THE BLACK-LITTERMAN
ASSET ALLOCATION MODEL
An Empirical Comparison to the Classical Mean-Variance Framework

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Abstract

Within the scope of this thesis, the Black-Litterman Asset Allocation Model (as presented in He & Litterman, 1999) is compared to the classical mean-variance framework by simulating past performance of portfolios constructed by both models using identical input data. A quantitative investment strategy which favours stocks with high dividend yield rates is used to generate private views about the expected excess returns for a fraction of the stocks included in the sample. By comparing the ex-post risk-return characteristics of the portfolios and performing ample sensitivity analysis with respect to the numerical values assigned to the input variables, we evaluate the two models’ suitability for different categories of portfolio managers. As a neutral benchmark towards which both portfolios can be measured, a third market-capitalization-weighted portfolio is constructed from the same investment universe. The empirical data used for the purpose of our simulations consists of total return indices for 23 of the 30 stocks included in the OMXS30 index as of the 21st of February 2014 and stretches between January of 2003 and December of 2013.

The results of our simulations show that the Black-Litterman portfolio has delivered risk-adjusted return which is superior not only to that of its market-capitalization-weighted counterpart but also to that of the classical mean-variance portfolio. This result holds true for four out of five simulated strengths of the investment strategy under the assumption of zero transaction costs, a rebalancing frequency of 20 trading days, an estimated risk aversion parameter of 2.5 and a five per cent uncertainty associated with the CAPM prior. Sensitivity analysis performed by examining how the results are affected by variations in these input variables has also shown notable differences in the sensitivity of the results obtained from the two models. While the performance of the Black-Litterman portfolio does undergo material changes as the inputs are varied, these changes are nowhere near as profound as those exhibited by the classical mean-variance portfolio.

In the light of our empirical results, we also conclude that there are mainly two aspects which the portfolio manager ought to consider before committing to one model rather than the other. Firstly, the nature behind the views generated by the investment strategy needs to be taken into account. For the implementation of views which are of an α-driven character, the dynamics of the Black-Litterman model may not be as appropriate as for views which are believed to also influence the expected return on other securities. Secondly, the soundness of using market-capitalization weights as a benchmark towards which the final solution will gravitate needs to be assessed. Managers who strive to achieve performance which is fundamentally uncorrelated to that of the market index may want to either reconsider the benchmark weights or opt for an alternative model.
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A Note on the Use of the Word ‘Return’

Throughout the rest of this thesis, ‘return’ and ‘expected return’ will be used to refer to ‘excess return’ and ‘expected excess return’ over the one-period risk-free interest rate unless otherwise stated.
1 Introduction

Before Markowitz presented his pioneering work on portfolio theory, not much was known about the benefits of diversification apart from the more obvious advantages of not putting all eggs in one basket. With Markowitz (1952) came a deeper understanding of how the underlying dynamics of price movements and the correlations among them govern not only the total return but also the total volatility of a portfolio. The key finding presented in this paper was that a well-chosen combination of assets can add up to a portfolio with higher expected return and lower volatility than any of its assets alone. Rational investors would then be expected to consistently prefer portfolios which generate higher ratios of return to risk and diversify accordingly.

Practical attempts to transform Markowitz’ mean-variance framework (henceforth the M-V framework) into ready-to-use portfolio optimisation techniques turned out to be more difficult than one would have hoped for, mainly due to difficulties in finding appropriate estimates for the necessary inputs (Michaud, 1989). These include expected returns and expected covariances for the coming holding period. The obstacles were not easily overcome even with the aid of the continual improvements in computing power that took place since the fundamentals of the model were established.

The aforementioned issues rendered the mean-variance approach to portfolio optimisation a rather frustrating topic, even for professional investment managers. Among these were the employees working at the fixed income research function at Goldman Sachs at the end of the 1980’s. Their job was, among other things, to offer advice to investors with global portfolios of highly correlated bonds and currencies from various markets (Litterman, 2003). The optimal portfolio weights for these securities were found to be extremely sensitive to even the most subtle changes in expected yields and made it all but easy for the advisors to come up with reasonable suggestions to their clients.

The proposal came from Fischer Black to incorporate the global CAPM equilibrium as a reference point into the optimisation process in order to make it better behaved and avoid unreasonable results (Litterman, 2003). The model that eventually emerged became known as the Black-Litterman Asset Allocation Model (henceforth the B-L model). It takes the global CAPM equilibrium as a starting point, letting the investor specify private views either as absolute return figures or relative values reflecting expected return differences between securities. The optimal portfolio is then derived roughly as a form of weighted average of the market portfolio (corresponding to the CAPM equilibrium weights) and the 'view portfolios', i.e. the portfolios that would have resulted had the views been incorporated into M-V framework one at a time.

The finesse of the B-L model is its capability to automatically adjust the entire vector of expected returns in accordance with views concerning only part of the vector. The combination of having the global CAPM equilibrium as a reference point and adjusting the entire vector when views are stated about one or more securities is reported by various researchers (see for instance Cheung, 2010; Martellini & Ziemann, 2007) to make the model suggest less extreme portfolio weights compared to the practical attempts to implement the standard M-V framework.
Whether the more ‘intuitive’ weight vector suggested by the B-L model actually yields a more attractive risk-return profile once it is taken out of its in-sample context and applied to real market data has not been covered very thoroughly in the academic literature. Such analysis would likely be of significant interest to professional fund managers who are in a constant struggle to maximize the performance of their portfolio. More often than not, portfolio managers find themselves being evaluated in terms of how well their fund performs in relation to some predefined benchmark index. Given such circumstances, it has not yet been established which of the two approaches is the most suitable for those who are willing to take the CAPM equilibrium as a reference point. Furthermore, there have been several studies aimed at presenting the mathematical underpinnings of some of the more frequently discussed variables of the B-L model. Various different suggestions as to how they are to be calculated have emerged, but the analysis is still lacking in discussion of the economic significance of such choices.

1.1 Purpose and Delimitations

In this study, we intend to illustrate and analyse the differences between portfolios generated by the classical M-V framework and the B-L model for an investor who wishes to base the portfolio selection decision partially on private views. We will also illustrate the sensitivity of the portfolio performance to the numerical values assigned to the more frequently discussed input variables of the B-L model.

The scope of our empirical study is limited to a comparison of simulated performance of portfolios consisting of Swedish equity over a ten-year holding period starting from the first trading day of 2004. The sample of securities included in the simulations is restricted to stocks included in the OMXS30 index as of the 21st of February 2014.

1.2 Hypothesizing Questions

- For an investor who has private views about the future performance of some of the securities included in his investment universe, what are the differences in risk-return characteristics between portfolios generated by the M-V framework and the B-L model?

- How sensitive are the risk-return characteristics with respect to the numerical values assigned to the input variables of the two models?

- Which aspects are the most important ones for a portfolio manager to consider before committing to one model rather than the other?

- How does the implied equilibrium returns approach compare to the risk-adjusted equal means approach with respect to risk-return and weighting characteristics?

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1 Chiefly $\delta$, $\tau$ and $\Omega$ as denoted by He and Litterman in their 1999 article.
1.3 Contribution

Since most of the prior studies of the B-L model have focused mainly on portfolio weights as such and/or the model’s mathematical underpinnings, we have chosen to focus our study on aspects which have so far not been covered anywhere near as extensively. By applying the model to historical market data and simulating how the resulting portfolio would have performed over an extended period of time, our empirical study will provide useful information on actual portfolio performance patterns. The simulated performance of a classical M-V portfolio using the same inputs as the B-L portfolio will enable us to distinguish what effects the unique dynamics of the B-L model have on the performance characteristics. This would not have been possible, had the M-V portfolio been derived from inputs other than those which are used in the B-L framework.

Extensive sensitivity analysis of how the performance of the B-L portfolio responds to changes in some of its input variables will also provide useful insights which can help portfolio managers determine how much resources to devote to the estimation of the numerical values of these variables. With the help of such results, it becomes possible to survey the two models’ appropriateness for different categories of investors.
2 Theoretical Framework

2.1 Portfolio Theory

In 1952, a paper by Harry Markowitz was published in The Journal of Finance that outlined the dynamic properties of diversification within the scope of asset management. He assumed that investors care not only about the expected return on their portfolio, but also about the return volatility. More specifically, it was assumed as a working hypothesis that most investors like return, dislike risk and avoid gambling, i.e. are risk-averse. With these assumptions as a starting point, Markowitz presented the relationship between the correlations of prices of different securities and the possibilities of combining them into a portfolio yielding a higher risk-adjusted return than any of the individual securities alone. This was shown to be true not just for assets with negatively correlated prices but for all assets whose prices are not perfectly positively correlated. The aim of the portfolio manager is to identify the investment universe, i.e. the various accessible assets, and then to find the combination of these that is expected to yield the highest volatility-adjusted return (Bodie, Kane & Marcus, 2011).

As to how the principles of finding the optimal portfolio are to be implemented in practice, Markowitz did not offer any concrete instructions. He did suggest, though, that there would perhaps be ways of combining statistical techniques and the judgment of experts to form reasonable estimates of expected returns and covariances. More specifically, using the observed values of returns and correlations for “some period of the past” (Markowitz, 1952, p.91) was mentioned as a possible approach. At the same time, however, Markowitz also made it clear that he believed there could be better methods that take more information into account and that a probabilistic reformulation of security analysis is essentially what was needed. Worth noting is that the mean-variance approach to portfolio optimisation relies upon several simplifying assumptions, among which are the absence of both transaction costs and differences in liquidity between different securities (Wilford, 2012).

2.2 The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) is a market equilibrium model built as an extension of the mean-variance framework presented by Markowitz. According to Craig W. French (2003), Treynor was the first to lay down the groundwork of the CAPM, although the most cited authors having contributed to it were Sharpe (1964), Lintner (1965; 1965) and Mossin (1966). The CAPM models the theoretical expected return of an asset as a function of its covariance with the market portfolio and the overall variance and expected return of this portfolio. In its original form, the CAPM models the market for tradable assets, including not only financial securities such as stocks, bonds and currencies but also fine art, collectable stamps and the like. The theoretically appropriate market portfolio thus consists of all such assets and is therefore practically unobservable. In many cases, however, the model is used in an exclusively financial context, restricting the universe of investable assets to the publicly traded securities on the world’s financial markets. Therefore, a broad market index is often used as a proxy for the theoretical market portfolio. The latter version of the CAPM is presented by Bodie, Kane and Marcus (2011).
and relies upon its own set of simplifying assumptions\(^2\). Its algebraic representation follows from equation 1.

\[
E(R_i) = R_f + \beta_i E(R_m - R_f), \quad \text{where}
\]

\[
E(R_i) = \text{the expected return on the } i\text{th asset (not expected excess return but expected total return)},
\]

\[
R_f = \text{the risk-free interest rate},
\]

\[
E(R_m) = \text{the expected return on the market portfolio (again, not excess but total return)},
\]

\[
\beta_i = \frac{\text{cov}(R_i, R_m)}{\text{var}(R_m)} = \text{the sensitivity of the expected return for the } i\text{th asset to expected market return}.
\]

In short, the model states that the higher the asset's covariance with the already diversified market portfolio, the higher the expected return will need to be in order for the asset to be attractive enough for investors to keep it on their books. This follows from the fact that the asset's \( \beta \) with the market portfolio is the only risk that the investor cannot eliminate by diversifying his portfolio. In such a simplified investment universe, the market-capitalization-weighted index consisting of all available securities will be mean-variance efficient. This means that the risk aversion of the average investor, along with the volatility of the market portfolio will determine the compensation an investor requires for bearing the risks inherent in the market portfolio. This relationship can be presented as:

\[
E(R_m) - R_f = \delta \sigma_m^2, \quad \text{where}
\]

\[
\delta = \text{a parameter representative of the risk aversion of the average investor in the market}.
\]

The return that can be expected for an individual security follows as the product of \( \beta \) and the expected risk premium of the market. Within the scope of the model, the end result is that all securities on the market will be held by all investors. If a security suddenly appears less attractive as an investment (in terms of the ratio of expected return to systematic risk), its equilibrium price will drop until it is once again considered equally as attractive as the other securities traded in the market place. The differences in risk aversion between investors only lead them to invest different fractions of their total wealth in the market portfolio while allocating the rest to the risk-free asset. The simple implications of the CAPM mean that in such an efficient market, there will be a linear relationship between the non-diversifiable risk of a given security and its expected return (Bodie, Kane & Marcus, 2011).

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\(^2\) (i) The market consists of a large number of investors, each with an insignificant power to affect the market prices of the securities being traded. (ii) All investors plan ahead for one and the same holding period. (iii) There are no taxes on returns and no transaction costs when trading the securities. (iv) All investors act in accordance with Markowitz' mean-variance framework and thereby opt to maximize the volatility-adjusted return on their portfolios. (v) Investors have homogeneous expectations – that is, when analysing assets in the investment universe, they all share the same basic view of the economic environment and thus arrive at the same fundamental conclusions. This means that given the same economic indicators, they will all end up feeding the same inputs into the Markowitz optimisation framework and thereby arrive at the same portfolio weights.
2.3 Bayes’ Theorem

The basic outlines of Bayes’ theorem were sketched by Thomas Bayes in the 18th century and then refined by Price and Laplace in the 19th century. Its use is to describe a conditional probability given some prior likelihood and another conditional probability (Walters, 2011). The fundamental contribution of this relationship is that it acts as a guiding principle as to how one should rationally alter a prior belief to take additional facts into consideration - something which lies at the very heart of the Black-Litterman Model. The formula describing the Bayesian relationship follows from equation 3.

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \]  

The left-hand side of the expression represents what is known as the posterior distribution, i.e. a probability distribution which is derived by taking all the available information into consideration. This information consists of the conditional probability distribution of B given A (known as the sampling distribution), the probability distribution of A (known as the prior distribution) and the probability of B, which serves as a normalising constant. A simplistic numerical example of how Bayes’ Theorem can be used in practice can be found in Appendix A.

