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Abstract

We make an initial investigation into the numerical efficiency of a fully discrete summation-by-parts approach for unsteady flows. As a model problem for the Navier-Stokes equations we consider a two-dimensional advection-diffusion problem with a boundary layer. The problem is discretized in space using finite difference approximations on summation-by-parts form together with weak boundary conditions, leading to optimal stability estimates. For the time integration part we consider various forms of high order summation-by-parts operators, and compare the results to other popular high order methods. To solve the resulting fully discrete equation system, we employ a multi-grid scheme with dual time stepping.

1. Introduction

Based on finite difference operators on summation-by-parts (SBP) form and the simultaneous-approximation-term (SAT) technique for boundary conditions, the SBP-SAT technique constitutes a robust framework for implementing high order finite difference schemes on complex geometries. By construction, it leads to discrete energy estimates that perfectly imitates the corresponding continuous estimates, and hence stability. This technique was recently extended to initial value problems [1, 2], now making it possible to formulate fully discrete SBP-SAT approximations with optimal energy estimates. The purpose of this work is to make an initial efficiency study of these schemes for a stiff model problem.

The numerical treatment of unsteady flow problems has gained increased attention in later years due to increased computer resources making realistic calculations more viable. However, the construction of efficient algorithms still remains a significant computational challenge. The two basic methods most commonly used employ Newton iteration and dual time stepping. While Newton iterations typically achieves better convergence rates for small error tolerances, dual time stepping is more reliable, at least for the initial iterations, and it can also be used for preconditioning purposes. Some studies indicate that a combination of both these techniques can be the most fruitful approach [3, 4, 5].

The organization of this report is as follows. In section 2 we review the basic properties of the SBP-SAT technique for time integration. In section 3 we analyze a linear model of the Navier-Stokes equations with a boundary layer. In section 4 a multi-grid dual time stepping scheme is formulated. Numerical results for the model problem is presented in section 5, and finally in section 6 we draw conclusions.

2. Summation-by-parts in time

We introduce the SBP-SAT technique for time integration by considering the test equation $u_t + \lambda u = 0$ with initial condition $u(0) = f$. Let $\vec{t} = (0, \Delta t, \dots, N\Delta t = T)$ be a uniform grid vector with $N + 1$ grid points, and let $(\vec{e}_j)_{j=0}^N$ denote the unit vectors in the standard basis. An SBP-SAT approximation to the test problem can then be written

$$P^{-1}Q\vec{u} + \lambda\vec{u} = P^{-1}\sigma(u_0 - f)\vec{e}_0. \quad (1)$$

The SAT penalty treatment on the right hand side of (1) forces the initial solution to the initial data, and the SBP properties of the first derivative operator $P^{-1}Q$ are

$$Q + Q^T = \vec{e}_N \vec{e}_N^T - \vec{e}_0 \vec{e}_0^T = \text{Diag}(-1, 0, \dots, 0, 1), \quad P = P^T > 0.$$

These properties lead in an automatic way to a clean, optimally sharp energy estimate. Especially, for $\sigma = -1$ we get after multiplying (1) with $\vec{u}^* P$ and adding the conjugate transpose:

$$|u_N|^2 + 2\text{Re}(\lambda) \|\vec{u}\|_P^2 = |f|^2 - |u_0 - f|^2,$$

which mimics the continuous estimate $|u(T)|^2 + 2\text{Re}(\lambda) \int_0^T |u|^2 dt = |f|^2$.

As an alternative to the global formulation (1), we may also consider a multi-stage version with $r + 1$ stages on a general grid vector $\vec{t} = (t^0, t^1, \dots, t^n, t^{n+1}, \dots)$:

$$(P^{-1}Q + \lambda I)\vec{u}^{n+1} = P^{-1}\sigma(u_0^{n+1} - u_r^n)\vec{e}_0, \quad (2)$$

where $\vec{u}^{n+1} = (u_0^{n+1}, u_1^{n+1}, \dots, u_r^{n+1})$ for $n = 0, 1, \dots$, and $P^{-1}Q$ is defined on the uniform subgrid $\vec{t}^{n+1} = (t^n, t^n + \Delta t/r, \dots, t^{n+1})$. Starting with $u_r^0 = f$, the scheme (2) then produces the sequence of solutions $u_r^0, u_r^1, \dots, u_r^n, u_r^{n+1}, \dots$ on $\vec{t}$.

