A Provable Stable and Accurate Davies-like Relaxation Procedure Using Multiple Penalty Terms for Lateral Boundaries in Weather Prediction

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Abstract

A lateral boundary treatment using summation-by-parts operators and simultaneous approximation terms is introduced. The method, that we refer to as the multiple penalty technique, is similar to Davies relaxation and have similar areas of application. The method is proven, by energy methods, to be stable. We show how to apply this technique on the linearized Euler equations in two space dimensions, and that it reduces the errors in the computational domain.

Keywords: Davies relaxation, lateral boundary conditions, summation-by-parts, weak boundary conditions, penalty terms

1. Introduction

Accurate numerical calculations on large domains are often unfeasible. To circumvent this issue, a coarse global mesh is used, and inter-spaced local domains are inserted where accurate results are wanted. These local meshes are typically employed where something interesting happens, and where fine meshes are needed to capture the phenomena with a given accuracy. This can, for example, be used to model local weather phenomena or turbulence around airplane wings. However, this model requires that the local domains are matched correctly to the global one. H.C Davies [1] introduced in 1976 a lateral boundary procedure for weather prediction models where the coarse grid numerical results are interpolated into the local fine grid domain, a
method later referred to as Davies relaxation. Other types of interpolation methods are also used within the weather prediction community, see [5],[2][3]. In most applications, one aims for accurate results in the local domains, while the accuracy in the global domain is considered given. Therefore, the coupling between the domains are in most cases neglected. There are methods, see [4], where the coupling between the meshes is considered, such that the results from the local domain influence the global one. The results from the global area can be transferred to the local one by using so-called window functions, see [6],[7], which are used to blend the results from the domains. These window functions can, for example, be based on Fourier extensions close to the interface. However, this coupling often introduces stability issues unless optimally done [9],[10],[11],[12]. In other cases, the global data is mixed into global domain by interpolation [1],[8], which is the technique we will adapt in this work. Moreover, we will neglect any coupling between the domains.

We demonstrate our technique on the linearized Euler equations, discretized using operators with the summation-by parts (SBP) property [14],[15] and augmented with simultaneous approximation terms (SAT) [16]. This is often referred to as the SBP-SAT technique and a comprehensive review is given in [13]. If well-posed boundary conditions are available, this technique yields stable numerical schemes [17].

In [18], it was shown that if data is available inside the computational domain, additional penalty terms can be applied at these points without ruining stability. This method, which we call the multiple penalty technique (MPT), can be used to assimilate global data into local area models, constructing non-reflecting boundaries, raising accuracy of the scheme and increasing the rate of convergence to steady state, see [18],[21]. More importantly, it can be done in a provable stable way. The MPT is similar to the Davies relaxation mentioned above, but can also be proven stable.

The rest of the paper will proceed as follows. In section 2, we derive well-posed boundary conditions for the linearized Euler equations and discretize it by using the SBP-SAT technique. Stability conditions are derived and multiple penalties are applied such that stability is preserved. In section 3, numerical experiments are performed to illustrate the increased rate of convergence to steady-state when using multiple penalties. Numerical experiments are also performed on time-dependent problems, illustrating an increased convergence rate when MPT is used. The results of this paper are concluded in section 4.
Figure 1: Illustration of the problem set-up. The calculations are performed in the central domain where boundary data has been interpolated from the coarse, global domain.

2. Well-posedness and stability

We consider the Euler equations in the dimensionless quantities \( \rho = \hat{\rho}/\hat{\rho}_\infty, \ u = \hat{u}/\hat{u}_\infty, \ v = \hat{v}/\hat{v}_\infty \) and \( p = \hat{p}/(\hat{\rho}_\infty c^2) \) where \( \hat{\rho} \) is the density, \( \hat{u}, \hat{v} \) the horizontal and vertical component of the velocity, \( \hat{p} \) the pressure and \( \hat{c} \) the speed of sound. The subscript \( \infty \) denotes a reference state. For simplicity, and without restriction, a square domain \( x, y \in [0, 1] \) is considered and \( t \geq 0 \). This domain is assumed to be contained within a global mesh, from which the boundary data is obtained. See Figure 1 for an illustration.

