+2}$ must be in G for the path to be $pa_G(A)$ -open. By repeating this reasoning, we can conclude that $A = V_1 - V_2 - \dots - V_m \rightarrow V_{m+1} \rightarrow \dots \rightarrow V_n = B$ is in G . However, this contradicts the assumption that $B \notin de_G(A)$.

Case 1.5: $A = V_1 - V_2 - \dots - V_m \leftrightarrow V_{m+1} \dots V_n = B$. Note that $V_m \notin pa_G(A)$ due to the constraint C1. Then, this path is not $pa_G(A)$ -open.

Case 1.6: $A = V_1 - V_2 - \dots - V_n = B$. This case contradicts the assumption that $B \notin de_G(A)$.

Case 2: $A \in de_G(B)$ and $B \in de_G(A)$, i.e. $uc_G(A) = uc_G(B)$. Then, there is an undirected path ρ between A and B in G . Then, every path between A and B in G falls within one of the following cases.

Case 2.1: $A = V_1 \leftarrow V_2 \dots V_n = B$. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open.

Case 2.2: $A = V_1 \leftrightarrow V_2 \dots V_n = B$. Note that $V_2 \neq V_n$ due to ρ and the constraints C1 and C2. Note also that $V_2 \notin ne_G(A) \cup pa_G(A \cup ne_G(A))$ due to the constraint

- C1. Then, $V_2 \rightarrow V_3$ must be in G for the path to be $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open. By repeating this reasoning, we can conclude that $A = V_1 \leftrightarrow V_2 \rightarrow V_3 \rightarrow \dots \rightarrow V_n = B$ is in G . However, this together with ρ violate the constraint C1.
- Case 2.3:** $A = V_1 - V_2 \leftarrow V_3 \dots V_n = B$. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open.
- Case 2.4:** $A = V_1 - V_2 \leftrightarrow V_3 \dots V_n = B$. Note that $V_3 \neq V_n$ due to ρ and the constraints C1 and C2. Note also that $V_3 \notin ne_G(A) \cup pa_G(A \cup ne_G(A))$ due to the constraints C1 and C2. Then, $V_3 \rightarrow V_4$ must be in G for the path to be $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open. By repeating this reasoning, we can conclude that $A = V_1 - V_2 \leftrightarrow V_3 \rightarrow \dots \rightarrow V_n = B$ is in G . However, this together with ρ violate the constraint C1.
- Case 2.5:** $A = V_1 - V_2 - V_3 \dots V_n = B$ st $sp_G(V_2) = \emptyset$. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open.
- Case 2.6:** $A = V_1 - V_2 - \dots - V_m - V_{m+1} \leftarrow V_{m+2} \dots V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq m$. Note that $V_i \in ne_G(V_1)$ for all $3 \leq i \leq m+1$ by the constraint C3. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open.
- Case 2.7:** $A = V_1 - V_2 - \dots - V_m - V_{m+1} \leftrightarrow V_{m+2} \dots V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq m$. Note that $V_{m+2} \neq V_n$ due to ρ and the constraints C1 and C2. Note also that $V_{m+2} \notin ne_G(A) \cup pa_G(A \cup ne_G(A))$ due to the constraints C1 and C2. Then, $V_{m+2} \rightarrow V_{m+3}$ must be in G for the path to be $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open. By repeating this reasoning, we can conclude that $A = V_1 - V_2 - \dots - V_m - V_{m+1} \leftrightarrow V_{m+2} \rightarrow \dots \rightarrow V_n = B$ is in G . However, this together with ρ violate the constraint C1.
- Case 2.8:** $A = V_1 - V_2 - \dots - V_m - V_{m+1} - V_{m+2} \dots V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq m$ and $sp_G(V_{m+1}) = \emptyset$. Note that $V_i \in ne_G(V_1)$ for all $3 \leq i \leq m+1$ by the constraint C3. Then, this path is not $(ne_G(A) \cup pa_G(A \cup ne_G(A)))$ -open.
- Case 2.9:** $A = V_1 - V_2 - \dots - V_n = B$ st $sp_G(V_i) \neq \emptyset$ for all $2 \leq i \leq n-1$. Note that $V_i \in ne_G(V_1)$ for all $3 \leq i \leq n$ by the constraint C3. However, this contradicts the assumption that A and B are non-adjacent in G . □

Proof of Theorem 6. We start by recalling some definitions from Andersson et al. (2001, Section 2). Let F be an AMP CG and F' the result of removing all the directed edges from F . Given a set $U \subseteq V$, let $W = U \cup san_G(U)$ and $W' = \bigcup_{A \in W} uc_G(A)$. Let $F[W]$ denote the graph whose nodes and edges are the union of the nodes and edges in F_W and F'_W . $F[W]$ is called an extended subgraph of F . An undirected graph is called complete if it has an edge between any pair of nodes. In an AMP CG, a triplex $(\{A, C\}, B)$ is an induced subgraph of the form $A \rightarrow B \leftarrow C$, $A \rightarrow B - C$, or $A - B \leftarrow C$. Augmenting a triplex $(\{A, C\}, B)$ means replacing it with the complete undirected graph over $\{A, B, C\}$. In an AMP CG G , a 2-biflag $(\{A, D\}, \{B, C\})$ is a subgraph of the form $A \rightarrow B - C \leftarrow D$ st A is not adjacent to C in G and B is not adjacent to D in G . Augmenting a 2-biflag $(\{A, D\}, \{B, C\})$ means replacing it with the complete undirected graph over $\{A, B, C, D\}$. Augmenting an AMP CG F , denoted as F^a , means augmenting all its triplexes and 2-biflags and converting the remaining directed edges into undirected edges. Note that $X \perp_{F^a} Y | Z$ iff X is separated from Y given Z in $F[X \cup Y \cup Z]^a$ (Levitz et al., 2001, Theorem 4.1).

Given an undirected graph F and a set $U \subseteq V$, let F^U denote the undirected graph over U resulting from adding an edge $A - B$ to F_U if F has a path between A and B whose only nodes in U are A and B . F^U is sometimes called the marginal graph of F for U .

Now, we start the proof per se. Let \widehat{G} denote the AMP CG obtained by replacing every bidirected edge $A \leftrightarrow B$ in G with $A \leftarrow L_{AB} \rightarrow B$. The node L_{AB} is called latent. Let $\overline{G} = (\widehat{G}[X \cup Y \cup Z]^a)^V$. As mentioned before, G and \widehat{G} represent the same separations over

V . Then, $X \perp_G Y | Z$ iff X is separated from Y given Z in \overline{G} . Note that the separations in \overline{G} coincide with the graphoid closure of the separations $A \perp_{\overline{G}} V(\overline{G}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A)$ for all $A \in V(\overline{G})$, where $V(\overline{G})$ denotes the nodes in \overline{G} (Bouckaert, 1995, Theorem 3.4). Therefore, to prove that $X \perp_{d(G)} Y | Z$, it suffices to prove that $A \perp_{d(G)} V(\overline{G}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A)$ for all $A \in V(\overline{G})$. Let K_1, \dots, K_n denote the connectivity components of $G_{V(\overline{G})}$ st if $A \rightarrow B$ is in $G_{V(\overline{G})}$, then $A \in K_i$ and $B \in K_j$ with $i < j$. Consider the following cases.

