Department of Mathematics

A New Well-posed Vorticity Divergence Formulation of the Shallow Water Equations

Jan Nordström, Sarmad Ghader

LiTH-MAT-R--2014/20--SE

Linköpings universitet
TEKNISKA HÖGSKOLAN

Department of Mathematics
Linköping University
S-581 83 Linköping, Sweden.

A new well-posed vorticity divergence formulation of the shallow water equations

JAN NORDSTRÖM

Department of Mathematics, Computational Mathematics, Linköping University, SE-581 83 Linköping, Sweden

SARMAD GHADER

Institute of Geophysics, University of Tehran, Tehran, Iran

ABSTRACT

A completely new vorticity-divergence formulation of the two-dimensional shallow water equations including boundary conditions is derived. The new formulation is necessary since the conventional one does not lead to a well-posed initial boundary value problem for limited area modelling.

The new vorticity-divergence formulation include four dependent variables instead of three, and require more equations and boundary conditions than the conventional formulation. On the other hand, it forms a symmetrizable hyperbolic set of equations with well defined boundary conditions that leads to a well-posed problem with a bounded energy.

1. Introduction

The vorticity-divergence form of the shallow water equations (SWE), are regularly used in global spectral modeling of the atmosphere. It has been demonstrated that using the vorticity and divergence as prognostic variables leads to advantages such as easy implementation of potential vorticity, potential enstrophy conservation principles and control of gravity waves via divergence damping. In addition, the vorticity and divergence are scalar variables in all coordinate systems (e.g., Randall (1994)).

Nevertheless, excluding the spectral method, the vorticity and divergence variables are seldom employed in computational algorithms developed for global and limited-area models. The main reasons are i) the difficulty in solving elliptic equations for the horizontal velocity components from the vorticity and divergence relations and ii) the lack of suitable boundary conditions to close the system for limited area domains. Efficient numerical algorithms such as the well known multigrid technique (e.g., Trottenberg et al. (2001)) or modern algorithms developed for solutions of linear systems (e.g., Boyd et al. (2013)) can potentially be used to overcome the first drawback. In this paper we focus on the second drawback.

Well-posed boundary conditions for the SWE in terms of horizontal velocity components

have been investigated by many researchers. For the one-dimensional SWE, well-posed boundary conditions have been derived by transforming them into a set of decoupled scalar equations (Durran (2010)). Oliger and Sundström (1978) derived well-posed boundary conditions for several sets of partial differential equations including the SWE by using the energy method. Ghader and Nordström (2014) derived a general form of well-posed open boundary conditions using similar techniques.

In this paper we derive a completely new vorticity-divergence formulation for the two-dimensional SWE including boundary conditions. The new formulation was derived since it was realized that the conventional one does not lead to a well-posed initial boundary value problem for limited area modeling. The core mathematical tool that we use is the energy method where one bounds the energy of the solution by choosing a minimal number of suitable boundary conditions (Gustafsson et al. (1995); Nordström and Svärd (2005); Nordström (2007); Gustafsson (2008)). In the initial stage of the analysis we also employ Fourier analysis.

The remainder of this paper is organized as follows. The three different forms of the SWE that will be discussed are given in section 2. As an introduction to the limited area problem, we briefly discuss the Cauchy problem in section 3. In section 4 we define well-posedness and show that the standard SWE on vorticity-divergence form does not lead to well-posedness. The core content of the paper is given in section 5 where we present the new SWE formulation, derive energy estimates, boundary conditions and show that it is well-posed. Finally, concluding remarks are given in section 6.

2. The shallow water equations

The inviscid single-layer shallow water equations (SWE), including the Coriolis term, are (Pedlosky (1987); Vallis (2006))

$$\frac{D\mathbf{V}}{Dt} + F\hat{\mathbf{k}} \times \mathbf{V} + G\nabla\tilde{h} = 0 \quad (1)$$

$$\frac{D\tilde{h}}{Dt} + \tilde{h}\nabla \cdot \mathbf{V} = 0 \quad (2)$$

where $\mathbf{V} = \tilde{u}\hat{\mathbf{i}} + \tilde{v}\hat{\mathbf{j}}$ is the horizontal velocity vector with $\tilde{u}$ and $\tilde{v}$ being the velocity components in the x and y directions, and $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors in the x and y directions, respectively. $\tilde{h}$ represents the surface height, $D()/Dt = \partial()/\partial t + (\mathbf{V} \cdot \nabla)()$ is the substantial time derivative, F is the Coriolis parameter and G is the acceleration due to gravity. The unit vector in the vertical direction is denoted by $\hat{\mathbf{k}}$. Here, we use the F -plane approximation where the Coriolis parameter is taken to be a constant.