The number $r + 1$ of intermediate stages clearly depends on the size of the matrix $P^{-1}Q$. The classical SBP operators are based on central finite difference stencils together with boundary closures. An example is the second order operator

$$P = \frac{\Delta t}{r} \begin{bmatrix} \frac{1}{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{bmatrix}, \quad Q = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Higher order operators have more extensive boundary closures, thus increasing the minimum number of stages required. Other operators on SBP form, e.g. based on Legendre spectral collocation [6] may alternatively be used to decrease the number of stages necessary. See also [7] for more information on possible ways to construct non-classical SBP operators.

The formal properties of the schemes obtained with the SBP-SAT technique are summarized below.

- The SBP-SAT schemes (2) are always A-stable and L-stable. If P is diagonal they are also B-stable and preserve energy stability. Moreover, they lead to optimally sharp fully discrete energy estimates.
- The order of convergence is given by the order of accuracy of the quadrature P . For classical SBP operators this coincides with the order of the interior scheme.
- The order of stiff convergence is given by the local order of consistency of the operator $P^{-1}Q$, the so-called stage order. Classical operators are thus limited by the accuracy of the boundary closures.

In this work we consider two types of SBP operators. First the classical operators with diagonal norms, denoted by $\text{SBP}(2s, s)$, and secondly the spectral element operators based on Gauss-Lobatto quadrature, denoted by $\text{GL}(2s, s)$. In both cases, $2s$ is the order of the quadrature P , and thus of the scheme itself, while s is the stage order.

3. A stiff flow model in two dimensions

As a model of the Navier-Stokes equation, we study a fluid undergoing advective flow past a plate with fixed value:

$$\begin{aligned}
u_t + u_x &= \varepsilon(u_{xx} + u_{yy}) + \psi & 0 \leq x, y \leq 1 & \quad t \geq 0 \\
u(0, x, y) &= f(x, y) & t &= 0 \\
u(t, 0, y) - \varepsilon u_x(t, 0, y) &= g_1(t, y) & \partial\Omega_1 &= \{(x, y) : x = 0\} \\
\varepsilon u(t, 1, y) &= g_2(t, y) & \partial\Omega_2 &= \{(x, y) : x = 1\} \\
u(t, x, 0) &= 0 & \partial\Omega_3 &= \{(x, y) : y = 0\} \\
u_y(t, x, 1) &= 0 & \partial\Omega_4 &= \{(x, y) : y = 1\},
\end{aligned} \tag{3}$$

where $\varepsilon = 0.01$. The solid boundary $\partial\Omega_3$ is associated with a boundary layer of width $\sqrt{\varepsilon}$ at $y = 0$, the inflow and outflow boundaries are $\partial\Omega_1$ and $\partial\Omega_2$ respectively, and $\partial\Omega_4$ is a far-field boundary. An exact manufactured solution can be imposed by appropriately specifying the forcing function ψ .

The energy method (multiplying the equation with u^* and integrating over the physical domain) applied to (3) yields the estimate

$$\begin{aligned}
\|u\|_t^2 + 2\varepsilon(\|u_x\|^2 + \|u_y\|^2) &= \int_{\partial\Omega_1} (g_1^2 - (u - g_1)^2) dS \\
&+ \int_{\partial\Omega_2} (g_2^2 - (u - g_2)^2) dS,
\end{aligned} \tag{4}$$

which shows that the problem (3) is well-posed.

In order to properly resolve the boundary layer at $y = 0$ in (3), we introduce a stretching function η of the vertical coordinate, given by

$$y = 1 + \frac{\tanh(B(\eta - 1))}{\tanh B},$$

where $B = 9/4$, leading to $\eta_y(0) = 1/\sqrt{\varepsilon}$, and the full stretching function is shown in Figure 1. See also Appendix A for a detailed discussion of this choice. After this change of coordinate, the model problem (3) becomes

Figure 1: Stretching of vertical coordinate to resolve the boundary layer.