Case 1: Assume that $A \in K_n$. Note that if $D \in ad_{\overline{G}}(A)$, then $D \in ne_G(A) \cup pa_G(A \cup ne_G(A)) \cup A' \cup ne_G(A') \cup pa_G(A' \cup ne_G(A'))$ for some $A' \in V$ st G has a path $A \dots \leftrightarrow A'$ whose every node is in K_n . To see it, note that if $D \in ad_{\overline{G}}(A) \setminus ne_G(A) \setminus pa_G(A \cup ne_G(A))$, then $\widehat{G}[X \cup Y \cup Z]^a$ must have a path between A and D whose every node except A and D is latent. Then, $\widehat{G}[X \cup Y \cup Z]$ must have a path of the form $A \dots L \rightarrow A' = D$, $A \dots L \rightarrow A' - D$, $A \dots L \rightarrow A' \leftarrow D$ or $A \dots L \rightarrow A' - A'' \leftarrow D$, where L is a latent node and every non-latent node between A and A' is in K_n . Note also that $A' \in de_G(A)$ iff $A \in de_G(A')$, because $A, A' \in K_n$. Let \mathcal{B} contain all those A' st $A' \in de_G(A)$, i.e. $uc_G(A) = uc_G(A')$. Let \mathcal{C} contain all those A' st $A' \notin de_G(A)$.

Consider any $A' \in ne_G(A)$. Note that $de_G(A) = de_G(A')$. Then, $pa_G(A \cup ne_G(A)) \subseteq nde_G(A) = nde_G(A')$. Therefore,

- (1) $A \perp_{d(G)} nde_G(A) \setminus pa_G(A) | pa_G(A)$ and
- (2) $A' \perp_{d(G)} nde_G(A') \setminus pa_G(A') | pa_G(A')$ follow from the pairwise separation base of G by repeated composition, which imply
- (3) $A \perp_{d(G)} nde_G(A) \setminus pa_G(A \cup ne_G(A)) | pa_G(A \cup ne_G(A))$ and
- (4) $A' \perp_{d(G)} nde_G(A') \setminus pa_G(A \cup ne_G(A)) | pa_G(A \cup ne_G(A))$ by weak union. Therefore,
- (5) $A \cup ne_G(A) \perp_{d(G)} nde_G(A) \setminus pa_G(A \cup ne_G(A)) | pa_G(A \cup ne_G(A))$ by repeated symmetry and composition, which implies
- (6) $A \perp_{d(G)} nde_G(A) \setminus pa_G(A \cup ne_G(A)) | ne_G(A) \cup pa_G(A \cup ne_G(A))$ by symmetry and weak union. Note that
- (7) $A \perp_{d(G)} uc_G(A) \setminus A \setminus ne_G(A) | ne_G(A) \cup pa_G(A \cup ne_G(A))$ follows from the pairwise separation base of G by repeated composition, which implies
- (8) $A \perp_{d(G)} [nde_G(A) \setminus pa_G(A \cup ne_G(A))] \cup [uc_G(A) \setminus A \setminus ne_G(A)] | ne_G(A) \cup pa_G(A \cup ne_G(A))$ by symmetry and composition on (6) and (7).

Consider any $B \in \mathcal{B} \setminus ne_G(A)$. By repeating the reasoning above, we can conclude

- (9) $B \perp_{d(G)} [nde_G(B) \setminus pa_G(B \cup ne_G(B))] \cup [uc_G(B) \setminus B \setminus ne_G(B)] | ne_G(B) \cup pa_G(B \cup ne_G(B))$.

Recall that $de_G(A) = de_G(B)$ and $uc_G(A) = uc_G(B)$. Then, $pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B})) \subseteq nde_G(A) = nde_G(B)$, and $ne_G(A \cup \mathcal{B}) \subseteq uc_G(A) = uc_G(B)$. Therefore,

- (10) $A \perp_{d(G)} [nde_G(A) \setminus pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))] \cup [uc_G(A) \setminus A \setminus \mathcal{B} \setminus ne_G(A \cup \mathcal{B})] | ne_G(A \cup \mathcal{B}) \cup pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))$ and
- (11) $B \perp_{d(G)} [nde_G(B) \setminus pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))] \cup [uc_G(B) \setminus A \setminus \mathcal{B} \setminus ne_G(A \cup \mathcal{B})] | ne_G(A \cup \mathcal{B}) \cup pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))$ by weak union and decomposition on (8) and (9). Therefore,
- (12) $A \cup \mathcal{B} \perp_{d(G)} [nde_G(A) \setminus pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))] \cup [uc_G(A) \setminus A \setminus \mathcal{B} \setminus ne_G(A \cup \mathcal{B})] | ne_G(A \cup \mathcal{B}) \cup pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))$ by repeated symmetry and composition, which implies
- (13) $A \perp_{d(G)} [nde_G(A) \setminus pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))] \cup [uc_G(A) \setminus A \setminus \mathcal{B} \setminus ne_G(A \cup \mathcal{B})] | \mathcal{B} \cup ne_G(A \cup \mathcal{B}) \cup pa_G(A \cup \mathcal{B} \cup ne_G(A \cup \mathcal{B}))$ by symmetry and weak union.

Finally, note that $\mathcal{C} \cup ne_G(\mathcal{C}) \cup pa_G(\mathcal{C} \cup ne_G(\mathcal{C})) \subseteq nde_G(A)$. Therefore,

- (14) $A \perp_{d(G)} [nde_G(A) \setminus \mathcal{C} \setminus ne_G(\mathcal{C}) \setminus pa_G(A \cup \mathcal{B} \cup \mathcal{C} \cup ne_G(A \cup \mathcal{B} \cup \mathcal{C}))] \cup [uc_G(A) \setminus A \setminus \mathcal{B} \setminus ne_G(A \cup \mathcal{B})] | \mathcal{B} \cup \mathcal{C} \cup ne_G(A \cup \mathcal{B} \cup \mathcal{C}) \cup pa_G(A \cup \mathcal{B} \cup \mathcal{C} \cup ne_G(A \cup \mathcal{B} \cup \mathcal{C}))$ by weak union on (13), which implies
- (15) $A \perp_{d(G)} V(\overline{G}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A)$ by decomposition.

Case 2: Assume that $A \in K_{n-1}$. Let $\overline{H} = (\widehat{G}[X \cup Y \cup Z \setminus K_n]^a)^V$. Note that \overline{H} is a subgraph of \overline{G} and, thus, $ad_{\overline{H}}(A) \subseteq ad_{\overline{G}}(A)$. Let $\mathcal{B} = ad_{\overline{G}}(A) \cap K_n$ and $\mathcal{C} = K_n \setminus \mathcal{B}$. Note that \mathcal{B} or \mathcal{C} may be empty. Note also that $pa_G(\mathcal{B}) \subseteq ad_{\overline{G}}(A) \cup A$. To see it, note that this is evident for any $D \in ch_G(A) \cup ne_G(ch_G(A))$. On the other hand, if $D \in \mathcal{B} \setminus ch_G(A) \setminus ne_G(ch_G(A))$ then $\widehat{G}[X \cup Y \cup Z]^a$ must have a path between A and D whose every node except A and D is latent. Then, $\widehat{G}[X \cup Y \cup Z]$ must have a path of the form $A \dots L \rightarrow D$ or $A \dots L \rightarrow A' - D$, where L is a latent node and A' is a non-latent node. Then, clearly $pa_G(D) \subseteq ad_{\overline{G}}(A) \cup A$.

Consider any $B \in \mathcal{B}$. Then,

- (1) $B \perp_{cl(G)} V(\overline{H}) \setminus pa_G(B) | pa_G(B)$ follows from the pairwise separation base of G by repeated composition, which implies
- (2) $B \perp_{cl(G)} V(\overline{H}) \setminus pa_G(\mathcal{B}) | pa_G(\mathcal{B})$ by weak union, which implies
- (3) $\mathcal{B} \perp_{cl(G)} V(\overline{H}) \setminus pa_G(\mathcal{B}) | pa_G(\mathcal{B})$ by repeated symmetry and composition, which implies
- (4) $\mathcal{B} \perp_{cl(G)} V(\overline{H}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A) \setminus \mathcal{B} \cup A$ by weak union and the fact, shown above, that $pa_G(\mathcal{B}) \subseteq ad_{\overline{G}}(A) \cup A$.

