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A Century of Evolution from Harris's Basic Lot Size Model: Survey and Research Agenda

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Abstract: Determining the economic lot size has always represented one of the most important issues in production planning. This problem has long attracted the attention of researchers, and several models have been developed to meet requirements at minimum cost. In this paper we explore and discuss the evolution of these models during one hundred years of history, starting from the basic model developed by Harris in 1913, up to today. Following Harris's work, a number of researchers have devised extensions that incorporate additional considerations. The evolution of EOQ theory strongly reflects the development of industrial systems over the past century. Here we outline all the research areas faced in the past by conducting a holistic analysis of 219 selected journal papers and trying to give a comprehensive view of past work on the EOQ problem. Finally, a new research agenda is proposed and discussed.

Keywords: Economic lot sizing, EOQ, EPQ, theoretical framework, survey, research agenda

1. Introduction

The economic order quantity (EOQ) model is undoubtedly one of the oldest models in the inventory analysis literature. The first who tackled the problem of determining the economic lot size in production systems was Ford Whitman Harris, born on August 8, 1877, and who passed away on October 27, 1962. In February 1913 at the age of 35, he proposed his formulation of this problem under the assumption of a continuous constant rate for demand and his recognition of the need to balance intangible inventory costs against tangible costs for ordering. Harris's solution has become the well-known "Square root formula". Even though its wide circulation, Harris's original paper was apparently unnoticed before its rediscovery in 1988 (Erlenkotter, 1989, 1990). In the first decades of the last century a large number of researchers formulated their own models, so that nowadays Harris's formula is also known as the "Wilson lot size formula" (Wilson 1934) or "Camp's formula" (Camp 1922), or the "Barabas formula" Erlenkotter (1989, 1990) provides an interesting historical account of the formula's early life including a biography of F.W. Harris.

The second major contribution focussing on this problem was authored by Taft (1918), who incorporated a finite production rate and developed the classical Economic Production Quantity (EPQ) model, the first in a long sequence of generalisations to come.

As reported by Best (1930) an EOQ formula was used at Eli Lilly and Company from 1917 onwards. The EPQ/EOQ inventory control models are still widely accepted by many industries today for their simplicity and effectiveness. However, these simple models have several weaknesses. The obvious one is the number of simplifying assumptions. In these traditional inventory models in fact the sole objective is to minimise the total inventory-related costs, typically holding cost and ordering cost. For this reason many researchers studied the EOQ extensively under real-life situations and provided mathematical models that more closely conform to actual inventories and respond to the factors that contribute to inventory costs. The result was a very vast literature on inventory and production models generalising the economic order quantity model in numerous directions, a major example being the famous *dynamic lotsizing* algorithm devised by Harvey M. Wagner and Thomson M. Whitin (1958) for solving the problem, in the case when requirements may vary between different discrete points in time, and this formulation has gained many followers.

The large number and broad range of papers using the EOQ inventory model have also raised important concerns about the state of the lot sizing literature stream. It is unclear what this large stream of papers has collectively accomplished. Now, after one century from the first EOQ model, there is a need to assess what our collective understanding of lot sizing appears to be at this point in time, and what directions might be fruitful for future research. We here explore how lot sizing research has built on Harris's basic model idea by analysing a selection of 219 papers published in relevant peer-reviewed management journals between 1913 and 2012 (see Table 1). We organise them in a new literature framework schema and we analyse how the EOQ/EPQ concept has been used by researchers. Second, we examine how cohesive the Lot Sizing research community is. This paper is organised as follows. In Section 2 we state the basic problem and in Section 3 the literature research methodology is explained. A new framework able to map Lot Sizing theory across the century 1913-2012 is proposed and discussed in Section 4. In order to complete the literature analysis, in Section 5 we report on the citation network analysis of the selected papers. Finally, Section 6 includes a background discussion on different topics related to lot-sizing, not covered or extracted from our survey, in particular (i) the possibility of including transportation cost considerations in inventory replenishment decisions, (ii) sustainable lot sizing procedures, and (iii) developing the NPV methodology for lot size decisions. This section ends with a summary of proposals for future research directions. Our conclusions are reported in Section 7.

A preliminary version of major parts of this paper were presented at the IFAC conference MIM 2013 (Andriolo, Battini, Gamberi, Sgarbossa, Persona 2013).

2. Preface to the state of the art

Following the well-known assumptions used by Harris (1913) but applying a more modern notation, we formulate the “classical” EOQ model (Harris's original notation and terminology within brackets):

Q	<i>order quantity</i> [size of order, lot size, X]
D	<i>annual demand</i> [number of units used per month, movement, M]
K	<i>cost of placing one order</i> [set-up cost, S]
c	<i>unit purchase/production cost per item</i> [quantity cost, without considering the set-up or getting-ready expense, or the cost of carrying the stock after it is made, C]

- h unit stock holding cost per item per year including interest and depreciation in stock [not given an own symbol in Harris's work, assumed to be 10% per annum on average value of stock, which makes $h = 0.1 \cdot C / (12M)$]
- Q^* optimal order/production quantity [lot size, which is most economical]
- C_{tot} total cost per unit [Y]

Harris developed his model assuming that the demand rate (“movement”) was known and constant, shortages were not allowed, and replenishments were instantaneous. Under these assumptions the total cost per unit consists of only three elements: inventory holding cost, ordering cost and purchase/production cost:

$$C_{tot} = \frac{(h/c) \cdot (cQ + K)}{2 \cdot D} + \frac{K}{Q} + c = \frac{h \cdot Q}{2 \cdot D} + \frac{K}{Q} + c + \frac{h \cdot K}{2 \cdot c \cdot D}. \quad (1)$$

In this original expression, Harris had added an interest charge also on the setup cost value of stocked items $h \cdot K / (2 \cdot c \cdot D)$, a practice which has later been abandoned in the literature. This additional term is constant and will not influence the optimal order quantity. The total cost is a continuous convex function of the order quantity as shown by Harris (1913). For this reason it can be differentiated to minimise the total cost. This operation leads to the well-known square root formula for determining the Economic Order Quantity, Q^* .

$$Q^* = \sqrt{\frac{2 \cdot K \cdot D}{h}}, \quad (2)$$

The same result can be derived using an algebraic method based on the observation that EOQ objective functions most often include pairs of terms of the type $(ax + b/x)$, such as in Eq. (1), where a and b are positive parameters and x a positive decision variable. The terms can be rewritten according to

$$ax + \frac{b}{x} = \frac{a}{x} \left(x - \sqrt{b/a} \right)^2 + 2\sqrt{ab}. \quad (3)$$

From the expression on the right-hand side, we immediately see that the positive quadratic term vanishes for $x = \sqrt{b/a}$ leaving $2\sqrt{ab}$ as the minimum value of the objective function regarding these two terms.

Roach (2005) points out that Kelvin's law minimising the cost of transmitting electricity is analogous with $(ax + b/x)$, x here denoting the cross-sectional area of the wire, and Grubbström (1974) notes that the total air resistance for an aeroplane has the same structure with x now interpreted as the dynamic pressure, both cases leading to square-root formulae for the optimum design decisions.

Minner (2007) proposed a different approach for obtaining the economic order quantity formula without taking derivatives, by using the cost comparisons in a finite horizon and analysing the limiting behaviour instead of performing algebraic manipulations of the average cost function and comparison of coefficients.

It is easy to note that the “Square root formula” Eq.(1) is completely characterised by three key parameters that we are going to briefly discuss below; the holding cost h , the order cost K and the demand rate D .

Holding costs are usually defined as the cost of holding inventory for one year. Typically they are expressed as a percentage of the price of the item, supposing that the large proportion of the holding cost is represented by the cost of capital. Inventory holding costs h cover not only the cost of capital tied up in inventory, but originally also the depreciation that inventory is subject to. But, as several authors have added, some cost items to be included are related to the value of inventory (such as insurance premiums), others to physical properties, such as handling, controlling, warehousing, etc., often named “out-of-pocket holding costs” (Azzi et al. 2014). Obviously, Harris took for granted that a good approximation for the aggregate costs should be an annual interest percentage charged on the value of the average physical level.

Despite the vast amount of literature on lot sizing developed during the last 100 years, the majority of contributions have been concerned with a total cost function definition from an economic point of view, following Harris’s basic approach by using a direct costing method, although financial considerations following an NPV (Net Present Value) evaluation have added new aspects during the latter half of this century. The cost-oriented aggregate approach was questioned as to its accuracy, for instance because it is insensitive to the temporal allocation of payments within a period. For this reason, Hadley (1964), Trippi and Levin (1974) and others (Aucamp and Kuzdral 1986, Kim, Philippatos, and Chung 1986, Klein Haneveld and Teunter 1998, Horowitz 2000, Van Delft and Vial 1996) followed up this approach by arguing that the discounted value offers a more correct logical basis for analysing effects from investments in inventory. In 1980, Grubbström introduced the Annuity Stream concept (a constant payment flow providing a given NPV value), providing comparisons with the average costs per time (AC). Comparing the AC and NPV approaches, he found that the holding cost h should be approximated by $h = \rho \cdot c$, when demand has a constant rate and by $h = \rho \cdot p$, when demand is in batches, where ρ is the continuous interest rate, c the unit production cost, and p the unit revenue (sales price). The main difference between the two approaches is that the traditional periodic cost minimisation is focussed on the average inventory level, whereas the discounted cash flow methodology instead focusses on payments and their timing. Although the two approaches are conceptually different, the optimum does not differ significantly over a wide range of the values of the pertinent parameters, other than under special circumstances, cf. Teunter and van der Laan (2002), Beullens and Janssens (2011).