$$\begin{aligned}
u_t + u_x &= \varepsilon(u_{xx} + \eta_y(\eta_y u_\eta)_\eta) + \psi & 0 \leq x, \eta \leq 1 \quad t \geq 0 \\
u(0, x, \eta) &= f(x, \eta) & t = 0 \\
u(t, 0, \eta) - \varepsilon u_x(t, 0, \eta) &= g_1(t, \eta) & \partial\Omega_1 = \{(x, \eta) : x = 0\} \\
\varepsilon u(t, 1, \eta) &= g_2(t, \eta) & \partial\Omega_2 = \{(x, \eta) : x = 1\} \\
u(t, x, 0) &= 0 & \partial\Omega_3 = \{(x, \eta) : \eta = 0\} \\
\eta_y u_\eta(t, x, 1) &= 0 & \partial\Omega_4 = \{(x, \eta) : \eta = 1\}.
\end{aligned} \tag{5}$$

3.1. Semi-discrete approximation

We follow the techniques presented in [8, 9] to discretize (5) in space. The solution vector U is organized in lexicographical order with respect to the indices of the grid vectors $\vec{x}$ and $\vec{\eta}$, and we let $(e_{xi})_{i=0}^{N_x}$ and $(\vec{e}_{\eta j})_{j=0}^{N_\eta}$ denote the standard unit vectors in the spaces corresponding to $\vec{x}$ and $\vec{\eta}$. We also define the solution at fixed coordinates as $U|_{x=x_i} = (\vec{e}_{xi}^T \otimes I_\eta)U$ and $U|_{\eta=\eta_j} = (I_x \otimes \vec{e}_{\eta j}^T)U$. Finally, we let $D_x = P_x^{-1}Q_x$ and $D_\eta = P_\eta^{-1}Q_\eta$ be SBP operators approximating the spatial derivatives, and define $U_x = (D_x \otimes I_\eta)U$, $U_\eta = (I_x \otimes D_\eta)U$ and $U_y = (I_x \otimes H_y D_\eta)U$, where $H_y = \text{Diag}(\vec{\eta}_y)$. The matrices P_x and P_η are both assumed to be diagonal. The SBP-SAT approximation

of (5) can then be written as

$$\begin{aligned}
U_t + (D_x \otimes I_\eta)U &= \epsilon((D_x^2 \otimes I_\eta)U + (I_x \otimes (H_y D_\eta)^2)U) \\
&\quad + (P_x^{-1} \otimes P_\eta^{-1} H_y)(\Sigma_x(t) + \Sigma_\eta(t)) + \Psi(t) \\
U(0) &= F,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\Psi(t) &= \psi(t, (\vec{x} \otimes \vec{1}_\eta), (\vec{1}_x \otimes \vec{\eta})) \\
\Sigma_x(t) &= \sigma_{0x}(\vec{e}_{x0} \otimes H_y^{-1} P_\eta(U|_{x=0} - \epsilon U_x|_{x=0} - \vec{g}_1(t))) \\
&\quad + \sigma_{1x}(\vec{e}_{xN_x} \otimes H_y^{-1} P_\eta(\epsilon U_x|_{x=1} - \vec{g}_2(t))) \\
\Sigma_\eta(t) &= \sigma_{0\eta}(P_x U|_{\eta=0} \otimes \vec{e}_{\eta 0}) + \sigma_{1\eta}(P_x U|_{\eta=1} \otimes \vec{e}_{\eta N_\eta}) \\
\vec{g}_1(t) &= g_1(t, (\vec{e}_{x0} \otimes \vec{\eta})) \\
\vec{g}_2(t) &= g_2(t, (\vec{e}_{xN_x} \otimes \vec{\eta})) \\
F &= f((\vec{x} \otimes \vec{1}_\eta), (\vec{1}_x \otimes \vec{\eta})).
\end{aligned}$$

The vectors $\vec{1}_x$ and $\vec{1}_\eta$, of length $N_x + 1$ and $N_\eta + 1$ respectively, have the value 1 on each position. We now apply the discrete energy method by multiplying (6) with $U^*(P_x \otimes H_y^{-1} P_\eta)$ and then adding the conjugate transpose.