Note that \overline{H} is a proper marginal augmented extended subgraph. Then,

- (5) $A \perp_{cl(G)} V(\overline{H}) \setminus A \setminus ad_{\overline{H}}(A) | ad_{\overline{H}}(A)$ follows from repeating Case 1 for \overline{H} , which implies
- (6) $A \perp_{cl(G)} V(\overline{H}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A) \setminus \mathcal{B}$ by weak union, which implies
- (7) $A \cup \mathcal{B} \perp_{cl(G)} V(\overline{H}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A) \setminus \mathcal{B}$ by symmetry and contraction on (4) and (6), which implies
- (8) $A \perp_{cl(G)} V(\overline{H}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A)$ by symmetry and weak union.

Finally, consider any $C \in \mathcal{C}$. Then,

- (9) $C \perp_{cl(G)} V(\overline{G}) \setminus C \setminus ad_{\overline{G}}(C) | ad_{\overline{G}}(C)$ by Case 1, which implies
- (10) $C \perp_{cl(G)} A | V(\overline{G}) \setminus A \setminus C$ by weak union, which implies
- (11) $\mathcal{C} \perp_{cl(G)} A | V(\overline{G}) \setminus A \setminus \mathcal{C}$ by repeated symmetry and intersection, which implies
- (12) $\mathcal{C} \perp_{cl(G)} A | V(\overline{H}) \setminus A \cup \mathcal{B}$, which implies
- (13) $\mathcal{C} \cup [V(\overline{H}) \setminus A \setminus ad_{\overline{G}}(A)] \perp_{cl(G)} A | ad_{\overline{G}}(A)$ by symmetry and contraction on (8) and (12), which implies
- (14) $A \perp_{cl(G)} V(\overline{G}) \setminus A \setminus ad_{\overline{G}}(A) | ad_{\overline{G}}(A)$ by symmetry.

Case 3: Assume that $A \in K_i$ with $1 \leq i \leq n-2$. Then, repeat Case 2 replacing K_{n-1} with K_i , and letting $\mathcal{B} = ad_{\overline{G}}(A) \cap K_{i+1}$ and $\mathcal{C} = K_{i+1} \cup \dots \cup K_n \setminus \mathcal{B}$.

□

Proof of Theorem 7. We first prove the “only if” part. Let G_1 and G_2 be two Markov equivalent MAMP CGs. First, assume that two nodes A and C are adjacent in G_2 but not in G_1 . If A and C are in the same undirected connectivity component of G_1 , then $A \perp C | ne_{G_1}(A) \cup pa_{G_1}(A \cup ne_{G_1}(A))$ holds for G_1 by Theorem 5 but it does not hold for G_2 , which is a contradiction. On the other hand, if A and C are in different undirected connectivity components of G_1 , then $A \notin de_{G_1}(C)$ or $C \notin de_{G_1}(A)$. Assume without loss of generality that $A \notin de_{G_1}(C)$. Then, $A \perp C | pa_{G_1}(C)$ holds for G_1 by Theorem 5 but it does not hold for G_2 , which is a contradiction. Consequently, G_1 and G_2 must have the same adjacencies.

Finally, assume that G_1 and G_2 have the same adjacencies but G_1 has a triplex $(\{A, C\}, B)$ that G_2 does not have. If A and C are in the same undirected connectivity component of G_1 , then $A \perp C | ne_{G_1}(A) \cup pa_{G_1}(A \cup ne_{G_1}(A))$ holds for G_1 by Theorem 5. Note also that $B \notin ne_{G_1}(A) \cup pa_{G_1}(A \cup ne_{G_1}(A))$ because, otherwise, G_1 would not satisfy the constraint C1 or C2. Then, $A \perp C | ne_{G_1}(A) \cup pa_{G_1}(A \cup ne_{G_1}(A))$ does not hold for G_2 , which is a contradiction. On the other hand, if A and C are in different undirected connectivity components of G_1 ,

then $A \notin de_{G_1}(C)$ or $C \notin de_{G_1}(A)$. Assume without loss of generality that $A \notin de_{G_1}(C)$. Then, $A \perp C | pa_{G_1}(C)$ holds for G_1 by Theorem 5. Note also that $B \notin pa_{G_1}(C)$ because, otherwise, G_1 would not have the triplex $(\{A, C\}, B)$. Then, $A \perp C | pa_{G_1}(C)$ does not hold for G_2 , which is a contradiction. Consequently, G_1 and G_2 must be triplex equivalent.

We now prove the ‘‘if’’ part. Let G_1 and G_2 be two triplex equivalent MAMP CGs. We just prove that all the non-separations in G_1 are also in G_2 . The opposite result can be proven in the same manner by just exchanging the roles of G_1 and G_2 in the proof. Specifically, assume that $\alpha \perp \beta | Z$ does not hold for G_1 . We prove that $\alpha \perp \beta | Z$ does not hold for G_2 either. We divide the proof in three parts.

Part 1

We say that a path has a triplex $(\{A, C\}, B)$ if it has a subpath of the form $A \leftrightarrow B \leftrightarrow C$, $A \leftrightarrow B - C$, or $A - B \leftrightarrow C$. Let ρ_1 be any path between α and β in G_1 that is Z -open st (i) no subpath of ρ_1 between α and β in G_1 is Z -open, (ii) every triplex node in ρ_1 is in Z , and (iii) ρ_1 has no non-triplex node in Z . Let ρ_2 be the path in G_2 that consists of the same nodes as ρ_1 . Then, ρ_2 is Z -open. To see it, assume the contrary. Then, one of the following cases must occur.

Case 1: ρ_2 does not have a triplex $(\{A, C\}, B)$ and $B \in Z$. Then, ρ_1 must have a triplex $(\{A, C\}, B)$ because it is Z -open. Then, A and C must be adjacent in G_1 and G_2 because these are triplex equivalent. Let ϱ_1 be the path obtained from ρ_1 by replacing the triplex $(\{A, C\}, B)$ with the edge between A and C in G_1 . Note that ϱ_1 cannot be Z -open because, otherwise, it would contradict the condition (i). Then, ϱ_1 is not Z -open because A or C do not meet the requirements. Assume without loss of generality that C does not meet the requirements. Then, one of the following cases must occur.

Case 1.1: ϱ_1 does not have a triplex $(\{A, D\}, C)$ and $C \in Z$. Then, one of the following subgraphs must occur in G_1 .³

$$A \overset{\circ}{\circ} B \overset{\circ}{\circ} C \rightarrow D \quad A \overset{\circ}{\circ} B \leftarrow C \overset{\circ}{\circ} D \quad A \rightarrow B \leftarrow C - D \quad A \leftrightarrow B \leftrightarrow C - D$$

However, the first three subgraphs imply that ρ_1 is not Z -open, which is a contradiction. The fourth subgraph implies that ϱ_1 is Z -open, which is a contradiction.

Case 1.2: ϱ_1 has a triplex $(\{A, D\}, C)$ and $C \notin Z \cup san_{G_1}(Z)$. Note that C cannot be a triplex node in ρ_1 because, otherwise, ρ_1 would not be Z -open. Then, one of the following subgraphs must occur in G_1 .