2.4 The Black-Litterman Asset Allocation Model

In this section, an overview of the B-L model and its workings will be presented. The approach is of a step-by-step character, beginning with the fundamentals and successively introducing details as they come into play. Since the model was introduced in the early 1990’s, a variety of researchers have created various different versions of the original model. Some of these have differed with respect to the theoretical approaches used to derive the original model’s variables while others have excluded one or more variables altogether in order to improve the model’s practical usability. The scope of the following presentation is restricted to the original form such as the one first introduced by Black and Litterman in 1992 and later described in greater detail by He and Litterman (1999; 2002).

2.4.1 Market Clearing Expected Returns

The first step of the B-L model is to derive an initial estimate of the vector of expected returns. Not arriving at an unreasonable estimate of this vector is of great importance for real-world investors who have little or no interest in highly skewed and unrealistic portfolio weights. Before the introduction of the B-L model in the early 1990’s, approaches often involved estimations based on historical averages of returns or the assumption that assets across different countries would yield equal mean returns (Black & Litterman, 1992). Black and Litterman pointed out that these methods were all flawed since they fail to take the supply side of the market equation into account and only considered the demand for risky assets. Black proposed using the CAPM equilibrium as a point towards which the weight vector would gravitate (Litterman, 2003).

3 A working example of the Black-Litterman model can be found in Appendix B.
meant using a vector of expected returns that would clear the market if all investors had identical views. The sensible feature of using this approach is that the investor who does not have private views about the market will always end up holding a market-capitalization-weighted portfolio. Black and Litterman (1992) calculated this vector of expected returns in accordance with equation 4.

$$\Pi = \delta \Sigma W_{mkt}$$, where

- $\Pi$ = column vector of CAPM equilibrium expected returns ($nx1$, where $n$ equals the number of securities),
- $\delta$ = scalar representing the risk aversion of the average investor in the market,
- $\Sigma$ = covariance matrix of returns believed to be representative of the intended holding period ($nxn$),
- $W_{mkt}$ = column vector of market capitalization weights of the assets ($nx1$), where

$$w_i = \frac{m_i}{\sum_{i=1}^{n}m_i} \quad \text{and}$$

$$w_i = \text{market weight of the } i\text{th asset and}$$

$$m_i = \text{market capitalization of the } i\text{th asset.}$$

The risk aversion parameter ($\delta$) deserves a more detailed explanation. It acts as a measure of the price of risk, i.e. the return the average investor demands as compensation for bearing the risk of the market portfolio. Algebraically, $\delta$ is defined according to equation 6.

$$\delta = \frac{E(R_m) - R_f}{\sigma_m^2}$$, where

- $E(R_m)$ = expected return on the market-capitalization-weighted portfolio (not excess return but total return),
- $R_f$ = return on the risk-free asset, i.e. the risk-free interest rate,
- $\sigma_m^2$ = expected variance of the returns on the market-capitalization weighted portfolio.

Although Black and Litterman (1992) were the ones to draw up the formulas for implied equilibrium expected returns and the risk aversion parameter, these formulas were still derived from the by then already well-known CAPM equilibrium. To see this more clearly, simply multiply both sides of equation 6 by $\sigma_m^2$. The resulting expression follows as equation 7.

$$E(R_m) - R_f = \delta \sigma_m^2 .$$

It can easily be seen that equation 7 is identical to equation 2 and that the expected return on the market portfolio follows as the product of the average risk aversion and the market variance. In other words, the expression for the risk aversion parameter comes directly from the assumption of the CAPM market equilibrium.
2.4.2 The Black-Litterman Master Formula

Since the original B-L model is of a Bayesian nature, the dynamics of it are best understood in the light of Bayes’ Theorem. In a Bayesian fashion, the model uses the CAPM to form the prior distribution - that is, an estimate of the expected returns implied by the market. This prior is then revised using the information contained in the view distribution to derive the posterior distribution, which serves as an estimate that has taken both public information and private views into account.

THE PRIOR DISTRIBUTION

A pivotal point of the B-L model is that expected returns are not considered to be observable fixed values. Instead, they are viewed as stochastic variables which are normally distributed around some population mean (He & Litterman, 1999). This means they have to be modelled using a probability distribution. In contrast, returns (not expected but actual) are viewed as directly observable random variables and can easily be observed in historical data. This separation of expected and actual returns is a crucial piece of the Black-Litterman puzzle. The vector of actual returns (denoted \( r \)) is assumed to be normally distributed around a mean vector (denoted \( \mu \)) with a covariance matrix (denoted \( \Sigma \)) (ibid). Algebraically,

\[
r \sim N(\mu, \Sigma),
\]

where

\[
\mu = \Pi + \varepsilon^{(e)},
\]

\[
\varepsilon^{(e)} \sim N(0, \tau\Sigma).
\]

In other words, the prior distribution of the B-L model states that expected return is best described as a normally distributed stochastic variable with a mean \( \Pi \) and a covariance matrix proportional to the covariance matrix of actual returns, namely \( \tau\Sigma \). The variable \( \tau \) is a scalar which represents the degree of uncertainty associated with the CAPM prior.

THE VIEW DISTRIBUTION

The mechanism used to express views is based upon three components – \( P \), \( Q \) and \( \Omega \). The first contains information about which security a view concerns and whether the view is stated in absolute or relative form. The second contains information about the strength of the view, i.e. the return that the investor expects a particular security to yield (known as an absolute view) or the expected return difference between securities (known as a relative view). Mathematically, a row in \( P \) will have a sum of one if the view that this row describes is stated in absolute form whereas the sum will instead be zero if the view is stated in relative form. Algebraically,

\[
P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{k,1} & \cdots & p_{k,n} \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 \\ \vdots \\ q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}.
\]

The third component that the model needs the investor to specify is the uncertainty associated with the views that are specified by \( P \) and \( Q \). This uncertainty measure takes the shape of the covariance matrix \( \Omega \). Algebraically,
\[ \mathbf{P} \mu = \mathbf{Q} + \mathbf{e}^{(v)}, \text{ where } \mathbf{e}^{(v)} \text{ is a vector of error terms – that is} \]

\[ \mathbf{e}^{(v)} \sim N(0, \mathbf{\Omega}). \]

In other words, the diagonal elements of \( \mathbf{\Omega} \) represent the expected variances of the elements in \( \mathbf{Q} \), whereas the off-diagonal elements represent their expected covariances. Algebraically,

\[ \mathbf{\Omega} = \begin{bmatrix} \omega_{1,1} & \cdots & \omega_{1,j} \\ \vdots & \ddots & \vdots \\ \omega_{i,1} & \cdots & \omega_{i,j} \end{bmatrix}. \]

Black and Litterman did not show any specific way of estimating \( \mathbf{\Omega} \) in their 1992 paper. The exact computational method that was used was nevertheless revealed by He and Litterman (1999). Employing this method, the elements of \( \mathbf{\Omega} \) are calculated in accordance with the following expression:

\[
\omega_{i,j} = \mathbf{P}(\tau \mathbf{\Sigma})\mathbf{P}' \quad \forall \ i = j
\]

\[
\omega_{i,j} = 0 \quad \forall \ i \neq j.
\]

Alternatively, this formula can be expressed as

\[ \mathbf{\Omega} = \text{diag}(\mathbf{P}(\tau \mathbf{\Sigma})\mathbf{P}') \quad (10) \]

(Walters, 2011).

This way of estimating \( \mathbf{\Omega} \) involves assuming that the uncertainty associated with the mean of the view distribution is proportional to the uncertainty associated with the mean of the prior distribution (i.e. \( \tau \mathbf{\Sigma} \)). Setting all off-diagonal elements to zero corresponds to the simplifying (but not necessary) assumption of the views being uncorrelated.

\textit{THE POSTERIOR DISTRIBUTION}

The algebraic expression used to blend the prior distribution and the view distribution to form the posterior distribution of expected returns is known as the Black-Litterman Master Formula. After combining the prior with the views in accordance with the original B-L framework, the expected returns (denoted \( \mu \)) is expressed as a normally distributed random vector with a mean equal to the vector \( \mathbf{\bar{\mu}} \) and a covariance matrix denoted \( \mathbf{M} \). Algebraically,

\[ \mathbf{\mu} \sim N(\mathbf{\bar{\mu}}, \mathbf{M}), \text{ where} \]

\[ (11) \]
\[ \tilde{\mu} = \left[ (\tau \Sigma)^{-1} + P'\Omega^{-1}P \right]^{-1} \left[ (\tau \Sigma)^{-1} \Pi + P'\Omega^{-1}Q \right] \]

(Black and Litterman, 1992)

When deriving \( \tilde{\mu} \), the uncertainty associated with the prior distribution (as measured by \( \tau \)) and the uncertainty associated with the view distribution (as measured by \( \Omega \)) are taken into account. The anatomy of the model ensures that there will always be an inverse relationship between the stated uncertainty of an input distribution and its impact on the mean of the posterior distribution. For example, if the uncertainty of the view distribution increases, the mean of the posterior distribution will move closer towards the estimated mean of the prior distribution (\( \Pi \)) and further away from the mean of the view distribution (\( Q \)). Correspondingly, if the uncertainty associated with the prior distribution increases, the mean of the posterior distribution will move closer towards the estimated mean of the view distribution and further away from the mean of the prior distribution. Black and Litterman’s original paper from 1992 does not present the algebraic expression for the covariance matrix that characterizes the posterior distribution of expected returns (i.e. \( M \)). Fortunately, such an expression is presented in the paper by He and Litterman from 2002. It is also restated in equation 13.

\[ M = \left[ (\tau \Sigma)^{-1} + P'\Omega^{-1}P \right]^{-1}. \]

Since expected returns themselves are assumed to be random variables in the B-L framework, the covariance matrix \( M \) can only be associated with the expected returns, i.e. the expected values of future returns. For an investor seeking to estimate the optimal portfolio weights, a crucial input needed to accomplish this is the covariances of the actual returns that are believed to prevail over the holding period. If the investor was to use the prior estimate of the covariance matrix of returns (i.e. \( \Sigma \)), he would fail to take into account that the expected returns themselves are not constants but random variables. The appropriate estimate of the posterior covariance matrix associated with the return distribution (denoted \( \tilde{\Sigma} \)) was shown by He and Litterman (2002) as the expression in equation 14.

\[ \tilde{\Sigma} = \Sigma + M. \]

The posterior covariance matrix takes into account that since the mean itself (\( \mu \)) is a random variable, there is an added source of uncertainty that needs to be considered by the investor when estimating the covariances of future market returns. The consideration given to the stochastic property of \( \mu \) means that the distribution of returns that is expected to prevail over the intended holding period can be described in accordance with equation 15.

\[ r \sim N(\tilde{\mu}, \Sigma). \]

---

4 \( \tilde{\mu} \) is a column vector representing the mean of the expected returns that is then used in the optimisation process (\( nx1 \)).

\( \tau \) is a scalar representing the proportionality constant between the covariances of returns and the covariances of expected returns.

\( \Sigma \) is a covariance matrix of returns about their means (\( nxn \)).

\( P \) is a link matrix connecting the views the investor holds about various securities and the securities themselves (\( kxn \), where \( k \) is the number of views held by the investor).

\( \Omega \) is a (diagonal) covariance matrix representing the uncertainty of the views held by the investor (\( kxk \)).

\( \Pi \) is a column vector of CAPM equilibrium expected returns (\( nx1 \), where \( n \) equals the number of securities).

\( Q \) is a column vector containing the estimated returns implied by the investor’s views about the \( k \) assets (\( kx1 \)).
2.4.3 The Uncertainty of the Prior Distribution

An important and frequently discussed feature of the model is the scalar representing the relationship between the covariance matrices associated with the estimates of returns and expected returns, respectively. The underlying assumption is that the variance of the returns ($r$) about the mean of returns ($\mu$) is higher than the variance of the CAPM expected returns ($\Pi$) about the mean of returns ($\mu$). While $\Sigma$ represents the covariance matrix of the former, the product $\tau \Sigma$ is assigned to represent the covariance matrix of the latter. In other words, $\tau$ serves as a measure of how inaccurate the prior estimate (i.e. the CAPM equilibrium mean returns) is expected to be in relation to the true population mean.

Setting $\tau$ to close to zero would imply that there is virtually no expected gap between the CAPM equilibrium estimate of mean returns and true mean returns. This does not imply, however, that future returns can be more or less perfectly forecasted using the CAPM; it only implies that the mean of the return distribution can be more or less perfectly forecasted using the CAPM. Setting $\tau$ close to unity, on the other hand, would imply that the CAPM estimate of the mean of the return distribution is virtually as uncertain as the estimate of returns themselves - a scenario which is not very likely in reality. Regardless of the value assigned to $\tau$, its impact on the final estimate of $\bar{\mu}$ is non-existent as long as $\Omega$ is set in proportion to the covariance matrix of the prior distribution, i.e. $(\tau \Sigma)$ (Walters, 2009). This means that the only way in which $\tau$ impacts the vector of optimal portfolio weights is through its influence on the posterior covariance matrix.
3 Prior Research on the Black-Litterman Model

While the previously conducted research on the B-L model has been extensive, the various different studies have been oriented towards different aspects of the model. Many researchers have focused on the theoretical grounds on which the model is based and presented a number of suggestions as to how the model can be modified to better suit various different investors. In this chapter, we present a brief overview focused on selected parts of the more empirically oriented research that has been conducted over the years.

3.1 Comparing B-L Portfolios to Equilibrium Weighting

In their original paper published in the Financial Analysts Journal (1992), Black and Litterman presented an introductory historical simulation aimed at exhibiting how three different investment strategies perform compared to an equilibrium-weighted portfolio. Although no comparison with the corresponding mean-variance portfolio was presented, the simulation was the first to demonstrate how the B-L framework can be used to generate carefully tilted portfolios that reflect different sets of strategic and/or tactical views held by the investor.