A stable scheme is obtained with the following choice of penalty parameters: $\sigma_{0x} = \sigma_{1x} = -1$, $\sigma_{1\eta} = -1/2$ and $\sigma_{0\eta} = -\varepsilon \eta_y(0)^2 / P_{\eta 00}$. Letting $\Psi(t) = 0$, this leads to the estimate

$$\begin{aligned}
&(\|U\|_{P_x \otimes (H_y^{-1} P_\eta)}^2)_t + 2\varepsilon(\|U_x\|_{P_x \otimes (H_y^{-1} P_\eta)}^2 + \|U_y\|_{P_x \otimes (H_y^{-1} P_\eta)}^2) \\
&\quad + 2\varepsilon \begin{pmatrix} U|_{\eta=0} \\ U_y|_{\eta=0} \end{pmatrix}^T (\mathcal{F} \otimes P_x) \begin{pmatrix} U|_{\eta=0} \\ U_y|_{\eta=0} \end{pmatrix} \\
&= \|\vec{g}_1(t)\|_{H_y^{-1} P_\eta}^2 - \|\vec{g}(t)_1 - U|_{x=0}\|_{H_y^{-1} P_\eta}^2 \\
&\quad + \|\vec{g}_2(t)\|_{H_y^{-1} P_\eta}^2 - \|\vec{g}(t)_2 - U|_{x=1}\|_{H_y^{-1} P_\eta}^2,
\end{aligned} \tag{7}$$

where

$$\mathcal{P}_\eta = P_\eta - P_{\eta 00} e_{0\eta} e_{0\eta}^T, \quad \mathcal{F} = \begin{pmatrix} \frac{\eta_y(0)}{P_{\eta 00}} & \frac{1}{2} \\ \frac{1}{2} & \frac{P_{\eta 00}}{\eta_y(0)} \end{pmatrix}.$$

We see that this estimate closely mimics the corresponding continuous estimate (4). Note that the value $P_{\eta 00}$ is “borrowed” from P_η into $\mathcal{F}$ in order to make the latter a positive definite matrix. This trick for imposing a no-slip wall boundary conditions was first described in [9].

3.2. Fully discrete approximation with SBP-SAT in time

Consider the semi-discrete energy estimate (7) again. Integrating over a time interval $t \in [a, b]$ yields

$$\begin{aligned}
& \|U(b)\|_{P_x \otimes (H_y^{-1} P_\eta)}^2 + 2\varepsilon \int_a^b (\|U_x\|_{P_x \otimes (H_y^{-1} P_\eta)}^2 + \|U_y\|_{P_x \otimes (H_y^{-1} P_\eta)}^2) dt \\
& + 2\varepsilon \int_a^b \begin{pmatrix} U|_{\eta=0} \\ U_y|_{\eta=0} \end{pmatrix}^T (\mathcal{F} \otimes P_x) \begin{pmatrix} U|_{\eta=0} \\ U_y|_{\eta=0} \end{pmatrix} dt \\
& = \int_a^b (\|\vec{g}_1(t)\|_{H_y^{-1} P_\eta}^2 - \|\vec{g}_1(t) - U|_{x=1}\|_{H_y^{-1} P_\eta}^2 \\
& \quad + \|\vec{g}_2(t)\|_{H_y^{-1} P_\eta}^2 - \|\vec{g}_2(t) - U|_{x=1}\|_{H_y^{-1} P_\eta}^2) dt \\
& + \|U(a)\|_{P_x \otimes (H_y^{-1} P_\eta)}^2.
\end{aligned} \tag{8}$$

This form of the estimate is what we want to preserve for the fully discrete solution.

We now consider the semi-discrete problem (6) written in a compact way as

$$\begin{aligned}
U_t + BU &= R(t) \quad 0 < t \leq T \\
U(0) &= F,
\end{aligned}$$

where

$$\begin{aligned}
B &= P_x^{-1} (Q_x - \sigma_{0x} \vec{e}_{x0} \vec{e}_{x0}^T - \epsilon (Q_x + \sigma_{0x} \vec{e}_{x0} \vec{e}_{x0}^T - \sigma_{1x} \vec{e}_{xN_x} \vec{e}_{xN_x}^T) D_x) \otimes I_\eta \\
&+ I_x \otimes P_\eta^{-1} (-\sigma_{0\eta} \vec{e}_{\eta 0} \vec{e}_{\eta 0}^T - \epsilon (H_y Q_\eta H_y + \sigma_{1\eta} \vec{e}_{\eta N_\eta} \vec{e}_{\eta N_\eta}^T H_y) D_\eta) \\
R(t) &= -\sigma_{0x} (P_x^{-1} \vec{e}_{x0} \otimes \vec{g}_1(t)) - \sigma_{1x} (P_x^{-1} \vec{e}_{xN_x} \otimes \vec{g}_2(t)) + \Psi(t).
\end{aligned}$$