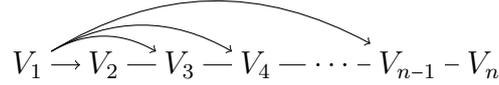
$$A \overset{\circ}{\circ} B \leftarrow C \leftarrow D \quad A \overset{\circ}{\circ} B \leftarrow C - D \quad A \leftrightarrow B \rightarrow C - D \quad A \rightarrow B \rightarrow C - D$$

However, the first and second subgraphs imply that $C \in Z \cup san_{G_1}(Z)$ because $B \in Z$, which is a contradiction. The third subgraph implies that $B - D$ is in G_1 by the constraint C3 and, thus, that the path obtained from ρ_1 by replacing $B - C - D$ with $B - D$ is Z -open, which contradicts the condition (i). For the fourth subgraph, assume that A and D are adjacent in G_1 . Then, one of the following subgraphs must occur in G_1 .

$$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \quad A \rightarrow B \rightarrow C \rightarrow D \leftarrow E \quad A \rightarrow B \rightarrow C \rightarrow D - E$$

³If ϱ_1 does not have a triplex $(\{A, D\}, C)$, then $A \leftarrow C$, $C \rightarrow D$ or $A - C - D$ must be in G_1 . Moreover, recall that B is a triplex node in ρ_1 . Then, $A \rightarrow B \leftarrow C$, $A \rightarrow B \leftrightarrow C$, $A \rightarrow B - C$, $A \leftrightarrow B \leftarrow C$, $A \leftrightarrow B \leftrightarrow C$, $A \leftrightarrow B - C$, $A - B \leftarrow C$ or $A - B \leftrightarrow C$ must be in G_1 . However, if $A \leftarrow C$ is in G_1 then the only legal options are those that contain the edge $B \leftarrow C$. On the other hand, if $A - C - D$ is in G_1 then the only legal options are $A \rightarrow B \leftarrow C$ and $A \leftrightarrow B \leftrightarrow C$.

However, the first subgraph implies that the path obtained from ρ_1 by replacing $A \rightarrow B - C - D$ with $A \rightarrow D$ is Z -open, because $D \notin Z$ since ρ_1 is Z -open. This contradicts the condition (i). The second subgraph implies that the path obtained from ρ_1 by replacing $A \rightarrow B - C - D$ with $A \rightarrow D$ is Z -open, because $D \in Z \cup \text{san}_{G_1}(Z)$ since ρ_1 is Z -open. This contradicts the condition (i). Therefore, only the third subgraph is possible. Thus, by repeatedly applying the previous reasoning, we can conclude without loss of generality that the following subgraph must occur in G_1 , with $n \geq 4$, $V_1 = A$, $V_2 = B$, $V_3 = C$, $V_4 = D$ and where V_1 and V_n are not adjacent in G_1 . Note that the subgraph below covers the case where A and D are not adjacent in the original subgraph by simply taking $n = 4$.



Since V_1 and V_n are not adjacent in G_1 , G_1 has a triplex $(\{V_1, V_n\}, V_{n-1})$ and, thus, so does G_2 because G_1 and G_2 are triplex equivalent. Then, one of the following subgraphs must occur in G_2 .



Note that V_1, \dots, V_n must be a path in G_2 , because G_1 and G_2 are triplex equivalent. Note also that this path cannot have any triplex in G_2 . To see it, recall that we assumed that ρ_2 does not have a triplex $(\{A, C\}, B)$. Recall that $V_1 = A$, $V_2 = B$, $V_3 = C$. Moreover, if the path V_1, \dots, V_n has a triplex $(\{V_i, V_{i+2}\}, V_{i+1})$ in G_2 with $2 \leq i \leq n-2$, then V_i and V_{i+2} must be adjacent in G_1 and G_2 , because such a triplex does not exist in G_1 , which is triplex equivalent to G_2 . Specifically, $V_i - V_{i+2}$ must be in G_1 because, as seen above, $V_i - V_{i+1} - V_{i+2}$ is in G_1 . Then, the path obtained from ρ_1 by replacing $V_i - V_{i+1} - V_{i+2}$ with $V_i - V_{i+2}$ is Z -open, which contradicts the condition (i). However, if the path V_1, \dots, V_n has no triplex in G_2 , then every edge in the path must be directed as \leftarrow in the case of the first and second subgraphs above, whereas every edge in the path must be undirected or directed as \leftarrow in the third subgraph above. Either case contradicts the constraint C1 or C2.

Case 2: Case 1 does not apply. Then, ρ_2 has a triplex $(\{A, C\}, B)$ and $B \notin Z \cup \text{san}_{G_2}(Z)$. Then, ρ_1 cannot have a triplex $(\{A, C\}, B)$. Then, A and C must be adjacent in G_1 and G_2 because these are triplex equivalent. Let ϱ_1 be the path obtained from ρ_1 by replacing the triplex $(\{A, C\}, B)$ with the edge between A and C in G_1 . Note that ϱ_1 cannot be Z -open because, otherwise, it would contradict the condition (i). Then, ϱ_1 is not Z -open because A or C do not meet the requirements. Assume without loss of generality that C does not meet the requirements. Then, one of the following cases must occur.

Case 2.1: ϱ_1 has a triplex $(\{A, D\}, C)$ and $C \notin Z \cup \text{san}_{G_1}(Z)$. Then, one of the following subgraphs must occur in G_1 .⁴

⁴If ϱ_1 has a triplex $(\{A, D\}, C)$, then $A \rightarrow C \leftarrow D$, $A \rightarrow C - D$, $A \leftrightarrow C \leftarrow D$, $A \leftrightarrow C - D$ or $A - C \leftarrow D$ must be in G_1 . Moreover, recall that B is not a triplex node in ρ_1 . Then, $A \leftarrow B \leftarrow C$, $A \leftarrow B \rightarrow C$, $A \leftarrow B \leftrightarrow C$, $A \leftarrow B - C$, $A \rightarrow B \rightarrow C$, $A \leftrightarrow B \rightarrow C$, $A - B \rightarrow C$ or $A - B - C$ must be in G_1 . However, if $A \rightarrow C$ is in G_1 then the only legal options are those that contain the edge $B \rightarrow C$. On the other hand, if $A \leftrightarrow C$ is in G_1 then the only legal option is $A \leftarrow B \rightarrow C$. Finally, if $A - C$ is in G_1 then the only legal options are $A \leftarrow B \rightarrow C$ and $A - B - C$.

$$\begin{array}{ccc}
A \overset{\curvearrowright}{\circ\circ} B \rightarrow C \leftarrow\circ D & A \overset{\curvearrowright}{\circ\circ} B \rightarrow C - D & A \overset{\curvearrowright}{\leftarrow} B \rightarrow C \leftarrow\circ D \\
A \overset{\curvearrowright}{\leftarrow} B \rightarrow C - D & A \overset{\curvearrowright}{\leftarrow} B \rightarrow C \leftarrow\circ D & A \overset{\curvearrowright}{-} B - C \leftarrow\circ D
\end{array}$$

However, this implies that C is a triplex node in ρ_1 , which is a contradiction because ρ_1 is Z -open but $C \notin Z \cup \text{san}_{G_1}(Z)$.

Case 2.2: ϱ_1 does not have a triplex $(\{A, D\}, C)$ and $C \in Z$. Then, $A \leftarrow C$, $C \rightarrow D$ or $A - C - D$.

Case 2.2.1: If $C \rightarrow D$ or $A - C - D$, then one of the following subgraphs must occur in G_1 .

$$A \overset{\circ}{\circ\circ} B \overset{\circ}{\circ\circ} C \rightarrow D \quad A \overset{\curvearrowright}{-} B - C - D \quad A \overset{\curvearrowright}{\leftarrow} B \rightarrow C - D$$

However, the first and second subgraphs imply that ρ_1 is not Z -open, which is a contradiction. The third subgraph implies that ϱ_1 is Z -open, which is a contradiction.