Using an investment universe consisting of bonds, currencies and equities from seven different countries, four different portfolios were constructed - each one with its own unique vector of expected returns. One of the portfolios, the so called equilibrium portfolio, was used as a benchmark as its vector of expected returns coincided with the vector of implied equilibrium returns. The other three portfolios differed with respect to the investment strategy being used to generate views about the securities. The ex-post performance of the four portfolios could thereby be directly compared since they were restricted to both the same ten-year holding period and the same investment universe.

The concrete investment strategies selected by Black and Litterman for these simulations were three well-known and simplistic investment strategies that they considered as representative of standard investment approaches of the time. These involved forming optimistic views about high-yielding currencies, high-yielding bonds and equities of countries with high ratios of dividend yield to bond yield. The simulations were performed by using ten years of historical price data to estimate a covariance matrix of security returns. For each of the portfolios, the vector of expected returns was estimated in accordance with the investment strategy in question. The four sets of portfolio weights were then optimised for a given level of portfolio risk without constraints. The optimisation process was repeated once a month over a decade-long holding period.

As pointed out in the paper, any simulation such as this one is a test not only of the asset allocation model itself but also of the strategy being used to generate views. In this case, Black and Litterman established that over this particular holding period, the only strategy resulting in a lower rate of return than the equilibrium portfolio was the strategy favouring high-yielding bonds. Both the other two strategies were found to have been noticeable more profitable than maintaining equilibrium portfolio weights. Although the tracking error volatilities of the three
portfolios were not presented, the similarities between their historical performance and that of the equilibrium portfolio indicate that they are essentially differently tilted versions of one and the same benchmark portfolio.

3.2 Implementing a Quantitative Macro Strategy

An empirical study that utilized the quantitative nature of the B-L model to incorporate views generated by an econometric model is the one presented by Beach and Orlov in their 2007 paper. In this study, an empirical simulation of an investment strategy based on an EGARCH-M specification was performed. More specifically, this model was used to estimate the one-step-ahead expected returns on 20 assets along with the expected variances of these returns. The explanatory variables employed in the prognostication process were a number of global and local macroeconomic indicators reported by some to have a significant ability to explain returns in developed markets. The overall aim of the study was to find a suitable econometric model that could accurately describe both the dynamics of returns on international portfolios and the volatility dynamics of these returns. Like Black and Litterman, Beach and Orlov used ten years of historical data to estimate the covariance matrix of returns, re-optimising the portfolio weights without constraints on a monthly basis. Unlike Black and Litterman, however, extensive attention was given to the means of generating views and the variances associated with those views.

Based on the simulated performance of this investment strategy between 1998 and 2003, the authors concluded that the B-L portfolio with EGARCH-M-generated views did manage to yield a noticeably higher rate of return than the corresponding market-capitalization-weighted portfolio. The performance also dominated that of a ‘traditional’ mean-variance optimal portfolio used as an additional benchmark. As pointed out by Beach and Orlov, historical statistics are the most widely used inputs for standard mean-variance optimisation – something which were shown to result in quite different risk-return characteristics.

Apart from presenting an example of how the B-L model’s ability to incorporate views generated by a quantitative strategy can be utilized, Beach and Orlov also showed a novel way of calibrating the portfolio weights to suit the investor’s desired level of portfolio risk. Since they estimated the optimal portfolio weights analytically, specifying the desired level of portfolio volatility as such was not possible. Instead, they suggested that $\tau$ be calibrated until the estimated portfolio volatility is on par with the risk appetite of the investor. If the investor is dissatisfied with the estimated portfolio volatility, the value assigned to $\tau$ can be lowered (raised) to force the model to suggest weights which imply lower (higher) portfolio volatility.

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5 Readers who would like to know more about GARCH models are referred to Verbeek (2012) for a more elaborated presentation.
3.3  Simplifying and Extending the Original Framework

When it comes to suggesting possible modifications and extensions to the canonical B-L framework, there is certainly no shortage of researchers who have contributed with new approaches. The different versions of the B-L model that have resulted from these academic breakthroughs have been categorized by Jay Walters (Blacklitterman.org) into two main forms of the model. Walters refers to these as the Canonical Reference Model (the one described in the previous chapter) and the Alternative Reference Model, which is the version used by Idzorek (2004), Meucci (2009) and others.

In Idzorek’s version from 2004, the confidences in the views are specified by the investor as percentages, which are then in turn translated into variances. In Meucci’s version from 2009, on the other hand, the uncertainty associated with the views is measured by a covariance matrix roughly like the one used by He and Litterman in their 1999 version of the Canonical Reference Model. The main difference between the two is that Meucci scales the covariance matrix by a parameter representing the best estimate of the overall level of confidence in views, whereas He and Litterman (1999) scales it by the parameter representing the level of uncertainty in the CAPM prior (i.e. $\tau$). Differences such as these render comparisons between empirical studies employing separate versions of the model challenging, since the settings used in one simulation do not necessarily let themselves be translated into the language of another.
4 Methodology

The empirical research conducted within the scope of our study consists of historical simulations aimed at uncovering how the B-L model and the standard mean-variance framework perform when applied to real market data over an extended period of time. Certain care is taken to closely follow the methodology used by Black and Litterman (1992) - more specifically the detailed methodology description provided by He and Litterman (1999; 2002). This chapter contains a detailed description of how the simulations are performed.

4.1 Data

All historical simulations are restricted to the Swedish equity market. Moreover, the investment simulations are performed using individual stocks rather than broad equity indices. On the one hand, this allows us to avoid difficulties in accounting for structural changes that have been made to the composition of such indices over time. On the other, individual stocks tend to show relatively larger differences with respect to return and volatility than broad equity indices – something which means that our results may become more sensitive to the portfolio weights as compared to a portfolio consisting of indices. To avoid results which are plagued by a lack of realism due to limited liquidity of the securities included in the sample, the sample is restricted to OMXS30 stocks only.

Like Black and Litterman (1992), we consider a holding period of ten years to be sufficient for the results to be meaningful. For the sole purpose of estimating the covariance matrix of returns, another year of price data is used, utilizing a total of eleven years of market data for every security. For the data to reflect actual returns rather than just the prices at which the stocks change hands, total return indices are used to estimate both returns and covariances. All data series are sourced from Datastream 5.0 but due to insufficient data on dividend yield rate and market capitalization for seven of the 30 stocks, the cross section of the sample is reduced accordingly.6 The final sample thus includes daily time series data on 23 different stocks stretching over roughly eleven years (2003-2013). The holding period is set to range from the first trading day of 2004 to the last trading day of 2013. Ideally, we would have liked to follow the example set by Black and Litterman (1992) and Beach and Orlov (2007) by using ten years of return data to estimate the covariance matrix. Due to limited availability of return data, however, keeping the holding period at ten years requires us to limit the amount of data used to estimate the covariance matrix to 235 trading days of daily return data.

4.2 Simulating Past Performance

Like the majority of the previously conducted research, the evaluation of the B-L and mean-variance models is based on simulated performance of differently constructed portfolios. In this particular study, a total of three different weighting schemes are used. Apart from comparing the

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6More specifically, the seven stocks excluded from the sample are ABB.ST, ALFA.ST, ATCOb.ST, AZN.ST, LUPE.ST, NOKL.ST and TLSN.ST.
performance of a B-L portfolio and the corresponding mean-variance portfolio (henceforth the *equilibrium portfolio*) constructed from the same 23-asset investment universe is used as an additional benchmark. This facilitates performance comparisons both in the way demonstrated by Black and Litterman in their 1992 paper (where B-L portfolios were compared to an equilibrium portfolio) and in the way presented by Beach and Orlov in their 2007 paper (where a B-L portfolio was compared to a standard mean-variance portfolio).

4.2.1 The Black-Litterman Portfolio

The B-L portfolio is constructed using the framework presented in the previous chapter to derive the vector of expected returns and the posterior covariance matrix. These two matrices are then fed into an iterative optimiser which estimates the portfolio weights that yield the highest expected portfolio Sharpe ratio subject to two constraints. Firstly, a normalizing constraint is used to ensure that the weights sum up to unity, i.e. that the investor is fully invested. Secondly, short selling is assumed to be infeasible, constraining the weight vector to non-negative values. Imposing such a constraint may render the resulting weight vector suboptimal as compared to its unconstrained counterpart. Nevertheless, results obtained under an investment management policy which does not allow short positions are likely to be relevant to a significantly broader spectrum of institutional investors than results which rely partially upon the portfolio manager taking short positions.

In order for the simulation to be realistic, private views are not being generated for all 23 stocks, but rather for eight of them. This ensures that there is a meaningful fraction of the securities in the portfolio about which the investor does not have any views – just like case of many institutional investors. Moreover, this fraction (15/23) is approximately equal to that of the simulation performed by Black and Litterman (1992), where views were held for 7 of the 21 assets at a time. Like Black and Litterman too, the views concern the same eight securities throughout the entire simulation. The eight stocks for which views are generated were randomly selected from the sample before the start of the simulation.

As a means to generate views, the equity strategy used by Black and Litterman was taken as a basis. The precise formula they used to adjust the expected returns from the $\Pi$ matrix is not employed due to the fact that it was intended for broad indices from different countries rather than individual stocks from one and the same country. Instead, a more simplistic formula is used to form views based strictly on the dividend yield rates of the securities – still favouring high-yielding stocks over lower-yielding ones. More concretely, for the eight stocks about which views are held, the elements of $Q$ are computed in accordance with the following formula:

$$q_i = \pi_i + \xi(d_i - d_{mkt}) , \text{ where}$$

$$d_i = \text{dividend yield rate of the } i\text{th asset},$$

$$d_{mkt} = \text{the market capitalization-weighted average of dividend yield rates},$$

$$\xi = \text{strength of views coefficient}.$$  

\[ (16) \]

$^7$ The eight stocks drawn are ATCOa.ST, ERICb.ST, HMb.ST, MTGb.ST, NDA.ST, SCAb.ST, SKFb.ST and VOLVb.ST.
Subsequently, the expected returns on the eight selected stocks are adjusted upwards if the stock in question has managed to deliver a dividend yield rate higher than the market-capitalization-weighted average and downwards in the opposite case. The variable $\xi$ is used to calibrate the strength of views, i.e. how strongly deviations from the average dividend yield rate influence the adjustment of the elements in $\Pi$.

4.2.2 The Mean-Variance Portfolio
For the comparison between the B-L and the M-V portfolios to be as informative as possible, the simulations of the latter are performed using the same set of market data as the former. Therefore, the implied equilibrium expected returns of the $\Pi$ matrix serve as a starting point even in the case of the M-V portfolio. In this portfolio, the final vector of expected returns consists of the eight $q$ values for the securities about which views are held and the 15 $\pi$ values for the others.\footnote{This calculation is demonstrated in greater detail in Appendix B.} The covariance matrix that is fed into the optimiser alongside this final vector of expected returns is simply the covariance matrix estimated from historical returns, i.e. $\Sigma$. The optimisation process itself is identical to that used in the case of the B-L portfolio.

4.2.3 The Equilibrium Portfolio
As a benchmark against which the two optimised portfolios can be measured, a market capitalization weighted portfolio constructed from the same 23 asset universe is used. As described by Black and Litterman in their 1992 paper, for the investor who has no private views about the future performance of the securities traded in the market, there is little or no reason to deviate from the vector of market-capitalization weights. This equilibrium portfolio (henceforth the $EQ$ portfolio) thus serves as a ‘passive’ alternative weighting scheme that only requires the investor to occasionally rebalance the portfolio in accordance with the directly observable market-capitalizations of the stocks included in the portfolio.

4.3 The Investment Process
Starting on the first trading day of January 2004 (let us denote this day $t_0$) the covariance matrix of returns ($\Sigma$) is estimated using continuous daily rates of return for a period of 235 trading days of the past – starting from the return observed on day $t_{235}$ and ending with the return observed on day $t_1$. When the covariance matrix has been estimated, the matrix of equilibrium expected returns, $\Pi$, is estimated for a given value of $\delta$ and the market-capitalization weights observed on day $t_1$. For the eight assets about which the investor has views, the final expected returns are calculated in accordance with equation 16 for the dividend yield rates observed on the same day as the market-capitalization weights, i.e. $t_1$.

The resulting vector of expected returns (either computed in accordance with the B-L or the Markowitz framework) is then fed into the optimiser along with the appropriate covariance matrix (either $\bar{\Sigma}$ or $\Sigma$). The weight vector returned by the optimiser is then taken into account
when buying shares of the securities at the prices prevailing on day $t_0$. In this process, a brokerage fee of $f$ per cent of the amount invested is subtracted from the value of the portfolio. This process is repeated after an investment horizon of $h$ days. When subsequently rebalancing the portfolio, both purchases and sales of shares are subject to the brokerage fee. In the case of the equilibrium portfolio, the investment process is slightly simpler since there is neither a need to estimate covariances of returns, nor a need to add views. Instead, the purchases of shares on day $t_0$ are simply made in accordance with the market-capitalization weights observed on day $t_0$. The portfolio is then rebalanced as frequently as the B-L and M-V portfolios, simply adjusting the actual positions to comply with the market-capitalization weights observed on the previous day.

4.4 A Framework for Sensitivity Analysis

For the results of our simulations to be relevant to a wide range of investors, the historical simulations of the B-L and M-V portfolios are performed using multiple numerical values for the input variables that typically have to be selected by the investor based on some form of judgement. This makes it possible to uncover how each variable affects the end results. To avoid excessive amounts of results, we perform sensitivity analysis by first defining default values for all variables and then altering the value of one variable at a time while keeping the others fixed at their respective default levels. The default values in question have been carefully selected to closely imitate the settings used by Black and Litterman (1992), He and Litterman (1999; 2002) and Beach and Orlov (2007).

**TRANSACTION COSTS**
Since different investor face different levels of transaction costs, sensitivity analysis is performed by measuring the portfolio performance statistics with a variable brokerage fee (denoted $f$). While this variable does not in itself affect the optimal portfolio weights, its effect on the ex-post portfolio performance is of utmost importance for the assessment of the robustness of the results. As the default setting, we follow Black and Litterman’s (1992) approach, set $f$ to 0 %. While transaction costs this low may not be a close approximation of the reality faced by all institutional investors, it is still the only default setting which ensures ample comparability between the results of our simulations and those of Black and Litterman (1992) and Beach and Orlov (2007).

