Department of Mathematics

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Muni S. Srivastava and Martin Singull

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Linköpings universitet
TEKNISKA HÖGSKOLAN

Department of Mathematics
Linköping University
S-581 83 Linköping, Sweden.

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Muni S. Srivastava* and Martin Singull†

*Department of Statistics, University of Toronto,
100 St. George Street, Toronto, Canada, M5S 3G3
E-mail: srivasta@utstat.toronto.edu

†Department of Mathematics, Linköping University,
SE-581 83 Linköping, Sweden
E-mail: martin.singull@liu.se

Abstract

In this paper, we consider the problem of estimating and testing a general linear hypothesis in a general multivariate linear model, the so called Growth Curve model, when the $p \times N$ observation matrix is normally distributed with an unknown covariance matrix.

The maximum likelihood estimator (MLE) for the mean is a weighted estimator with the inverse of the sample covariance matrix which is unstable for large p close to N and singular for p larger than N . We modify the MLE to an unweighted estimator and propose a new test which we compare with the previous likelihood ratio test (LRT) based on the weighted estimator, i.e., the MLE. We show that the performance of this new test based on the unweighted estimator is better than the LRT based on the MLE.

For the high-dimensional case, when p is larger than N , we construct two new tests based on the trace of the variation matrices due to the hypothesis (between sum of squares) and the error (within sum of squares).

To compare the performance of all four tests we compute the attained significance level and the empirical power.

Keywords: Asymptotic distribution; High dimension; GMANOVA; Growth Curve Model; Estimation; Hypothesis testing; Power comparison.

1 Introduction

In this paper, we consider the problem of estimating and testing a general linear hypothesis in a general multivariate linear model, the so called Growth Curve Model, when the $p \times N$ observation matrix is normally distributed with an unknown covariance matrix. The Growth Curve model introduced by Potthoff and Roy (1964) has been extensively studied over many years. It is a generalized multivariate analysis of variance model (GMANOVA) which belongs to the curved exponential family. The mean structure for the Growth Curve model is bilinear in contrary to the ordinary multivariate analysis of variance (MANOVA) model where it is linear. For more details about the Growth Curve model see e.g., Kollo and von Rosen (2005); Kshirsagar and Smith (1995); Srivastava and Khatri (1979); Srivastava and von Rosen (1999).

For the Growth Curve model, when no assumption about the covariance matrix was made, Potthoff and Roy (1964) originally derived a class of weighted estimators for the mean parameter matrix. Khatri (1966) extended this result and showed that the MLE also is a weighted estimator. Under a certain covariance structure, Rao (1967) and Reinsel (1982) have shown that the unweighted estimator also is the MLE.

The comparison of the mean vectors of several groups with independently normally distributed observation vectors is discussed in many papers, see for example the review paper by Srivastava (2013) and the references therein. Here below follows a short summary.

A test if two mean vectors are equal has been proposed by Dempster (1958), under the assumption that the two distributions have the same covariance matrix. Bai and Saranadasa (1996) proposed another asymptotically equivalent test which does not require the assumption of normality but have asymptotically the same power as the one proposed by Dempster (1958). Srivastava (2007) proposed a Hotelling's T^2 type test, by using Moore-Penrose inverse of the sample covariance matrix instead of the inverse when N is smaller than p . It may be noted that all the above discussed tests are invariant under the group of orthogonal matrices. A test that is invariant under the group of non-singular diagonal matrices has recently been proposed by Srivastava and Du (2008) under the normal distribution and Srivastava (2009) under non-normality. It may be noted that this test is not invariant under the transformation by orthogonal matrices.

In MANOVA, when testing the equality of the mean vectors of several groups, it is assumed that all the groups have the same covariance matrix. Fujikoshi et al. (2004) generalize the two-sample tests given by Dempster (1958) to the MANOVA problem, under the assumption that $(p/n) \rightarrow c \in (0, \infty)$. Other tests that do not require this assumption have been proposed

by Srivastava and Fujikoshi (2006). Srivastava and Fujikoshi (2006) and Srivastava (2007) generalized the test given by Bai and Saranadasa (1996) for two-sample to MANOVA and gave its asymptotic distribution without any assumption on the ratio of convergence of (p/n) . Schott (2007) proposed the same test as proposed by Srivastava and Fujikoshi (2006) and Srivastava (2007) but required the assumption that $(p/n) \rightarrow c \in (0, \infty)$ to obtain the asymptotic distribution of the test statistic. This is a severe restriction.

The above tests are, however, not invariant under the transformation by non-singular diagonal matrices. A test that has this property for the MANOVA problem has been recently proposed by Yamada and Srivastava (2012) under normality assumption, and Srivastava and Kubokawa (2013) under non-normality assumption.

In this paper we will generalize the idea given by Yamada and Srivastava (2012) to the Growth Curve model. We will start with considering the problem of estimating the parameters with focus on the mean parameter matrix. The MLE for the mean is a weighted estimator with the inverse of the sample covariance matrix which is very unstable for p close to N and singular for N less than p . We modify the MLE to an unbiased and unweighted estimator, just by removing the inverse of the sample covariance matrix. We show that for large N , the new unweighted estimator of the mean parameter matrix has a larger covariance than the weighted one, as expected since the weighted estimator is the MLE. But, when also p is large, still less than N , we show that the covariance for the weighted estimator (MLE) can actually be larger than the covariance of the unweighted estimator.

The organization of the paper is as follows. In Section 2, we give the Growth Curve model and the basic idea how to estimate the parameters. The MLE and the new unweighted estimator are given in Section 3, where also a comparison of the two estimators is discussed. In Section 4 we give the LRT and the new test based on the unweighted estimator. When p is larger than N none of the tests given in Section 4 are applicable. Two new tests are proposed in Section 5 which cover the high-dimensional case for p larger than N . In Section 6 the attained significance levels and the empirical powers are compared. The paper concludes in Section 7.

2 Model

In this section, some general ideas of how to estimate parameters in the Growth Curve model will be presented. The Growth Curve model was introduced by Potthoff and Roy (1964) and is defined as follows.

Definition Let $\mathbf{X} : p \times N$ and $\boldsymbol{\xi} : q \times m$ be the observation and parameter

matrices, respectively, and let $\mathbf{B} : p \times q$ and $\mathbf{A} : m \times N$ be the within and between individual design matrices, respectively. Suppose that $q \leq p$ and $r + p \leq N$, where $r = \text{rank}(\mathbf{A})$. The Growth Curve model is given by

$$\mathbf{X} = \mathbf{B}\boldsymbol{\xi}\mathbf{A} + \boldsymbol{\varepsilon},$$

where the columns of $\boldsymbol{\varepsilon}$ are assumed to be independently p -variate normally distributed with mean zero and an unknown positive definite covariance matrix $\boldsymbol{\Sigma}$, i.e.,

$$\boldsymbol{\varepsilon} \sim N_{p,N}(\mathbf{0}, \boldsymbol{\Sigma}, \mathbf{I}_N),$$

where $N_{p,N}(\bullet, \bullet, \bullet)$ stands for the matrix normal distribution.