Case 2.2.2: If $A \leftarrow C$ then $(\{A, D\}, C)$ is not a triplex in ϱ_1 . However, note that ρ_1 must have a triplex $(\{B, D\}, C)$, because ρ_1 is Z -open and $C \in Z$. Then, one of the following subgraphs must occur in G_1 .

$$A \overset{\curvearrowright}{\leftarrow} B \overset{\circ}{\circ} C \leftarrow\circ D \quad A \overset{\curvearrowright}{\leftarrow} B \overset{\circ}{\circ} C - D \quad A \overset{\curvearrowright}{\leftarrow} B - C \leftarrow\circ D$$

Assume that A and D are adjacent in G_1 . Then, $A \leftarrow D$ must be in G_1 . Moreover, $D \in Z$ because, otherwise, we can remove B and C from ρ_1 and get a Z -open path between A and B in G_1 that is shorter than ρ_1 , which contradicts the condition (i). Then, D must be a triplex node in ρ_1 . Then, one of the following subgraphs must occur in G_1 .

$$\begin{array}{ccc}
A \overset{\curvearrowright}{\leftarrow} B \overset{\circ}{\circ} C \leftrightarrow D \leftarrow\circ E & A \overset{\curvearrowright}{\leftarrow} B \overset{\circ}{\circ} C \leftrightarrow D - E & A \overset{\curvearrowright}{\leftarrow} B \overset{\circ}{\circ} C - D \leftarrow\circ E \\
A \overset{\curvearrowright}{\leftarrow} B - C \leftrightarrow D \leftarrow\circ E & A \overset{\curvearrowright}{\leftarrow} B - C \leftrightarrow D - E &
\end{array}$$

Thus, by repeatedly applying the previous reasoning, we can conclude without loss of generality that the following subgraph must occur in G_1 , with $n \geq 4$, $V_1 = A$, $V_2 = B$, $V_3 = C$, $V_4 = D$ and where V_1 and V_n are not adjacent in G_1 . Note that the subgraph below covers the case where A and D are not adjacent in the original subgraph by simply taking $n = 4$.

$$V_1 \overset{\curvearrowright}{\leftarrow} V_2 \quad \cdots \quad V_{n-1} \overset{\circ}{\circ} V_n$$

Note that V_i is a triplex node in ρ_1 for all $3 \leq i \leq n-1$. Then, $V_i \in Z$ for all $3 \leq i \leq n-1$ by the condition (ii) because ρ_1 is Z -open. Then, V_i must be a triplex node in ρ_2 for all $3 \leq i \leq n-1$ because, otherwise, Case 1 would apply instead of Case 2. Recall that $V_2 = B$ is also a triplex node in ρ_2 . Note that G_1 does not have a triplex $(\{V_1, V_n\}, V_{n-1})$ and, thus, G_2 does not have it either because these are triplex equivalent. Then, one of the following subgraphs must occur in G_2 .

$$V_1 \overset{\circ}{\curvearrowright} \cdots \overset{\circ}{\curvearrowright} V_{n-1} \rightarrow V_n \quad V_1 \overset{\leftarrow}{\curvearrowright} \cdots \overset{\leftarrow}{\curvearrowright} V_{n-1} \circ \circ V_n \quad V_1 \overset{\curvearrowright}{\curvearrowright} \cdots \overset{\curvearrowright}{\curvearrowright} V_{n-1} - V_n$$

However, the first subgraph implies that V_{n-1} is not a triplex node in ρ_2 , which is a contradiction. The second subgraph implies that G_2 has a cycle that violates the constraint C1. To see it, recall that V_i is a triplex node in ρ_2 for all $2 \leq i \leq n-1$ and, thus, $V_i \leftarrow V_{i+1}$ is not in G_2 for all $1 \leq i \leq n-2$. The third subgraph implies that $V_{n-2} \leftrightarrow V_{n-1}$ is not in G_2 because, otherwise, V_1 and V_n would be adjacent by the constraint C3. Therefore, $V_{n-2} \rightarrow V_{n-1}$ must be in G_2 because V_{n-1} is a triplex node in ρ_2 . However, this implies that V_{n-2} is not a triplex node in ρ_2 , which is a contradiction.

Part 2

Let ρ_1 be any of the shortest Z -open paths between α and β in G_1 st all its triplex nodes are in Z . Let ρ_2 be the path in G_2 that consists of the same nodes as ρ_1 . We prove below that ρ_2 is Z -open. We prove this result by induction on the number of non-triplex nodes of ρ_1 that are in Z . If this number is zero, then Part 1 proves the result. Assume as induction hypothesis that the result holds when the number is smaller than m . We now prove it for m .

Let $\rho_1^{A:B}$ denote the subpath of ρ_1 between the nodes A and B . Let C be any of the non-triplex nodes of ρ_1 that are in Z . Note that there must exist some node $D \in pa_{G_1}(C) \setminus Z$ for ρ_1 to be Z -open. If D is in ρ_1 , then $\rho_1^{\alpha:D} \cup D \rightarrow C \cup \rho_1^{C:\beta}$ or $\rho_1^{\alpha:C} \cup C \leftarrow D \cup \rho_1^{D:\beta}$ is a Z -open path between α and β in G_1 that has fewer than m non-triplex nodes in Z . Then, the result holds by the induction hypothesis. On the other hand, if D is not in ρ_1 , then $\rho_1^{\alpha:C} \cup C \leftarrow D$ and $D \rightarrow C \cup \rho_1^{C:\beta}$ are two paths. Moreover, they are Z -open in G_1 and they have fewer than m non-triplex nodes in Z . Then, by the induction hypothesis, there are two Z -open paths $\rho_2^{\alpha:D}$ and $\rho_2^{D:\beta}$ in G_2 st the former ends with the nodes C and D and the latter starts with these two nodes. Now, consider the following cases.

Case 1: $\rho_2^{\alpha:D}$ ends with $A - C \leftarrow D$. Then, $\rho_2^{D:\beta}$ starts with $D \rightarrow C - B$ or $D \rightarrow C \leftrightarrow B$.

Then, $\rho_2 = \rho_2^{\alpha:C} \cup \rho_2^{C:\beta}$ is Z -open a path in either case.

Case 2: $\rho_2^{\alpha:D}$ ends with $A - C \leftrightarrow D$. Then, $\rho_2^{D:\beta}$ starts with $D \leftrightarrow C - B$ or $D \leftrightarrow C \leftrightarrow B$.

Then, $\rho_2 = \rho_2^{\alpha:C} \cup \rho_2^{C:\beta}$ is Z -open a path in either case.

Case 3: $\rho_2^{\alpha:D}$ ends with $A \circ \rightarrow C - D$. Then, $\rho_2^{D:\beta}$ starts with $D - C \leftrightarrow B$, or $D - C - B$ st there is some node $E \in pa_{G_2}(C) \setminus Z$. Then, $\rho_2 = \rho_2^{\alpha:C} \cup \rho_2^{C:\beta}$ is Z -open a path in either case.

Case 4: $\rho_2^{\alpha:D}$ ends with $A \circ \rightarrow C \leftrightarrow D$. Then, $\rho_2^{D:\beta}$ starts with $D \circ \rightarrow C - B$ or $D \circ \rightarrow C \leftrightarrow B$.

Then, $\rho_2 = \rho_2^{\alpha:C} \cup \rho_2^{C:\beta}$ is Z -open a path in either case.

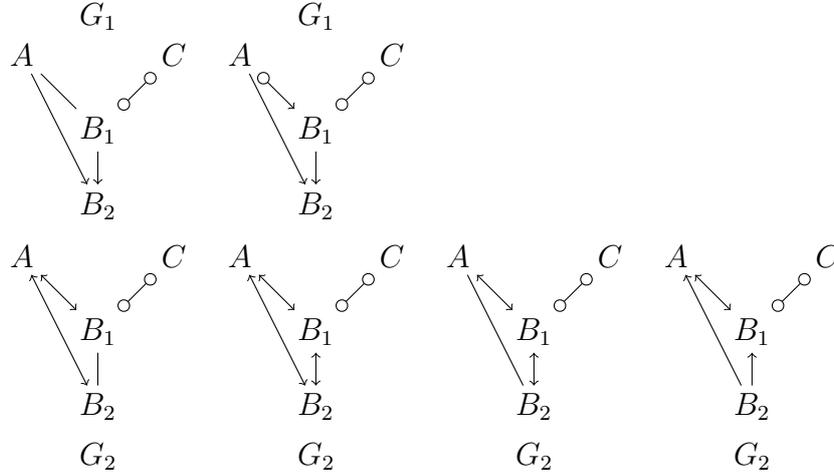
Case 5: $\rho_2^{\alpha:D}$ ends with $A - C - D$ st there is some node $E \in pa_{G_2}(C) \setminus Z$. Then, $\rho_2^{D:\beta}$ starts with $D - C \leftrightarrow B$, or $D - C - B$ st there is some node $F \in pa_{G_2}(C) \setminus Z$. Then, $\rho_2 = \rho_2^{\alpha:C} \cup \rho_2^{C:\beta}$ is a Z -open path in either case.