**INVESTMENT HORIZON**
A particularly delicate decision faced by the investor is how often to rebalance the portfolio. On the one hand, more frequent rebalancing of the portfolio will allow the investor to account for new information and/or to comply with new investment policies with shorter delay. On the other, the higher number of transactions needed to rebalance the portfolio more often may give rise to a significant increase in financial and/or operational transaction costs. To reflect differing opinions among investors as to which rebalancing frequency is considered the optimal trade-off, the simulations are performed using multiple values of $h$. Like Black and Litterman (1992), He and Litterman (1999; 2002) and Beach and Orlov (2007) the default value selected for the investment horizon is set to one month, i.e. 20 trading days.
THE RISK AVERSION PARAMETER AND THE HIGH-YIELD STRATEGY

As for the numerical value assigned to the risk aversion parameter ($\delta$), the approach used by He and Litterman in their 1999 paper is taken as a basis. Like He and Litterman, we use $\delta = 2.5$ as the default setting. Instead of keeping $\delta$ fixed at this level, however, we perform the simulations for several different values, thereby examining how varying degrees of market optimism on the part of the investor translate into differences in the portfolio’s risk-return characteristics. In the same fashion, the variable used to calibrate the strength of views, $\xi$, is varied to demonstrate the role played by the magnitude of the investor’s private views when generated by the particular investment strategy considered in our study. As the default setting, we set $\xi = 3.0$ to ensure that the views are strong enough to actually impact the optimal portfolio weights yet still modest enough to be regarded as realistic.

THE VIEWS

To examine how strongly private views affect the portfolio characteristics more generally, we perform additional sensitivity analysis similar to that of Bertsimas, Gupta and Paschalidis (2012). In their study, they analyse the sensitivity of the ex-post portfolio performance to changes in the views by first setting the elements of $Q$ to their equilibrium levels (as defined by $\Pi$). The resulting ‘neutral’ vector of expected returns is then tilted both in a positive and a negative direction to illustrate how such alterations affect the optimal portfolio weights and the resulting performance. In other words, the method involves recording the changes in the portfolio Sharpe ratio and tracking error volatility that result as the elements of $Q$ are pushed further and further away from their equilibrium levels. This is a simple yet effective way of illustrating how sensitive each portfolio is with respect to the strength of the views without the results being dependent upon any particular investment strategy.

THE UNCERTAINTY OF THE CAPM PRIOR

One of the more confusing decisions faced by the investor is what numerical value to assign to $\tau$. Since the $\Omega$ matrix is set proportional to $\tau$, we already know beforehand that the final vector of expected returns will not be affected by the value assigned to it. In contrast, the posterior covariance matrix ($\tilde{\Sigma}$) is affected by its value. The impact that variations in this parameter actually has on the resulting portfolio performance is therefore all but obvious. Although the previously conducted research has contributed with a number of ways to quantify and interpret $\tau$, there is still no single way of determining its appropriate numerical value that has become industry standard. Our simulations are carried out using an approach similar to that of He and Litterman (2002). Like them, we set $\tau = 0.05$ as the default setting. Instead of fixing its value at this level, however, sensitivity analysis is performed by also letting $\tau$ take on a range of values near 0.05.

IMPLIED EQUILIBRIUM EXPECTED RETURNS

To facilitate the assessment of how sound a starting point the global CAPM equilibrium is for an investor seeking to construct a mean-variance efficient portfolio based partially on private views, the M-V portfolio will also be simulated using the so called risk-adjusted equal means approach. With this approach, the investor assumes that the risk-adjusted expected returns of all securities are

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9 Since the empirical study presented in their paper does not involve any comparison between the canonical B-L model and the Markowitz framework, we select not to present it in detail.
equal. Since it was described by Black and Litterman (1992) as one of several ‘naive’ approaches that an investor might use to derive the expected returns on the assets in the considered investment universe, it will serve as a benchmark against which the CAPM approach can be evaluated.

When implementing this approach, the historical Sharpe ratio of the Swedish equity market is taken as a rough estimate of what the investor can expect in terms of risk-adjusted return. To arrive at a reasonable figure, this Sharpe ratio is estimated using return data for ten years prior to the intended holding period (resulting in an estimate of roughly 0.2093). The expected return of an individual asset is then computed by scaling this estimate by the latest year’s return volatility of the asset in question. The covariance matrix used in the optimisation process is the same as for the ordinary M-V portfolio, i.e. $\Sigma$. Simulating portfolio performance with this approach enables us to assess how this approach measures up to the approach suggested by Black and Litterman (1992) – i.e. to use the global CAPM equilibrium to estimate implied expected returns. Ideally, simulations of other ‘alternative’ approaches would have facilitated this assessment, but due to the limited amount of time available for this study, we have selected only to test one alternative to the CAPM approach. The reason for choosing the risk-adjusted equal means approach as an alternative to implied equilibrium returns is that it was presented by Black and Litterman (1992) as one of three approaches that were widely used in the industry. Furthermore, of the three approaches that were presented, this is the only one which accounts for the fact that different securities are characterized by different levels of volatility without requiring extensive time series of historical return data on the 23 stocks included in our sample.
5 Results and Analysis

Within the scope of our empirical study, we have run no less than 55 simulations of how differently constructed portfolios would have performed between 2004 and 2013. In this chapter, we present a summary of how the Black-Litterman and mean-variance portfolios would have measured up to their market-capitalization-weighted counterpart.

5.1 Optimal Portfolio Allocations – Simulating a High-Yield Strategy

In table 1 and figure 1, the results of the simulations performed with the variables kept at their respective default levels are presented.

Table 1 - (f = 0 %; b = 20 days; δ = 2.5; ξ = 3.0; ε = 0.05 )

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>EQ</th>
<th>B-L</th>
<th>M-V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>13.50%</td>
<td>14.79%</td>
<td>14.17%</td>
</tr>
<tr>
<td>Volatility</td>
<td>23.66%</td>
<td>23.89%</td>
<td>26.40%</td>
</tr>
<tr>
<td>-Highest Volatility</td>
<td>40.06%</td>
<td>41.88%</td>
<td>49.30%</td>
</tr>
<tr>
<td>-Lowest Volatility</td>
<td>11.91%</td>
<td>11.32%</td>
<td>12.39%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.4925</td>
<td>0.5419</td>
<td>0.4668</td>
</tr>
<tr>
<td>-Highest Sharpe Ratio</td>
<td>2.6927</td>
<td>2.3135</td>
<td>1.7906</td>
</tr>
<tr>
<td>-Median Sharpe Ratio</td>
<td>1.0110</td>
<td>0.9785</td>
<td>0.9950</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td>5.01%</td>
<td>10.99%</td>
<td></td>
</tr>
<tr>
<td>Information Ratio</td>
<td>0.2918</td>
<td>0.0421</td>
<td></td>
</tr>
<tr>
<td>Alpha Compared with EQ Portfolio (CAR)</td>
<td>1.46%</td>
<td>0.46%</td>
<td></td>
</tr>
<tr>
<td>Beta Compared with EQ Portfolio</td>
<td>0.9873</td>
<td>1.0153</td>
<td></td>
</tr>
<tr>
<td>Correlation with EQ Portfolio</td>
<td>0.9780</td>
<td>0.9100</td>
<td></td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>3.81%</td>
<td>6.00%</td>
<td>10.17%</td>
</tr>
<tr>
<td>Largest Cumulative Loss</td>
<td>-55.30%</td>
<td>-55.32%</td>
<td>-67.39%</td>
</tr>
<tr>
<td>Largest Cumulative Gain</td>
<td>291.73%</td>
<td>342.89%</td>
<td>398.33%</td>
</tr>
</tbody>
</table>

As can be seen from the summary statistics in table 1, both the risk and return figures for the three portfolios are fairly similar.\(^{10}\) More concretely, the correlation with the EQ portfolio amount to more than 0.90 for both the B-L and the M-V portfolios. Over this particular holding period, the B-L portfolio has managed to yield an ex-post Sharpe ratio which is higher than those of both the other portfolios. The differences are, however, quite modest - something which is not too surprising given the fact that views are held only for a relatively small fraction of the portfolio and the fact that the strength of views (ξ) is set to no more than 3.0.

\(^{10}\) The portfolio return is measured in terms of a continuously compounded annual rate (CAR), while the risk (volatility) is measured in terms of yearly standard deviation over the entire holding period.
Another plausible reason for the similarity of the results is the fact that market-capitalization weights are used as a basis for all three portfolios. The differences in terms of deviations from the market-capitalization-weighted benchmark are indicated by the tracking error volatility (henceforth $TEV$). This measure acts as an intuitive indicator of to what extent the portfolio weights deviate from their respective market-capitalization levels and is thus a valuable piece of information for assessing the skewness of the weights suggested by each model. Estimating the vector of expected returns in accordance with the B-L framework leaves the $TEV$ figure at a mere 5.01 %, whereas the M-V portfolio exhibits a $TEV$ of 10.99 % despite the fact that these two portfolios are constructed using fundamentally identical inputs. This difference demonstrates the B-L model’s signifying feature, namely to see to it that the optimised weight vector gravitates towards market-capitalization weights for the securities about which no views are held. Summarizing the results obtained for the three portfolios over a holding period of ten years is of course not without difficulties. A more detailed tabulation of the performance figures is provided in Appendix C, where statistics for each year are presented.

### 5.2 Transaction Costs

Within the scope of our simulations, the only transaction cost considered is a financial transaction cost in the form of a brokerage fee. For real-world asset managers, however, the transaction cost structure can of course take on radically different shapes. The costs associated with the buying and selling of shares need not be proportional to the value of the shares changing hands. Furthermore, the main costs associated with managing an equity portfolio may arise from other sources than the transactions. For instance, the costs of staffing may far exceed the costs of the transactions. To keep the sensitivity analysis comprehensible, a simple brokerage fee is used as a proxy for the cost structure of the portfolio manager. This approximation may be more or less accurate depending on the scale of operation.

The volatility of the actual portfolio weights can be taken as a rough indicator of the dispersion of the portfolio weights over the holding period. Table 1 shows that the M-V portfolio, as
expected, is associated with relatively larger swings in the portfolio weights. To better understand how the portfolios differ with respect to the changes in portfolio weights dictated by the three different investment schemes, the performance statistics is presented in Table 2 for three levels of transaction costs.

Table 2 - \( b = 20 \) days; \( \delta = 2.5; \xi = 3.0; \tau = 0.05 \)

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>( f )</th>
<th>0.0%</th>
<th>0.5%</th>
<th>1.0%</th>
<th>1.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>EQ</td>
<td>13.50%</td>
<td>13.37%</td>
<td>13.24%</td>
<td>13.11%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>14.79%</td>
<td>13.71%</td>
<td>12.62%</td>
<td>11.53%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>14.17%</td>
<td>12.36%</td>
<td>10.54%</td>
<td>8.72%</td>
</tr>
<tr>
<td>Volatility</td>
<td>EQ</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.67%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>23.89%</td>
<td>23.89%</td>
<td>23.90%</td>
<td>23.92%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>26.40%</td>
<td>26.41%</td>
<td>26.44%</td>
<td>26.49%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>EQ</td>
<td>0.4925</td>
<td>0.4870</td>
<td>0.4815</td>
<td>0.4759</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>0.5419</td>
<td>0.4963</td>
<td>0.4505</td>
<td>0.4045</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>0.4668</td>
<td>0.3979</td>
<td>0.3287</td>
<td>0.2592</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td>EQ</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>5.01%</td>
<td>5.02%</td>
<td>5.06%</td>
<td>5.12%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>10.99%</td>
<td>11.01%</td>
<td>11.07%</td>
<td>11.16%</td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>EQ</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>6.00%</td>
<td>6.00%</td>
<td>6.00%</td>
<td>6.00%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>10.17%</td>
<td>10.18%</td>
<td>10.18%</td>
<td>10.19%</td>
</tr>
</tbody>
</table>

As expected, the portfolio characterized by the largest swings in the portfolio weights (the M-V portfolio) is also the most sensitive to transaction costs. As far as risk-adjusted return is concerned, the break-even brokerage fee that equalizes the Sharpe ratios of the B-L and the EQ portfolios has been calculated to approximately 0.7%. While brokerage fees of less than 0.7% may well be unrealistic for most private investors, it is by no means obvious that large institutional portfolio managers would face transaction costs this high.
5.3 Investment Horizon

Table 3 summarizes the essential performance figures for the three portfolios for four different rebalancing frequencies.