The Euler equations with boundary and initial conditions are

\[
U_t + A(U)U_x + B(U)U_y = 0
\]
\[
L_w U(0, y, t) = g_w(y, t)
\]
\[
L_s U(x, 0, t) = g_s(x, t)
\]
\[
L_e U(1, y, t) = g_e(y, t)
\]
\[
L_n U(x, 1, t) = g_n(x, t)
\]
\[
u(x, y, 0) = f(x, y)
\]

where \( U = [\rho, u, v, p]^T \), \( L_{w,s,e,n} \) are the boundary operators to be determined and \( g_{w,s,e,n} \) are the boundary data. The matrices are

\[
A(U) = \begin{bmatrix}
u & \rho & 0 & 0 \\
0 & u & 0 & 1/\rho \\
0 & 0 & u & 0 \\
0 & \rho c^2 & 0 & u
\end{bmatrix}, \quad B(U) = \begin{bmatrix}
u & 0 & \rho & 0 \\
0 & v & 0 & 0 \\
0 & 0 & v & 1/\rho \\
0 & 0 & \rho c^2 & v
\end{bmatrix}.
\]
The problem (1) is linearized by considering small deviations from the reference state; that is, we insert $U = U_0 + U'$ where $U_0$ is the reference state and $U'$ is small. The linearized problem is then transformed according to $w' = T_s U' = [p'/\rho_\infty \bar{c}, u', v', p' - \rho' \bar{c}^2]^T$, with

$$T_s = \begin{bmatrix}
0 & 0 & 0 & 1/\rho_\infty \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\bar{c}^2 & 0 & 0 & 1
\end{bmatrix},$$

which leads to the symmetric and linearized form of the Euler equations,

$$w'_t + A_s(U_0)w'_x + B_s(U_0)w'_y = 0$$

$$L_w w'(0, y, t) = g'_w(y, t)$$

$$L^*_x w'(x, 0, t) = g'_w(x, t)$$

$$L^*_y w'(1, y, t) = g'_e(y, t)$$

$$L^*_x w'(x, 1, t) = g'_n(x, t)$$

$$w'(x, y, 0) = f'(x, y)$$

where $f'(x, y)$ is the perpetuated initial condition, $g'_{w,s,e,n}$ are the perpetuated boundary data and

$$A_s = T_s A T_s^{-1} = \begin{bmatrix}
\bar{u} & \bar{c} & 0 & 0 \\
\bar{c} & \bar{u} & 0 & 0 \\
0 & 0 & \bar{u} & 0 \\
0 & 0 & 0 & \bar{u}
\end{bmatrix}, \quad B_s = T_s B T_s^{-1} = \begin{bmatrix}
\bar{v} & 0 & \bar{c} & 0 \\
0 & \bar{v} & 0 & 0 \\
\bar{c} & 0 & \bar{v} & 0 \\
0 & 0 & 0 & \bar{v}
\end{bmatrix}.$$

Note that $A_s$ and $B_s$ are now constant symmetric matrices and $\bar{u}, \bar{v}$ are the normalized reference state velocities. A general technique to symmetrize the Euler and Navies-Stokes equations can be found in [20].

2.1. Well-posedness

We now determine the boundary operators such that (2) becomes well-posed. If one multiply (2) with $w'^T$ from the left, integrate over the domain and apply Greens theorem, one arrives at

$$\|w'\|^2_t = -\iint w'^T A_s w' + w'^T B_s w' \cdot \hat{n} ds$$

(3)
where $\hat{n} = \begin{bmatrix} dy, -dx \end{bmatrix}^T / \sqrt{dx^2 + dy^2}$ is the outward pointing normal to the domain and $ds = \sqrt{dx^2 + dy^2}$ is the differential along the boundary. In (3), the norm is defined as $||w'||^2 = \int_\Omega (w')^2 dxdy$. In our simplified model, the domain $\Omega$ is a square.