Part 3

Assume that Part 2 does not apply. Then, every Z -open path between α and β in G_1 has some triplex node B_1 that is outside Z because, otherwise, Part 2 would apply. Note that for the path to be Z -open, G_1 must have a subgraph $B_1 \rightarrow \dots \rightarrow B_n$ st $B_1, \dots, B_{n-1} \notin Z$ but $B_n \in Z$. Let us convert every Z -open path between α and β in G_1 into a route by replacing each of its triplex nodes B_1 that are outside Z with the corresponding route $B_1 \rightarrow \dots \rightarrow B_n \leftarrow \dots \leftarrow B_1$. Let ϱ_1 be any of the shortest routes so-constructed. Let ρ_1 be the path from which ϱ_1 was constructed. Note that ρ_1 cannot be Z -open because, otherwise, Part 2 would apply. Let W denote the set of all the triplex nodes in ρ_1 that are outside Z . Then, ρ_1 is one of the shortest $(Z \cup W)$ -open paths between α and β in G_1 st all its triplex nodes are in $Z \cup W$. To see it, assume to the contrary that ρ'_1 is a $(Z \cup W)$ -open path between α and β in G_1 that is shorter than ρ_1 and st all the triplex nodes in ρ'_1 are in $Z \cup W$. Let ϱ'_1 be the route resulting

from replacing every node B_1 of ρ'_1 that is in W with the route $B_1 \rightarrow \dots \rightarrow B_n \leftarrow \dots \leftarrow B_1$ that was added to ρ_1 to construct ϱ_1 . Clearly, ϱ'_1 is shorter than ϱ_1 , which is a contradiction. Let ϱ_2 and ρ_2 be the route and the path in G_2 that consist of the same nodes as ϱ_1 and ρ_1 . Note that ρ_2 is $(Z \cup W)$ -open by Part 2.

Consider any of the routes $B_1 \rightarrow \dots \rightarrow B_n \leftarrow \dots \leftarrow B_1$ that were added to ρ_1 to construct ϱ_1 . This implies that ρ_1 has a triplex $(\{A, C\}, B_1)$. Assume that $B_1 \rightarrow B_2$ is in G_1 but $B_1 - B_2$ or $B_1 \leftarrow B_2$ is in G_2 . Note that $A \circ B_1$ or $B_1 \leftarrow C$ is in G_2 because, as noted above, ρ_2 is $(Z \cup W)$ -open. Assume without loss of generality that $A \circ B_1$ is in G_2 . Then, $A - B_1 \rightarrow B_2$ or $A \circ B_1 \rightarrow B_2$ is in G_1 whereas $A \circ B_1 - B_2$ or $A \circ B_1 \leftarrow B_2$ is in G_2 . Therefore, A and B_2 must be adjacent in G_1 and G_2 because these are triplex equivalent. This implies that $A \rightarrow B_2$ is in G_1 . Moreover, $A \in Z$ because, otherwise, we can construct a route that is shorter than ϱ_1 by simply removing B_1 from ϱ_1 , which is a contradiction. This implies that $A \leftrightarrow B_1$ is in G_2 because, otherwise, ρ_2 would not be $(Z \cup W)$ -open. This implies that $A \leftrightarrow B_1 - B_2$ or $A \leftrightarrow B_1 \leftarrow B_2$ is in G_2 , which implies that $A - B_2$ or $A \leftarrow B_2$ is in G_2 . The situation is depicted in the following subgraphs.



Now, let A' be the node that precedes A in ρ_1 . Note that $A' \leftarrow A$ cannot be in ρ_1 or ρ_2 because, otherwise, these would not be $(Z \cup W)$ -open since $A \in Z$. Then, $A' - A$ or $A' \circ A$ is in G_1 and G_2 . Then, $A' - A \rightarrow B_2$ or $A' \circ A \rightarrow B_2$ is in G_1 whereas $A' - A \leftarrow B_2$, $A' - A - B_2$, $A' \circ A \leftarrow B_2$ or $A' \circ A - B_2$ is in G_2 . These four subgraphs of G_2 imply that A' and B_2 must be adjacent in G_1 and G_2 : The second subgraph due to the constraint C3 because $A \leftrightarrow B_1$ is in G_2 , and the other three subgraphs because G_1 and G_2 are triplex equivalent. By repeating the reasoning in the paragraph above, we can conclude that $A' \rightarrow B_2$ is in G_1 , which implies that $A' \in Z$, which implies that $A' - A$ or $A' \leftrightarrow A$ is in G_2 , which implies that $A' - B_2$ or $A' \leftarrow B_2$ is in G_2 .

By repeating the reasoning in the paragraph above,⁵ we can conclude that $\alpha \rightarrow B_2$ is in G_1 and, thus, we can construct a route that is shorter than ϱ_1 by simply removing some nodes from ϱ_1 , which is a contradiction. Consequently, $B_1 \rightarrow B_2$ must be in G_2 .

Finally, assume that $B_1 \rightarrow B_2 \rightarrow B_3$ is in G_1 but $B_1 \rightarrow B_2 - B_3$ or $B_1 \rightarrow B_2 \leftarrow B_3$ is in G_2 . Then, B_1 and B_3 must be adjacent in G_1 and G_2 because these are triplex equivalent. This implies that $B_1 \rightarrow B_3$ is in G_1 , which implies that we can construct a route that is shorter than ϱ_1 by simply removing B_2 from ϱ_1 , which is a contradiction. By repeating this reasoning, we can conclude that $B_1 \rightarrow \dots \rightarrow B_n$ is in G_2 and, thus, that ρ_2 is Z -open. \square

⁵Let A'' be the node that precedes A' in ρ_1 . For this repeated reasoning to be correct, it is important to realize that if $A' - A$ is in G_2 , then $A'' \leftrightarrow A'$ must be in G_2 , because $A' \in Z$ and ρ_2 is $(Z \cup W)$ -open.

Proof of Lemma 1. Assume to the contrary that there are two such sets of directed node pairs. Let the MAMP CG G contain exactly the directed node pairs in one of the sets, and let the MAMP CG H contain exactly the directed node pairs in the other set. For every $A \rightarrow B$ in G st $A - B$ or $A \leftrightarrow B$ is in H , replace the edge between A and B in H with $A \rightarrow B$ and call the resulting graph F . We prove below that F is a MAMP CG that is triplex equivalent to G and thus to H , which is a contradiction since F has a proper superset of the directed node pairs in H .

First, note that F cannot violate the constraints C2 and C3. Assume to the contrary that F violates the constraint C1 due to a cycle ρ . Note that none of the directed edges in ρ can be in H because, otherwise, H would violate the constraint C1, since H has the same adjacencies as F but a subset of the directed edges in F . Then, all the directed edges in ρ must be in G . However, this implies the contradictory conclusion that G violates the constraint C1, since G has the same adjacencies as F but a subset of the directed edges in F .