The semi-discrete spectrum of a fifth order discretization (i.e. SBP(8,4) in space) with $N_x = N_\eta = 95$ is shown in Figure 2. The spectral radius of almost 10^5 indicates that an explicit time marching scheme would be an inefficient way to solve this problem. Instead we employ an implicit SBP-SAT time integration scheme with $r + 1$ stages:

$$(P^{-1}Q \otimes I_B) \vec{U}^{n+1} + (I_t \otimes B) \vec{U}^{n+1} = (P^{-1}\sigma \vec{e}_0) \otimes (U_0^{n+1} - U_r^n) + \vec{R}^{n+1} \tag{9}$$

where

$$\begin{aligned}
\vec{U}^{n+1} &= (U_0^{n+1}, U_1^{n+1}, \dots, U_r^{n+1}) \\
\vec{R}^{n+1} &= (R(t^n), R(t^n + \Delta t/r), \dots, R(t^n + \Delta t = t^{n+1})),
\end{aligned}$$

Figure 2: Left: spectrum of the semi-discrete spectrum. Right: The computational grid with stretched vertical coordinate.

and as before we let $\sigma = -1$. We can now employ a fully discrete energy method by multiplying (9) with $(\vec{U}^{n+1})^*(P \otimes P_x \otimes H_y^{-1}P_\eta)$, and then adding the conjugate transpose, leading to the estimate

$$\begin{aligned}
& \|\vec{U}_r^{n+1}\|_{P_x \otimes (H_y^{-1}P_\eta)}^2 + 2\varepsilon(\|\vec{U}_x^{n+1}\|_{P \otimes P_x \otimes (H_y^{-1}P_\eta)}^2 + \|\vec{U}_y^{n+1}\|_{P \otimes P_x \otimes (H_y^{-1}P_\eta)}^2) \\
& + 2\varepsilon \begin{pmatrix} \vec{U}^{n+1}|_{\eta=0} \\ \vec{U}_y^{n+1}|_{\eta=0} \end{pmatrix}^T (P \otimes \mathcal{F} \otimes P_x) \begin{pmatrix} \vec{U}^{n+1}|_{\eta=0} \\ \vec{U}_y^{n+1}|_{\eta=0} \end{pmatrix} \\
& = \|\vec{G}_1^{n+1}\|_{P \otimes (H_y^{-1}P_\eta)}^2 - \|\vec{G}_1^{n+1} - \vec{U}^{n+1}|_{x=1}\|_{P \otimes (H_y^{-1}P_\eta)}^2 \\
& + \|\vec{G}_2^{n+1}\|_{P \otimes (H_y^{-1}P_\eta)}^2 - \|\vec{G}_2^{n+1} - \vec{U}^{n+1}|_{x=1}\|_{P \otimes (H_y^{-1}P_\eta)}^2 \\
& + \|U_r^n\|_{P_x \otimes (H_y^{-1}P_\eta)}^2 - \|U_0^{n+1} - U_r^n\|_{P_x \otimes (H_y^{-1}P_\eta)}^2,
\end{aligned}$$

where

$$\begin{aligned}
\vec{G}_1^{n+1} &= (\vec{g}_1(t^n), \vec{g}_1(t^n + \Delta t/r), \dots, \vec{g}_1(t^{n+1})) \\
\vec{G}_2^{n+1} &= (\vec{g}_2(t^n), \vec{g}_2(t^n + \Delta t/r), \dots, \vec{g}_2(t^{n+1})).
\end{aligned}$$

This perfectly imitates the semi-discrete estimate (8), only adding a small additional damping term. It of course also mimics the continuous estimate (4).