In this paper, for the sake of convenience and without loss of generality, we will assume that the design matrices $\mathbf{B}$ and $\mathbf{A}$ are of full rank, i.e., $\text{rank}(\mathbf{B}) = q \leq p$ and $r = \text{rank}(\mathbf{A}) = m \leq N$.

Our starting point for estimating parameters in the Growth Curve model are the two jointly sufficient statistics, the "mean" $\mathbf{X}\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}$ and the sum of squares matrix

$$\mathbf{V} = \mathbf{X} \left(\mathbf{I} - \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A} \right) \mathbf{X}'. \quad (1)$$

The distribution of the "mean" and the sum of squares matrix are given by

$$\mathbf{X}\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A} \sim N_{p,N} \left(\mathbf{B}\boldsymbol{\xi}\mathbf{A}, \boldsymbol{\Sigma}, \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A} \right)$$

and

$$\mathbf{V} \sim W_p(\boldsymbol{\Sigma}, n),$$

where $W_p(\bullet, \bullet)$ stands for the Wishart distribution with $n = N - m$ degrees of freedom. Observe that $\mathbf{V}$ is independent of the parameter $\boldsymbol{\xi}$.

If $\boldsymbol{\Sigma}$ is known, by minimizing

$$\text{tr} \boldsymbol{\Sigma}^{-1}(\mathbf{X} - \mathbf{B}\boldsymbol{\xi}\mathbf{A})(\mathbf{X} - \mathbf{B}\boldsymbol{\xi}\mathbf{A})'$$

the MLE of $\mathbf{B}\boldsymbol{\xi}\mathbf{A}$ is

$$\mathbf{B}\tilde{\boldsymbol{\xi}}\mathbf{A} = \mathbf{P}_B^{\boldsymbol{\Sigma}} \mathbf{X} \mathbf{P}_{A'}, \quad (2)$$

where

$$\mathbf{P}_B^{\boldsymbol{\Sigma}} = \mathbf{B}(\mathbf{B}'\boldsymbol{\Sigma}^{-1}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}^{-1}, \quad (3)$$

$$\mathbf{P}_{A'} = \mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}. \quad (4)$$

We make the following important observation: The role of Σ is twofold; it is used as a weight matrix in $\mathbf{B}(\mathbf{B}'\Sigma^{-1}\mathbf{B})^{-1}\mathbf{B}'\Sigma^{-1}$ in order to obtain an estimator of ξ , and it describes the variation in data.

In estimator $\mathbf{B}\tilde{\xi}\mathbf{A}$ given in equation (2) there are two orthogonal projectors involved, namely $\mathbf{P}_B^\Sigma$ and $\mathbf{P}_{A'}$. Thus, tests and estimates can also be obtained by a decomposition of the whole tensor space as done in Seid Hamid and von Rosen (2006), von Rosen (1995) and Ohlson and von Rosen (2010).

In the next section, we give the MLE of ξ as well as that of Σ .

3 Estimators

In this section we will give the MLE of the mean parameter ξ , when the covariance matrix Σ is unknown, and propose a new unbiased and unweighted estimator of ξ . The MLE of Σ is also given. Finally we will compare the MLE and the new unweighted estimator.

3.1 MLEs of the parameters ξ and Σ

The MLEs of ξ and Σ were given by Khatri (1966). The MLE of ξ is given by

$$\hat{\xi}_{MLE} = (\mathbf{B}'\mathbf{V}^{-1}\mathbf{B})^{-1}\mathbf{B}'\mathbf{V}^{-1}\mathbf{X}\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}, \quad (5)$$

where $\mathbf{V}$ is given in equation (1). The MLE of Σ is

$$N\hat{\Sigma}_{MLE} = (\mathbf{X} - \mathbf{B}\hat{\xi}_{MLE}\mathbf{A})(\mathbf{X} - \mathbf{B}\hat{\xi}_{MLE}\mathbf{A})' = \mathbf{V} + \hat{\mathbf{R}}_1\hat{\mathbf{R}}_1', \quad (6)$$

where

$$\hat{\mathbf{R}}_1 = \mathbf{X}\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{A} - \mathbf{B}\hat{\xi}_{MLE}\mathbf{A} = (\mathbf{I}_p - \mathbf{P}_B^\mathbf{V})\mathbf{X}\mathbf{P}_{A'}, \quad (7)$$

and where $\mathbf{P}_B^\mathbf{V}$ and $\mathbf{P}_{A'}$ are given in (3) and (4), respectively. The mean and covariance matrix for the estimator $\hat{\xi}_{MLE}$ are given in Kollo and von Rosen (2005) as

$$\begin{aligned} \mathbb{E}(\hat{\xi}_{MLE}) &= \xi, \\ \text{cov}(\hat{\xi}_{MLE}) &= \frac{n-1}{n-1-(p-q)}(\mathbf{A}\mathbf{A}')^{-1} \otimes (\mathbf{B}'\Sigma^{-1}\mathbf{B})^{-1}, \end{aligned}$$

if $n-1-(p-q) > 0$. Furthermore, since $q \leq p$, we note that

$$\frac{n-1}{n-1-(p-q)} \geq 1.$$

Following Gleser and Olkin (1970), several authors have described the distribution of $\hat{\xi}_{MLE}$, see for example Kabe (1975) and Kenward (1986) for different techniques and expressions. Furthermore, Fujikoshi (1985, 1987) derived asymptotic expansions for the density of linear combinations of the elements in $\hat{\xi}_{MLE}$. The asymptotic expansion given by Fujikoshi are based on a decomposition of $\hat{\xi}_{MLE}$ into a sum of a normally distributed variable and another independently distributed random variable.

Later Fujikoshi and Shimizu (1989) extended the result of Fujikoshi (1985) and considered particular mixtures of distributions. They used a result given by Gleser and Olkin (1970) where it was observed that the distribution of $\hat{\xi}_{MLE}$ was a function of three independent random variables.

Recently, Kollo et al. (2007) derived an Edgeworth-type approximation of the density of $\hat{\xi}_{MLE}$ as a mixture of a normal and a Kotz-type distribution, i.e., an elliptical distribution.

All the different ways above to describe the distribution of $\hat{\xi}_{MLE}$ are cumbersome and difficult to use in practice. In the next section we will give a new unweighted estimator of ξ which will directly follows a normal distribution.