Ordering costs, also called set-up costs, are the sum of all costs incurred as a result from ordering items. While holding cost includes all those costs that are proportional to the amount of inventory on hand, the cost of placing one order K is traditionally considered as a fixed cost of each batch, thus it is independent of the amount ordered/produced. However, it could be argued that in real applications, the order cost K has two components as stated in Eq. (4): a fixed cost k_f that is incurred independently of the size of the order and a variable cost k_v that is incurred on a per-unit basis.

$$K = k_f + Q \cdot k_v \quad (4)$$

The fixed costs would include the costs of contacting the supplier and the cost of invoicing in case of external purchasing, or the setup cost of the production system in case of internal

production. The variable costs may include the cost for transporting, handling and inspecting the goods.

Eq. (2) and Eq. (4) show that the Economic Order Quantity increases with the order cost. This simple but important observation induced several authors to study the effect of setup cost reductions (e.g, Porteus 1985, Porteus 1986). The reduction of setup times from investment or a more clever design, reduces setup costs and leads towards applying the Japanese Just-In-Time JIT philosophy (Shingo 1981), according to which work-in-process inventories are not desirable and inventory should be reduced to its bare minimum. So EOQ and JIT theory are strictly linked to each other in this sense, and there should be strength in incorporating the lean manufacturing paradigm into Lot Sizing theory.

The third key parameter in Eq. (2) is the *demand rate* D . Assumptions made about the pattern and characteristics of demand often turn out to be significant in determining the complexity of the inventory control model. The simplest models such as the basic EOQ, assume a constant deterministic demand rate. The literature on extensions to stochastic demand is immense. For example, interest has been devoted to analysing the size of errors incurred when replacing stochastic demand by its expected value in the model, e.g. (Zheng, 1992) and Axsäter (1996). Demand may be sequences of discrete demand events with variable size and in-between time intervals. Roundy (1985) studied how well models assuming systems with one warehouse and two retailer operated when using different order size policies. A further related branch of studies concerns assumptions of the demand process belonging to a specific class of stochastic processes, such as Poisson demand (Presman and Sethi, 2006).

When shortages are allowed in an inventory model, similarly as for holding costs, we may make a distinction between the capital costs for backlogging or lost sales, i.e. the consequence of postponing or losing the revenue in-payment, and other costs related to loss of goodwill, etc.

A classification scheme for inventory models has been developed by Chikán and his associates, cf. (Chikán 1990).

3. Review methodology and descriptive analysis

As stated above, F.W. Harris has undoubtedly provided the earliest and most important contribution in lot sizing theory. For this reason he can be considered as the Father of EOQ theory. Thus, we attempted to investigate how lot sizing research has emanated from the first Harris model published in 1913. By using the Scopus and Google Scholar data bases to locate papers citing Harris's (1913) article,, we found 177 pieces of work ranging from 1996 to 2012 in Scopus and 517 pieces of work in Google Scholar, in the same time period (Figure 1). From this analysis, it becomes evident that in the many years since it was introduced, the EOQ/EPQ construct has been used in around 700 peer-reviewed journal papers. This number would be very much higher if we also included conference papers and books. The instantaneous picture reported in Figure 1 makes clearly understandable to the reader the necessity of the authors to develop a selection procedure to analyze only a sub-set of papers as sample of the entire literature. Moreover, the Harris paper citation count was not sufficient to review the whole literature on EOQ and EPQ problems, since it was published in a time when citations were not fully registered. Since we have no data available to verify the true number of citations to Harris's model before 1996, we felt a need to apply a different research approach for this earlier period. Therefore we decided to use different search techniques depending on the different periods of time. Table 1 lists the research steps applied and

keywords adopted. In particular the Scopus database allowed us to find the most important literature from 1996 until today. We used the keywords “EOQ”, “EPQ”, “Economic order quantity”, “Economic production quantity” in field “Title”. We then excluded titles containing the specific words “Review”, ”Survey” or ”Case study”, in order to confine the literature selection only to the paper developing new methods and models rather than analysis or application of existing theory.

Figure 1. Papers citing Harris’s work retrieved by Scopus and by Google Scholar on May 10, 2013 for the period 1996-2012.

STEP	YEARS	KEYWORDS	EXCLUSION CRITERIA	SELECTION CRITERIA	PAPERS FOUND
1	1996-2009	Title= "EOQ" or Title= "EPQ or Title="Economic Order Quantity" or Title="Economic Production Quantity"			352
2	1996-2009		and not Title= "Review" and not Title=" Survey" and not Title= "Case Study"		349
3	1996-2009			LIMIT-TO Subject Area = "Decisions Science" or "mathematics" or "engineering" or "business management and accounting" or "economic econometrics and finance" or "multidisciplinary"	311
4	1996-2009		LIMIT-TO Document Type = Article		259
5	1996-2009			Citation number: papers contributing to the 90% of total citations	95
6	2010-2012	Title= "EOQ" or Title= "EPQ or Title="Economic Order Quantity" or Title="Economic Production Quantity"			208
7	2010-2012		and not Title= "Review" and not Title=" Survey" and not Title= "Case Study"		181
8	2010-2012			LIMIT-TO Subject Area = "Decisions Science" or "mathematics" or "engineering" or "business management and accounting" or "economic econometrics and finance" or "multidisciplinary"	169
9	2010-2012		LIMIT-TO Document Type = Article		133
10	2010-2012			Papers classification applying a 4 point scale evaluation according to the level of centrality of the "EOQ/EPQ construct" (according to Lane et al., 2006)	49
Total 1996-2012					144
11	1913-2012	Title= "Title 1", "Title 2", "Title 3", ...all belonging to the reference lists of the 144 selected papers	none	Citation number and journal relevance (Impact Factor)	75
Total 1913-2012					219
<i>*From Scopus: steps 1-10, from Google Scholar: step 11</i>					

Table 1. Review methodology: keywords and the 11 search steps adopted.

After this, we limited our search to the following subject areas: Decision Sciences, Engineering, Business Management and Accounting, Economics, Econometrics and Finance, Multidisciplinary. Finally, we filtered our results in order to exclude conference papers, articles in press, reviews, letters, notes. This resulted in 259 papers that were made subject to a further analysis of their abstracts and contents, and ranked for a citation count in Scopus. To consider only the most relevant pieces of work, we selected the 95 papers (out of 259) that represented 90% of the total count of citations. The results of this analysis are shown in the Pareto Curve reported in Figure 2.

Figure 2. Pareto analysis of the 259 papers selected in step 4 of our research procedure.

Then, in order to identify the most relevant contribution in the recent period 2010-2012, according to the approach presented by Lane et al. (2006), each paper selected by Scopus with the same keywords has been read and classified according to the “*centrality of the EOQ/EPQ construct*”. In particular, we applied a 4 point scale evaluation in accordance with the methodology applied by Lane et al. (2006) by using the following four categories for evaluating each paper:

- (1) The paper extends the EOQ/EPQ construct’s definition by developing new EOQ/EPQ models and methods,
- (2) The paper is centred on the subject (EOQ/EPQ) and on its dynamics by further investigating or extending previous EOQ/EPQ models and methods including new criteria or input parameters,
- (3) The EOQ/EPQ construct is a necessary part of the paper’s hypotheses,
- (4) The EOQ/EPQ construct is only instrumental (not necessary) in developing the logic for the paper’s propositions, or the paper uses the EOQ/EPQ construct to explain the results, or the paper uses the EOQ/EPQ construct as a minor citation with little or no discussion,

By selecting articles with an evaluation of (1), (2) or (3) and by excluding articles belonging to the fourth set, the final output consists of a set of 49 papers published in 2010-2012. These, added to the previous selected pieces of work, gave a total set of 144 papers on EOQ and lot sizing theory published in the period 1996-2012.

To recover the older articles, a snowball-approach was performed by checking articles that were cited in the 144 previously selected pieces of work and where the citation received together with the journal relevance (impact factor) indicated that the paper might be relevant for this review. These papers and their citation counts were found using the Google Scholar database.

We focussed our attention on the EOQ and EPQ problems and their extensions, and therefore exclude work dealing with the Joint Economic Lot Size determination except for the first contribution given by Goyal (1976). For further insight into the JELS literature we refer the reader to the recent work of Glock (2012). In this stage we identified 75 relevant papers published in ISI journals. In total, 219 relevant papers were finally identified after this stage and subsequently included in our analysis.

Figure 3 illustrates that the number of pieces of work on EOQ/EPQ models has increased considerably over the last years, highlighting the importance that researchers have attributed to this topic especially in the current business environment. Certainly, this is also due to the constant increment in the interest in publishing.

Figure 3. The distribution of the 219 selected papers over the time (year of publication).

Table 2 shows the list of the ten journals to which the major part of the selected articles belong, ranked in descending order of paper published. It can be stated that these ten journals published more than half of the total number of paper considered. To keep the length of this review within reasonable limits, we will not discuss all 219 papers in detail in the following, but restrict ourselves to discussing those papers that are most representative according to the existing literature, and enable the reader to clarify rather exhaustively the scenario that has been created in a century of research in the field of economic lot sizing computation. All the papers considered in this review are listed in the reference section of this paper. A thematic classification of the EOQ models can be found in Section 4.

Journal	Number of papers
<i>International Journal of Production Economics</i>	40
<i>Journal of the Operational Research Society</i>	32
<i>European Journal of Operational Research</i>	9
<i>International Journal of Systems Science</i>	7
<i>Production Planning and Control</i>	7
<i>Applied Mathematical Modelling</i>	6
<i>Computers and Operations Research</i>	6
<i>International Journal of Information and Management Sciences</i>	6
<i>Management Science</i>	6
<i>Omega</i>	6

Table 2. The top 10 journals publishing 125 of the 219 papers selected in the period 1913-2012.

4. State of the art and classification framework

A deeper analysis of the 219 papers selected for the period 1913-2012 leads immediately to a first classification into three major sub-systems in relation to the type of input data considered in the models:

- Deterministic models: All the input data are completely known a priori. Due to the easiness in dealing with known parameters, the majority of the existing literature consists of deterministic models. Some of these try to give an optimal solution of the problem, others give some heuristic approach in order to gain good results for practical situations.
- Stochastic models: Some input data are described by a known/unknown probability density function.
- Fuzzy models: Some input data belong to a set of variables having degrees of membership according to Fuzzy Set theory (Zadeh 1965).