The relation (3) can under these circumstances be written out explicitly as

$$||w'||^2 = -\oint_\omega \begin{bmatrix} w'^T A_s w', w'^T B_s w' \end{bmatrix} \cdot \hat{n} ds =$$

$$\int_{x=0}^1 w'(x, 0, t)^T B_s w'(x, 0, t) dx - \int_{x=0}^1 w'(x, 1, t)^T B_s w'(x, 1, t) dx$$

$$\int_{y=0}^1 w'(0, y, t)^T A_s w'(0, y, t) dy - \int_{y=0}^1 w'(1, y, t)^T A_s w'(1, y, t) dy. \quad (4)$$

All boundary operators that bounds the right-hand side of (4) results in an energy estimate, and well-posedness follows if the correct number of boundary conditions is used.

By considering the eigenvalue decomposition of $A_s = S_A \Lambda_A S_A^T$ and $B_s = S_B \Lambda_B S_B^T$ where $\Lambda_{A,B}$ are the eigenvalue matrices of $A_s$ and $B_s$ and $S_{A,B}$ the corresponding similarity transformations, the most direct way to construct well-posed boundary conditions is then to choose

$$L^+_w = S_A \Lambda_A^+ S_A^T = A^+_s$$
$$L^+_s = S_B \Lambda_B^+ S_B^T = B^+_s$$
$$L^-_e = S_A \Lambda_A^- S_A^T = A^-_s$$
$$L^-_n = S_B \Lambda_B^- S_B^T = B^-_s. \quad (5)$$

In (5), $\Lambda^+_A$ and $\Lambda^-_A$ refers to the positive and negative parts of the eigenvalue matrices, respectively. Problem (2) can with the operators in (5) be written

$$w'_t + A_s(U_0) w'_x + B_s(U_0) w'_y = 0$$

$$A^+_s w'(0, y, t) = g'_w(y, t)$$
$$B^+_s w'(x, 0, t) = g'_s(x, t)$$
$$A^-_s w'(1, y, t) = g'_e(y, t)$$
$$B^-_s w'(x, 1, t) = g'_n(x, t)$$
$$w'(x, y, 0) = f'(x, y). \quad (6)$$
By using the eigenvalue decomposition as in (5) to describe the boundary conditions in (6), one obtains

\[
\begin{align*}
\frac{dw_t}{dt} + A_s(U_0)w_x + B_s(U_0)w_y &= 0 \\
A_A^+ \dot{w}(0, y, t) &= \dot{g}_w(y, t) \\
A_B^+ \dot{w}(x, 0, t) &= \dot{g}_s(x, t) \\
A_A^- \dot{w}(1, y, t) &= \dot{g}_e(y, t) \\
A_B^- \dot{w}(x, 1, t) &= \dot{g}_n(x, t) \\
w(x, y, 0) &= f'(x, y)
\end{align*}
\]

(7)

where \( \dot{w}' = S A w \), \( \dot{w}' = S B w \), \( \dot{g}_w, e = S A g_w, e \) and \( \dot{g}_s, n = S B g_s, n \). Here, we study only subsonic velocities, such that \( u^2 + v^2 < c^2 \). Using the boundary operators (7) in (4), one obtains