Second, assume to the contrary that G (and, thus, H) has a triplex $(\{A, C\}, B)$ that F has not. Then, $\{A, B\}$ or $\{B, C\}$ must be a directed node pair in G because, otherwise, F would have a triplex $(\{A, C\}, B)$ since F would have the same induced graph over $\{A, B, C\}$ as H . Specifically, $A \rightarrow B$ or $B \leftarrow C$ must be in G because, otherwise, G would not have a triplex $(\{A, C\}, B)$. Moreover, neither $A \leftarrow B$ nor $B \rightarrow C$ can be in H because, otherwise, H would not have a triplex $(\{A, C\}, B)$. Therefore, if $A \rightarrow B$ or $B \leftarrow C$ is in G and neither $A \leftarrow B$ nor $B \rightarrow C$ is in H , then $A \rightarrow B$ or $B \leftarrow C$ must be in F . However, this implies that $B \rightarrow C$ or $A \leftarrow B$ must be in F because, otherwise, F would have a triplex $(\{A, C\}, B)$ which would be a contradiction. However, this is a contradiction since neither $B \rightarrow C$ nor $A \leftarrow B$ can be in G or H because, otherwise, neither G nor H would have a triplex $(\{A, C\}, B)$.

Finally, assume to the contrary that F has a triplex $(\{A, C\}, B)$ that G has not (and, thus, nor does H). Then, $A - B - C$ must be in H because, otherwise, $A \leftarrow B$ or $B \rightarrow C$ would be in H and, thus, F would not have a triplex $(\{A, C\}, B)$. However, this implies that $A \rightarrow B$ or $B \leftarrow C$ is in G because, otherwise, F would not have a triplex $(\{A, C\}, B)$. However, this implies that $B \rightarrow C$ or $A \leftarrow B$ is in G because, otherwise, G would have a triplex $(\{A, C\}, B)$. Therefore, $A \rightarrow B \rightarrow C$ or $A \leftarrow B \leftarrow C$ is in G and, thus, $A \rightarrow B \rightarrow C$ or $A \leftarrow B \leftarrow C$ must be in F since $A - B - C$ is in H . However, this contradicts the assumption that F has a triplex $(\{A, C\}, B)$. \square

Proof of Lemma 2. Assume to the contrary that there are two such sets of bidirected edges. Let the MDCG G contain exactly the bidirected edges in one of the sets, and let the MDCG H contain exactly the bidirected edges in the other set. For every $A \leftrightarrow B$ in G st $A - B$ is in H , replace $A - B$ with $A \leftrightarrow B$ in H and call the resulting graph F . We prove below that F is a MDCG that is triplex equivalent to G , which is a contradiction since F has a proper superset of the bidirected edges in G .

First, note that F cannot violate the constraint C1. Assume to the contrary that F violates the constraint C2 due to a cycle ρ . Note that all the undirected edges in ρ are in H . In fact, they must also be in G , because G and H have the same directed node pairs and bidirected edges. Moreover, the bidirected edge in ρ must be in G or H . However, this is a contradiction. Now, assume to the contrary that F violates the constraint C3 because $A - B - C$ and $B \leftrightarrow D$ are in F but A and C are not adjacent in F (note that if A and C were adjacent in F , then they would not violate the constraint C3 or they would violate the constraint C1 or C2, which is impossible as we have just shown). Note that $A - B - C$ must be in H . In fact, $A - B - C$ must also be in G , because G and H have the same directed node pairs and bidirected edges. Moreover, $B \leftrightarrow D$ must be in G or H . However, this implies that A and C are adjacent in G or H by the constraint C3, which implies that A and C are adjacent in G and H because they are triplex equivalent and thus also in F , which is a contradiction. Consequently, F is a MAMP CG, which implies that F is a MDCG because it has the same directed edges as G and H .

Second, note that all the triplexes in G are in F too.

Finally, assume to the contrary that F has a triplex $(\{A, C\}, B)$ that G has not (and, thus, nor does H). Then, $A - B - C$ must be in H because, otherwise, $A \leftarrow B$ or $B \rightarrow C$ would be in H and thus F would not have a triplex $(\{A, C\}, B)$. However, this implies that F has the same induced graph over $\{A, B, C\}$ as G , which contradicts the assumption that F has a triplex $(\{A, C\}, B)$. \square

Proof of Theorem 8. It suffices to show that every Z -open path between α and β in G can be transformed into a Z -open path between α and β in G' and vice versa, with $\alpha, \beta \in V$ and $Z \subseteq V \setminus \alpha \setminus \beta$.

Let ρ denote a Z -open path between α and β in G . We can easily transform ρ into a path ρ' between α and β in G' : Simply, replace every maximal subpath of ρ of the form $V_1 \mapsto V_2 \mapsto \dots \mapsto V_{n-1} \mapsto V_n$ ($n \geq 2$) with $V_1 \leftarrow \epsilon^{V_1} \mapsto \epsilon^{V_2} \mapsto \dots \mapsto \epsilon^{V_{n-1}} \mapsto \epsilon^{V_n} \rightarrow V_n$. We now show that ρ' is Z -open.

Case 1.1: If $B \in V$ is a triplex node in ρ' , then ρ' must have one of the following subpaths:

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \leftarrow \epsilon^B \mapsto \epsilon^C \quad \epsilon^A \mapsto \epsilon^B \rightarrow B \leftarrow C$$

with $A, C \in V$. Therefore, ρ must have one of the following subpaths (specifically, if ρ' has the i -th subpath above, then ρ has the i -th subpath below):

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \mapsto C \quad A \mapsto B \leftarrow C$$

In either case, B is a triplex node in ρ and, thus, $B \in Z \cup \text{san}_G(Z)$ for ρ to be Z -open.

Then, $B \in Z \cup \text{san}_{G'}(Z)$ by construction of G' and, thus, $B \in D(Z) \cup \text{san}_{G'}(D(Z))$.

Case 1.2: If $B \in V$ is a non-triplex node in ρ' , then ρ' must have one of the following subpaths:

$$A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B \leftarrow \epsilon^B \mapsto \epsilon^C \quad \epsilon^A \mapsto \epsilon^B \rightarrow B \rightarrow C$$

with $A, C \in V$. Therefore, ρ must have one of the following subpaths (specifically, if ρ' has the i -th subpath above, then ρ has the i -th subpath below):

$$A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B \mapsto C \quad A \mapsto B \rightarrow C$$

In either case, B is a non-triplex node in ρ and, thus, $B \notin Z$ for ρ to be Z -open.

Since Z contains no error node, Z cannot determine any node in V that is not already in Z . Then, $B \notin D(Z)$.

Case 1.3: If ϵ^B is a triplex node in ρ' , then ρ' must have one of the following subpaths:

$$\epsilon^A \leftrightarrow \epsilon^B \mapsto \epsilon^C \quad \epsilon^A - \epsilon^B \leftrightarrow \epsilon^C$$

Therefore, ρ must have one of the following subpaths (specifically, if ρ' has the i -th subpath above, then ρ has the i -th subpath below):

$$A \leftrightarrow B \mapsto C \quad A - B \leftrightarrow C$$

In either case, B is a triplex node in ρ and, thus, $B \in Z \cup \text{san}_G(Z)$ for ρ to be Z -open. Then, $\epsilon^B \in Z \cup \text{san}_{G'}(Z)$ by construction of G' and, thus, $\epsilon^B \in D(Z) \cup \text{san}_{G'}(D(Z))$.