In the following, we are going to give a holistic description of the existing literature on lot sizing. In particular we focus our attention particularly on deterministic models that represent the vast part of the research in this field. Figure 4 anticipates the considerations made in the next paragraphs, by outlining the evolution from Harris's basic model along the last century.

Figure 4. EOQ literature historical evolution in relation to the aspects and assumptions considered (the analysis is based on the 219 papers selected).

Figure 5 provides a classification framework including the 219 selected papers, that permits the reader to easily identify the main research directions developed inside the EOQ theory along the entire century. This framework incorporates also a classification of each of the 219 papers. According to the three major sub-systems previously identified, the framework is organized in deterministic, stochastic and fuzzy models.

Figure 5. EOQ-theory classification framework related to the 219 selected papers: starred numbers refer to the most representative work in the corresponding area.

4.1. Deterministic models

One of the classic EOQ assumptions is that each replenishment happens instantaneously at the moment at which the retailer places the order. In the industrial real world indeed, a lead time occurs between the two moments, and in case of limited production capacity of the production plants, the replenishments are made gradually. To consider this fact, Taft (1918) proposed to modify the square root formula adding a parameter, to represent the ratio between the demand rate and the production capacity in the same period. This assumption led to the EPQ model and it has been taken into account by a multitude of researchers after Taft. During the life cycle of the product, the assumption of a constant demand rate is never met. This assumption was ~~first~~ relaxed by Donaldson (1977) offering an important special case of dynamic lotsizing (see above), when demand changes with a linear trend.

Donaldson ~~considered the case of a linear trend in demand, and he~~ established a key property of this optimal replenishment pattern, namely that *“the quantity ordered at a replenishment point, i.e. a point at which actual inventory becomes zero, should be the product of the current instantaneous demand rate and the elapsed time since the last replenishment”* Donaldson (1977). Finally, he used this property to determine for a given demand pattern and horizon length, the best locations in time of a given number of replenishments. ~~Notice that his approach was fully analytical and required computational efforts for obtaining a solution.~~ Silver (1979) adapted the Silver-Meal heuristic (Silver and Meal 1973) to develop an approximate solution procedure for the positive linear trend, in an effort to reduce the computational load needed in Donaldson's work. Many other researchers dealt with the problem of a linearly changing demand because of its limited complexity. Barbosa and Friedman (1978) presented a continuous-time inventory model with known time-varying demand, and they provided a complete solution for demand functions of the type $D(t) = kt^\beta$ with $k > 0$ and $\beta > -2$. For $\beta = 0$, the solution reduces to the classical "square root law", Eq. (2), for infinite horizons. More generally, for β integer, the solution can be expressed in a " $(\beta + 2)$ root law". The power form of this model is widely applicable because many real-life demand patterns are well-approximated by appropriately adjusting the parameters k and β .

However, the life cycle of many products can be portrayed as a period of growth, followed by a period of relatively level demand and finishing with a period of decline. Ritchie (1980) considered appropriate policies for a linear increase in demand followed by a period of steady demand, and this model was generalised by Hill (1995), who considered a general power function for demand during the growth phase $D(t) = k(t/b)^\beta$ of which a linear increase is a special case ($\beta = 1$). Sana (2008) proposed an Economic Order Quantity model for seasonal goods using a sine function to model the seasonal demand rate.

One of the most well-known contributions is the dynamic version of the economic order quantity mentioned above (Wagner and Whitin 1958). This dynamic lot-sizing model, using dynamic programming, is a generalisation of the basic EOQ model, so it pursues the goal to minimise the sum of setup costs and inventory holding costs, but it allows the demand for the product to vary over time. The algorithm requires a forecast of product demand over a relevant time horizon, and it then determines the optimal replenishment policy for all periods. Recently Grubbström (2012, 2013) has proposed a dynamic EPQ model with NPV as the objective, in which the assumption of instantaneous replenishments is relaxed, including an

algorithm leading to optimality. This model includes the Average Cost approach as an approximation, earlier published by Hill (1997).

A third deficiency of the classical model encountered is that goods may be stored indefinitely to meet future demand. However, in many real-world situations this is not accurate because of the effect of deterioration, which is vital in many inventory systems and cannot be disregarded. Food, blood, photo films, pharmaceuticals and other chemicals, and radioactive substances are examples. Deterioration is defined as decay, damage, spoilage, evaporation, obsolescence, loss of utility or loss of marginal value of a commodity resulting in decreased effectiveness from original (Wee 1993). The earliest work describing the deterioration problem was authored by Ghare and Schrader (1963). They observed that certain commodities shrink with time by a proportion which can be approximated by a negative exponential function of time. Therefore, they considered a constant deterioration rate θ that models the situation in which a constant fraction of the on-hand inventory level deteriorates with time. This type of deteriorating process may be described by the differential equation:

$$\frac{dI}{dt} = -D(t) - \theta I(t), \quad (5)$$

where θ is the constant deterioration rate, $I(t)$ the inventory level at time t , and $D(t)$ the demand rate at time t . This formula describes the situation in which the inventory level is depleted simultaneously by the demand rate and by an exponential deterioration process. Assuming $D(t)$ to be constant and equal to D , denoting T to be the moment at which the inventory level reaches zero, and Q the batch size, with the boundary conditions that initial inventory is Q , $I(0) = Q$, and that final inventory at T is zero, $I(T) = 0$, the solution to Eq. (5) is given by:

$$I(t) = \frac{D}{\theta} (e^{\theta(T-t)} - 1), 0 \leq t \leq T, \quad (6)$$

The order quantity becomes:

$$Q = I(0) = \frac{D}{\theta} (e^{\theta T} - 1). \quad (7)$$

From (5), we obtain the time interval T for a batch of Q units to meet a demand of DT to be:

$$T = \frac{1}{\theta} \ln \left(\frac{D}{\theta} Q + 1 \right). \quad (8)$$

Since the length of all time intervals are the same, we have:

$$I(t + kT) = \frac{D}{\theta} (e^{\theta(T-t)} - 1), 0 \leq k \leq n-1, 0 \leq t \leq T. \quad (9)$$

A further development of this model is treated in Section 6.5 below.

In the literature many authors have dealt with the problem of deteriorating items using a constant deterioration rate. Examples are Hariga (1996), Dave and Patel (1981), Chang and Dye (1999), Chang (2004), Ouyang, Chang and Teng (2005), Chakrabarti and Chaudhuri (1997), Bose, Goswami and Chaudhuri (1995), and recently Mahata (2011, 2012).

The deterioration rate can also be assumed to vary with time according to some function $\theta(t)$. This assumption causes difficult mathematical calculations and closed-form solutions are generally impossible, so algorithms providing a numerical solution must be developed. Covert

and Philip (1973) extended Ghare and Schrader's model and obtained an EOQ model for a variable rate of deterioration by assuming a two-parameter Weibull distribution $\theta(t) = \alpha\beta t^{\beta-1}$, where α and β are the scale and shape parameters of the Weibull distribution. It is simple to observe that the case of exponential distribution is a special case of the Weibull distribution with $\beta = 1$. Philip (1974) generalised Covert and Philip's EOQ model using a three-parameter Weibull distribution $\theta(t) = \alpha\beta(t - \gamma)^{\beta-1}$, which takes into account the impact of the already deteriorated items that are received into an inventory system as well as those items that may start deteriorating in the future. A few years later Tadikamalla (1978) developed an EOQ model for deteriorating items, using the gamma distribution to representing the time to deterioration. More recently, many authors have dealt with the economic lot size problem with deteriorating items using the Weibull distribution. Examples are Jalan, Giri and Chaudhuri (1996), Chang and Dye (2000), Wu, Lin, Tan and Lee (2000), and Wu (2001).

In real industrial situations the supplier often offers the retailer a fixed delay period in paying for the amount of purchasing cost, in order to stimulate the demand of his commodities. In this way, before the end of the trade credit period, the retailer can sell the goods and accumulate revenue and earn interest. A higher interest is charged, if the payment is not settled by the end of the trade credit period. The first basic work on this topic was provided by Goyal (1985). In his model, Goyal established two cases:

Case I: Replenishment period T exceeds the trade credit period t ($T \geq t$). In this case the customer has to pay an interest charge for items kept in stock, for the time that exceeds the trade credit period. We adopt similar notation as in Section 2 with C_{tot} now referring to total annual costs, adding the following variables:

- I_c interest charges per dollar in stocks per year
- I_d interest which can be earned per dollar in a year

The total average cost per year, including the interest payable and the interest earned per year is given by formula (10):

$$C_{\text{tot}} = \frac{K}{T} + \frac{D \cdot t \cdot h}{2} + \frac{D \cdot c \cdot t \cdot I_c}{2} + \frac{D \cdot c \cdot t^2 \cdot I_c}{2T} - D \cdot c \cdot t \cdot I_c - \frac{D \cdot c \cdot t^2 \cdot I_d}{2T}. \quad (10)$$

By deriving formula (10) and equating it to zero, he derived the mathematical expression that minimise the total variable cost (11):

$$Q^* = \sqrt{\frac{D(2K + D \cdot c \cdot t^2 \cdot (I_c - I_d))}{h + c \cdot I_c}}. \quad (11)$$

Case II: The trade credit period t exceeds replenishment period T ($t > T$). In this case, no interest charges are paid for the items kept in stock.

The total variable cost in this case is given by (12):

$$C_{tot} = \frac{K}{T} + \frac{D \cdot T \cdot (h + c \cdot I_d)}{2} - D \cdot c \cdot t \cdot I_d. \quad (12)$$

The economic order quantity in this case is:

$$Q^* = \sqrt{\frac{2K \cdot D}{h + c \cdot I_d}}. \quad (13)$$

This is one of the most studied topics concerning the EOQ problem, so there were several interesting and relevant papers related to trade credits, extending Goyal's model in many directions, and a forerunner is found in (Grubbström 1980, Figure 3). Some examples are Aggarwal and Jaggi (1995), Teng (2002), Huang (2003), Chung and Huang (2003).