\[
\begin{align*}
\|w'\|_t^2 &= \int_{x=0}^{1} \dot{g}_e^T \begin{bmatrix}
\frac{1}{\nu}
0
0
0
0
0
0
0
0
\end{bmatrix} \dot{g}_e + \int_{x=0}^{1} w'(x, 0, t)B_w^- w'(x, 0, t)dx \\
- \int_{x=0}^{1} \dot{g}_n^T \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \dot{g}_n + \int_{x=0}^{1} w'(x, 1, t)B_s^+ w'(x, 1, t)dx \\
+ \int_{y=0}^{1} \dot{g}_w^T \begin{bmatrix} \frac{1}{\nu} & 0 & 0 & 0 \\
0 & \frac{1}{\nu} & 0 & 0 \\
0 & 0 & \frac{1}{\nu} & 0 \\
0 & 0 & 0 & \frac{1}{\nu} & 0 \\
\end{bmatrix} \dot{g}_w + \int_{y=0}^{1} w'(0, y, t)A_w^- w'(0, y, t)dx \\
- \int_{y=0}^{1} \dot{g}_e^T \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \dot{g}_e - \int_{y=0}^{1} w'(1, y, t)A_e^+ w'(1, y, t)dx
\end{align*}
\]

(8)

where we have used \( \Lambda_A = \text{diag}(\bar{u}, \bar{u}, \bar{u} + \bar{c}, \bar{u} - \bar{c}) \), \( \Lambda_B = \text{diag}(\bar{v}, \bar{v}, \bar{v} + \bar{c}, \bar{v} - \bar{c}) \) and \( \bar{u}^2 + \bar{v}^2 < \bar{c}^2 \). Equation (8) states that the norm \( \|w\|_t \) is bounded by data. Since we also have used the correct number of boundary conditions, (2) is strongly well-posed.

We state this result in the following proposition.
Proposition 1. The problem (2) is strongly well-posed if

\[ L^s_w = A^+_s, \quad L^s_e = B^+_s, \quad L^s_s = A^-_s, \quad L^s_n = B^-_s \]

2.2. Stability and error estimates

The semi-discrete approximation of (6) using the SBP-SAT technique is

\[ v_t + P_x^{-1} Q_x \otimes I_y \otimes A_x v + I_x \otimes P_y^{-1} Q_y \otimes B_s v = \]

\[ (P_x^{-1} E_0 \otimes P_y^{-1} \otimes I_4) \Sigma_w (\tilde{A}_s^+ v - g_w^{'}) + \]

\[ (P_x^{-1} E_N \otimes P_y^{-1} \otimes I_4) \Sigma_e (\tilde{A}_s^- v - g_e^{'}) + \]

\[ (P_x^{-1} \otimes P_y^{-1} E_0 \otimes I_4) \Sigma_s (\tilde{B}_s^+ v - g_s^{'}) + \]

\[ (P_x^{-1} \otimes P_y^{-1} E_N \otimes I_4) \Sigma_n (\tilde{B}_s^- v - g_n^{'}) \]

\tag{9}

where \( P_{x,y} \) are symmetric and positive definite and \( Q_{x,y} \) satisfy the SBP-property, \( Q_{x,y} + Q_{x,y}^T = B_{x,y} = -E_0 + E_N = diag(-1,0,...,0,1) \). In (9), \( \tilde{A}_S^\pm = I_x \otimes I_y \otimes A_S^\pm \) and \( \tilde{B}_S^\pm = I_x \otimes I_y \otimes B_S^\pm \). The matrices \( E_{0,N} \) are projection matrices, such that \( (E_0)_{11} = (E_N)_{NN} = 1 \) and zero otherwise. The penalty matrices \( \Sigma_{w,e,s,n} \) will be determined such that stability is obtained and \( g_{w,e,s,n}' \) are the boundary data. By multiplying (9) with \( v^T P_x \otimes P_y \otimes I_4 \) from the left, one obtains

\[ (||v||^2_{P_x \otimes P_y \otimes I_4})_t + v^T (B_x \otimes P_y \otimes A_s) v + v^T (P_x \otimes B_y \otimes B_s) v = \]

\[ 2\alpha_1 v^T (E_0 \otimes P_y \otimes I_4) (\tilde{A}_s^+ v - g_w^{'}) + \]

\[ 2\alpha_2 v^T (E_N \otimes P_y \otimes I_4) (\tilde{A}_s^- v - g_e^{'}) + \]

\[ 2\alpha_3 v^T (P_x \otimes E_0 \otimes I_4) (\tilde{B}_s^+ v - g_s^{'}) + \]

\[ 2\alpha_4 v^T (P_x \otimes E_N \otimes I_4) (\tilde{B}_s^- v - g_n^{'}) \]