3.2 Unweighted estimator of the parameter ξ

A natural alternative to the MLE would be an unweighted estimator of ξ given by

$$\hat{\xi} = (B'B)^{-1} B' X A' (A A')^{-1}. \quad (8)$$

This estimator is simpler than the MLE, given in (5), since we do not need to calculate the inverse of the sum of squares matrix V^{-1} . Using the properties for the matrix normal distribution it follows directly that the distribution of the estimator is given by

$$\hat{\xi} \sim N_{q,m}(\xi, (B'B)^{-1} B' \Sigma B (B'B)^{-1}, (A A')^{-1}),$$

i.e., we have

$$\begin{aligned} E(\hat{\xi}) &= \xi, \\ \text{cov}(\hat{\xi}) &= (A A')^{-1} \otimes (B'B)^{-1} B' \Sigma B (B'B)^{-1}. \end{aligned}$$

Also, an unbiased and consistent estimator of the covariance matrix Σ is given by

$$n\hat{\Sigma} = V \sim W_p(\Sigma, n),$$

with $n = N - m$ degrees of freedom

3.3 Compare the estimators $\widehat{\xi}_{MLE}$ and $\widehat{\xi}$

Both $\widehat{\xi}_{MLE}$ given in (5), and $\widehat{\xi}$, given in (8), are unbiased. The covariances for $\widehat{\xi}_{MLE}$ and $\widehat{\xi}$ respectively are given by

$$\begin{aligned}\text{cov}\left(\widehat{\xi}_{MLE}\right) &= \frac{n-1}{n-1-(p-q)}(\mathbf{A}\mathbf{A}')^{-1} \otimes (\mathbf{B}'\Sigma^{-1}\mathbf{B})^{-1}, \\ \text{cov}\left(\widehat{\xi}\right) &= (\mathbf{A}\mathbf{A}')^{-1} \otimes (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\Sigma\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}.\end{aligned}$$

To compare the two estimators we want to compare their covariances, i.e., we want to compare $(\mathbf{B}'\Sigma^{-1}\mathbf{B})^{-1}$ and $(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\Sigma\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}$. Following Rao (1967) (Lemma 2.c) one can show that

$$(\mathbf{B}'\Sigma^{-1}\mathbf{B})^{-1} \leq (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\Sigma\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}$$

with equality if and only if $\mathcal{C}(\Sigma^{-1}\mathbf{B}) = \mathcal{C}(\mathbf{B})$, where $\mathcal{C}(\mathbf{B})$ stands for the linear space generated by the columns of $\mathbf{B}$. The inequality is with respect to the Loewner partial ordering, i.e., $(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\Sigma\mathbf{B}(\mathbf{B}'\mathbf{B})^{-1} - (\mathbf{B}'\Sigma^{-1}\mathbf{B})^{-1}$ is nonnegative definite. So for large n , the unweighted unbiased estimator of ξ has a larger covariance than the weighted one, as expected since the weighted estimator is the MLE. When also p is large, but still less than n , the factor $(n-1)/(n-1-(p-q))$ can be much greater than one and the covariance for the weighted estimator can actually be larger than the covariance for the unweighted estimator.

4 Two different likelihood ratio tests

The general problem of testing the hypothesis

$$H : \mathbf{L}\xi\mathbf{M} = \mathbf{0} \quad \text{vs.} \quad A : \mathbf{L}\xi\mathbf{M} \neq \mathbf{0}$$

is equivalent to testing the hypothesis

$$H : \xi = \mathbf{L}_1\delta\mathbf{M}_1 \quad \text{vs.} \quad A : \xi \neq \mathbf{L}_1\delta\mathbf{M}_1,$$

for some $\mathbf{L}_1$ and $\mathbf{M}_1$ depending on $\mathbf{L}$ and $\mathbf{M}$ and their dimension. Since $\mathbf{L}_1$ and $\mathbf{M}_1$ can be combined with $\mathbf{B}$ and $\mathbf{A}$, without loss of generality, we shall consider only the hypothesis

$$H : \xi = \mathbf{0} \quad \text{vs.} \quad A : \xi \neq \mathbf{0}.$$

Given the MLEs (5) and (6), the LRT is given in Srivastava and Khatri (1979) Chapter 6.4.4-6.4.5 (page 193-195) as

$$\lambda_{MLE}^{2/N} = \frac{|\mathbf{V} + \widehat{\mathbf{R}}_1 \widehat{\mathbf{R}}_1'|}{|\mathbf{V} + \mathbf{V}_1|},$$

where $\mathbf{V}$ and $\widehat{\mathbf{R}}_1$ are given in equation (1) and (7), respectively and $\mathbf{V}_1 = \mathbf{X} \mathbf{A}' (\mathbf{A} \mathbf{A}')^{-1} \mathbf{A} \mathbf{X}'$. Using Box's method to approximate the distribution of $T_1 = -\kappa_1 \log \lambda_{MLE}^{2/N}$, one can show that for large N ,

$$P_0(T_1 > c) = P(\chi_f^2 > c),$$

where $\kappa_1 = n - p + q - (q - m + 1)/2$ and $f = qm$. See Srivastava and Khatri (1979) Chapter 6.3.6-6.3.7 (page 177-186) for more details.

Equivalently testing the hypothesis $H : \boldsymbol{\xi} = \mathbf{0}$ one can test the hypothesis

$$H : \boldsymbol{\eta} = \mathbf{0} \quad \text{vs.} \quad A : \boldsymbol{\eta} \neq \mathbf{0},$$

where $\boldsymbol{\eta} = (\mathbf{B}' \mathbf{B})^{1/2} \boldsymbol{\xi} (\mathbf{A} \mathbf{A}')^{1/2}$. Using the unweighted estimator $\widehat{\boldsymbol{\xi}}$ given in (8) the distribution of $\widehat{\boldsymbol{\eta}} = (\mathbf{B}' \mathbf{B})^{1/2} \widehat{\boldsymbol{\xi}} (\mathbf{A} \mathbf{A}')^{1/2}$ is

$$\widehat{\boldsymbol{\eta}} = (\mathbf{B}' \mathbf{B})^{1/2} \widehat{\boldsymbol{\xi}} (\mathbf{A} \mathbf{A}')^{1/2} \sim N_{q,m}(\boldsymbol{\eta}, \boldsymbol{\Delta}, \mathbf{I}_m),$$

where

$$\boldsymbol{\Delta} = (\mathbf{B}' \mathbf{B})^{-1/2} \mathbf{B}' \boldsymbol{\Sigma} \mathbf{B} (\mathbf{B}' \mathbf{B})^{-1/2}.$$

Furthermore, we estimate $\boldsymbol{\Delta}$ with $\widehat{\boldsymbol{\Delta}} = \frac{1}{n} \mathbf{V}^*$ which is distributed as

$$\mathbf{V}^* = (\mathbf{B}' \mathbf{B})^{-1/2} \mathbf{B}' \mathbf{V} \mathbf{B} (\mathbf{B}' \mathbf{B})^{-1/2} \sim W_q(\boldsymbol{\Delta}, n).$$