In reality, a supplier is often willing to offer the purchaser a permissible delay of payments if the purchaser orders a large quantity which is greater than or equal to a predetermined quantity. If the order is less than this quantity, the purchaser must pay for the items received immediately (see Chang, Ouyang, and Teng (2003) or Chang (2004), Chung and Liao (2009), Mahata (2011, 2012)). Below, we will use the models of Ghare and Schrader (1963) and Goyal (1985), combined in Mahata (2012), to demonstrate research opportunities using the NPV combined with Laplace transform methodology.

Harris's model as well as many other inventory models before 1975 do not consider the inflation effect. Inflation can be defined as a general rise in prices, or conversely as a general decrease in purchasing power. To compensate this erosion of purchasing power, the market interest rate includes an inflation premium. These findings make it very important to investigate how time-value of money influences various inventory policies. The first attempt to consider inflation and time value of money in the Lot Sizing field has been reported by Buzacott (1975) that dealt with the EOQ problem with inflation subject to different types of pricing policies in order to investigate how time-value of money affect inventory policies. Misra (1979) developed a discounted-cost model that included internal and external inflation rates for different costs associated with inventories.

In order to accommodate the common industrial policies, some researchers developed EOQ models that incorporate two types of quantity discounts: all-units quantity discounts and incremental quantity discounts, as stated in the framework (Figure 5). In the former case the supplier provides a discount for all the items sold to the customer, if the quantity purchased exceeds predetermined quantities called price-break quantities. As a consequence, this policy results in discontinuities in the purchase cost function.

In the latter case the supplier provides a discount only for the items that exceed a predetermined level, and he sells the remaining items at the usual price. In this case the purchase cost function is continuous. Tersine and Barman (1991) studied the problem of scheduling replenishment orders under the classical EOQ model when both quantity and freight rate discounts are encountered.

Carlson, Miltenburg and Rousseau (1996) examined the optimal order quantity under both all-units and incremental-quantity discounts, using a discounted cash flow methodology. Khouja and Mehrez (1996) investigated four different supplier credit policies which included both of these situations and provided closed-form solutions in all cases. Chang(2002) provided a

model to determine an optimal ordering policy under a permissible delay of payment and/or cash discount for the customer. Huang and Chung (2003) extended Goyal's model incorporating the cash discount policy for an early payment. They developed two theorems to determine the optimal cycle time, optimal order quantity and optimal payment policy. Ouyang, Chang and Teng (2005) provided the optimal policy for the customer in presence of a permissible delay and cash discount, and they also presented an easy-to-use algorithm to find the optimal order quantity and replenishment time.

The basic assumption that shortages are not permitted is restrictive in real industrial situations. Furthermore, the inclusion of a shortage cost and the possibility of backlogging might lead to lower total inventory costs. The risk of shortages is currently a common assumption for most researchers dealing with Lot sizing models. Grubbström and Erdem (1999) derived the EOQ formula including backlogging without reference to the use of derivatives, neither for necessary conditions nor for second-order sufficient conditions, as is basically illustrated by Eq. (3). The same approach was extended to the economic production quantity (EPQ) model with shortages by Cárdenas-Barrón (2001). For models where shortages are allowed, complete backlogging, or complete loss of unsatisfied demand are two extreme cases. Deb and Chaudhuri (1987) modified Silver's (1979) procedure by including shortages which were completely backordered. They followed a replenishment policy that allowed shortages in all cycles except the final one. Each of the cycles during which shortages were permitted starts with a replenishment and stocks were built up for a certain length of time which was followed by a period of stockout. Many researchers have turned their attention to models that allow partial backlogging. Examples are Wee (1995), Chang and Dye (1999), Yan and Cheng (1998).

Another relevant aspect is that manufacturing facilities in practice do not function perfectly during all production runs. Process deterioration added to other factors, inevitably generates imperfect quality items. Although this type of situation can be faced more accurately following a stochastic approach, some interesting deterministic models have been developed. These assume that the fraction of defective items is known and constant in each production cycle. Jamal, Sarker and Mondal (2004) considered an EPQ model under two policies. With the first policy, defective items were reworked within the same cycle. With the second policy alternative, the defective items were accumulated until a number of cycles were completed, after which the defective parts were processed. Cárdenas-Barrón (2009) generalised the Jamal et al. (2004) EPQ inventory model with planned backorders, under the first policy. Kevin Hsu and Yu (2009) investigated an inventory model with imperfect quality under a one-time-only discount, where a 100% screening process was performed on the received lot and the defectives were assumed to be sold in a single batch by the end of the screening process. Jaber, Goyal and Imran (2008) assumed that the percentage of defective items per lot diminishes according to a learning curve.

4.2. Stochastic models

In a century of history from Harris's model, only a limited number of articles have directly faced the case of EOQ when some parameters are uncertain, an exception being uncertainty in demand mentioned above in Section 2. This is probably because of two reasons. First, the total inventory cost has a very low sensitivity to inventory cost parameters (Axsäter, 1996). A second reason is the complexity of algebraic operations among random parameters with a probability distribution. However, a stochastic approach is often desirable for dealing with real industrial problems, in which input data are not known a priori and have random properties.

A typical example is the EOQ model with products of imperfect quality, in which the production process produces a fraction of defective items. Since it is impossible to know the fraction of defective items a priori, this parameter can be modelled using a probability density function with known or unknown parameters. Porteus (1986) incorporated the effect of defective items into the basic EOQ model, assuming that there was a probability q that the process would go out of control while producing one unit of the product. Rosenblatt and Lee (1986) assumed that the time from the beginning of the production run, until the process goes out of control is exponentially distributed and that defective items could be reworked instantaneously at a cost. In a subsequent paper, Lee and Rosenblatt (1987) considered process inspection during the production run so that the shift to out-of-control state could be detected earlier. Haneveld and Teunter (1998) determined the optimal ordering quantity where demand is modelled by a Poisson process. Salameh and Jaber, (2000) considered a production/inventory situation where items, received or produced, were not of perfect quality, with a known density function. Items of imperfect quality could be used in a less restrictive situation. Papachristos and Konstantaras (2006) extended this model to the case in which withdrawing takes place at the end of the planning horizon. Goyal and Cárdenas-Barrón (2002) developed a simple practical approach that is easier to implement as compared to the optimal approach. Maddah and Jaber (2008) analysed the effect of screening speed and variability of the supply process on the order quantity. Eroglu and Ozdemir (2007) developed an EOQ model with defective items and shortages backordered. Liberatore, (1979) developed a stochastic lead-time generalisation of the EOQ model with backlogging of demand. Hariga and Haouari (1999) presented a general formulation of the inventory lot sizing model with random supplier capacity. Horowitz (2000) considered the effects of inflation on inventory, when the rate of inflation was not known with certainty.

4.3. Fuzzy models

A different approach to deal with uncertain parameter is the application of fuzzy set theory (Zadeh 1965). In the classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$. For this reason fuzzy set theory is usually employed in a wide range of domains in which information is incomplete. However, this methodology requires considerable computational efforts and it is often accused of introducing unnecessary complexity reducing the transparency of results. As a consequence, Fuzzy set theory does not appear to find wide applications in real industrial environments, where managers ask for easy-to-use models. Some examples of Fuzzy EOQ models developed in the past are found among the following references. Vujosevic et al. (1996) considered trapezoidal fuzzy inventory costs, providing four ways of determining the EOQ in the fuzzy sense. Yao and Lee (1998) investigated a computing schema for the EPQ in the fuzzy case, describing demand quantity and production quantity per day with triangular fuzzy numbers. Chang (2004) presented a model with a fuzzy defective rate and fuzzy annual demand. Chen, Wang and Chang (2007) proposed a Fuzzy Economic Production Quantity (FEPQ) model with imperfect products that could be sold at a discounted price, and where costs and quantities were expressed as trapezoidal fuzzy numbers. Further exemplifications can be found in Wang Tang and Zhao (2007), Halim, Giri and Chaudhuri (2008) and in Björk (2009).

5. Citation network analysis

We believe that the most relevant EOQ literature can be analysed by examining citation patterns between the papers selected. One of the characteristics of a well-defined community of researchers is to present a network of citations, among their papers, that centers on a core set of works delineating the basic concepts, theories, and methodologies shared by the community itself (Garfield 1979, Kuhn 1970, Merton 1973). We also believe that the less tightly a paper is linked into a research community's citation network, the greater the risk that the authors are deviating from the community's norms regarding its core concepts and basic hypotheses.

We have analysed the central contributors to this research (the “pillars”), how tightly interlinked our set of 219 papers is, and the presence of different “schools of analysis”. Using the Scopus and the Google Scholar databases, we downloaded the references for each paper. We then tabulated these data by using Excel in order to determine how frequently each paper (i) has cited other papers in the set and (ii) has been cited by other papers within the set. Accordingly, we derived two measures of centrality to determine how tightly linked each paper is to the rest of the literature: the number of times a given paper cites the older ones in the set of 219 papers (“sent citations”) and the number of times a paper is cited by the later ones in the same set (“received citations”). The former represents the authors’ positioning of the paper relative to the EOQ/EPQ literature, whereas the latter reflects other authors’ perceptions of that paper’s contribution to the literature. While both measures have temporal biases (papers that were published earlier have a better chance of getting cited, and those published later have more opportunities to cite other papers), these biases offset each other. Next, we created a third centrality measure by adding each paper’s number of sent and received citations and then normalizing the sum by dividing it by 100 (the number of years covered by the sample, according to Lane et al. (2006) This process provides an index denoting the average annual number of links to and from the EOQ literature. These three measures are listed in descending order in Table 3, for the pieces of work that we found to be the most tightly linked to the core set.