\tag{10}

where we have used \( \Sigma_{w,e} = \alpha_{1,2} I_x \otimes P_y \otimes I_4, \Sigma_{s,n} = \alpha_{3,4} P_x \otimes I_y \otimes I_4 \). The constants \( \alpha_{1,2,3,4} \) are to be determined such that stability is obtained. In (10), we have used the SBP-property of \( Q_{x,y} \): \( Q_{x,y} + Q_{x,y}^T = B_{x,y} = -E_0 + E_N \). By
putting $\alpha_{1,3} = -1$ and $\alpha_{2,4} = 1$, (10) becomes

$$
(||v||_{P_x \otimes P_y \otimes I_4})^2_t = -v^T(E_0 \otimes P_y \otimes I_4)\tilde{A}^+_s v
+ 2v^T(E_0 \otimes P_y \otimes I_4)g'_w + v^T(E_0 \otimes P_y \otimes A^+_s)v
+ v^T(E_N \otimes P_y \otimes I_4)\tilde{A}^-_s v
- 2v^T(E_N \otimes P_y \otimes I_4)g'_w - v^T(E_N \otimes P_y \otimes A^-_s)v
-v^T(P_x \otimes E_0 \otimes I_4)\tilde{B}^+_s v
+ 2v^T(P_x \otimes E_0 \otimes I_4)g'_s + v^T(P_x \otimes E_0 \otimes B^-_s)v
+ v^T(P_x \otimes E_N \otimes I_4)\tilde{B}^-_s v
- 2v^T(P_x \otimes E_0 \otimes I_4)g'_n - v^T(P_x \otimes E_N \otimes B^+_s)v.
\tag{11}
$$

By completing squares, (11) finally becomes

$$
(||v||_{P_x \otimes P_y \otimes I_4})^2_t = -(\tilde{A}^+_s v - g'_w)^T(E_0 \otimes P_y \otimes (A^-_s)^+)(\tilde{A}^+_s v - g'_w)
+ g'_w^T(E_0 \otimes P_y \otimes (A^-_s)^+)g'_w + v^T(E_0 \otimes P_y \otimes A^-_s)v
+ (\tilde{A}^-_s v - g'_w)^T(E_N \otimes P_y \otimes (A^-_s)^-)(\tilde{A}^-_s v - g'_w)
- g'_w^T(E_N \otimes P_y \otimes (A^-_s)^-)g'_w - v^T(E_N \otimes P_y \otimes A^+_s)v
-(\tilde{B}^+_s v - g'_s)^T(P_x \otimes E_0 \otimes (B^-_s)^+)(\tilde{B}^+_s v - g'_s)
+ g'_s^T(P_x \otimes E_0 \otimes (B^-_s)^+)g'_s + v^T(P_x \otimes E_0 \otimes B^-_s)v
-(\tilde{B}^-_s v - g'_n)^T(P_x \otimes E_N \otimes (B^-_s)^-)(\tilde{B}^-_s v - g'_n)
-g'_n^T(P_x \otimes E_N \otimes (B^-_s)^-)g'_n - v^T(P_x \otimes E_N \otimes B^+_s)v.
\tag{12}
$$

where $(A^-_s)^\pm$ and $(B^-_s)^\pm$ refers to the negative and positive parts of the inverses of $A_s$ and $B_s$, respectively. Since (12) is bounded by the data on the right-hand side, the semi-discrete formulation (9) is stable. Note that the terms $g^T_{w,e}(E_0 \otimes P_y \otimes (A^-_s)^\pm)g^T_{w,e}$ and $g^T_{s,n}(P_x \otimes (A^-_s)^\pm)g^T_{y,w}$ mimics the integrals in (8) containing the boundary data and the terms $v^T(E_0 \otimes P_y \otimes A^+_s)v$ and $v^T(P_x \otimes E_0 \otimes B^+_s)v$ mimics the remaining integrals. Hence, (12) mimics the continuous energy estimate, given by (8), if the small damping terms are neglected.
2.3. Error estimates