Also, $\widehat{\boldsymbol{\eta}}$ and $\mathbf{V}$ (or $\mathbf{V}^*$) are independently distributed. The likelihood function of $\boldsymbol{\eta}$ and $\boldsymbol{\Delta}$ is given by

$$L(\boldsymbol{\eta}, \boldsymbol{\Delta} | \widehat{\boldsymbol{\eta}}, \mathbf{V}^*) = c |\boldsymbol{\Delta}|^{-N/2} |\mathbf{V}^*|^{(n-q-1)/2} \text{etr} \left\{ \frac{1}{2} \boldsymbol{\Delta}^{-1} (\mathbf{V}^* + N(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})') \right\},$$

where c is a constant, $\text{etr} \{\mathbf{Q}\} = \exp \{\text{tr} \mathbf{Q}\}$ and the notation $(\mathbf{Q})(\cdot)'$ is used instead of $(\mathbf{Q})(\mathbf{Q})'$ for some matrix $\mathbf{Q}$. Thus, another LRT is given by

$$\lambda^{2/N} = \frac{|\mathbf{V}^*|}{|\mathbf{V}^* + \widehat{\boldsymbol{\eta}} \widehat{\boldsymbol{\eta}}'|}.$$

Again using Box's method to approximate the distribution, one can show that for large N , the distribution of $T_2 = -\kappa_2 \log \lambda^{2/N}$ is given by

$$P_0(T_2 > c) = P(\chi_f^2 > c),$$

where $\kappa_2 = n - (q - m + 1)/2$ and $f = qm$. See Srivastava and Khatri (1979) Chapter 6.3.6-6.3.7 (page 177-186) for more details.

5 Large p and small N

For high dimensions, when $N - m < p$, then $\mathbf{V}$ is singular and none of the tests given in Section 4 are applicable. We will propose two new tests.

Now consider again the variable

$$\widehat{\boldsymbol{\eta}} \sim N_{q,m}(\boldsymbol{\eta}, \boldsymbol{\Delta}, \mathbf{I}_m).$$

Under the hypothesis $H : \boldsymbol{\xi} = \mathbf{0}$, i.e., $\boldsymbol{\eta} = \mathbf{0}$ we have

$$\mathbf{W} = \widehat{\boldsymbol{\eta}}\widehat{\boldsymbol{\eta}}' \sim W_q(\boldsymbol{\Delta}, m).$$

We see that $\mathbf{W}$ and $\widehat{\boldsymbol{\Sigma}}$ are independently distributed.

5.1 Test statistic based on $\text{tr}\mathbf{W}$

The mean and variance of the statistic $\text{tr}\mathbf{W}$, under the hypothesis $H : \boldsymbol{\xi} = \mathbf{0}$, are given by

$$\begin{aligned} \text{E}(\text{tr}\mathbf{W}) &= m \text{tr}\boldsymbol{\Delta}, \\ \text{var}(\text{tr}\mathbf{W}) &= 2m \text{tr}\boldsymbol{\Delta}^2. \end{aligned}$$

Under the assumption of normality, unbiased and consistent estimators of $\text{tr}\boldsymbol{\Delta}$ and $\text{tr}\boldsymbol{\Delta}^2$ are given by

$$\begin{aligned} \widehat{\text{tr}\boldsymbol{\Delta}} &= \frac{1}{n} \text{tr}\mathbf{V}^*, \\ \widehat{\text{tr}\boldsymbol{\Delta}^2} &= \frac{1}{(n-1)(n+2)} \left(\text{tr}\mathbf{V}^{*2} - \frac{1}{n} (\text{tr}\mathbf{V}^*)^2 \right), \end{aligned}$$

respectively, see Srivastava (2005) for details. Furthermore, we know that under the hypothesis $H : \boldsymbol{\xi} = \mathbf{0}$,

$$\tilde{T}_3 = \frac{1}{\sqrt{m}} \frac{\text{tr}\mathbf{W} - m \text{tr}\boldsymbol{\Delta}}{\sqrt{2 \text{tr}\boldsymbol{\Delta}^2}} \rightarrow N(0, 1).$$

Substituting unbiased and consistent estimators of $\text{tr}\boldsymbol{\Delta}$ and $\text{tr}\boldsymbol{\Delta}^2$ we get a test statistic, proposed by Srivastava and Fujikoshi (2006) and Srivastava (2007), which is given by

$$T_3 = \frac{\text{tr}\mathbf{W} - \frac{m}{n} \text{tr}\mathbf{V}^*}{\sqrt{\frac{2m}{(n-1)(n+2)} \left(\text{tr}\mathbf{V}^{*2} - \frac{1}{n} (\text{tr}\mathbf{V}^*)^2 \right)}} \rightarrow N(0, 1).$$

The test statistic T_3 is invariant under the group of orthogonal transformations, but not invariant under the units of measurements, which is an undesirable feature. That is, the test is not invariant under a diagonal transformation and the test statistic T_3 changes.

5.2 Test statistic based on $\text{tr} \mathbf{W} \mathbf{D}_{\hat{\Delta}}^{-1}$

In this section we will propose a test statistic which is invariant under diagonal transformations. The test statistic will be based on the quantity $\text{tr} \mathbf{W} \mathbf{D}_{\hat{\Delta}}^{-1}$, where $\mathbf{D}_{\hat{\Delta}} = \text{diag}(\hat{\Delta})$, the diagonal matrix with the diagonal elements of $\hat{\Delta}$. More precisely the test statistic is based on $\text{tr} \mathbf{W} \mathbf{D}_{\mathbf{V}^*}^{-1}$ and given as

$$T_4 = \frac{n \text{tr} \mathbf{W} \mathbf{D}_{\mathbf{V}^*}^{-1} - n q m / (n - 2)}{\sqrt{2m(\text{tr} \mathbf{R}^2 - q^2/n) c_{q,n}}},$$

where $\mathbf{R} = \mathbf{D}_{\mathbf{V}^*}^{-1/2} \mathbf{V}^* \mathbf{D}_{\mathbf{V}^*}^{-1/2}$ and $c_{q,n} = 1 + \text{tr} \mathbf{R}^2 / q^{3/2}$ is an adjustment factor converging to 1 in probability as $(n, q) \rightarrow \infty$, $n = O(q^\delta)$, $\delta > 1/2$ proposed by Srivastava and Du (2008). Define the population correlation matrix as

$$\mathcal{R} = \mathbf{D}_{\Delta}^{-1/2} \Delta \mathbf{D}_{\Delta}^{-1/2}.$$

It has been shown by Srivastava and Du (2008) that a consistent estimator of $\text{tr} \mathcal{R}^2 / q$ is given by

$$\frac{1}{q} \left(\text{tr} \mathbf{R}^2 - \frac{q^2}{n} \right).$$

Hence, for large n and q we have

$$T_4 \stackrel{d}{=} \frac{(n \text{tr} \mathbf{W} \mathbf{D}_{\mathbf{V}^*}^{-1} - q m) / \sqrt{q}}{\sqrt{2m \text{tr} \mathcal{R}^2 / q}}.$$

We will give the asymptotic distribution of the test statistic T_4 under the following assumptions

A0: $n = O(q^\delta)$, $\delta > 1/2$,

A1: $\frac{1}{q} \text{tr} \mathcal{R}^2 = O(1)$ as $q \rightarrow \infty$,

A2: $\frac{1}{q^2} \text{tr} \mathcal{R}^4 = o(1)$ as $q \rightarrow \infty$.