Paper	Sent Citations <i>S</i>	Received Citations <i>R</i>	Index Sent+Received $(S+R)/100$
Donaldson, W.A. (1977)	0	45	0,45
Ghare, P.M., Schrader, G.F. (1963)	0	44	0,44
Goyal, S.K. (1985)	0	41	0,41
Hariga, M. (1996)	24	16	0,4
Covert, R.B., Philip, G.S. (1973)	1	37	0,38
Dave, U., Patel, L.K. (1981)	0	33	0,33
Aggarwal, S.P., Jaggi, C.K. (1995)	4	29	0,33
Silver E.A. (1979)	2	31	0,33
Jamal, A.M., Sarker, B.R., Wang, S. (1997)	7	24	0,31
Salameh, M. K., Jaber, M. Y. (2000)	5	26	0,31
Mishra, R.B. (1975)	0	29	0,29
Bose, S., Goswami, A., Chaudhuri, K.S. (1995)	18	9	0,27
Chang, H.C., (2004)	13	14	0,27
Manna, S. K., Chaudhuri, K. S. (2001)	21	6	0,27
Rosenblatt, M.J., Lee, H.L., (1986)	0	26	0,26

Manna S.K., Chiang C.(2010)	25	1	0,26
Goswami, A., Chaudhuri, K.S. (1991)	1	24	0,25
Ritchie, E. (1984)	2	22	0,24
Harris, F.W. (1913)	0	23	0,23
Chang, C., Ouyang, L., Teng, J. (2003)	15	8	0,23
Deb, M., Chaudhuri, K.S. (1987)	4	19	0,23
Sana, S. S. (2008)	21	2	0,23
Porteus, E.L. (1986)	1	21	0,22
Goyal, S. K., Cárdenas-Barrón, L. E. (2002)	0	21	0,21
Chakrabarti, T., Chaudhuri, K. S. (1997)	13	8	0,21
Ghosh, S. K., Chaudhuri, K. S. (2006)	21	0	0,21
Liao, H.C., Tsai, C.H., Su, C.T. (2000)	10	11	0,21
Mahata G.C. (2011)	21	0	0,21
Murdeswar, T.M. (1988)	4	17	0,21
Chang, H.J., Hung, C.H., Dye, C.Y. (2001)	12	8	0,2
Chung, K.J. (1998)	3	17	0,2
Giri, B. C., Goswami, A., Chaudhuri, K. S. (1996)	16	4	0,2
Hill, R.M. (1995)	13	7	0,2
Huang, Y. (2003)	10	10	0,2
Manna, S. K., Chaudhuri, K. S., Chiang, C. (2007)	18	2	0,2
Roy T., Chaudhuri K. (2011)	20	0	0,2

Table 3. Most linked papers out of the 219 mapped in the citation network

Figure 6 presents the citation network derived from the relative citation count among the 219 selected papers. Each of these papers is represented by a node, and each of the 1296 relations among the papers is represented by a directional arrow. This network is built using the PAJEK software. In order to identify the importance of each paper, the size of each node is made proportional to its number of citations received in the network. Additionally, we have divided the century into five periods of equal length, assigning these five different colours.

Examining the network, we can identify strong connections between the nodes but we can also notice a few clusters concentrated around some central pieces of work. These clusters are densely woven and it is difficult to clearly identify all their relations. By deepening our analysis of the network, we have identified four main groups focussing on different modelling aspects:

1. GROUP 1: Papers focussed on “imperfect quality items” issues. This cluster incorporates among others Porteus (1996), Rosenblatt and Lee (1986) and Salameh and Jaber (2000).
2. GROUP 2: Papers focussed on “permitted delays in payment” issues. Goyal (1985), Aggarwal and Jaggy (1995), Jamal et al. (1997) are included;
3. GROUP 3: Papers focussed on “deteriorating items” issues - next to Ghare and Shrader (1963) and Covert and Philips (1973), we find Mishra (1975) and Dave and Patel (1981);

4. GROUP 4: Papers focussed on “time varying demand” problems - in this cluster we find Donaldson (1977), Silver (1979) and Ritchie (1984).

The central parts of these clusters are identified in Figure 6 by four coloured circles. There are also papers belonging to several clusters, such as Mahata (2011) belonging to Groups 2, 3, and 4.

In order to understand the effectiveness of this citation network in the determination of the literature pillars, we designed the following table, showing the number of citations received in the network and the number of citations received in the Google Scholar database.

Author	Year	Title	Journal	Citation received in the network	Citation received in Google Scholar
Donaldson, W.A.	1977	Inventory replenishment policy for a linear trend in demand - An analytical solution.	Operational Research Quarterly	45	301
Ghare, P.M., Schrader, G.F.	1963	A model for an exponentially decaying inventory.	Journal of Industrial Engineering	44	681
Goyal, S.K.	1985	Economic order quantity under conditions of permissible delay in payments.	Journal of the Operational Research Society	41	584
Covert, R.B., Philip, G.S.	1973	An EOQ model with Weibull distribution deterioration.	AIIE Transactions	37	385
Dave, U., Patel, L.K.	1981	(T, S1) policy inventory model for deteriorating items with time proportional demand.	Journal of the Operational Research Society	33	88
Silver, E.A.	1979	A Simple Inventory Replenishment Decision Rule for a Linear Trend in Demand .	Journal of the Operational Research Society	31	189
Aggarwal, S.P., Jaggi, C.K.	1995	Ordering policies of deteriorating items under permissible delay in payments.	Journal of the Operational Research Society	29	412
Mishra, R.B.	1975	Optimum production lot size model for a system with deteriorating inventory.	International Journal of Production Research	29	217
Rosenblatt, M.J., Lee, H.L.	1986	Economic production cycles with imperfect production processes.	IIE Transactions	26	583
Salameh, M.K., Jaber, M.Y.	2000	Economic production quantity model for items with imperfect quality.	International Journal of Production Economics	26	339
Goswami, A., Chaudhuri, K.S.	1991	An EOQ model for deteriorating items with shortages and a linear trend in demand.	Journal of the Operational Research Society	24	127
Jamal, A.M., Sarker, B.R., Wang, S.	1997	An ordering policy for deteriorating items with allowable shortage and permissible delay in payment.	Journal of the Operational Research Society	24	363
Harris, F.W.	1913	How Many Parts to Make at Once	Factory, The Magazine of Management	23	547
Ritchie, E.	1984	The EOQ for linear increasing demand - a simple optimal solution.	Journal of the Operational Research Society	22	130
Goyal, S.K., Cárdenas-Barrón, L.E.	2002	Note on: Economic production quantity model for items with imperfect quality - A practical approach.	International Journal of Production Economics	21	139
Porteus, E.L.	1986	Optimal lot sizing, process quality improvement and setup cost reduction.	Operations Research	21	685

Table 4. Most cited papers according to the citation count in the network of Figure 10 and in the Google Scholar database.

After a detailed analysis of the data in Table 4, we immediately observe some peculiarities. The ranking of the analysed papers is made in order of number of citations received in the network, but using the number of citation received in Google Scholar we would have obtained a rather different order. In detail, we can notice two considerable cases: Dave and Patel (1981) has the lowest Google Scholar citation count although it is fifth in our ranking; Porteus (1986) has the highest count, but it appears in the last position of the list. The reason for this apparent inconsistency lies in the fact that a research paper simultaneously can deal with more than one topic. Thus, a paper that received a large number of citations in the Google Scholar database while only a few in the network, is relevant in different research fields besides the one that we are considering. Therefore it is cited by many papers that do not appear in our set. On the contrary, a paper that received many citations in the network, but has a low count in Google Scholar, is strongly centred in our EOQ construct. In order to highlight the EOQ pillars, the list has been ranked according to number of citations received in the network

6. Background considerations for a future research agenda

6.1. Introduction

In this section we make an attempt to paint a background for a future research agenda. As a first starting point, we analyze keywords used in the subset of articles pertaining to the period 1996-2012.

We then turn to two areas of importance related to new or recently established aspects of lot sizing, the first focussing on transportation, the second on sustainability. A major part of the related literature falls outside of the set of 219 papers, due to novelty and an inability to extract these areas by applying the procedure reported in Table 1.

6.2. Keyword implications

A first background for future research proposals may be found by studying trends in the keywords used in the set of articles chosen from the period 1996-2012 covered by Scopus.

All keywords applied in the articles have been analysed and grouped into five main sets:

Set 1: Keywords related to economic and financial aspects: costs, cost accounting, cost comparison optimisation, cost effectiveness, cost benefit analysis, cost minimisation models, cost oriented model, cost parameters.

Set 2: Keywords related to quality problems: imperfect production, imperfect production process, imperfect production system, imperfect quality items, imperfect quality, inspection, imperfect reworking, defective items.

Set 3: Reuse and waste disposal aspects: Return policy, reuse, reverse logistics, rework, rework and salvage, rework process, repair, waste disposal, remanufacturing, recovery, product reuse.

Set 4: Social sustainability and impact on human workforce.

Set 5: Environmental impact, environmental problems, green supply chain, carbon emission.

Table 5 identifies on the one hand a continuous, although variable use over time of terms related to economic and financial aspects (cost accounting, cost minimisation models, cost effectiveness) that are still the most important drivers in lot sizing theory, and, on the other, a remarkable increase in the use of keywords related to aspects of imperfect product quality, repair, reuse, waste, and disposal. It also highlights the introduction of environmental impact aspects during the last few years. The last two columns of Table 5 show the total count of the aforementioned keywords in the respective periods, and their averages. These averages are determined as $Average = TotK / Time\ interval$, where $TotK$ is the total count of the keywords, and $Time\ interval$ represents the length of the interval, in which the keyword is used from its first year until its last year.