By inserting the analytic solution, \( u \), into (9) and subtracting it from (9) with the numerical solution \( v \), one obtains

\[
e_t + P_x^{-1}Q_x \otimes I_y \otimes A_x e + I_x \otimes P_y^{-1}Q_y \otimes B_x e =
(P_x^{-1}E_0 \otimes P_y^{-1} \otimes I_4) \Sigma_w A^+_s e +
(P_x^{-1}E_N \otimes P_y^{-1} \otimes I_4) \Sigma_e A^-_s e +
(P_x^{-1}E_0 \otimes I_4) \Sigma_n B^+_s e +
(P_x^{-1}P_y^{-1}E_N \otimes I_4) \Sigma_n B^-_s e + Te \tag{13}
\]

where \( e = v - u \) and \( Te \) is the truncation error.

By multiplying (13) with \( e^T(P_x \otimes P_y \otimes I_4) \) from the left, using the SBP-property of \( Q_{x,y} \) and the same penalty matrices as in (10), one obtains

\[
(|e|^2)_t \leq -e^T(E_0 \otimes P_y \otimes A^+_s)e + e^T(E_N \otimes P_y \otimes A^-_s)e
-e^T(P_x \otimes E_0 \otimes B^+_s)e + e^T(P_x \otimes E_N \otimes B^-_s)e + 2e^T(P_x \otimes P_y \otimes I_4)Te. \tag{14}
\]

In [20], it was shown that equation (14) can be rewritten as

\[
||e||_t \leq \frac{e^T(Pen)e}{2||e||^2} ||e|| + ||Te|| \tag{15}
\]

where the matrix \( Pen = Pen^T \leq 0 \) denotes the sum of all penalty matrices and boundary terms in (14). By assuming that \( \eta = \frac{e^T(Pen)e}{2||e||^2} < \eta_0 < 0 \) for some constant \( \eta_0 \), (15) leads to

\[
||e|| \leq e^{\eta_0 t} \int_0^t e^{-\eta_0 \tau} ||Te|| d\tau \leq \frac{||Te||_{max}}{\eta_0}. \tag{16}
\]

The error \( ||e|| \) is thus bounded by the truncation error according to (16), and the error (13) is therefore bounded in time. It can be shown, see [20], that even the case where \( \eta \) is occasionally zero leads to an error bound.

We conclude these results in the following proposition

**Proposition 2.** The semi-discrete formulation (9) with the boundary operators given by (5) is error bounded in time if

\[
\Sigma_{e,w} = \alpha_{1,2}I_x \otimes P_y \otimes I_4, \quad \Sigma_{s,n} = \alpha_{3,2}P_x \otimes I_y \otimes I_4
\]

where \( \alpha_{1,3} = -1 \) and \( \alpha_{2,4} = 1 \).
3. The multiple penalty technique

Next, we move on to the MPT. We consider a local domain where the boundary data has been incorporated from global results. The MPT is applied at several grid points close to the boundaries where the applied data has been interpolated from the global mesh. This is similar to the method introduced Davies [1]. Here, the data is applied weakly with the SAT technique such that stability is preserved. This technique has been studied earlier in [18]. The technique will be demonstrated close to the eastern boundary; the analysis is similar at the other boundaries, and therefore neglected. Equation (13) with the MPT on the eastern boundary is

$$e_t + (P^{-1}_x Q_x \otimes I_y \otimes A_x) e + (I_x \otimes P^{-1}_y Q_y \otimes B_x) e = (Pen) e + P^{-1}_x \otimes P^{-1}_y \otimes I_4 \sum_{i=1}^{N_p} \phi_{N-i}((E_{N-i} \otimes \Sigma_{\epsilon}^{N-i} \otimes L^*) e$$