Theorem 5.1 *Suppose that the assumptions A0-A2 hold. Then under the null hypothesis $H : \boldsymbol{\xi} = \mathbf{0}$ the following is true*

$$\lim_{(n,q) \rightarrow \infty} P(T_4 \leq x) = \Phi(x),$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution function.

The proof for the asymptotic distribution for the test statistic T_4 is very similar to the result given Yamada and Srivastava (2012) and will be omitted here.

6 Compare the performance

To compare the four tests we can compute the attained significance level (ASL) and the empirical power. Let c be the critical value from the distribution considered for the test statistics. With 10 000 simulated replications under the null hypothesis, the ASL is computed as

$$\hat{\alpha} = \frac{(\# \text{ of } t_H \geq c)}{(\# \text{ simulated replications})},$$

where t_H is the values of the test statistics derived from the simulated data under the null hypothesis. We set the nominal significance level to $\alpha = 0.05$.

For the simulation study let

$$\mathbf{B} = (b_{ij}), \quad b_{ij} \sim U(0, 1), \quad i = 1, \dots, p, \quad j = 1, \dots, q,$$

and assume that we have two groups such that

$$\mathbf{A} = \begin{pmatrix} \mathbf{1}'_{N_1} & \mathbf{0}'_{N_2} \\ \mathbf{0}'_{N_1} & \mathbf{1}'_{N_2} \end{pmatrix},$$

i.e., $m = 2$. For simplicity we will put $N_1 = N_2 = N/2$. Furthermore, N , p and q will vary depending on which asymptotic is considered.

Since the covariances for the estimators in Section 3 depending on $\boldsymbol{\Sigma}$, we will use three different covariance matrices for the simulation study. The first one is identity, i.e., $\boldsymbol{\Sigma}_1 = \mathbf{I}_p$. Furthermore, let $\mathbf{D}_j = \text{diag}(\sigma_1^{(j)}, \dots, \sigma_p^{(j)})$, for $j = 2, 3$, be two different diagonal matrices. Define $\sigma_i^{(2)} = 2 + (p - i + 1)/p$, for $i = 1, \dots, p$, and $\sigma_i^{(3)}$, for $i = 1, \dots, p$, are independent observations from $\sqrt{U} [0, 2]$, respectively. Also, let $\mathbf{R} = (\rho_{ij})$, where $\rho_{ij} = (-1)^{i+j} r^{|i-j|^f}$. The other two covariance matrices that we will use are given as

$$\boldsymbol{\Sigma}_j = \mathbf{D}_j \mathbf{R} \mathbf{D}_j, \quad \text{with } r = 0.2, \quad f = 0.1, \quad \text{for } j = 2, 3.$$

	q	ASL		Power		ASL		Power	
		T_1	T_2	T_1	T_2	T_3	T_4	T_3	T_4
(I)	4	5.43	5.42	36.83	82.06	8.01	5.69	86.83	85.35
	6	5.20	5.29	61.02	96.70	7.72	6.00	98.59	98.45
	10	5.51	5.37	95.23	100.00	6.86	5.44	100.00	100.00
	14	6.34	5.36	99.84	100.00	6.68	5.58	100.00	100.00
	18	6.86	5.95	100.00	100.00	6.23	4.98	100.00	100.00
	24	9.82	8.21	100.00	100.00	6.54	5.62	100.00	100.00
(II)	4	4.81	4.81	24.91	51.81	7.40	4.92	38.00	39.13
	6	5.17	4.65	39.86	75.92	7.41	5.43	65.52	67.01
	10	5.87	5.03	77.23	98.71	6.89	5.58	98.60	98.48
	14	6.83	5.60	95.66	99.97	6.69	5.17	99.99	100.00
	18	7.42	6.05	99.81	100.00	7.55	5.96	100.00	100.00
	24	9.37	7.77	100.00	100.00	7.61	5.91	100.00	100.00
(III)	4	5.09	5.23	53.61	91.99	7.73	5.39	85.18	87.24
	6	5.15	5.08	77.53	99.39	7.35	5.38	99.30	99.33
	10	5.84	5.15	99.00	100.00	7.22	5.58	100.00	100.00
	14	6.79	5.45	100.00	100.00	6.90	5.22	100.00	100.00
	18	7.61	6.04	100.00	100.00	6.70	5.45	100.00	100.00
	24	9.52	7.77	100.00	100.00	7.70	5.87	100.00	100.00

Table 1: ASL and empirical power for T_1 , T_2 , T_3 and T_4 for different q when $p = 30$ and $N = 50$.

Hence, we have three cases

$$(I) \ \Sigma = \Sigma_1 = \mathbf{I}, \quad (II) \ \Sigma = \Sigma_2 = \mathbf{D}_2 \mathbf{R} \mathbf{D}_2 \quad \text{and} \quad (III) \ \Sigma = \Sigma_3 = \mathbf{D}_3 \mathbf{R} \mathbf{D}_3.$$

We can compute the empirical power using two different critical values. We can either use the critical value c from the asymptotic distribution, or we can use the estimated critical value $\hat{c}$ calculated from the simulated data under the null hypothesis, i.e., the critical value calculated from the empirical null distribution. We will use the estimated critical value since the ASL is greatly affected for some tests.

The empirical power is calculated from 10 000 new replications simulated under the alternative hypothesis when $\boldsymbol{\xi} = (\xi_{ij})$ and $\xi_{ij} = 0.1$ if $i + j$ is even and zero otherwise. Let t_A be the value of the test statistic derived from the simulated data under the alternative hypothesis. The empirical power is given as

$$\hat{\beta} = \frac{(\# \text{ of } t_A \geq \hat{c})}{(\# \text{ simulated replications})}.$$

Comparisons

From Table 1 we see that for larger q , the significance level of test T_1 and T_2 are greatly affected, while the ASL of T_3 is affected for smaller q and slightly larger than the true level for the rest. The ASL of test T_4 follows the true level. From the empirical power we see that T_2 performs much better than T_1 , i.e., the unweighted estimator is preferred compared to the weighted estimator given by the MLE. Comparing the empirical power of all four tests, we see that T_2 , T_3 and T_4 are comparable and perform better than T_1 . Since the test T_4 has both good ASL and power that test should be preferred. In the next comparisons we choose smaller q , namely $q = 4$ and $q = 10$.

Comparing test T_1 and T_2 for $q = 4$ and $q = 10$ one can see in Table 2 that the ASL of T_1 is greatly affected when p is close to N while the ASL of T_2 is affected just when p is close to N for smaller N and larger q , i.e., when $q = 10$. Also in Table 2 one can see that the empirical power of the test T_2 is similar or better than T_1 . The test T_1 has poor power for the cases when p is close to N while T_2 has good power in these cases. It is worth to note that the empirical power of the test T_1 is almost never greater than the empirical power of T_2 . Hence, again the unweighted estimator is preferred compared to the weighted estimator given by the MLE.