	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	Total	Average
Costs, Cost accounting, Cost comparisons optimization, Cost effectiveness, Cost benefit analysis, Cost minimization models, Cost oriented model, Cost parameters	2	2	3	1	1	1	4	7	5	1	6	7	1	6	12	6	6	61	3.59
Imperfect production, Imperfect production process, Imperfect production system, Imperfect quality items, imperfect quality, 100% inspection, Imperfect reworking, Defective items	0	0	0	0	0	0	0	2	1	0	0	2	3	10	7	4	7	34	3.40
Return policy, Reuse, reverse logistics, Rework, Rework and salvage, Rework process, Repair, Waste disposal, remanufacturing, recovery, Product reuse	0	2	0	3	0	0	0	0	0	0	0	0	5	6	2	4	9	31	1.94
Social sustainability	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	3	0.38
Environmental impact, Environmental problems, Green supply chain, Carbon emissions	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	4	4	10	3.33

Table 5. Count of keywords used in the period 1996-2012.

6.3. Transportation cost considerations in inventory replenishment decisions

Transportation costs are becoming increasingly important in inventory replenishment decisions and, in practice, lotsizing decisions are strongly affected by material handling equipment, transportation flow paths, transportation mode and technical constraints. Companies within a global sourcing context daily experience the cost of transportation as playing a major rôle in total purchasing costs. A small group of authors recently investigated the transportation cost computation problem in lot sizing decisions (Battini et al. 2012).

As illustrated by Figure 7, transportation costs as a function of batch size can often behave in a discontinuous way, which cannot be differentiated during the whole interval. Moreover, they depend on the number of different vehicle types used in the transportation (for example different containers with different capacities) and in practice, more types of vehicles are available with different capacity and different costs.

Furthermore, transportation and handling activities have a great impact on total emissions generated, whereas other activities, such as ordering, warehousing and disposing of waste, have much lower incidence on the total environmental impact.

Figure 7. Cost function behaviour in a purchasing order cycle (Source: Battini et al, 2013).

Vroblefski et al. (2000) proposed a model where transportation costs are considered. They showed that the total cost of the system is a piecewise convex function of the ordering levels with discontinuities at the cost breaks, unlike the traditional model where the total cost is convex over the entire range of ordering levels. Swenseth and Godfrey (2002) incorporate the transportation cost into the total annual logistics cost function. Their model also includes a freight rate per pound for a given shipping weight over a given route. Zaho et al. (2004) considered both the fixed transportation cost and the variable transportation cost in the problem of deciding the optimal ordering quantity and frequency for a supplier–retailer logistic system. A multiple use of vehicles is also considered. They provided an EOQ-modified model and an algorithm to determine the optimal solution that minimises the whole average cost of the logistic system in the long-run planning horizon. Mendoza and Ventura (2008) extended the Economic Order Quantity model by introducing all-units and incremental quantity discount structures into their analysis. Birbil et al. (2009) provided EOQ models in which the impact of transportation costs is considered. They studied a subclass of problems that also includes the well-known carload discount schedule.

Finally, the impact of transportation costs has a significant influence not only from an economic point of view, but also for its environmental impact, which is explained in the following section. Battini et al. (2013) investigate internal and external transportation costs according to the vendor and supplier position and the different freight vehicle utilization ratios in order to provide an easy-to-use methodology for sustainable lot-sizing. Freight discounts and their effect on lot sizing are also investigated in Burwell, et al. (1997) and further followed up by Chang (2013).

We may also refer to an early basic lot sizing model, in which the inventory of goods-in-transit is considered, cf. (Axsäter and Grubbström 1979). The decision variables would be the lot size, the safety stock level, and the speed (mode) of transportation. A valuable item ties up more capital during transportation than a cheaper one. A more speedy mode of transport would lower capital costs, but direct transportation costs would increase. The length of transport also influences the lead-time and thereby safety stock conditions.

Here, for simplicity, we confine our attention to a simplest case, when only capital holding costs are considered and with no consideration of safety stock consequences. Let K denote the fixed ordering cost, c the economic production/purchase cost per item, ρ the opportunity cost of capital, Q the order quantity, v the velocity of transportation, $\hat{s}$ the transportation distance, and m the weight (alternatively volume) of an item. The direct transportation cost for a batch of weight mQ in the simplest case is assumed to be proportional to weight and to transportation distance $\hat{s}$, $c_{\text{transport}}(v) \cdot mQ\hat{s}$, where $c_{\text{transport}}(v)$ is the cost per distance and weight unit for moving one item at a speed v , which would be expected to be an increasing function of v ; in the simplest case proportional, $c_{\text{transport}}(v) = \hat{c} \cdot v$. Assuming one stocking point en route, the total cost per unit will be

$$C_{\text{tot}} = c + \frac{K}{Q} + \frac{Q\rho c}{2D} + \frac{\rho c \hat{s}}{v} + \hat{c} v m \hat{s}, \quad (14)$$

where the second and third terms are traditional inventory-related costs per item with $h = \rho c$ representing the inventory holding cost parameter, the fourth term being the capital cost per item during transit ($\hat{s}/v$ being time in transit), and the fifth term the direct transportation cost per item. In this simple approach, the optimal choice of Q is independent of the choice of v , and the resulting solution in Q is the traditional one, Eq. (2). However, concerning transportation speed, we again have two terms of the type in Eq. (3) with x now interpreted as v , and therefore the optimal solution:

$$v^* = \sqrt{\frac{\rho c}{\hat{c} m}}, \quad (15)$$

which shows the optimal speed v^* to be proportional to the square root of the value density of the item c/m . Gold should be moved by air freight, and brick by sea.

6.4. Sustainability issues in lot sizing

Sustainability is probably one of the most over-used expressions in recent years, it is a very broad term, formally defined by the Brundtland Report (UN Documents 1987) as “the kind of development that meets the needs of the present without compromising the ability of future generations to meet their own needs”. As stressed by the European Union targets (for instance in the Horizon 2020 program), an industrial sector that is economically, socially and environmental sustainable will contribute to EU policy targets and company competitiveness.

The international increasing concern on environmental problems stresses the need to treat inventory management decisions as a whole by integrating economic, environmental and social objectives. Today modern societies have begun to consider environmental and social aspects on top of economic and financial goals. These different aspects contribute to the same target of improving the quality of life of people. In the last three decades new business terms such as recycling, remanufacturing (see Richter 1997, Richter and Dobos 1999), “sustainable development” and ergonomics, the latter having a slightly older history, have been introduced in our economy. Recently, several world class corporations have declared themselves to be committed environmentalists and they have been integrating environmental and ergonomic aspects in the development of their strategies. Consumers are often willing to pay extra for recycled products, recyclable and non-damaging to the environment and human health. They are also demanding more detailed information about the processes used by their suppliers. Social Sustainability is a core element of Sustainability and has considerably grown in

importance during the last few years. Social sustainability pursues the goal of creating and maintaining quality of life for people, protecting the mental and physical health of all stakeholders, encouraging community and treating all stakeholders fairly. These elements are essential because a healthy society cannot be developed and maintained, if the population is in poor health. Environmental and social problems are an area of steadily increasing concern in modern society.

For this reason, research on environmental and social sustainability has considerably enriched the lot sizing literature during the last three years. Benjaafar et al. (2010) incorporated carbon emission constraints on single and multi-stage lot-sizing models with a cost minimisation objective. They considered four different policies based respectively on a strict carbon cap, a tax on the amount of emissions, the cap-and-trade system and the possibility to invest in carbon offsets to mitigate carbon caps. Arslan and Turkay (2010) included environmental and social criteria in addition to conventional economics. In order to consider the social impact in their EOQ model they used working hours as social metric. Hua et al. (2011) extended the EOQ model taking carbon emissions into account under the cap and trade system. Bonney and Jaber (2011) discussed a range of inventory and lot sizing problems that are not considered appropriately in traditional inventory models, for example the effects on the environment from packaging choices, location of stores, and waste. They focussed their attention on the environmental impact of logistic operations, and outlined a hierarchy of the players involved in inventory management. They also suggested some metric categories that could be used to assess inventory performance related to its environmental impact. In the last part of their paper, they formulated a model called “Enviro-EOQ”, that considers environmental costs of vehicle emissions, providing a closed-form solution for the optimal lot size. Wahab et al. (2011) formulated a model for a two-level supply chain that determines the optimal production-shipment policy for items with imperfect quality and incorporating the environmental impact by taking into account fixed and variable carbon emission costs. Battini et al. (2012) provided a “Sustainable EOQ model” that from an economic point of view incorporates and investigates the environmental impact of transportation and inventory. In particular internal and external transportation costs, vendor and supplier location and the different freight vehicle utilisation ratios are considered in order to provide an easy-to-use methodology. In the second step of their research (Battini et al. 2013), the authors underline the necessity to investigate the optimal purchasing/production lot size by a multi-objective optimization approach: the optimal lot-sizing decision depends on both costs and emission functions and how these two functions depend on quantity.

Recently Bouchery et al. (2012) have proposed a model that reformulates the classical economic order quantity problem into a multi-objective problem. Economic, environmental and social impacts are there considered, using the traditional formulas proposed by Harris’s original work. They developed an interactive procedure allowing to quickly identify the best option among the three solutions. A similar approach is applied also in Andriolo et al. (2013), in which the lot sizing problem in material purchasing is studied from a different perspective: the social impact due to manual handling activities necessary for stock keeping, unit lifting and storing. The ergonomic risk function is here applied to measure the influence on the work conditions of human operators.

We also need to mention that the consideration of recycling and remanufacturing activities have brought new terminology and distinctions into inventory research. For instance, we must now distinguish between *recoverable inventory*, having an upward slope for a more or less continuous input flow, and with withdrawals from inventory taking place in batches, on the one hand, and *serviceable inventory*, describing the stock of available refurbished

components, on the other. Also extra care is needed when considering holding costs, if the loop has been closed, cf. Teunter and van der Laan (2002).