(17)

where $\phi_{N-i} > 0$ are arbitrary chosen coefficients, $Pen$ is the standard penalty matrices and $E_{N-i}$ a projection matrix with the entry $N - i$, $N - i$ equal to one and zero otherwise and $\Sigma_{\epsilon}^{N-i}$ are the additional penalty matrices, where the superscript indicate at which grid points the penalties are applied. Note that the truncation error is neglected in (17).

By multiplying (17) with $e^T (P_x \otimes P_y \otimes I_4)$, one arrives at

$$||e||^2 \leq \sum_i e^T \phi_i (P_x \otimes E_{N-i} \otimes (\Sigma_{\epsilon}^{N-i} + (\Sigma^{N-i})^T) \otimes L^*) e$$

(18)

since the rest of the terms only contributes with a decay of energy, as concluded in the previous section. According to (5), the boundary operator $L^*$ is symmetric and negative semi-definite, so the norm in (18) will remain bounded if the penalty matrices have a positive semi-definite symmetric part, $\Sigma_{\epsilon}^{N-i} + (\Sigma^{N-i})^T \geq 0$. The general form of the stability condition is

$$(\Sigma_{\epsilon,w,s,n}^{N-i} + (\Sigma^{N-i})^T_{e,w,s,n}) \otimes L^*_{e,w,s,n} \leq 0.$$

We summarize the results in the following proposition

**Proposition 3.** The MPT applied close to the boundaries as in (17) with penalty matrices $\Sigma_{\epsilon,w,s,n}^{N-i}$ preserves stability if

$$(\Sigma_{\epsilon,w,s,n}^{N-i} + (\Sigma^{N-i})^T_{e,w,s,n}) \otimes L^*_{e,w,s,n} \leq 0.$$
4. Numerical results for the steady-state problem

In our test-cases we have chosen the reference state \( u_\infty = v_\infty = 1, \bar{c} = 2 \), \( \rho_\infty = 1 \) and introduced a disturbance in form of a Gaussian pressure pulse centered in the middle of the local domain, that is

\[
p'(x, y, 0) = e^{-100(x-1/2)^2-100(y-1/2)^2},
\]

see Figure 2 for an illustration. This local domain is contained in a coarse, global mesh, as illustrated in Figure 1. Since the pulse illustrate a disturbance or an error, we want it to disappear as fast as possible.

The multiple penalties are applied according to the procedure described in section 3, such that stability is preserved. A square, uniform mesh is used and the number of multiple penalties close to each boundary is \( N_{MP} = rN \), where \( r \) is a number between zero and one and \( N \) is the number of grid points. The solution, \( u \), is assumed to be known in these domains. For the numerical approximation, a SBP scheme with third order overall accuracy is used and a fourth order Runge-Kutta method is used to integrate in time.

Figure 2 shows the pressure in the local domain at \( T = 0 \), which is advected out of the domain when integrated in time. The pulse will eventually be reflected at the boundaries, and parts of the error remain in the domain of interest.

Figure 3 shows the error at \( T = 2.1 \) when standard penalties are used, and Figure 4 how the amplitude of the error is reduced by using multiple penalties. In Figures 5 and 6, we observe how the error decreases as one integrate in time. This is clarified by displaying the time-dependence of the error in Figures 7-9. The increased rate of convergence to steady-state when applying multiple penalties is consistent with previous work in [18].

4.1. Time-dependent solutions

We will in this section demonstrate how the numerical errors are reduced also for time-dependent boundary data when multiple penalties are applied.
Figure 2: Initial condition in the local domain for the linearized pressure variable inside the local domain. In our example, we consider a uniform mesh with $N = 40$.