Table 3 - \( f = 0 \); \( \delta = 2.5; \xi = 3.0; \tau = 0.05 \)

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>( b )</th>
<th>20</th>
<th>50</th>
<th>125</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>EQ</td>
<td>13.50%</td>
<td>13.64%</td>
<td>12.73%</td>
<td>12.18%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>14.79%</td>
<td>15.22%</td>
<td>13.73%</td>
<td>13.06%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>14.17%</td>
<td>14.90%</td>
<td>12.94%</td>
<td>14.24%</td>
</tr>
<tr>
<td>Volatility</td>
<td>EQ</td>
<td>23.66%</td>
<td>24.27%</td>
<td>24.47%</td>
<td>24.46%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>23.89%</td>
<td>24.27%</td>
<td>24.75%</td>
<td>25.14%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>26.40%</td>
<td>26.62%</td>
<td>27.14%</td>
<td>28.00%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>EQ</td>
<td>0.4925</td>
<td>0.4857</td>
<td>0.4447</td>
<td>0.4224</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>0.5419</td>
<td>0.5510</td>
<td>0.4799</td>
<td>0.4460</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>0.4668</td>
<td>0.4903</td>
<td>0.4086</td>
<td>0.4426</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td>EQ</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>5.01%</td>
<td>4.83%</td>
<td>5.30%</td>
<td>5.39%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>10.99%</td>
<td>10.94%</td>
<td>11.63%</td>
<td>12.66%</td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>EQ</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.80%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>6.00%</td>
<td>5.96%</td>
<td>6.07%</td>
<td>6.03%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>10.17%</td>
<td>10.07%</td>
<td>10.61%</td>
<td>11.15%</td>
</tr>
<tr>
<td>Largest Cumulative Loss</td>
<td>EQ</td>
<td>-55.30%</td>
<td>-53.90%</td>
<td>-56.81%</td>
<td>-57.81%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>-55.32%</td>
<td>-52.64%</td>
<td>-55.21%</td>
<td>-58.42%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>-67.39%</td>
<td>-60.72%</td>
<td>-60.04%</td>
<td>-60.11%</td>
</tr>
<tr>
<td>Largest Cumulative Gain</td>
<td>EQ</td>
<td>291.73%</td>
<td>296.99%</td>
<td>262.63%</td>
<td>243.29%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>342.89%</td>
<td>362.17%</td>
<td>298.08%</td>
<td>272.54%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>398.33%</td>
<td>353.71%</td>
<td>278.24%</td>
<td>330.90%</td>
</tr>
</tbody>
</table>

As the figures suggest, extending the investment horizon beyond 50 days results in lower Sharpe ratios for all three portfolios. Varying the horizon does of course affect the volatility of returns somewhat, but these differences remain quite small. The results of rebalancing the portfolios less frequently are thus captured mainly by the worsening of the return figures. This seems to hold true for all investment horizons except 50 days, which (in terms of outright return) has been the optimal horizon for all three portfolios. The real-world investor should of course bear in mind that less frequent rebalancing can aid in lowering the transaction costs. Since different investors face different transaction cost structures, it is difficult to point to a single rebalancing frequency which would have been ‘optimal’ for a real-world investor over this particular holding period. Given zero transaction costs though, the best Sharpe ratio was observed for an investment horizon of 20 days for the EQ portfolio and 50 days for the B-L and M-V portfolios.

Regardless of the weighting scheme employed by the investor, more frequent rebalancing of the portfolio implies that the allocation of assets will account for the most recent market data for a
larger part of the time. If the investor believes the market to be as efficient as advocated by the CAPM, he would naturally like to minimize the degree to which the allocation deviates from the equilibrium benchmark. Given the fact that new issues of shares and share repurchases do occur from time to time, this can be accomplished in two ways. The investor can either select to maintain a high-enough rebalancing frequency or he can make sure to rebalance the portfolio whenever share repurchases and/or new issues of shares take place. To illustrate the differences between the two most extreme rebalancing frequencies, the performance of the EQ portfolio is presented in table 4 figure 2 for \( b = 20 \) days (still denoted \( EQ \)) and for \( b = 10 \) years (denoted \( STATIC \)).

**Table 4 - (f = 0 %)**

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>EQ</th>
<th>STATIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>13,50%</td>
<td>11,30%</td>
</tr>
<tr>
<td>Volatility</td>
<td>23,66%</td>
<td>24,54%</td>
</tr>
<tr>
<td>-Highest Volatility</td>
<td>40,06%</td>
<td>42,46%</td>
</tr>
<tr>
<td>-Lowest Volatility</td>
<td>11,91%</td>
<td>12,45%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0,4925</td>
<td>0,3850</td>
</tr>
<tr>
<td>-Highest Sharpe Ratio</td>
<td>2,6927</td>
<td>2,1505</td>
</tr>
<tr>
<td>-Median Sharpe Ratio</td>
<td>1,0110</td>
<td>1,0490</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td></td>
<td>6,68%</td>
</tr>
<tr>
<td>Information Ratio</td>
<td></td>
<td>-0,3271</td>
</tr>
<tr>
<td>Alpha Compared with EQ Portfolio (CAR)</td>
<td></td>
<td>-0,0218</td>
</tr>
<tr>
<td>Beta Compared with EQ Portfolio</td>
<td></td>
<td>0,9982</td>
</tr>
<tr>
<td>Correlation with EQ Portfolio</td>
<td></td>
<td>0,9625</td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>3,81%</td>
<td>4,15%</td>
</tr>
</tbody>
</table>

**Figure 2 - (f = 0 %)**

As can be seen from figure 2, the regularly rebalanced EQ portfolio outperforms its static counterpart for the period as a whole despite underperforming somewhat during the first few years. Not rebalancing the portfolio for ten years means that the actual weights in the static portfolio do, at times, deviate significantly from their respective market-capitalization levels. For example, the weights for ERICb.ST, ELUXb.ST and SWEDa.ST in the static portfolio deviate as
much as 4.7, 2.7 and 3.0 percentage points from their equilibrium levels. These deviations are not at all negligible given the fact that the actual market-capitalization weights for these three stocks range from 1 to 23 per cent of the considered investment universe. Another way of measuring these deviations is to consider the TEV figure for the static portfolio. Over the entire holding period, the TEV for the static portfolio amounts to 6.68 % - in other words a notable deviation from the benchmark. Measuring the deviations for each separate year, however, the TEV figures range from 2.08 to as much as 13.69 %. Such deviations suggest that an investor who tries to cut corners by not rebalancing the portfolio weights regularly can expect the performance to differ considerably from that of the corresponding dynamic portfolio as the years go by.

As noted earlier, the rebalancing frequency must of course be selected with the transaction cost structure in mind. To illustrate how high the brokerage fee would have to be for the static alternative to yield a Sharpe ratio equal to that of the 20-day EQ portfolio, the break-even brokerage fee has been calculated to approximately 15.4 %. For this particular investment universe and this particular holding period then, the EQ portfolio is superior in terms of risk-adjusted return compared to its static counterpart - even for investors facing very high transaction costs.

### 5.4 The Risk Aversion Parameter

Another quite difficult decision for the investor is what numerical value to assign to $\delta$. This value should represent the portfolio manager's best estimate of the future risk aversion of the average investor in the market, i.e. the best estimate of the future risk-adjusted return on the market portfolio. To illustrate how large an impact this decision can have, portfolio performance statistics for five different values of $\delta$ are provided in table 5.

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>$\delta$</th>
<th>1.0</th>
<th>1.5</th>
<th>2.5</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>EQ</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>16.33%</td>
<td>15.60%</td>
<td>14.79%</td>
<td>14.37%</td>
<td>14.23%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>20.77%</td>
<td>20.39%</td>
<td>14.17%</td>
<td>13.55%</td>
<td>14.03%</td>
</tr>
<tr>
<td>Volatility</td>
<td>EQ</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>25.02%</td>
<td>24.46%</td>
<td>23.89%</td>
<td>23.71%</td>
<td>23.67%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>32.97%</td>
<td>29.51%</td>
<td>26.40%</td>
<td>25.89%</td>
<td>25.72%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>EQ</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>0.5788</td>
<td>0.5619</td>
<td>0.5419</td>
<td>0.5279</td>
<td>0.5230</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>0.5737</td>
<td>0.6283</td>
<td>0.4668</td>
<td>0.4517</td>
<td>0.4736</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td>EQ</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>9.21%</td>
<td>7.38%</td>
<td>5.01%</td>
<td>3.79%</td>
<td>3.38%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>23.08%</td>
<td>17.46%</td>
<td>10.99%</td>
<td>9.66%</td>
<td>9.04%</td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>EQ</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>9.22%</td>
<td>7.65%</td>
<td>6.00%</td>
<td>5.21%</td>
<td>4.97%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>12.56%</td>
<td>11.51%</td>
<td>10.17%</td>
<td>9.22%</td>
<td>8.66%</td>
</tr>
</tbody>
</table>
Since $\delta$ does not in any way affect the allocation decision for the EQ portfolio, the values in the rows denoted EQ remain constant as $\delta$ is altered. For the B-L and M-V portfolios, on the other hand, the value of $\delta$ has a notable effect on the return figures. In our empirical simulations, it can be seen that lower values of $\delta$ has generally led to higher returns than higher values. Lowering the numerical value of $\delta$ has also given rise to a fairly modest increase in the volatility of the B-L portfolio, whereas the volatility of the M-V portfolio has increased dramatically. For the B-L portfolio then, the Sharpe ratio has seen consistent improvements as the value of $\delta$ has been lowered. Due to less consistent changes in the return figures for the M-V portfolio, its Sharpe ratio has not increased as consistently as that of the B-L portfolio. Lowering the $\delta$ has also gone hand-in-hand with higher volatility of the portfolio weights for both portfolios. Not surprisingly, this has pushed the TEV figures upwards. At first glance, these results may not seem all that intuitive. There is no ‘natural’ economic principle that explains why less optimistic expectations on the part of a single investor would translate into better or worse performance on that investor’s portfolio. If one stops to consider the mechanism by which the vectors of expected returns are estimated, however, the results are no longer so surprising.

For simplicity, let us consider the case of the M-V portfolio first. Since $\delta$ in effect represents the expected risk-adjusted return on a market-capitalization-weighted portfolio, higher values will, ceteris paribus, raise\textsuperscript{11} all values of the elements in $\Pi$ proportionately. For the eight assets about which the investor has views, the expected returns are calculated in accordance with equation 16. As can be seen from this equation, the amount by which the expected return is adjusted away from the equilibrium level stands in proportion to the values of the dividend yield rates and the market capitalizations of the assets under consideration – not in proportion to the $\pi$ values themselves. This means that the adjustments made to these eight $\pi$ values are, in absolute terms, independent of the value assigned to $\delta$. A higher value of $\delta$ - that is, (higher) expected returns on the 15 assets about which no views are held - will therefore render the adjustments made to the other eight assets relatively smaller. This means that for as long as the investor has views which are not set in proportion to the values in $\Pi$, the proportions among the expected returns that are fed into the optimiser will be affected by the numerical value assigned to $\delta$.

In other words, higher $\delta$ values mean that the views will have a less significant impact on the final vector of expected returns that is fed into the optimiser. This holds true for the particular view generating process employed in our simulations, i.e. equation 16. Had the process been otherwise configured, the adjustments could have been made in proportion to the $\pi$ values in question. In general, the investor should always bear in mind that what matters to the outcome of the optimisation process (except the covariance matrix) is the proportions among the expected returns. If $\delta$ affects these proportions, it will also affect the weight vector returned by the optimiser. In the case of the B-L model, the mechanism by which the final vector of expected returns is estimated is of course slightly different. Nevertheless, the principle still holds true - something which explains why decreases in $\delta$ have given rise to consistently higher TEV figures for both the B-L and the M-V portfolios.

\textsuperscript{11} Individual elements of $\Pi$ will either increase or decrease depending on the values in the covariance matrix. Since positive covariances are noticeably more commonly observed than negative ones, the elements of $\Pi$ will on most occasions increase as $\delta$ is raised.
While the anatomy of the view generating process explains why the TEV figures of both portfolios increase consistently as $\delta$ is lowered, it does not explain the observed changes in the Sharpe ratios of the two portfolios. The Sharpe ratio of the B-L portfolio has consistently increased as $\delta$ is lowered. In contrast, the Sharpe ratio of the M-V portfolio does not change anywhere near as consistently. This makes it more difficult to determine whether or not the strategy has added value. When implemented via the B-L model, the views have undoubtedly managed to increase the performance of the investment. When put to work in the classical M-V framework, however, the results are mixed. A plausible explanation to these results is the fact that the algebraic backbone of the B-L model is designed to limit the impact that the investor’s private views have on the vector of expected returns. Since the M-V framework does not come with any such mechanism, the portfolio weights of the M-V portfolio are bound to respond more dramatically as the proportions among the expected returns are altered. This may well account for part of the reason as to why the Sharpe ratio of the M-V portfolio does not undergo consistent changes as $\delta$ is lowered.

5.5 The Investment Strategy

To gain a better understanding of to what extent the views have added value to the investment, the portfolio performance statistics have been simulated for five values of $\xi$. Since higher values imply stronger views on the eight selected stocks, raising this variable is a simple yet effective way of ‘amplifying’ the investment strategy’s influence on the asset allocation decision. The relevant statistics are tabulated in table 6.

Table 6 - ($f = 0\%; b = 20$ days; $\delta = 2.5; \tau = 0.05$)

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>$\xi$</th>
<th>1.0</th>
<th>1.5</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>EQ</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>13.83%</td>
<td>14.02%</td>
<td>14.79%</td>
<td>15.23%</td>
<td>15.59%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>14.34%</td>
<td>13.94%</td>
<td>14.17%</td>
<td>16.34%</td>
<td>20.16%</td>
</tr>
<tr>
<td>Volatility</td>
<td>EQ</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>23.60%</td>
<td>23.63%</td>
<td>23.89%</td>
<td>24.15%</td>
<td>24.46%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>25.10%</td>
<td>25.70%</td>
<td>26.40%</td>
<td>27.31%</td>
<td>29.56%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>EQ</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>0.5079</td>
<td>0.5150</td>
<td>0.5419</td>
<td>0.5540</td>
<td>0.5619</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>0.4975</td>
<td>0.4702</td>
<td>0.4668</td>
<td>0.5308</td>
<td>0.6192</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td>EQ</td>
<td>1.91%</td>
<td>2.77%</td>
<td>5.01%</td>
<td>6.27%</td>
<td>7.38%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>6.47%</td>
<td>8.27%</td>
<td>10.99%</td>
<td>13.11%</td>
<td>17.46%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>EQ</td>
<td>4.25%</td>
<td>4.64%</td>
<td>6.00%</td>
<td>6.86%</td>
<td>7.65%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>6.99%</td>
<td>8.11%</td>
<td>10.17%</td>
<td>10.94%</td>
<td>11.56%</td>
</tr>
<tr>
<td></td>
<td>M-V</td>
<td>6.99%</td>
<td>8.11%</td>
<td>10.17%</td>
<td>10.94%</td>
<td>11.56%</td>
</tr>
</tbody>
</table>
Since $\xi$ does not in any way influence the allocation in the EQ portfolio, the figures in these rows remain constant. The performance figures for the B-L portfolio confirm that the view generating process employed in our simulations does indeed add value to this portfolio by pushing its risk-adjusted return to higher levels as $\xi$ is raised. This relationship does not hold true for the M-V portfolio, whose Sharpe ratio increases both when $\xi$ is raised and lowered. The results for the M-V portfolio are not too surprising given the inconsistent changes seen in its Sharpe ratio as $\delta$ was altered. It can also be concluded that the amplifying this particular investment strategy hardly seems to affect the volatility figures for the B-L portfolio. In contrast, the volatility of the M-V portfolio undergoes considerably larger increases as $\xi$ is raised.