If we compare T_3 and T_4 we see in Table 3 that the ASL for both the tests are similar meanwhile the ASL for T_4 seems to follow the true level slightly better. The empirical power of the two test are similar, even if the power for test T_3 is marginally better. Observe that these two test also work and behave good when the dimension p is larger than the sample size N .

Comparing all four tests we see, Table 2 and 3, that the test T_2 and T_4 have best ASL. All four test are greatly affected when p is close to N for smaller N and larger q but T_3 and T_4 not as much as T_1 and T_2 . The empirical power of test T_2 , T_3 and T_4 are similar and all better than T_1 .

$q = 4$			ASL		Power		$q = 10$			ASL		Power	
	p	N	T_1	T_2	T_1	T_2	p	N	T_1	T_2	T_1	T_2	
(I)	5	8	9.60	7.74	4.96	5.36	10	12	44.11	44.11	6.79	6.79	
	5	12	5.54	5.31	6.58	6.73	10	16	11.44	11.44	12.67	12.67	
	5	16	5.30	5.21	7.59	7.68	10	20	7.21	7.21	21.22	21.22	
	5	32	5.21	5.06	11.03	11.65	10	30	5.11	5.11	46.89	46.89	
	10	12	16.76	5.49	5.67	8.14	20	22	44.07	6.98	8.23	56.05	
	10	16	5.99	5.35	6.89	9.15	20	26	10.97	5.88	25.21	74.46	
	10	20	5.60	5.16	8.36	11.90	20	32	7.07	5.66	55.95	90.41	
	10	30	5.16	4.91	14.09	18.52	20	40	5.49	5.11	85.42	98.01	
	20	22	16.63	5.19	5.12	25.50	30	32	45.42	5.86	8.63	98.71	
	20	26	6.51	4.85	8.99	33.18	30	36	11.34	5.54	35.46	99.68	
	20	32	5.67	5.22	15.65	40.71	30	40	7.16	5.47	66.10	99.94	
	20	40	5.11	5.35	28.55	52.59	30	50	5.60	5.06	97.19	100.00	
	50	52	17.45	5.13	7.28	98.40	50	52	44.97	5.40	10.96	100.00	
	50	56	5.89	5.03	16.34	99.08	50	56	10.76	5.04	52.40	100.00	
	50	70	5.12	5.17	62.91	99.87	50	70	5.76	5.10	99.65	100.00	
	50	100	4.91	5.50	98.30	100.00	50	100	5.01	5.16	100.00	100.00	
(II)	5	8	9.53	7.44	5.39	5.37	10	12	44.66	44.66	6.06	6.06	
	5	12	5.60	5.43	5.98	5.94	10	16	11.22	11.22	9.08	9.08	
	5	16	5.56	5.29	5.83	6.21	10	20	7.40	7.40	12.42	12.42	
	5	32	4.82	4.57	8.98	9.25	10	30	5.92	5.92	22.26	22.26	
	10	12	17.36	6.20	5.20	5.62	20	22	46.44	6.95	6.68	33.22	
	10	16	6.31	5.13	5.80	7.26	20	26	11.38	6.19	15.19	45.01	
	10	20	5.76	5.21	6.81	7.67	20	32	6.56	6.01	36.27	62.33	
	10	30	4.86	4.95	9.97	10.78	20	40	5.97	5.25	60.09	82.00	
	20	22	16.80	5.16	5.05	14.19	30	32	44.73	6.24	7.92	81.63	
	20	26	6.34	4.77	7.01	17.73	30	36	11.07	5.80	24.67	90.41	
	20	32	4.88	5.19	12.03	20.39	30	40	7.50	5.77	43.44	94.57	
	20	40	4.80	5.03	18.28	27.11	30	50	5.64	5.28	82.46	99.33	
	50	52	17.24	4.66	6.23	81.24	50	52	44.86	5.29	9.02	99.99	
	50	56	5.70	4.96	12.15	85.12	50	56	11.25	5.25	33.78	100.00	
	50	70	5.27	5.05	41.07	93.37	50	70	5.67	4.84	95.07	100.00	
	50	100	5.28	4.82	88.23	99.22	50	100	5.23	5.35	100.00	100.00	
(III)	5	8	9.36	7.25	5.72	5.57	10	12	43.89	43.89	7.03	7.03	
	5	12	5.72	5.70	6.57	6.65	10	16	10.59	10.59	16.36	16.36	
	5	16	5.67	5.40	7.01	7.27	10	20	7.55	7.55	23.36	23.36	
	5	32	4.72	4.71	11.43	11.12	10	30	5.30	5.30	53.32	53.32	
	10	12	16.17	5.63	5.97	8.80	20	22	44.41	6.57	8.68	68.94	
	10	16	6.18	5.54	7.45	10.77	20	26	11.09	5.95	34.18	86.22	
	10	20	5.49	5.33	10.50	14.40	20	32	6.41	5.35	74.80	97.34	
	10	30	5.44	5.46	18.65	22.01	20	40	5.11	5.12	96.34	99.73	
	20	22	16.07	5.04	6.82	27.38	30	32	44.64	5.82	9.81	99.81	
	20	26	6.09	5.09	10.10	33.79	30	36	11.03	5.17	45.58	99.96	
	20	32	5.52	4.96	19.29	43.59	30	40	7.13	4.83	78.61	100.00	
	20	40	5.38	5.02	32.80	56.44	30	50	6.04	5.19	99.46	100.00	
	50	52	16.40	5.01	6.79	99.60	50	52	44.58	5.41	12.55	100.00	
	50	56	5.81	5.10	22.03	99.87	50	56	11.22	4.78	68.40	100.00	
	50	70	5.44	5.39	76.21	99.98	50	70	5.82	4.93	99.99	100.00	
	50	100	4.95	5.44	99.78	100.00	50	100	5.19	5.09	100.00	100.00	

Table 2: ASL and empirical power of T_1 and T_2 for different p and N when $q = 4$ and 10.