4. The dual time stepping scheme

Consider a compact formulation of (9):

$$\tilde{B}\vec{U}^{n+1} = \tilde{R} \quad (10)$$

where

$$\begin{aligned} \tilde{B} &= P^{-1}(Q - \sigma e_0 e_0^T) \otimes I_B + I_t \otimes B \\ \tilde{R} &= -(P^{-1}\sigma \vec{e}_0) \otimes U^n + \vec{R}^{n+1}. \end{aligned}$$

We employ a multi-grid cycle for solving (10), where the smoothing step consists of stepping forward in pseudo-time toward steady-state. Thus, we add a pseudo time derivative to (10):

$$\frac{d\vec{U}^{n+1}}{d\tau} + \tilde{B}\vec{U}^{n+1} = \tilde{R}. \quad (11)$$

The solution to (10) is now given by the steady-state solution to (11). To march forward in pseudo-time, we use an explicit s -stage low storage Runge-Kutta smoother:

$$\begin{aligned} \vec{W}_0^{n+1,m+1} &= \vec{V}^{n+1,m} \\ \vec{W}_p^{n+1,m+1} &= \vec{V}^{n+1,m} + \Delta\tau\alpha_p(\tilde{R} - \tilde{B}\vec{W}_{p-1}^{n+1,m+1}), \quad p = 1, \dots, s \\ \vec{V}_s^{n+1,m+1} &= \vec{W}_s^{n+1,m+1} \end{aligned} \quad (12)$$

The stability function of this scheme is given by $S(z) = (1 + \alpha_s z(1 + \alpha_{s-1} z(\dots(1 + \alpha_1 z)\dots)))$. To match the semi-discrete spectrum in Figure 2, we use the 4-stage smoother $\alpha = (0.0178571, 0.0568106, 0.174513, 1)$ proposed for viscous problems in [10]. The stability region is shown in Figure 4.

Let $\vec{V}^{n+1}$ be an approximation to the fully discrete solution $\vec{U}^{n+1}$. Taking M steps in pseudo-time with the explicit scheme (12) to update $\vec{V}^{n+1}$ is defined by the function call $\vec{V}^{n+1} = S^M(\vec{V}^{n+1}, \tilde{B}, \tilde{R})$. A full multi-grid cycle for (10) with L levels of grid coarsening can now be written as the fixed point iteration scheme $\vec{V}^{n+1} = \text{MG}(\vec{V}^{n+1}, \tilde{R}, L)$, where the multi-grid step function $\vec{x} = \text{MG}(\vec{x}, \vec{b}, l)$ is defined recursively as:

- $\vec{x} = S^\nu(\vec{x}, \tilde{B}_l, \vec{b})$ (presmoothing)
- if $l > 0$, do:

Figure 3: Left: the semi-discrete spectrum of (6). Right: Stability region of Runge-Kutta smoother.

- $\vec{r} = \text{Restrict}(\vec{v} - \tilde{B}_l \vec{x})$ (restriction)
- $\vec{e} = \text{MG}(0, \vec{r}, l - 1)$ (coarse grid correction)
- $\vec{e} = \text{Prolong}(\vec{e})$ (prolongation)
- $\vec{x} = \vec{x} + \vec{e}$ (correction)
- $\vec{x} = S^\nu(\vec{x}, \tilde{B}_l, \vec{b})$. (postsmoothing)

5. Numerical results

We employ the manufactured solution $u = \sin(2\pi(x - t))e^{\frac{1-y}{\sqrt{\epsilon}}}$ to (3), and compare the numerical results for a selection of high order temporal schemes. The SBP operators used are the following: SBP(4,2) with 8 implicit stages, SBP(8,4) with 16 implicit stages, GL(4,2) with 3 implicit stages, and GL(8,4) with 5 implicit stages. For the classical operators we use the minimum number of stages possible, i.e. only the boundary closures are included. For comparison we add the fourth order diagonally implicit Runge-Kutta scheme ESDIRK4, with a stage order of 2, and 5 implicit stages.

In order to minimize the spatial error component we use the fifth order discretization with $N_x = N_\eta = 95$ with spectrum shown in Figure 2. Note that the spectral radius of almost 10^5 is a result both of the boundary layer and the overresolution in space. The time integration is carried out using a multi-grid V-cycle on three grid levels, with refinement in the vertical

Figure 4: Accuracy of the high order temporal schemes.