6.5 Methodological developments analysing cash flows related to inventory

In this section we attempt to show that a few of the basic models found in our previous analysis, may be tied closely together within the framework of a Net Present Value approach, in which *Laplace transform* methodology plays a central rôle. As mentioned earlier, applying the Net Present Value (NPV) principle has been regarded to be superior to the Average Cost approach (AC) by several authors, for instance by Trippi and Levin (1974) and Grubbström (1980) and completely essential on occasion (Teunter and van der Laan 2002, Grubbström and Kingsman 2004). Referring to the physically measurable cash flow, rather than costs and revenues, brings the model closer to reality, it is argued.

The practical AC model has often been shown to be a good approximation of NPV, when the discount rate ρ is small and the holding cost parameter is interpreted as $h = \rho c$, where c is unit production (or purchase) cost, cf. Beullens and Janssens (2012).

Laplace transform methodology is widely applied in different branches of engineering and statistics, in particular in electrical engineering and it may be applied successfully also in inventory theory, cf. (Grubbström 2007). Here we make an attempt to tie together a few of the central contributions from our findings above. The details of our analysis are relegated to an Appendix.

We illustrate this use by departing from the original Ghare and Schrader (1963) inventory model with exponential depletion at a rate of θ , cf. Eq. (5), this model recently extended by for instance Mahata (2012) to include a finite production rate q combined with trade credits of the Goyal (1985) type. Since this model is extremely rich, we only have space for some basic points in this context. We therefore refrain from analysing any optimisation conditions, leaving this to future research.

As demonstrated in the Appendix, if applying transform theory to Eq. (5) and adopting the standard assumption for finite rate models (as in Mahata 2012) that each cycle starts with a production ramp with slope q , lasting Q/q time units, where Q is the batch size, physical inventory level during a cycle will follow

$$I_{\text{cycle}}(t) = \begin{cases} \frac{(q-D)}{\theta}(1-e^{-\theta t}), & 0 \leq t \leq Q/q, \\ \frac{D}{\theta}(e^{\theta(T-t)} - 1), & Q/q < t \leq T, \end{cases} \quad (16)$$

which is the same as in (Mahata 2012, Eqs (5)-(6)), and it collapses into Eq. (7) above when $q \rightarrow \infty$. Continuity at $t = Q/q$ requires $q(e^{\theta Q/q} - 1) = D(e^{\theta T} - 1)$, showing the relation between Q and T , collapsing into Eqs (7)-(8) above, as $q \rightarrow \infty$:

$$Q = \frac{q}{\theta} \ln(1 + (D/q)(e^{\theta T} - 1)), \quad T = \frac{1}{\theta} \ln(1 + (q/D)(e^{\theta Q/q} - 1)). \quad (17)$$

In the Appendix, we also derive expressions for the annuity streams related to out-of-pocket inventory holding costs (as distinguished from capital costs from holding inventories) $AS_{\text{out-of-pocket}}$ and of sales revenues less production costs AS_{profits} , both shown to approach Mahata's AC expressions if the discount rate is considered negligible $\rho \rightarrow 0$.

In the problem reported by Mahata (2012) generalised from Ghare and Schrader (1963) and Goyal (1985), there are two credits (loans): (i) the supplier is willing to provide the retailer with a full trade credit for payments for a period M with the interest I_c , and (ii) a partial trade credit is offered to customers for a period N at the interest rate I_e . We make use of an NPV theorem concerning the cash flow associated with a loan, Eq (A.8) in the Appendix and derive expressions for the annuity streams related to the supplier's credit $AS_{\text{supplier's credit}}$, on the one hand, and to the credit to customers $AS_{\text{credit to customers}}$, on the other. The supplier's credit is interpreted as paying the supplier at time M rather than during the initial production ramp.

Concerning the customer credit with which an interest rate I_e is associated, a partial payment is made when items are sold. The customer must then pay off the remaining balance at the end of the offered trade credit period N . It is easily shown that the supplier's credit offers an advantage (a positive addition to the annuity stream), when the opportunity cost of capital exceeds the charged interest rate $\rho > I_c$, and vice versa. A first-order Maclaurin expansion in ρ of $AS_{\text{supplier's credit}}$ (neglecting terms with $I_c\rho$) leads to an average cost expression for the supplier $AC_{\text{supplier's credit}}$, demonstrating how this average cost depends on the length of the credit period M compared to the cycle length Q/q .

It is also shown that $AS_{\text{credit to customers}}$ has the same sign as $(I_e - \rho)$, so if the interest rate paid by the customers exceeds the opportunity cost $I_e > \rho$, the credit is favourable, and vice versa, just as expected.

It appears not to be fruitful to further compare our results on credit consequences with those of Mahata, since the capital cost for operations funded by equity are not explicitly considered in Mahata (2012). They are there probably indirectly assumed to be part of the average holding cost expression. Also it appears as if Mahata attaches the credits to items in inventory after depletion.

In this sub-section, we have attempted to show that it is possible to bring together several of the aspects included in inventory model extensions proposed through at least the recent half century, beginning with the NPV formulation by Hadley (1958), the first credit extension by Goyal (1985), and later a second credit extension, in the meantime applying the Taft (1918) extension from EOQ to EPQ, and the extension to include depletions by Ghare and Schrader (1963).

The opportunity for introducing upstream and downstream credits, a common NPV evaluation focussing on cash flows, measurable in reality, and finally to use a simple, but an often neglected opportunity to apply engineering mathematics in the form of transforms, together this should certainly offer new research opportunities. An additional area for methodological research could be in lot-sizing for general multi-echelon systems, relaxing limitations to general assembly systems (Grubbström, Bogataj and Bogataj 2010).

6.6 *Summary and proposals*

Therefore in the light of these considerations, we have made an attempt to understand what directions that might be fruitful for future research on EOQ in order to address the current problems of supply chains not yet assessed by the large traditional literature. In particular the following aspects would be of interest to be investigated (arising from the literature review presented in this work together with our previous discussion):

1. To reconsider our inventory management concepts and provide new models able to include environmental aspects right from the beginning of the analysis. The future models must design responsible inventory systems (according to Bonney and Jaber 2011), i.e. systems that reflect the needs of the environment. In this case it is necessary to analyse right from the beginning of the lot sizing problem the amount of green-house gases entering the atmosphere from deliveries, warehousing, and waste disposal/recycling. Since in most cases the cost and emission functions yield different optimal solutions, the cost trade-offs are different from the emission trade-offs. The limits of a direct accounting method when externalities need to be quantified are becoming evident in the recent literature, pushing for future research to find its best way to couple the cost function quantification with the emission function analysis, for instance according to a multi-objective optimisation approach.
2. To develop new methodological approaches capable of also treating new additional criteria that are difficult to quantify by traditional economical approaches, such as the social impact of inventory and purchasing decisions. Social criteria in addition to conventional economics and environmental aspects need to be assessed by applying specific approaches not always based on mathematical computations of quantitative entities. In this context, a lack of investigation is also identified in linking ergonomic aspects with lot-sizing decisions, both in production and in purchasing (Andriolo et al., 2013). Ergonomic aspects, in fact, are related to material handling/lifting activities usually performed by humans in many production systems, and even today they are most often not considered in EOQ/EPQ models. A report conducted by Denis et al. (2006) in two Québec warehouse superstores of a North American company specialising in the sales of office supplies, demonstrated that the disequilibrium between the amount of stock and the available storage space, results in the increase of three negative aspects: risk factors related to the development of musculoskeletal disorders; risks of accidents particularly related to loss of balance and falls from heights; and impacts on productivity and quality of service offered to customers in the form of time wasted and stock losses.
3. An integrated point of view makes it necessary to assess the lot sizing problem in closed-loop supply chains. New important challenges are expected for sustainable supply chains in the near future.
4. We have also made an attempt to show that several avenues of research found in our survey can be aligned into the same framework by using the NPV principle beginning with (Hadley, 1958) together with applying some engineering mathematics. Making practical and effective use of transform methodology, combines the solution of differential equations with NPV analysis as shown in Section 6.5 and the Appendix, and this combined methodology we add to our proposed agenda.

7. Conclusions

The presented survey and analysis finds strong evidence that the EOQ literature is a mirror image of the last century: it reflects the evolution of operations management techniques in worldwide industrial systems over the last 100 years. Incremental steps have been made from 1913 till today in an actualisation process of the first analytical model developed by Harris, this process aiming at better incorporating aspects of real industrial problems and constraints. Several researchers have added new incremental conditions attempting to reflect real case problems as well as to change the emphasis on different model conditions and constraints.

The work done after Harris has generally focussed on:

- Developing new cost and revenue functions for specific applications and for various types of systems in practice;
- Categorising new input data and factors to be included in the analysis;
- Relaxing constraints.

The lot sizing literature from the last century, easily demonstrates the impossibility for a single model to describe “perfectly” the reality under study (as with any road map). Efforts to obtain a “perfect model” have often brought researchers to increase model complexity and consequently the time spent for computations. A model may become so complex as to be computationally infeasible to use, and may be accurate only at a particular moment in time (Tukey 1962, pp. 13-14). Box and Draper (1987) remind us that “all models are wrong, some are useful”. Grubbström (2001) discusses this trade-off between applicability, simplicity and level of technique of scientific models. The trade-off concept, always present in mathematical modelling, has been fully demonstrated by the Harris EOQ model. Even if we today can benefit from a large and highly connected literature, the basic model of Harris is always attractive for its simplicity and the minimal amount of data needed. For this reason, it is frequently applied by managers today, even 100 years after its creation.

Important new challenges are expected for sustainable supply chains in the near future. We need only think of the planning, implementation and management of reverse and sustainable supply networks, in which waste material becomes the raw material input to the echelon following, and when CO₂ emissions must be minimised. In this context, more than in others, supply chain actors need to collaborate in sharing costs and deciding common lot sizing strategy policies in order to jointly achieve an overall sustainable development. In particular the following aspects need to be investigated in the close future: to couple the total cost function with emission consequences, to take into consideration social aspects linked with material purchasing and handling, and to create new approaches in modelling lot sizing activities in the context of closed-loop supply chains.