Figure 3: The error of the linearized pressure variable at $T = 2.1$ when standard penalties are used. $N = 20$
Figure 4: The error for the linearized pressure variable at $T = 2.1$ when multiple penalties $(r = 0.2)$ are used. $N = 40$

Figure 5: The error of the linearized pressure variable at $T = 8.1$ when standard penalties are used. Note that the z-axis has been re-scaled when comparing with Figure 3. $N = 20$
Figure 6: The error for the linearized pressure variable at $T = 8.1$ when multiple penalties ($r = 0.2$) are used. Note that the z-axis has been re-scaled when comparing with Figure 4. $N = 40$

Figure 7: Error for the quantity $p'$ as a function of time. The number of grid points is $N = 20$ in each space direction. $r = N_{MP}/N$
Figure 8: Error for the quantity $p'$ as a function of time. The number of grid points is $N = 40$ in each space direction. $r = N_{MP}/N$

Figure 9: Error for the quantity $p'$ as a function of time. The number of grid points is $N = 80$ in each space direction. $r = N_{MP}/N$
We consider a slightly modified version of (2)

\[ \begin{align*}
    w_t' + A_s(U_0)w_x' + B_s(U_0)w_y' &= F(x, y, t) \\
    L_w^* w'(0, y, t) &= g_w'(y, t) \\
    L_w^* w'(x, 0, t) &= g_w'(x, t) \\
    L_w^* w'(1, y, t) &= g_w'(y, t) \\
    L_n^* w'(x, 1, t) &= g_n(x, t) \\
    w'(x, y, 0) &= f'(x, y)
  \end{align*} \]  

(19)

where \( F(x, y, t) \) is a known forcing function. We create a manufactured analytic solution given by

\[ u(x, y, t) = \exp(-10r^2)\sin(2\pi(r^2 - t)) \]

(20)

by choosing \( F = u_t + A_s u_x + B_s u_y \) and the boundary data accordingly. In (20), \( r^2 = (x - 0.5)^2 + (y - 0.5)^2 \) and the function describes an oscillating pulse centered at \((0.5, 0.5)\), as illustrated in Figure 10. We then start from the initial condition \( f'(x, y) = 0 \) and measure how fast the numerical solution converges to (20). In Figure 11-13, the difference between the numerical solution and the manufactured one is measured as a function of time. Since a (20) cannot, unlike the trivial solution in the previous section, be approximated exactly by the SBP-operators, a small error will always remain. However, by adding multiple penalties, the error will drop to the minimal level faster, as illustrated in Figure 11-13.

5. Conclusion

Using the SBP-SAT technique, a lateral boundary procedure similar to the one introduced by H.C Davies [1] has been applied to the linearized Euler equations in two space dimensions. The technique is straightforward to implement and stability is easily shown by energy methods.

These additional penalty terms contributes with a decay of energy and increases the rate of convergence to steady-state. The lateral boundaries also works as buffer zones, where spurious reflections are reduced.
Figure 10: The time-dependent solution of $p'$ at $T = 1.7$. $N = 20$ in each space-direction.

Figure 11: Error for the quantity $p'$ as a function of time for time-dependent boundary data. $N = 20$ and $r = N_{MP}/N$. 

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Figure 12: Error for the quantity \( p' \) as a function of time for time-dependent boundary data. \( N = 40 \) and \( r = \frac{N_{MP}}{N} \).

Figure 13: Error for the quantity \( p' \) as a function of time for time-dependent boundary data. \( N = 80 \) and \( r = \frac{N_{MP}}{N} \).
The effect is in our examples large; the errors can in some cases be reduced by several orders of magnitude, both for the steady-state problem and with time-dependent boundary data.

In the cases studied in this work, all data from the global mesh is considered given with arbitrary accuracy, and the domains are assumed to be uncoupled.

References