As expected, the TEV figures for both portfolios increase consistently as the views are amplified. This effect stems from the fact that specifying more extreme views (i.e. raising $\xi$) alters the proportions among the expected returns that are fed into the optimiser. This gives rise to portfolio weights which deviate further from their respective market-capitalization levels. As noted earlier, the B-L model is designed to be less sensitive to variations in the private views held by the investor. Therefore, one might expect the Sharpe ratio of the B-L portfolio not to respond as sharply as that of the M-V portfolio as $\xi$ is altered. This is precisely what can be seen from the results. For the five different values of $\xi$ which have been tested, the Sharpe ratio of the B-L portfolio ranges between 0.5079 and 0.5619, whereas the corresponding interval for the M-V portfolio stretches between 0.4668 and 0.6192. On the one hand, the M-V portfolio has given the investor the opportunity to reach higher risk-adjusted return than the B-L portfolio. On the other, its sensitivity towards the choice of $\xi$ has also meant that such superior performance is highly dependent upon the investor selecting the ‘right’ value of $\xi$. With the B-L framework, the Sharpe ratio remains superior to that of the EQ portfolio for all five levels of $\xi$. The performance relative to the EQ portfolio is of course dependent upon the accuracy of the investment strategy. Since the investment strategy has resulted in Sharpe ratios higher than that of the EQ portfolio for all five B-L portfolios and three of the five M-V portfolios, the investment strategy must obviously have contributed with some form of informational edge to the investment.

5.6 The Views

Another simple and intuitive way of testing how sensitive the performance of the B-L and M-V portfolios are to variations in the views held by the investor is to follow the methodology presented by Bertsimas, Gupta and Paschalidis (2012). Instead of varying the strength of views generated by a specific investment strategy, the values assigned to the views are set to linear combinations of the implied equilibrium returns. In other words, the values assigned to the elements of $Q$ are the corresponding values from $\Pi$ multiplied by a ‘tilting factor’ (henceforth denoted $\theta$). Setting this factor to values below unity corresponds to the investor having negative views about the eight stocks in question, whereas values above unity correspond to positive views. This form of sensitivity analysis allows us to thoroughly investigate the consequences of consistently positive and negative views on the part of the investor. In table 7, key performance statistics for the three portfolios are presented for twelve different values of $\theta$. 

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Starting with \( \theta = 1 \), i.e. no views, it can be seen that the return and volatility of the B-L and M-V portfolios lie very close to the figures for the EQ portfolio. The reason the figures for the B-L portfolio are not exactly equal to the figures for the M-V portfolio stems principally from the fact that the covariance matrices that are fed into the optimiser are not entirely equal (\( \Sigma \neq \Sigma \)). The fact that the optimal portfolio weights have been estimated using an iterative optimiser may also help to explain why the figures do not match perfectly. This stems from the fact that rounding errors always plague the solution to some extent. Although small, such rounding errors may well give rise to the discrepancies between the figures for the EQ, B-L and M-V portfolios when compounded over ten years. Nevertheless, the TEV figures for the B-L and M-V portfolios still amount to no more than 0.14 and 1.41 per cent, respectively.

The figures in the other columns show that the return on the M-V portfolio is noticeably more sensitive to the value of \( \theta \) than the return on the B-L portfolio. The same is true for the volatility of the two, although these changes are considerably less noticeable. As a result, the Sharpe ratio of the B-L portfolio remains fairly close to the EQ level as \( \theta \) is altered, whereas the Sharpe ratio of the M-V portfolio shows considerably larger variations. The relationship between the choice of \( \theta \) and the resulting ex-post Sharpe ratio of the three different portfolios is plotted in figure 3.

**Figure 3** - (\( f = 0 \% ; b = 20 \) days; \( \delta = 2.5 \); \( \tau = 0.05 \))
The graph clearly shows that for the θ values studied within the scope of our sensitivity analysis, the B-L portfolio’s Sharpe ratio never deviates very far from that of the EQ benchmark. For the particular securities about which views are held, increasing the expected returns has consistently managed to increase the Sharpe ratio of the B-L portfolio. While there is no economic theory that would explain why increased optimism on the part of the investor has managed to lead to better performance, the resulting portfolio weights must obviously have moved in a direction which turned out to be favourable. In contrast, the M-V portfolio’s Sharpe ratio ranges from values both lower and higher than that of the benchmark. For the specific holding period and the specific investment universe considered in our simulations then, the M-V model gives rise to performance similar to that of the benchmark set by the EQ portfolio only for quite modest views (0.90 ≤ θ ≤ 1.10). In contrast, the B-L portfolio’s Sharpe ratio remains quite similar to that of the EQ portfolio even for considerably more extreme views (0.125 ≤ θ ≤ 2.000). A plausible reason for the striking differences between the Sharpe ratios of the B-L and M-V portfolios is that the portfolio weights suggested by the M-V framework is known to be considerably more sensitive to variations in the proportions among the expected returns used in the optimisation process.

Not surprisingly, the TEV figures for the M-V portfolio rise to values above 16 %, whereas the corresponding figures for the B-L portfolio remain well below 6 %. For most values of θ then, the choice of the model used to estimate the optimal portfolio weights has a significant impact on the degree to which the portfolio performance deviates from that of the benchmark. In figure 4, the TEV figures for the two portfolios are presented graphically for twelve values of θ.

Figure 4 - (f = 0 %; h = 20 days; δ = 2.5; τ = 0.05)

Like the relationship between the Sharpe ratios of the portfolios and the values assigned to θ, the two portfolios’ TEV figures respond quite differently to changes in θ. The TEV of the B-L portfolio responds only quite modestly as the θ value is altered. The TEV of the M-V portfolio, on the other hand, shows a tendency to increase quite swiftly as θ is pushed away from unity.
5.7 The Uncertainty of the CAPM Prior

As mentioned by other researchers (see for instance Mankert, 2010; Walters, 2011; Walters, 2013), deciding what numerical value to assign to $\tau$ is not without difficulties. To shed some light on what effect variations in this variable have on the outcome of the investment, sensitivity analysis has been performed by letting $\tau$ take on four different values which all lie within a reasonable range. In table 8, the resulting performance statistics for the B-L portfolio are presented alongside the EQ benchmark.

Table 8 - ($f = 0 \, \%; \, b = 20 \, \text{days}; \, \delta = 2.5; \, \xi = 3.0$ )

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>$\tau$</th>
<th>0.025</th>
<th>0.050</th>
<th>0.075</th>
<th>0.100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>EQ</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
<td>13.50%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>14.78%</td>
<td>14.79%</td>
<td>14.78%</td>
<td>14.79%</td>
</tr>
<tr>
<td>Volatility</td>
<td>EQ</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>23.90%</td>
<td>23.89%</td>
<td>23.88%</td>
<td>23.87%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>EQ</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
<td>0.4925</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>0.5413</td>
<td>0.5419</td>
<td>0.5417</td>
<td>0.5420</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td>EQ</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>5.04%</td>
<td>5.01%</td>
<td>4.99%</td>
<td>4.96%</td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>EQ</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
</tr>
<tr>
<td></td>
<td>B-L</td>
<td>6.00%</td>
<td>6.00%</td>
<td>5.99%</td>
<td>5.99%</td>
</tr>
</tbody>
</table>

Taking into account that the figures have been computed for a holding period of ten years, all of the changes in the portfolio performance statistics are extremely small and can hardly be considered economically significant. These are precisely the results one would expect, given the fact that the uncertainty of the views (i.e. $\Omega$) has been specified in proportion to $\tau$. Specifying $\Omega$ in accordance with equation 10 essentially means that the views (given by $Q$) and the implied equilibrium expected returns (given by $\Pi$) are given equal importance and that the final vector of expected returns becomes independent of $\tau$ (Walters, 2011). The only way in which $\tau$ can affect the optimal portfolio weights is thus through its influence on the posterior covariance matrix, i.e. $\tilde{\Sigma}$. As can be seen from table 8, changing $\tau$ only has very modest effects on the resulting portfolio characteristics.

The results indicate that for those investors who follow He and Litterman’s (1999) lead and set $\Omega$ proportional to $\tau$, there is certainly no need to expend scarce resources on the estimation of the true uncertainty of the CAPM. Eliminating this difficulty comes at a cost, however - namely the need to assume that the variances of the views are determined by the variance of returns and the uncertainty of the prior. Although assuming that $\Omega$ is proportional to these two variables may well be a theoretically reasonable starting point, the investor still misses out on the opportunity to express that some views are associated with extraordinarily high (low) uncertainty. For an investor who does not have such specific opinions about the uncertainty of the views, missing out on this opportunity may not pose any problems. For an investor who uses an investment
strategy which relies upon its own unique $\Omega$ matrix, however, it is important to recognize that the final vector of expected returns is no longer independent of $\tau$. The use of such investment strategies may therefore call for careful estimation of the true uncertainty of the CAPM.

5.8 Risk-Adjusted Equal Means – An Alternative Approach

For the purpose of evaluating how sound a starting point the CAPM equilibrium approach is, two additional portfolios have been constructed. One, denoted $M$-$V$-$IR$, is a portfolio which is constructed using the vector of implied equilibrium returns ($\Pi$) and the covariance matrix $\Sigma$, i.e. the mean-variance efficient portfolio generated by the M-V framework when the investor has no views. The other, denoted $M$-$V$-$RAEM$, is the mean-variance efficient portfolio generated for the investor who estimates the vector of expected returns in accordance with the risk-adjusted equal means approach. The summary statistics for these two portfolios are presented alongside the corresponding figures for the EQ benchmark portfolio in table 9. A more thorough presentation of the results can be found in Appendix D, where statistics for each of the ten years are presented.

Table 9 - ($f = 0\%$; $b = 20$ days; $\delta = 2.5$; $\tau = 0.05$)

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>EQ</th>
<th>M-V-IR</th>
<th>M-V-RAEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>13.50%</td>
<td>13.96%</td>
<td>17.22%</td>
</tr>
<tr>
<td>Volatility</td>
<td>23.66%</td>
<td>23.70%</td>
<td>18.95%</td>
</tr>
<tr>
<td>-Highest Volatility</td>
<td>40.06%</td>
<td>40.07%</td>
<td>32.69%</td>
</tr>
<tr>
<td>-Lowest Volatility</td>
<td>11.91%</td>
<td>11.91%</td>
<td>9.92%</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.4925</td>
<td>0.5108</td>
<td>0.8113</td>
</tr>
<tr>
<td>-Highest Sharpe Ratio</td>
<td>2.6927</td>
<td>2.6982</td>
<td>3.5379</td>
</tr>
<tr>
<td>-Median Sharpe Ratio</td>
<td>1.0110</td>
<td>1.0136</td>
<td>0.9728</td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td></td>
<td>1.41%</td>
<td>11.04%</td>
</tr>
<tr>
<td>Information Ratio</td>
<td></td>
<td>1.0196</td>
<td>2.1785</td>
</tr>
<tr>
<td>Alpha Compared with EQ Portfolio</td>
<td></td>
<td>0.45%</td>
<td>7.60%</td>
</tr>
<tr>
<td>Beta Compared with EQ Portfolio</td>
<td></td>
<td>1.0001</td>
<td>0.7127</td>
</tr>
<tr>
<td>Correlation with EQ Portfolio</td>
<td></td>
<td>0.9982</td>
<td>0.8898</td>
</tr>
<tr>
<td>Volatility of PortfolioWeights</td>
<td>3.81%</td>
<td>3.81%</td>
<td>6.58%</td>
</tr>
<tr>
<td>Largest Cumulative Loss</td>
<td>-55.30%</td>
<td>-55.33%</td>
<td>-40.62%</td>
</tr>
<tr>
<td>Largest Cululative Gain</td>
<td>291.73%</td>
<td>310.02%</td>
<td>502.77%</td>
</tr>
</tbody>
</table>

As can be seen, the M-V-IR portfolio shows performance figures very similar to those of the EQ portfolio. Theoretically, the two should yield identical results since the expected returns are derived from the market-capitalization weights and the covariance matrix $\Sigma$. Because the optimal portfolio weights have not been computed analytically but estimated using an iterative optimiser, rounding errors along the way are bound to lead to minor discrepancies in the portfolio weights. A ten-year TEV figure of a mere 1.41% shows that such discrepancies are indeed very small.

In contrast, the M-V-RAEM portfolio shows very different performance characteristics compared to the EQ benchmark. Both its return and risk measures look noticeably more
favourable than those of the benchmark. A TEV figure of no less 11.04% indicates that the risk-adjusted equal means approach has led the investor to portfolio weights which are quite different from those generated by the CAPM approach. While such a weighting scheme may not pose any practical problems for an asset manager responsible for a portfolio of large-cap stocks, it is not necessarily a realistic approach if the investment universe is confined to less liquid securities and/or small-cap stocks. If the amount of capital being invested is large, the manager of such a fund may struggle to take on large positions in specific securities without encountering liquidity-related obstacles.

The M-V-RAEM portfolio also shows a portfolio weight volatility figure which is noticeably higher than that of the benchmark. This indicates that the M-V-RAEM portfolio might be prone to lose performance quickly if the transaction costs rise. To uncover just how sensitive its performance really is, the performance has been simulated for seven different levels of brokerage fees. The results of these simulations are provided in table 10.