Figure 5: The amount of work required to resolve each implicit stage.

coordinate only. On each grid, 10 steps of the explicit Runge-Kutta scheme is used as smoother, and the pseudo-time step size is chosen to make the Runge-Kutta scheme stable on each respective grid. The number of pseudo-time iterations is set to make the iteration error small compared with the error from the physical time integration.

Figure 6: Efficiency of the high order temporal schemes.

Figures 4 through 6 show the numerical results for a selection of temporal schemes. The accuracy is measured at $t = 1$, and work is defined as the total number of multi-grid cycles summed over all implicit stages required to solve the problem up to that point in time. In all cases we observe some level of order reduction, with convergence rates somewhere between the stage order and the order of the scheme.

6. Conclusions and further work

We have investigated the efficiency of fully discrete SBP-SAT discretizations for unsteady flow calculations. A stiff linear model problem was considered, and a basic dual time-stepping scheme was employed, with no attempt made at optimizing the smoother. The numerical results indicate that SBP-SAT time stepping schemes can compete with ESDIRK4 for efficiency. Also, no advantage could be seen for schemes of higher than fourth order.

Future work will aim at developing more efficient multi-grid schemes, including optimization of the Runge Kutta smoother used for pseudo-time stepping. Non-linear model problems will also be considered, as well as combining the dual time stepping technique with Newton iteration.

Appendix A. Boundary layer and coordinate stretching

The solid wall boundary at $y = 0$ in (3) causes the solution to change rapidly within a thin layer close to this boundary. To study this, we consider a simplified steady version of (3):

$$\begin{aligned} u_x &= \epsilon(u_{xx} + u_{yy}) \quad 0 < x, y < \infty \\ u(x, 0) &= 0 \\ u(0, y) &= 0. \end{aligned}$$

Note that this problem has a discontinuity at $x = y = 0$. In order to study what happens close to the solid boundary, we make the coordinate transformation $y = \sqrt{\epsilon}Y$, which gives

$$u_x = \epsilon u_{xx} + u_{YY}.$$

We now expand u with a regular perturbation in powers of ϵ as $u = u_0 + \sqrt{\epsilon}u_1 + \epsilon u_2 + \dots$. The equation then becomes

$$u_{0x} + \sqrt{\epsilon}u_{1x} + \epsilon u_{2x} + \dots = \epsilon u_{0xx} + u_{0YY} + \sqrt{\epsilon}u_{1YY} + \epsilon u_{2YY} + \dots$$

Collecting all terms of the same order and setting each to zero now yields the sequence of equations

$$\begin{aligned} u_{0x} &= u_{0YY} \\ u_{1x} &= u_{1YY} \\ u_{2x} &= u_{0xx} + u_{2YY} \\ &\dots \end{aligned}$$

In particular, the full equation for the leading zeroth order term in the expansion is

$$\begin{aligned} u_{0x} &= u_{0YY} \\ u_0(0, Y) &= 1 \\ u_0(x, 0) &= 0. \end{aligned}$$

This problem has the following analytical solution:

$$u_0 = \epsilon r f\left(\frac{Y}{\sqrt{4x}}\right) = \epsilon r f\left(\frac{y}{\sqrt{4\epsilon x}}\right)$$

where erf is the so-called error function, defined by:

$$\text{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-s^2} ds,$$

The error function becomes almost constant for values of $\xi > 2$, making u_0 approximately constant for $y > 4\sqrt{\frac{\epsilon}{x}}$. For a fixed value of x , the boundary layer width is thus proportional to $\sqrt{\epsilon}$, and the derivative of u_0 with respect to y is moreover proportional to $1/\sqrt{\epsilon}$.

In order to resolve the boundary layer at $y = 0$ in, we can introduce a stretching η of the vertical coordinate such that $\eta_y(0)$ is proportional $1/\sqrt{\epsilon}$. A possible choice with this property is

$$y = 1 + \frac{\tanh(B(\eta - 1))}{\tanh B},$$

where $B = 9/4$, leading to $\eta_y(0) = 1/\sqrt{\epsilon}$. The full stretching function is shown in Figure 1.

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