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Appendix-

The Laplace Transform of a time function $a(t)$ written $\tilde{a}(s)$ or $\mathcal{L}\{a(t)\}$ is defined as $\mathcal{L}\{a(t)\} = \tilde{a}(s) = \int_{t=0}^{\infty} a(t)e^{-st} dt$, which transforms the time function $a(t)$ from the time domain into the frequency domain represented by the complex Laplace frequency s . In all normal cases of practical significance, there is a one-to-one correspondence between a time function and its transform, cf. Aseltine (1958).

But the NPV of a time dependent cash flow $a(t)$ has the following general form, which simply may be interpreted as the Laplace transform of $a(t)$, when the frequency s is exchanged for the continuous discount rate ρ :

$$\text{NPV} = \int_{t=0}^{\infty} a(t)e^{-\rho t} dt = \left[\int_{t=0}^{\infty} a(t)e^{-st} dt \right]_{s=\rho} = \tilde{a}(\rho). \quad (\text{A.1})$$

The cash flow may include discrete payments as well as continuous payment streams. In order to compare NPV expressions with average costs per time unit, one may make use of the *annuity stream*, which is the constant flow of cash providing a given NPV. With an infinite horizon, the Annuity Stream is the interest rate times the NPV value:

$$\text{AS} = \rho \cdot \text{NPV} = \rho \tilde{a}(\rho). \quad (\text{A.2})$$

Using Laplace transforms of the cash flows involved thereby opens up the whole of the extensive Laplace transform theory for application. Some applications may be found in Grubbström (2007).

We make use of a few theorems from Laplace transform theory, such as the transform of a time derivative being s times the transform of the function, the transform of a constant being the constant divided by s , the transform of a negative exponential time function $e^{-\theta t}$ being the inverse of the sum of s and the coefficient in the exponent θ , i.e. $1/(s + \theta)$, and the transform of an infinite chain of equal cycles being the transform of the first cycle divided by $(1 - e^{-sT})$, where T is the length of each cycle.

Developing generalisations of the basic model of Ghare and Schrader (1963), we take the transform of the left-hand member of Eq. (5) we have $\mathcal{L}\{dI/dt\} = s\tilde{I}(s)$, and of the right-hand member $-\mathcal{L}\{D(t) + \theta I(t)\} = -(D/s + \theta\tilde{I}(s))$. But replenishments/production are not included in Eq. (5). Assuming the standard case for finite rate models (as in Mahata 2012, originally from Taft 1918) that each cycle starts with a production ramp with slope q , lasting Q/q time units, where Q is the batch size, the inflow into inventory will have the transform $q(1 - e^{-sQ/q})/(s(1 - e^{-sT}))$, where T is the cycle length. Adding this inflow to the right-hand member gives us the equation

$$s\tilde{I}(s) = -\left(\frac{D}{s} + \theta\tilde{I}(s)\right) + \frac{q(1 - e^{-sQ/q})}{s(1 - e^{-sT})}, \quad (\text{A.3})$$

from which the inventory level $\tilde{I}(s)$ is solved:

$$\tilde{I}(s) = \frac{1}{s(s+\theta)} \left(\frac{q(1-e^{-sQ/q})}{1-e^{-sT}} - D \right). \quad (\text{A.4})$$

This is an infinite sequence of cycles, each cycle having the transform

$$\begin{aligned} \tilde{I}_{\text{cycle}}(s) &= \tilde{I}(s)(1-e^{-sT}) = \frac{1}{s(s+\theta)} \left(q(1-e^{-sQ/q}) - D(1-e^{-sT}) \right) = \\ &= \frac{1}{\theta} \left(\frac{1}{s} - \frac{1}{s+\theta} \right) \left(q(1-e^{-sQ/q}) - D(1-e^{-sT}) \right). \end{aligned} \quad (\text{A.5})$$

Translating this equation from the frequency domain into the time domain, we arrive at the time function shown in Eq.(16). To prove this, we may take the integral $\tilde{I}_{\text{cycle}}(s) = \int_{t=0}^{Q/q} \frac{(q-D)}{\theta} (1-e^{-\theta t}) e^{-st} dt + e^{\theta T} \int_{t=Q/q}^T \frac{D}{\theta} (e^{-\theta t} - e^{-\theta T}) e^{-st} dt$, which evaluated provides just (A.5). The one-to-one property of the transform and its corresponding time function then establishes the proof.

Introducing out-of-pocket holding costs corresponding to out-payments proportional to the physical inventory level and using the out-of-pocket holding cost parameter $\hat{h}$, by Eqs (A.2) and (A.4) the Annuity Stream of this part of the cash flow will be

$$\text{AS}_{\text{out-of-pocket}} = \rho \text{NPV}_{\text{out-of-pocket}} = \rho \hat{h} \tilde{I}(\rho) = \frac{\hat{h}}{(\rho+\theta)} \left(\frac{q(1-e^{-\rho Q/q})}{1-e^{-\rho T}} - D \right). \quad (\text{A.6})$$

This expression may be compared with Mahata's AC expression $(\hat{h}/(\theta T))(qQ/q - DT)$, which is the limit of (A.6) when the discount rate is neglected $\rho \rightarrow 0$.

Similarly, the Annuity Stream of sales revenues less production costs we find from

$$\text{AS}_{\text{profits}} = \rho \text{NPV}_{\text{profits}} = rD - cq \frac{1-e^{-\rho Q/q}}{1-e^{-\rho T}} = (r-c)D - c \left(\frac{q(1-e^{-\rho Q/q})}{(1-e^{-\rho T})} - D \right), \quad (\text{A.7})$$

where the first term shows average profits, if there were no depletion, and the second the annuity stream of expenses due to depleted items. This second term collapses into Mahata's AC expression $(c/T)(pQ/q - DT)$ when $\rho \rightarrow 0$.

As for trade credits, we can make use of the following NPV theorem concerning the cash flow associated with a loan. Let $\text{NPV}_{\text{principal}}$ denote the NPV of the loaned amount and the repayments (the latter with opposite sign and they eventually add up to the loaned amount), but excluding interest payments. It is then straightforward to show that the NPV of the loan also including consideration to interest payments written NPV_{loan} may be expressed as

$$\text{NPV}_{\text{loan}} = (1 - \hat{\rho}/\rho) \text{NPV}_{\text{principal}} \quad (\text{A.8})$$

where $\hat{\rho}$ is the interest rate attached to the loan contract, and ρ the discount rate representing the opportunity cost of capital, so $(-\hat{\rho}/\rho)\text{NPV}_{\text{principal}}$ is the NPV of interest charges. To prove this, we evaluate the following integral. Let the cash flow of loan and repayments be $a(t)$, hence $\text{NPV}_{\text{principal}} = \tilde{a}(\rho)$. Then after subtracting the interest payments on the outstanding loan amount at t from $a(t)$, we have $\text{NPV}_{\text{loan}} = \int_{t=0}^{\infty} \left(a(t) - \hat{\rho} \int_{\tau=0}^t a(\tau) d\tau \right) e^{-\rho t} dt = \tilde{a}(\rho) - \hat{\rho} \int_{t=0}^{\infty} \int_{\tau=0}^t a(\tau) e^{-\rho t} d\tau dt = \tilde{a}(\rho) - \hat{\rho} \int_{\tau=0}^{\infty} a(\tau) \int_{t=\tau}^{\infty} e^{-\rho t} dt d\tau = (1 - \hat{\rho}/\rho) \text{NPV}_{\text{principal}}$, where the order of integration has been changed in the fourth member

Eq. (A.8) is valid for any structure of the loan, i.e. the loaned amount may be distributed in portions over time, and the repayments spread out unevenly, as long as the cumulative cash flow eventually becomes zero. For a standard type of loan with in-payments (the borrowed money) coming earlier than the repayments, $\text{NPV}_{\text{principal}}$ must always be positive. Then, obviously the loan is favourable to take, if $\hat{\rho} < \rho$, and vice versa. When the interest rate of the loan equals the opportunity cost $\hat{\rho} = \rho$, NPV_{loan} is zero, so taking the loan or not would be of equal preference.

For the supplier's credit in the problem reported by Mahata (2012), generalised from Ghare and Schrader (1963) and Goyal (1985), we develop the following expression for the annuity stream related to supplier credit payments:

$$\text{AS}_{\text{supplier's credit}} = \rho \cdot \text{NPV}_{\text{supplier's credit}} = \frac{c\rho(1 - I_c/\rho)}{1 - e^{-\rho T}} \cdot \begin{cases} \left(q(1 - e^{-\rho M}) / \rho - qMe^{-\rho M} \right), & M < Q/q, \\ \left(q(1 - e^{-\rho Q/q}) / \rho - Qe^{-\rho M} \right), & Q/q \leq M. \end{cases} \quad (\text{A.9})$$

A first-order approximation in ρ (and neglecting terms with the product $I_c\rho$) gives the following AC expression (after changing sign):

$$\text{AC}_{\text{supplier's credit}} = - \left[\text{AS}_{\text{supplier's credit}} \right]_{1st \text{ order}} = \frac{(I_c - \rho)cq}{T} \begin{cases} M^2/2, & M < Q/q, \\ (Q/q)(M - (Q/q)/2), & Q/q \leq M. \end{cases} \quad (\text{A.10})$$

The second credit concerns the customer credit. For this loan, the annuity stream of the related payments amounts to

$$\text{AS}_{\text{credit to customers}} = \rho \cdot \text{NPV}_{\text{credit to customers}} = \frac{\rho(1 - I_e/\rho)}{1 - e^{-\rho T}} \cdot \begin{cases} \left(-r\alpha D(1 - e^{-\rho N}) / \rho + r\alpha DNe^{-\rho N} \right), & N < T, \\ \left(-r\alpha D(1 - e^{-\rho T}) / \rho + r\alpha DTe^{-\rho N} \right), & T \leq N, \end{cases} \quad (\text{A.11})$$

where both expressions to the right are easily shown to be non-positive.