<table>
<thead>
<tr>
<th>SUMMARY STATISTICS</th>
<th>/</th>
<th>0.0%</th>
<th>0.5%</th>
<th>1.0%</th>
<th>1.5%</th>
<th>2.0%</th>
<th>2.1%</th>
<th>2.3%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return (CAR)</td>
<td>EQ</td>
<td>13.50%</td>
<td>13.37%</td>
<td>13.24%</td>
<td>13.11%</td>
<td>12.98%</td>
<td>12.96%</td>
<td>12.90%</td>
</tr>
<tr>
<td>M-V-IR</td>
<td>13.96%</td>
<td>13.79%</td>
<td>13.61%</td>
<td>13.44%</td>
<td>13.27%</td>
<td>13.23%</td>
<td>13.16%</td>
<td></td>
</tr>
<tr>
<td>M-V-RAEM</td>
<td>17.22%</td>
<td>15.77%</td>
<td>14.31%</td>
<td>12.84%</td>
<td>11.37%</td>
<td>11.08%</td>
<td>10.49%</td>
<td></td>
</tr>
<tr>
<td>Volatility</td>
<td>EQ</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.66%</td>
<td>23.67%</td>
<td>23.67%</td>
<td>23.67%</td>
<td>23.67%</td>
</tr>
<tr>
<td>M-V-IR</td>
<td>23.70%</td>
<td>23.70%</td>
<td>23.71%</td>
<td>23.71%</td>
<td>23.71%</td>
<td>23.72%</td>
<td>23.72%</td>
<td></td>
</tr>
<tr>
<td>M-V-RAEM</td>
<td>18.95%</td>
<td>18.96%</td>
<td>18.98%</td>
<td>19.01%</td>
<td>19.05%</td>
<td>19.06%</td>
<td>19.08%</td>
<td></td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>EQ</td>
<td>0.4925</td>
<td>0.4870</td>
<td>0.4815</td>
<td>0.4759</td>
<td>0.4703</td>
<td>0.4692</td>
<td>0.4670</td>
</tr>
<tr>
<td>M-V-IR</td>
<td>0.5108</td>
<td>0.5035</td>
<td>0.4962</td>
<td>0.4889</td>
<td>0.4815</td>
<td>0.4800</td>
<td>0.4770</td>
<td></td>
</tr>
<tr>
<td>M-V-RAEM</td>
<td>0.8113</td>
<td>0.7340</td>
<td>0.6563</td>
<td>0.5782</td>
<td>0.4999</td>
<td>0.4842</td>
<td>0.4528</td>
<td></td>
</tr>
<tr>
<td>Tracking Error Volatility</td>
<td>EQ</td>
<td>1.41%</td>
<td>1.41%</td>
<td>1.41%</td>
<td>1.42%</td>
<td>1.42%</td>
<td>1.42%</td>
<td>1.42%</td>
</tr>
<tr>
<td>M-V-IR</td>
<td>11.04%</td>
<td>11.05%</td>
<td>11.08%</td>
<td>11.12%</td>
<td>11.18%</td>
<td>11.20%</td>
<td>11.23%</td>
<td></td>
</tr>
<tr>
<td>M-V-RAEM</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td></td>
</tr>
<tr>
<td>Volatility of Portfolio Weights</td>
<td>EQ</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
<td>3.81%</td>
</tr>
<tr>
<td>M-V-IR</td>
<td>6.58%</td>
<td>6.58%</td>
<td>6.59%</td>
<td>6.59%</td>
<td>6.60%</td>
<td>6.60%</td>
<td>6.60%</td>
<td></td>
</tr>
<tr>
<td>M-V-RAEM</td>
<td>6.58%</td>
<td>6.58%</td>
<td>6.59%</td>
<td>6.59%</td>
<td>6.60%</td>
<td>6.60%</td>
<td>6.60%</td>
<td></td>
</tr>
</tbody>
</table>

It can quickly be concluded that the performance of the M-V-RAEM portfolio is indeed considerably more sensitive to brokerage fees than that of the EQ portfolio. It does, however, take quite high levels of transaction costs for the Sharpe ratio of the M-V-RAEM to come close to that of the EQ portfolio. More specifically, it takes a brokerage fee of somewhere between 2.10 and 2.30% to equalize the Sharpe ratios of the two. To further investigate the robustness of the results, sensitivity analysis has also been performed by varying the investment horizon. A summary of the results of these simulations are presented in table 11.
The simulations indicate that longer investment horizons generally go hand-in-hand with slightly lower return and slightly higher volatility. Nevertheless, the Sharpe ratio of the M-V-RAEM portfolio still remains well above that of the EQ and M-V-IR portfolios for all four investment horizons. The TEV figures suggest that the degree to which the M-V-RAEM portfolio deviates from the EQ benchmark hardly changes as the investment horizon is altered. The simulations generally show higher performance for shorter investment horizons. Since we have concluded that the break-even brokerage fee of the M-V-RAEM portfolio is no less than around 2.20% for a 20-day investment horizon, portfolios rebalanced less frequently can be expected to be even more resistant to transaction costs. It can therefore be concluded that the past performance of the M-V-RAEM approach is by no means dependent upon an assumption of unrealistically low transaction costs. Unlike portfolios constructed using the CAPM approach, however, a higher rebalancing frequency does not necessarily imply that the portfolio weights of an M-V-RAEM portfolio will be more in tune with the prevailing market-capitalization weights. The occurrence of new issues of shares and share repurchases is therefore not in itself a deciding factor for determining the most suitable investment horizon when using the risk-adjusted equal means approach.

To shed some additional light on the differences between portfolios generated by the two approaches, the average portfolio weights of the M-V-IR and M-V-RAEM portfolios have been compared to the averages of the actual market-capitalization weights. As expected, there are hardly any differences between the average market-capitalization weights and the average portfolio weights of the M-V-IR portfolio. In contrast, the holding period averages of the portfolio weights of the M-V-RAEM portfolio are considerably more skewed. The differences can be seen clearly in figure 5.
The performance figures for the M-V-RAEM portfolio can now be viewed in a different light. Although the ex-post volatility of the portfolio is still lower than that of the M-V-IR portfolio, the M-V-RAEM portfolio takes on significantly larger exposures to idiosyncratic risk. The most notable example is the position in SWMA.ST, which has constituted an average of more than a quarter of the portfolio. The corresponding figure for the M-V-IR portfolio is 1.71% - that is, the same as the average market-capitalization weight for this security. The M-V-RAEM portfolio is also characterized by relatively large positions in GET1b.ST and TELE2b.ST. Other securities such as NDA.ST and VOLVb.ST are given average weights which are between five and ten percentage points lower than suggested by their market-capitalizations. Moreover, many securities have at one point or another been completely excluded from the portfolio. On an average day, no less than 6.13 of the 23 assets have been excluded from the portfolio. One stock, namely INVEb.ST, has never been included in this portfolio. The investor who uses the risk-adjusted equal means approach should therefore beware of its inherent tendency to lead to highly skewed portfolio weights which bear little or no relation to the actual capitalizations of the securities in question. While such skewed portfolio weights may yield high performance, the large exposures to individual securities mean that it is especially important for the investor to recognize the presence of risks which are more subtle than the day-to-day volatility of the portfolio value.
6 Practical Considerations for a Portfolio Manager

The results of our simulated portfolios leave little doubt that the B-L and M-V approaches to portfolio optimisation are indeed characterized by quite important differences in terms of what results they generate from identical inputs. In this section, we illuminate which factors a real-world portfolio manager ought to consider when deciding which approach to opt for.

6.1 The Usefulness of the Black-Litterman Framework

Having implemented the B-L model in practice, we can conclude that fully understanding the derivations and interpretations of the model’s variables can be quite a time-consuming task. Such deep understanding may still be necessary for financial engineers seeking to tailor the model to certain specific areas of use (see for instance the version presented in Meucci, 2009). After using the model on real market data, we can establish that such deep understanding of the model’s theoretical underpinnings is not always necessary for portfolio managers looking to construct well-diversified portfolios based partially on private views. Even for users with little or moderate understanding of the model’s theoretical roots, it still offers an effective remedy to the classical M-V framework’s tendency to suggest portfolio weights which are highly sensitive to variations in the input data. As shown by our sensitivity analysis, some of the model’s variables which are widely discussed in academic papers need not pose any practical problems for real-world portfolio managers. As an example, take $\tau$ and $\Omega$ - two of the trickiest variables of the model. Estimating the true uncertainty of the CAPM prior and the covariances of one’s private views would probably demand a great deal of research.Specifying $\Omega$ in accordance with He and Litterman’s (1999) quite reasonable formula lets the investor get round both these problems at once. By performing careful sensitivity analysis, we have shown that for investors who follow this approach, the numerical value assigned to $\tau$ has virtually no effect on the resulting portfolio characteristics.

The finesse of the model is that it can be used to implement both quantitatively and qualitatively oriented investment strategies. A portfolio manager responsible for implementing an investment strategy based on quantitative analysis can utilize the model’s ‘quant-friendly’ format to specify $\tau$ and $\Omega$ in accordance with his research. For the implementation of a more qualitatively oriented investment strategy, the portfolio manager may instead opt for He and Litterman’s simplifying approach without upsetting the reasonableness of the results. We would like to argue that even with the restrictions imposed by such a simplifying assumption, the inputs used to estimate the optimal portfolio weights are still sound – even from a theoretical point of view. The only real restriction placed upon the inputs is the assumption of the uncertainty of one’s private views being proportionate to the uncertainty of the CAPM and the historical volatility of the securities which the views concern. Making such a simplifying assumption enables the manager to save precious time and avoid constructing the portfolio based on bias-prone qualitative estimations of how accurate the views are expected to be.

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12 The covariance matrix used, i.e. $\Sigma$, need not be estimated from historical return data. It is of course supposed to represent the best feasible estimate of future return covariances.
6.2 Selecting the Model that Best Befits the Investment Philosophy

From the results of our simulations, it has become clear that the B-L and M-V models generate quite different portfolios despite the use of identical input data. Ample sensitivity analysis has also enabled us to confirm that the results of the M-V model are much more sensitive to variations in the input data than those of the B-L model (as shown by the M-V portfolios’ higher TEV figures). Based on these observations, we can establish that the two models are suited for different areas of use and that the B-L model exhibits characteristics which make it more appropriate for some applications than others. The results have shown that the B-L model leads the investor to portfolio performance which is more similar to that of a market-capitalization-weighted portfolio than the M-V model does.

If there is one type of investor that the B-L model is particularly suited for, it is the investor who believes that the CAPM is good for forecasting expected returns on financial assets in the long run, but who wishes to use private views to exploit perceived short-run mispricing. This could for instance be the case for mutual funds who charge their clients for active management but who do not wish to take on large exposures to individual securities or individual markets. By first defining the investment universe and a suitable investment strategy and then using that strategy to generate views, the portfolio weights can be tilted away from their market-capitalization levels accordingly. By simulating how the portfolio would have performed historically, the manager can get an idea of by how much its performance deviates from that of the benchmark. The desired deviation from the benchmark can then be specified in terms of a target TEV figure.

On the one hand, the B-L model’s tendency to generate portfolio performance which resembles that of a market-capitalization weighted index can be quite attractive to managers of portfolios whose performance is evaluated in relation to that of the index. On the other, this feature may pose a significant impediment if the portfolio manager’s aim is to design a fund whose performance is supposed to be dissimilar to that of the index. Hedge fund managers who seek to offer their clients performance which is fundamentally uncorrelated to the market may be better off using other means – such as the classical M-V framework – to construct their portfolios.

Another conceivable alternative for fund managers who for one reason or another do not regard the CAPM equilibrium as a suitable benchmark is to use the B-L model and estimate the \( \Pi \) matrix using benchmark weights which do not coincide with the market-capitalization weights. Instead, the elements of \( W_{\text{mkt}} \) may be set to values which the portfolio manager regards as ‘appropriate’ benchmark weights. The portfolio optimisation can then be carried out in the usual manner and the model will see to it that the weights suggested by the optimiser gravitate towards the pre-set levels specified by the manager. With such an alternative approach, the weight vector suggested by the optimiser will of course no longer be anchored to the CAPM equilibrium. Nevertheless, fund managers who do not subscribe to the theoretical soundness of the CAPM equilibrium may still find the dynamic properties of the B-L model quite useful for constructing portfolios whose performance characteristics gravitate towards those of a predefined benchmark. Specifying benchmark weights which are tailored to suit the investment philosophy might also be helpful to portfolio managers who face constraints. By specifying benchmark weights which lie within the acceptable range, the manager may still use the B-L framework to derive the ‘optimal’
portfolio weights without having to constrain the optimisation process itself. Unfortunately, since our empirical study has not covered such an alternative approach, we are unable to comment on how appropriate it is in comparison with simply imposing the constraints on the optimisation process itself.

6.3 The CAPM as a Benchmark Portfolio

An important aspect that the portfolio manager ought to consider is which factors are believed to account for the expected performance of the securities about which private views are held. Since the original B-L model has close ties with the CAPM, it is of crucial importance to consider the underlying dynamics of the CAPM and its theoretical underpinnings. According to the CAPM, the expected return on an individual asset follows as the product of its \( \beta \) and the expected return on the already diversified market portfolio. For an investor who believes that the expected return on a certain security will be, let us assume, one percentage point above its implied return, it is therefore important to consider why. If the extra percentage point is believed to stem from an event that it is prone to also influence the expected returns on securities about which the investor has not specified any specific views, adjusting the expected returns for these securities in accordance with the covariance matrix may well be a sound move. If instead the extra percentage point is believed to stem from an event that is believed to be fundamentally unique to the security in question, i.e. that it will yield positive \( \alpha \) over the intended holding period, adjusting the expected returns on other assets no longer makes sense.