Case 1.4: If ϵ^B is a non-triplex node in ρ' , then ρ' must have one of the following subpaths:

$$\begin{aligned} A \rightarrow B \leftarrow \epsilon^B \vdash \epsilon^C \quad \epsilon^A \vdash \epsilon^B \rightarrow B \leftarrow C \quad \alpha = B \leftarrow \epsilon^B \vdash \epsilon^C \quad \epsilon^A \vdash \epsilon^B \rightarrow B = \beta \\ A \leftarrow B \leftarrow \epsilon^B \vdash \epsilon^C \quad \epsilon^A \vdash \epsilon^B \rightarrow B \rightarrow C \quad \epsilon^A - \epsilon^B - \epsilon^C \end{aligned}$$

with $A, C \in V$. Recall that $\epsilon^B \notin Z$ because $Z \subseteq V \setminus \alpha \setminus \beta$. In the first case, if $\alpha = A$ then $A \notin Z$, else $A \notin Z$ for ρ to be Z -open. Then, $\epsilon^B \notin D(Z)$. In the second case, if $\beta = C$ then $C \notin Z$, else $C \notin Z$ for ρ to be Z -open. Then, $\epsilon^B \notin D(Z)$. In the third and fourth cases, $B \notin Z$ because $\alpha = B$ or $\beta = B$. Then, $\epsilon^B \notin D(Z)$. In the fifth and sixth cases, $B \notin Z$ for ρ to be Z -open. Then, $\epsilon^B \notin D(Z)$. The last case implies that ρ has the following subpath:

$$A - B - C$$

Thus, B is a non-triplex node in ρ , which implies that $B \notin Z$ or $pa_G(B) \setminus Z \neq \emptyset$ for ρ to be Z -open. In either case, $\epsilon^B \notin D(Z)$ (recall that $pa_{G'}(B) = pa_G(B) \cup \epsilon^B$ by construction of G').

Finally, let ρ' denote a Z -open path between α and β in G' . We can easily transform ρ' into a path ρ between α and β in G : Simply, replace every maximal subpath of ρ' of the form $V_1 \leftarrow \epsilon^{V_1} \vdash \epsilon^{V_2} \vdash \dots \vdash \epsilon^{V_{n-1}} \vdash \epsilon^{V_n} \rightarrow V_n$ ($n \geq 2$) with $V_1 \vdash V_2 \vdash \dots \vdash V_{n-1} \vdash V_n$. We now show that ρ is Z -open. Note that all the nodes in ρ are in V .

Case 2.1: If B is a triplex node in ρ , then ρ must have one of the following subpaths:

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \vdash C \quad A \vdash B \leftarrow C \quad A \leftrightarrow B \vdash C \quad A - B \leftrightarrow C$$

with $A, C \in V$. Therefore, ρ' must have one of the following subpaths (specifically, if ρ has the i -th subpath above, then ρ' has the i -th subpath below):

$$A \rightarrow B \leftarrow C \quad A \rightarrow B \leftarrow \epsilon^B \vdash \epsilon^C \quad \epsilon^A \vdash \epsilon^B \rightarrow B \leftarrow C \quad \epsilon^A \leftrightarrow \epsilon^B \vdash \epsilon^C \quad \epsilon^A - \epsilon^B \leftrightarrow \epsilon^C$$

In the first three cases, B is a triplex node in ρ' and, thus, $B \in D(Z) \cup san_{G'}(D(Z))$ for ρ' to be Z -open. Since Z contains no error node, Z cannot determine any node in V that is not already in Z . Then, $B \in D(Z)$ iff $B \in Z$. Since there is no strictly descending route from B to any error node, then any strictly descending route from B to a node $D \in D(Z)$ implies that $D \in V$ which, as seen, implies that $D \in Z$. Then, $B \in san_{G'}(D(Z))$ iff $B \in san_{G'}(Z)$. Moreover, $B \in san_{G'}(Z)$ iff $B \in san_G(Z)$ by construction of G' . These results together imply that $B \in Z \cup san_G(Z)$.

In the last two cases, ϵ^B is a triplex node in ρ' and, thus, $B \in D(Z) \cup san_{G'}(D(Z))$ for ρ' to be Z -open because Z contains no error node. Therefore, as shown in the previous paragraph, $B \in Z \cup san_G(Z)$.

Case 2.2: If B is a non-triplex node in ρ , then ρ must have one of the following subpaths:

$$A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B \vdash C \quad A \vdash B \rightarrow C \quad A - B - C$$

with $A, C \in V$. Therefore, ρ' must have one of the following subpaths (specifically, if ρ has the i -th subpath above, then ρ' has the i -th subpath below):

$$\begin{aligned} A \rightarrow B \rightarrow C \quad A \leftarrow B \rightarrow C \quad A \leftarrow B \leftarrow C \quad A \leftarrow B \leftarrow \epsilon^B \vdash \epsilon^C \quad \epsilon^A \vdash \epsilon^B \rightarrow B \rightarrow C \\ \epsilon^A - \epsilon^B - \epsilon^C \end{aligned}$$

In the first five cases, B is a non-triplex node in ρ' and, thus, $B \notin D(Z)$ for ρ' to be Z -open. Since Z contains no error node, Z cannot determine any node in V that is not already in Z . Then, $B \notin Z$. In the last case, ϵ^B is a non-triplex node in ρ' and, thus, $\epsilon^B \notin D(Z)$ for ρ' to be Z -open. Then, $B \notin Z$ or $pa_{G'}(B) \setminus \epsilon^B \setminus Z \neq \emptyset$. Then, $B \notin Z$ or $pa_G(B) \setminus Z \neq \emptyset$ (recall that $pa_{G'}(B) = pa_G(B) \cup \epsilon^B$ by construction of G'). \square

Proof of Theorem 9. We find it easier to prove the theorem by defining separation in MAMP CGs in terms of routes rather than paths. A node B in a route ρ in a MAMP CG G is called a triplex node in ρ if $A \leftrightarrow B \leftarrow C$, $A \leftrightarrow B - C$, or $A - B \leftarrow C$ is a subroute of ρ (note that maybe $A = C$ in the first case). Note that B may be both a triplex and a non-triplex node in ρ . Moreover, ρ is said to be Z -open with $Z \subseteq V$ when

- every triplex node in ρ is in $D(Z)$, and
- no non-triplex node in ρ is in $D(Z)$.

When there is no Z -open route in G between a node in X and a node in Y , we say that X is separated from Y given Z in G and denote it as $X \perp_G Y | Z$. It is straightforward to see that this and the original definition of separation in MAMP CGs introduced in Section 4 are equivalent, in the sense that they identify the same separations in G .

We prove the theorem for the case where L contains a single node B . The general case follows by induction. Specifically, given $\alpha, \beta \in V \setminus L$ and $Z \subseteq V \setminus L \setminus \alpha \setminus \beta$, we show below that every Z -open route between α and β in $[G']_L$ can be transformed into a Z -open route between α and β in G' and vice versa.

First, let ρ denote a Z -open route between α and β in $[G']_L$. We can easily transform ρ into a Z -open route between α and β in G' : For each edge $A \rightarrow C$ or $A \leftarrow C$ with $A, C \in V \cup \epsilon$ that is in $[G']_L$ but not in G' , replace each of its occurrence in ρ with $A \rightarrow B \rightarrow C$ or $A \leftarrow B \leftarrow C$, respectively. Note that $B \notin D(Z)$ because $B, \epsilon^B \notin Z$.

Second, let ρ denote a Z -open route between α and β in G' . Note that B cannot participate in any undirected or bidirected edge in G' , because $B \in V$. Note also that B cannot be a triplex node in ρ , because $B \notin D(Z)$ since $B, \epsilon^B \notin Z$. Note also that $B \neq \alpha, \beta$. Then, B can only appear in ρ in the following configurations: $A \rightarrow B \rightarrow C$, $A \leftarrow B \leftarrow C$, or $A \leftarrow B \rightarrow C$ with $A, C \in V \cup \epsilon$. Then, we can easily transform ρ into a Z -open route between α and β in $[G']_L$: Replace each occurrence of $A \rightarrow B \rightarrow C$ in ρ with $A \rightarrow C$, each occurrence of $A \leftarrow B \leftarrow C$ in ρ with $A \leftarrow C$, and each occurrence of $A \leftarrow B \rightarrow C$ in ρ with $A \leftarrow \epsilon^B \rightarrow C$. In the last case, note that $\epsilon^B \notin D(Z)$ because $B, \epsilon^B \notin Z$. \square

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