	$q = 4$		ASL		Power		$q = 10$		ASL		Power	
	p	N	T_3	T_4	T_3	T_4	p	N	T_3	T_4	T_3	T_4
(I)	5	8	13.30	13.75	6.49	6.66	10	12	9.84	10.97	22.81	18.86
	5	16	9.92	7.95	9.47	8.88	10	20	8.32	7.93	41.63	36.91
	5	32	8.06	5.93	16.95	16.00	10	30	7.54	6.38	65.02	62.63
	10	12	10.88	9.85	8.47	7.86	20	22	7.76	7.27	93.66	91.20
	10	20	9.05	6.91	11.27	10.93	20	32	7.35	6.18	99.55	99.34
	10	30	8.56	6.36	16.34	15.24	20	40	6.94	5.75	99.97	99.96
	20	22	8.42	6.74	28.00	26.19	30	32	7.28	6.31	99.97	99.94
	20	32	8.05	5.87	41.60	40.64	30	40	6.83	5.53	100.00	100.00
	20	40	8.04	5.64	52.21	51.33	30	50	6.84	5.26	100.00	100.00
	50	52	7.32	5.01	98.66	98.39	50	52	7.05	5.86	100.00	100.00
	50	70	7.50	4.90	99.83	99.84	50	70	6.71	5.16	100.00	100.00
	50	100	6.92	4.53	100.00	100.00	50	100	6.92	5.10	100.00	100.00
	100	50	7.84	5.37	100.00	100.00	100	50	7.43	6.10	100.00	100.00
	100	70	7.70	5.16	100.00	100.00	100	70	6.99	5.39	100.00	100.00
	200	100	7.13	4.34	100.00	100.00	200	100	6.99	5.00	100.00	100.00
	200	150	7.29	4.58	100.00	100.00	200	150	6.86	5.10	100.00	100.00
(II)	5	8	14.70	14.09	5.02	4.99	10	12	10.93	10.68	8.58	8.83
	5	16	10.51	8.59	5.83	5.87	10	20	8.74	7.25	13.04	14.54
	5	32	8.48	5.64	7.64	8.00	10	30	7.71	6.08	18.77	23.06
	10	12	10.65	9.80	6.10	5.78	20	22	8.22	7.12	47.07	47.20
	10	20	9.30	7.11	6.60	7.13	20	32	7.93	6.46	70.65	71.38
	10	30	8.19	5.66	8.28	8.87	20	40	7.47	5.82	84.82	84.95
	20	22	9.14	7.13	12.53	12.70	30	32	6.78	6.03	90.79	89.65
	20	32	7.62	5.62	19.41	19.37	30	40	6.99	5.75	96.96	96.54
	20	40	7.77	5.38	22.85	23.12	30	50	7.33	5.76	99.40	99.40
	50	52	7.55	5.41	67.50	66.56	50	52	7.17	5.74	100.00	100.00
	50	70	7.11	4.50	85.17	85.42	50	70	7.20	5.47	100.00	100.00
	50	100	7.33	4.71	96.42	96.51	50	100	6.72	4.88	100.00	100.00
	100	50	7.56	5.12	97.82	97.57	100	50	7.20	5.34	100.00	100.00
	100	70	7.58	4.94	99.79	99.80	100	70	6.75	4.94	100.00	100.00
	200	100	6.79	4.29	100.00	100.00	200	100	6.86	4.94	100.00	100.00
	200	150	6.98	4.59	100.00	100.00	200	150	6.79	4.72	100.00	100.00
(III)	5	8	14.26	13.61	6.81	5.88	10	12	10.22	10.33	19.61	19.59
	5	16	9.98	8.38	8.29	8.41	10	20	8.65	7.69	35.87	38.91
	5	32	8.40	6.01	13.11	14.44	10	30	8.14	6.37	60.29	64.69
	10	12	11.40	9.67	7.79	8.02	20	22	8.41	7.09	92.87	92.49
	10	20	9.36	7.20	9.82	10.33	20	32	7.86	6.29	99.48	99.55
	10	30	8.74	6.06	13.13	14.62	20	40	7.14	5.45	99.98	99.99
	20	22	8.76	6.86	28.05	26.93	30	32	7.83	6.47	99.99	99.98
	20	32	7.96	5.67	46.67	45.95	30	40	7.30	5.58	100.00	100.00
	20	40	7.66	5.43	58.70	58.21	30	50	7.00	5.42	100.00	100.00
	50	52	7.94	5.42	99.46	99.35	50	52	6.96	5.26	100.00	100.00
	50	70	7.30	4.75	99.97	99.96	50	70	6.99	5.29	100.00	100.00
	50	100	6.75	4.55	100.00	100.00	50	100	6.34	4.68	100.00	100.00
	100	50	8.31	5.56	100.00	100.00	100	50	6.78	5.12	100.00	100.00
	100	70	7.10	4.65	100.00	100.00	100	70	6.30	4.65	100.00	100.00
	200	100	6.88	4.41	100.00	100.00	200	100	7.01	5.11	100.00	100.00
	200	150	7.05	4.20	100.00	100.00	200	150	6.72	4.80	100.00	100.00

Table 3: ASL and empirical power of T_3 and T_4 for different p and N when $q = 4$ and 10.

$q = 10$			Normal		$\chi^2(2)$		$\chi^2(8)$	
	p	N	T_3	T_4	T_3	T_4	T_3	T_4
(I)	10	12	9.05	10.49	7.21	9.05	8.65	10.15
	10	20	7.68	7.35	6.17	6.32	7.32	7.07
	10	30	7.16	6.28	6.10	5.71	7.37	6.41
	20	22	7.74	7.14	7.14	6.78	7.69	7.01
	20	32	7.12	6.01	6.67	5.85	7.20	5.96
	20	40	7.06	5.69	6.65	5.55	6.87	5.83
	30	32	7.26	6.34	7.38	6.23	7.17	6.25
	30	40	7.32	6.13	6.91	5.53	7.43	5.96
	30	50	6.97	5.45	6.53	5.15	6.73	5.35
	50	52	7.11	5.25	6.39	5.07	6.78	5.72
	50	70	6.91	5.55	6.41	5.11	6.66	5.15
	50	100	6.30	4.66	6.44	4.74	6.49	4.80
	100	50	7.09	5.61	6.71	5.39	6.94	5.59
	100	70	6.81	5.09	6.53	4.99	7.09	5.09
	200	100	6.30	4.65	6.71	4.93	6.63	4.95
	200	150	6.70	5.00	6.34	4.55	6.61	4.78
(II)	10	12	10.45	10.57	9.00	8.83	9.37	9.91
	10	20	8.66	7.75	7.33	6.34	8.16	7.27
	10	30	8.30	6.73	7.32	5.48	7.91	6.40
	20	22	8.18	7.33	7.45	6.39	8.23	7.21
	20	32	7.54	5.91	7.36	6.10	7.76	6.26
	20	40	7.31	5.77	6.94	5.44	7.65	5.84
	30	32	7.83	6.24	7.42	5.92	7.37	5.85
	30	40	7.55	6.16	7.45	5.30	7.51	5.84
	30	50	7.39	5.35	7.14	5.16	7.09	5.36
	50	52	7.51	5.49	7.47	5.52	7.20	5.56
	50	70	7.53	5.59	7.12	5.14	8.06	5.72
	50	100	7.05	4.88	6.63	4.57	6.42	4.53
	100	50	7.33	5.51	7.09	5.31	7.29	5.57
	100	70	7.35	5.37	7.01	5.22	7.22	5.31
	200	100	6.80	5.04	6.59	4.72	6.61	5.07
	200	150	6.49	4.67	6.51	4.70	6.48	4.67
(III)	10	12	10.09	10.90	8.55	9.28	9.44	10.12
	10	20	8.73	7.69	7.42	6.62	8.54	7.34
	10	30	8.17	6.44	6.89	5.28	7.46	6.04
	20	22	7.78	6.84	7.31	6.36	8.18	7.27
	20	32	7.71	6.24	6.93	5.57	7.85	6.66
	20	40	7.28	5.58	7.24	5.98	7.17	5.56
	30	32	7.42	6.04	7.54	6.03	7.75	6.05
	30	40	7.46	5.63	7.11	5.52	7.66	6.19
	30	50	7.44	5.61	7.11	5.32	6.91	5.23
	50	52	6.74	5.12	7.38	5.77	6.65	5.42
	50	70	7.09	4.99	6.92	4.95	6.78	5.09
	50	100	6.56	4.45	7.28	5.17	6.95	4.86
	100	50	7.33	5.60	7.41	5.91	7.10	5.01
	100	70	7.04	5.04	6.83	4.87	7.24	5.19
	200	100	6.43	4.63	6.64	4.80	6.44	4.93
	200	150	6.68	4.85	5.87	4.10	7.00	4.95