What the CAPM equilibrium approach does is provide the investor with a set of portfolio weights backed by solid theoretical anchorage. When Black and Litterman (1992) presented the implied equilibrium returns approach, they pointed out that alternative approaches such as the equal means, risk-adjusted equal means and historical average approaches are all flawed since they fail to take the supply of assets into account. In our empirical study, we have illustrated how the risk-adjusted equal means approach perform when applied to real market data. After reviewing the results of this simulation, we were quickly able to conclude that this approach was prone to lead investors to high exposures to individual securities. While the ex-post Sharpe ratio and most other performance figures of the risk-adjusted equal means portfolio were far better than for the EQ, M-V and B-L portfolios, the highly skewed portfolio weights need to be taken into account when assessing the true risks inherent in this portfolio. Exposing a large fraction of the portfolio to the unique risks associated with an individual security is probably not even an option for many institutional investors. The fact that some of the risks associated with an individual security are likely not to be reflected in its historical volatility is a problem which becomes even more evident if the investment universe consists of entirely different asset classes. If the assets considered range from stocks to bonds to highly complex derivatives, an approach which only considers the past volatility of each asset may lead the investor to large exposures to assets whose risks are more subtle than their day-to-day volatilities. High-yield bonds with low credit ratings and a high risk of default is one example of an asset class which can yield high rates of return for a long period of time while exhibiting only moderate volatility – only to lose a considerable part of its market value (and/or liquidity) but a moment later. The CAPM equilibrium approach as such cannot of course capture the subtle risks associated with such securities. Nevertheless, taking the
market-capitalization weights as a starting point can still prevent the investor from taking on excessive exposures to individual securities – provided that the considered investment universe is comprehensive enough.
7 Conclusions

For an investor who has private views about the future performance of some of the securities included in the investment universe, the main differences between the risk-return characteristics of Black-Litterman and classical mean-variance portfolios can be summarized in three points. Firstly, the Black-Litterman model tends to lead the investor to portfolios, whose risk-return characteristics are more similar to those of a market capitalization-weighted portfolio. Secondly, the performance of Black-Litterman portfolios is noticeably less sensitive to variations in the input data. Thirdly, for the market data used in our simulations, the classical mean-variance portfolios are often characterized by higher volatility and inferior Sharpe ratios than their Black-Litterman counterparts.

Thorough sensitivity analysis has revealed that the input variables which have the greatest effect on the resulting portfolio performance are the strength of views coefficient ($\xi$), the risk aversion parameter ($\delta$) and the rebalancing frequency ($b$). We have also shown that the performance of portfolios generated by the Black-Litterman model is typically less sensitive to transaction costs than that of standard mean-variance portfolios. Another fact that has emerged during the course of our simulations is that increasing the strength of a successful investment strategy need not translate into equally predictable changes in the performance of the two types of portfolios.

For a portfolio manager who is to select which model to use for the implementation of his investment strategy, there are two aspects which we would recommend him to consider with extra care. Firstly, the manager ought to consider the nature of the views generated by the investment strategy. If the views are of an $\alpha$-driven character, the algebraic anatomy of the Black-Litterman model may not be appropriate. Secondly, the manager needs to establish whether market-capitalization-weighting the securities included in the investment universe constitutes a suitable benchmark. If so, the gravitating effect of the Black-Litterman model may be a significant advantage over the classical mean-variance framework. If not, this effect may on the contrary pose an impediment to the asset allocation decision unless the manager decides to define his own set of benchmark weights.

The results of our simulations of the risk-adjusted equal means approach has shown that it is not quite a comparable alternative to the implied equilibrium returns approach for an investor who is looking to construct a well-diversified portfolio. While the risk-adjusted equal means portfolio did deliver performance superior to that of the implied equilibrium returns portfolio, its tendency to suggest highly skewed portfolio weights renders its practical usefulness as a ‘neutral’ starting point rather doubtful. Nevertheless, its attractive ex-post performance figures suggest that it may be interesting to investigate further – not perhaps as an alternative to the implied equilibrium returns approach, but possibly as a view generating process.
8 Suggestions for Further Research

A topic which is particularly interesting is how to tackle the problems faced by portfolio managers who for one reason or another are unable or unwilling to take certain positions. As suggested in Chapter 6, a portfolio manager who does not feel comfortable having the optimisation results gravitating towards the CAPM equilibrium can specify his own set of benchmark weights towards which the final solution will gravitate. Whether or not this can be regarded as a sound alternative to simply adding extra constraints to the optimisation process may constitute an interesting subject to investigate.

The more mathematically inclined researchers may take an interest in developing a framework for distinguishing between which fractions of the elements in $Q$ that are believed to be $\alpha$-driven and market-driven, respectively. An algebraic framework capable of letting the portfolio manager take a probabilistic approach in specifying the expected fractions might expand the model's versatility.
9 Bibliography


10 Appendix A

A Working Example of Bayes’ Theorem

Suppose one is standing in a large greenhouse containing nothing but a wide variety of apple and pear trees, hearing (but not seeing) a fruit falling to the ground. Suppose also that one has observed that the historical aggregate output of the greenhouse has consisted of around 60% pears and 40% apples and finds it reasonable to assume that this fraction will still hold true. Now, suppose one wishes to estimate the likelihood that the fruit that fell is an apple (denoted $P(\text{apple})$). Not having access to any additional information, the likelihood that the fallen fruit is an apple would be estimated as:

$$P(\text{apple}) = 40\%,$$
and conversely

$$P(\text{pear}) = 1 - P(\text{apple}) = 60\%.$$ 

Assume now that one is given some additional factual information to work with, namely that 75% of the apples are red ($P(\text{red} | \text{apple}) = 75\%$) whereas only 5% of the pears are red ($P(\text{red} | \text{pear}) = 5\%$) and that the fruit that fell was indeed red. With this additional information, the likelihood that the fallen fruit was an apple can be re-estimated in accordance with Bayes’ Theorem as:

$$P(\text{apple} | \text{red}) = \frac{P(\text{red} | \text{apple})P(\text{apple})}{P(\text{red})} = \frac{0.75 \times 0.4}{0.75 \times 0.4 + 0.05 \times 0.6} \approx 91\%$$
and conversely,

$$P(\text{pear} | \text{red}) = 1 - P(\text{apple} | \text{red}) \approx 9\%.$$ 

The example illustrates how one can use additional factual information to improve the accuracy of an estimate. In the example, the fact that 40% of the fruits were apples represents the prior – i.e. an assumption which is regarded as realistic from the very beginning. The fact that 75% of the apples are red corresponds the sampling distribution, and the fraction of all fruits that are in fact red ($P(\text{red})$) serves as the normalizing constant. The final estimate (called the posterior) of approximately 91% utilizes not only the prior, but also the sampling distribution and the normalizing constant.
Appendix B

A Working Example of the Black-Litterman Model

Assume that an investor wishes allocate a certain amount of capital to a risky portfolio consisting of three different stocks (denoted A, B and C). Suppose also that the investor has observed the current market capitalizations of these stocks and calculates the market capitalization weights for each of the three securities in accordance with equation 5:

\[ w_A = 0.2890, \quad w_B = 0.3996, \quad \text{and} \quad w_C = 0.3114 \]

The column vector \( \mathbf{W}_{\text{mkt}} \) is thereby equal to

\[
\begin{bmatrix}
0.2890 \\
0.3996 \\
0.3114
\end{bmatrix}
\]

The investor then proceeds to estimate the covariance matrix of returns, \( \Sigma \). Suppose the investor conducts the analysis needed and arrives at the following estimate of \( \Sigma \):

\[
\Sigma \approx \begin{bmatrix}
0.0753 & 0.0217 & 0.0788 \\
0.0217 & 0.0902 & 0.0334 \\
0.0788 & 0.0334 & 0.1672
\end{bmatrix}
\]

The last step needed to compute the vector of implied equilibrium returns is to estimate the scalar that represents the level of risk aversion (\( \delta \)) that is expected to be representative of the average investor in the market over the intended holding period. Let us further assume that the investor estimates the values of the risk-free interest rate, the market expected return and the market variance as 1.50%, 10% and 3.40% respectively. This means that there is reason to believe that the best estimate of \( \delta \) follows as:

\[
\delta = \frac{0.10 - 0.015}{0.034} = 2.5.
\]

The vector of equilibrium expected returns can now be estimated as:

\[
\mathbf{\Pi} = \delta \Sigma \mathbf{W}_{\text{mkt}} \approx \begin{bmatrix}
0.1374 \\
0.1317 \\
0.2204
\end{bmatrix}
\]

The next step is for the investor to state private views and the uncertainty associated with them. Let us assume that the investor has private views about A and B, but not C. For the sake of simplicity, assume also that both these views are of an absolute nature - that is, that specific figures for the expected returns of the two companies are stated. More specifically, the investor has access to information which suggests that the expected return for A is most likely to be 16% and that the corresponding figure for B is 10%. Since the views only concern two securities, the link matrix, \( \mathbf{P} \), will be a simple 2x3 matrix in which each row has a sum of one. The \( \mathbf{Q} \) and \( \mathbf{P} \) matrices follow as:
The final component of the private views that has to be stated by the investor is the uncertainty associated with the views, i.e. the matrix denoted $\Omega$. Assume for simplicity that the investor chooses to estimate $\Omega$ in the same fashion as He and Litterman (1999) - including the assumption that the errors are independently normally distributed so that all off-diagonal elements are equal to zero. This implies that $\Omega$ will be computed as follows:

$$\Omega = \text{diag}(P(\tau \Sigma)P^\prime) \approx \begin{bmatrix} 0.0019 & 0 \\ 0 & 0.0023 \end{bmatrix}.$$

As can be seen, the estimated variance of the view for A is (approximately) equal to 0.19 %, whereas the corresponding figure for the view concerning B is 0.23 %. Before the investor can arrive at a vector of expected returns that is ready to be used in an optimiser, a value must be assigned to the scalar $\tau$. For simplicity, let’s assume the investor has come to the conclusion that $\tau$ should be set to (0.025). All inputs needed to blend the prior distribution with the view distribution have now been computed. The next step is to calculate $\bar{\mu}$, $M$ and $\bar{\Sigma}$:

$$\bar{\mu} = \begin{bmatrix} (\tau \Sigma)^{-1} + P' \Omega^{-1} P \end{bmatrix}^{-1} \begin{bmatrix} (\tau \Sigma)^{-1} \Pi + P' \Omega^{-1} Q \end{bmatrix} \approx \begin{bmatrix} 0.1465 \\ 0.1178 \\ 0.2279 \end{bmatrix};$$

$$M = \begin{bmatrix} (\tau \Sigma)^{-1} + P' \Omega^{-1} P \end{bmatrix}^{-1} \approx \begin{bmatrix} 0.0009 & 0.0001 & 0.0010 \\ 0.0001 & 0.0011 & 0.0003 \\ 0.0010 & 0.0003 & 0.0031 \end{bmatrix};$$

$$\bar{\Sigma} = \Sigma + M \approx \begin{bmatrix} 0.0762 & 0.0218 & 0.0798 \\ 0.0218 & 0.0913 & 0.0337 \\ 0.0798 & 0.0337 & 0.1703 \end{bmatrix}.$$

The matrices $\bar{\mu}$ and $\bar{\Sigma}$ can now be fed into an optimiser of the investor’s choice in order to estimate the optimal portfolio weights. Suppose the investor wishes to maximize the expected Sharpe ratio of the portfolio subject to short selling restrictions. The resulting estimate of the vector of optimal portfolio weights, $\bar{W}$ (depending on the optimisation algorithm used) can look as follows:

$$\bar{W} = \begin{bmatrix} 0.3304 \\ 0.3077 \\ 0.3619 \end{bmatrix}.$$

For comparison (recall that we specified a positive view for asset A, and a negative view for asset B),

$$W_{\text{mkt}} = \begin{bmatrix} 0.2890 \\ 0.3996 \\ 0.3114 \end{bmatrix}.$$
In the case of the M-V portfolio, the final expected returns that are fed into the optimiser are computed by replacing the first and second elements of $\Pi$ with the elements of $Q$. In other words, the final vector of expected returns (denoted $\mu_{M-V}$) follows as:

$$\mu_{M-V} = \begin{bmatrix} 0.1600 \\ 0.1000 \\ 0.2204 \end{bmatrix}.$$ 

Optimising is then performed by feeding this vector into the optimiser along with $\Sigma$. 
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**Variables**

- **Return on Investment (ROI)**
- **Volatility**
- **Correlation with EQ Portfolio**
- **Risk-Free Interest Rate (CAR)**
- **Rate of Return (EAR)**
- **Sharpe Ratio**

**Formula**

\[
\delta = 2.5; \tau = 0.05.
\]
14 Appendix E

Equations

\[ E(R_i) = R_f + \beta_i (E(R_m) - R_f) \]  
(1)

\[ E(R_i) - R_f = \delta \sigma_m^2 \]  
(2)

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]  
(3)

\[ \Pi = \delta \Sigma W_{mkt}, \text{ where} \]  
(4)

\[ w_i = \frac{m_i}{\Sigma_{i=1}^n m_i} \text{ and} \]  
(5)

\( w_i = \) market weight of the \( i \)th asset
\( m_i = \) market capitalization of the \( i \)th asset.

\[ \delta = \frac{E(R) - R_f}{\sigma_m^2} \]  
(6)

\[ E(R_m) - R_f = \delta \sigma_m^2. \]  
(7)

\[ r \sim N(\mu, \Sigma) \]  
(8)

\[ P\mu = Q + \varepsilon^{(v)}, \text{ where } \varepsilon^{(v)} \text{ is a vector of error terms} \]  
(9)

\[ \omega_{i,j} = P(\tau\Sigma)P' \quad \forall \ i = j \]  
(10)

\[ \mu \sim N(\bar{\mu}, M), \text{ where} \]  
(11)

\( \bar{\mu} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q] \)  
(12)

\[ M = [(\tau\Sigma)^{-1} + P'\Omega P]^{-1} \]  
(13)

\[ \Sigma = \Sigma + M \]  
(14)

\[ r \sim N(\bar{\mu}, \bar{\Sigma}). \]  
(15)

\[ q_i = \pi + \xi(d_i - d_{mkt}), \text{ where} \]  
(16)

\( d_i = \) dividend yield of the \( i \)th asset,
\( d_{mkt} = \) market capitalization weighted dividend yield,
\( \xi = \) strength of views.