Table 4: ASL of T_3 and T_4 under non-normality and when $q = 10$.

$q = 10$			Normal		$\chi^2(2)$		$\chi^2(8)$	
	p	N	T_3	T_4	T_3	T_4	T_3	T_4
(I)	10	12	28.80	23.54	24.95	14.85	26.75	19.63
	10	20	53.57	48.42	52.05	40.35	53.83	45.40
	10	30	77.86	74.75	80.73	72.12	78.40	73.91
	20	22	90.85	87.63	93.38	88.60	91.15	87.97
	20	32	98.96	98.46	99.60	99.20	99.34	98.91
	20	40	99.88	99.77	99.98	99.94	99.98	99.92
	30	32	99.98	99.97	100.00	99.99	100.00	100.00
	30	40	100.00	100.00	100.00	100.00	100.00	100.00
	30	50	100.00	100.00	100.00	100.00	100.00	100.00
	50	52	100.00	100.00	100.00	100.00	100.00	100.00
	50	70	100.00	100.00	100.00	100.00	100.00	100.00
	50	100	100.00	100.00	100.00	100.00	100.00	100.00
	100	50	100.00	100.00	100.00	100.00	100.00	100.00
	100	70	100.00	100.00	100.00	100.00	100.00	100.00
	200	100	100.00	100.00	100.00	100.00	100.00	100.00
	200	150	100.00	100.00	100.00	100.00	100.00	100.00
(II)	10	12	10.22	10.15	8.36	6.70	9.79	8.40
	10	20	14.75	14.59	13.80	12.44	14.06	13.42
	10	30	23.34	24.97	21.62	21.13	22.60	22.69
	20	22	35.98	35.03	34.60	29.83	34.03	32.16
	20	32	57.16	58.39	57.13	51.99	57.18	55.52
	20	40	72.60	72.44	73.57	71.25	72.62	72.00
	30	32	87.50	88.73	88.29	88.67	89.02	89.35
	30	40	95.37	96.03	96.91	97.26	96.21	96.73
	30	50	99.07	99.28	99.43	99.66	99.46	99.52
	50	52	100.00	100.00	100.00	100.00	100.00	100.00
	50	70	100.00	100.00	100.00	100.00	100.00	100.00
	50	100	100.00	100.00	100.00	100.00	100.00	100.00
	100	50	100.00	100.00	100.00	100.00	100.00	100.00
	100	70	100.00	100.00	100.00	100.00	100.00	100.00
	200	100	100.00	100.00	100.00	100.00	100.00	100.00
	200	150	100.00	100.00	100.00	100.00	100.00	100.00
(III)	10	12	22.40	20.35	19.45	13.82	20.40	17.08
	10	20	41.85	41.02	39.43	34.77	41.53	39.08
	10	30	68.90	69.35	71.21	69.40	69.84	69.00
	20	22	91.80	89.30	94.35	90.83	92.71	89.06
	20	32	99.24	98.97	99.84	99.68	99.55	99.24
	20	40	99.97	99.94	99.99	99.97	99.97	99.95
	30	32	99.98	99.96	99.99	99.98	99.99	99.99
	30	40	100.00	100.00	100.00	100.00	100.00	100.00
	30	50	100.00	100.00	100.00	100.00	100.00	100.00
	50	52	100.00	100.00	100.00	100.00	100.00	100.00
	50	70	100.00	100.00	100.00	100.00	100.00	100.00
	50	100	100.00	100.00	100.00	100.00	100.00	100.00
	100	50	100.00	100.00	100.00	100.00	100.00	100.00
	100	70	100.00	100.00	100.00	100.00	100.00	100.00
	200	100	100.00	100.00	100.00	100.00	100.00	100.00
	200	150	100.00	100.00	100.00	100.00	100.00	100.00

Table 5: Empirical power of T_3 and T_4 under non-normality and when $q = 10$.

In Table 4 and 5 we compare test statistic T_3 and T_4 under normality and non-normality. Assume that the data is generated from the model

$$\mathbf{X} = \mathbf{B}\boldsymbol{\xi}\mathbf{A} + \boldsymbol{\Sigma}^{1/2}\mathbf{Z},$$

where the elements $\mathbf{Z} = (z_{ij})$ are independently distributed as either of the following three distributions

- (i) $z_{ij} \sim N(0, 1)$ (as above),
- (ii) $z_{ij} \sim \frac{\chi^2(2) - 2}{2},$
- (iii) $z_{ij} \sim \frac{\chi^2(8) - 8}{4}.$

Observe that the skewness and kurtosis of $\chi^2(m)$ is, respectively, $\sqrt{8/m}$ and $3+12/m$. Hence, $\chi^2(2)$ has higher skewness and kurtosis, 2 and 9 respectively, compare to $\chi^2(8)$ with 1 and 4.5, respectively. From Table 4 and 5 we see that the test statistic T_3 and T_4 are robust under non-normality.

7 Concluding Remarks

In this paper we generalize the idea given by Yamada and Srivastava (2012) to the Growth Curve model. The MLE for the mean for a Growth Curve model is a weighted estimator with the inverse of the sample covariance matrix, which is very unstable for p close to N and singular for N less than p . This fact makes the MLE not suitable for '*large p and small N* '. We have modified the MLE to an unbiased and unweighted estimator, just by removing the inverse of the sample covariance matrix.

We have proposed three new test statistic, T_2 which is based on the unweighted estimator and T_3 and T_4 which can handle the high-dimensional case, when p is larger than N . T_3 and T_4 are based on the trace of the variation matrices due to the hypothesis and the error. The test statistic T_3 is invariant under the group of orthogonal transformations, but not invariant under a diagonal transformation, which is an undesirable feature. Test statistic T_4 has the benefit of being invariant under diagonal transformations.

We compare the LRT T_1 , based on the MLE, i.e., the weighted estimator, with the three new tests T_2 , T_3 and T_4 proposed in this paper. We have shown by simulation that these three test, based on the unweighted estimator, are preferred compared to T_1 .

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