a. The linearized SWE in terms of velocities and height

The vector form of the two-dimensional SWE, linearized around a constant basic state, can be written as

$$\tilde{\mathbf{u}}_t + \tilde{\mathbf{A}}\tilde{\mathbf{u}}_x + \tilde{\mathbf{B}}\tilde{\mathbf{u}}_y + \tilde{\mathbf{C}}\tilde{\mathbf{u}} = 0 \quad (3)$$

where $\tilde{\mathbf{u}} = (u, v, h)^T$ and the subscripts t, x, y denote the derivatives with respect to time and space respectively. The matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are

$$\tilde{\mathbf{A}} = \begin{pmatrix} U & 0 & G \\ 0 & U & 0 \\ H & 0 & U \end{pmatrix}, \tilde{\mathbf{B}} = \begin{pmatrix} V & 0 & 0 \\ 0 & V & G \\ 0 & H & V \end{pmatrix}, \tilde{\mathbf{C}} = \begin{pmatrix} 0 & -F & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, u and v are the perturbation velocity components and h is the perturbation height. In addition, U , V and H represent the constant mean fluid velocity components and height.

b. The SWE in terms of vorticity-divergence and height

The perturbation vorticity $\zeta = v_x - u_y$ and perturbation divergence $\delta = u_x + v_y$ in Cartesian coordinates (x, y) can be used as prognostic variables instead of the two components of the velocity. By differentiating the first two equations in (3) and combining them, the SWE in terms of vorticity, divergence and height becomes

$$\zeta_t + U\zeta_x + V\zeta_y + F\delta = 0 \quad (4)$$

$$\delta_t + U\delta_x + V\delta_y - F\zeta = -G(h_{xx} + h_{yy}) \quad (5)$$

$$h_t + Uh_x + Vh_y + H\delta = 0. \quad (6)$$

Equations (4)-(6) can be written in the following vector form

$$\bar{\mathbf{u}}_t + \bar{\mathbf{A}}\bar{\mathbf{u}}_x + \bar{\mathbf{B}}\bar{\mathbf{u}}_y + \bar{\mathbf{C}}\bar{\mathbf{u}} + \bar{\mathbf{D}}\bar{\mathbf{u}}_{xx} + \bar{\mathbf{E}}\bar{\mathbf{u}}_{yy} = 0 \quad (7)$$

where $\bar{\mathbf{u}} = (\zeta, \delta, h)^T$ and the matrices $\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}$ and $\bar{\mathbf{E}}$ are

$$\bar{\mathbf{A}} = \begin{pmatrix} U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{pmatrix}, \bar{\mathbf{B}} = \begin{pmatrix} V & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & V \end{pmatrix}, \bar{\mathbf{C}} = \begin{pmatrix} 0 & F & 0 \\ -F & 0 & 0 \\ 0 & H & 0 \end{pmatrix}$$

$$\bar{\mathbf{D}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & G \\ 0 & 0 & 0 \end{pmatrix}, \bar{\mathbf{E}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & G \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 1. *The differentiation of the first two equations in (3) introduce the Laplacian of the height and the divergence as a zero order term. This removes the clean hyperbolic character of the formulation, which as we will show below, leads to significant stability problems.*

c. The SWE in terms of vorticity-divergence and gradients of height

For reasons that will be explained in detail below, we introduce yet another form of the SWE, where we use the gradients of height as new variables. By differentiating also

the height equation (6) with respect to x and y and combining the equations for the new variables h_x, h_y with (4)-(5) we obtain the new extended system

$$\zeta_t + U\zeta_x + V\zeta_y + F\delta = 0 \quad (8)$$

$$\delta_t + U\delta_x + V\delta_y + G((h_x)_x + (h_y)_y) - F\zeta = 0 \quad (9)$$

$$(h_x)_t + U(h_x)_x + V(h_x)_y + H\delta_x = 0 \quad (10)$$

$$(h_y)_t + U(h_y)_x + V(h_y)_y + H\delta_y = 0. \quad (11)$$

Just as in the formulations above, equations (8)-(11) can be written in vector form as

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y + \mathbf{C}\mathbf{u} = 0 \quad (12)$$

where $\mathbf{u} = (\zeta, \delta, h_x, h_y)^T$ and the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are

$$\mathbf{A} = \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & U & G & 0 \\ 0 & H & U & 0 \\ 0 & 0 & 0 & U \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & V & 0 & G \\ 0 & 0 & V & 0 \\ 0 & H & 0 & V \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & F & 0 & 0 \\ -F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 2. *By differentiating the height equation in (3) we remove the Laplacian of the height appearing in (7) and re-introduce the hyperbolic character of the governing system. This re-stabilize the previously de-stabilized SWE in vorticity-divergence and height form.*

The standard formulation (3) in terms of velocities and height of the shallow water equations is well established, and energy estimates as well as a general set of boundary conditions for the initial boundary value problem can be found, see for example (Ghader and Nordström (2014)). For that reason, we will in this paper focus on the formulation (7) and (12). We will point out the significant problems with the standard formulation (7) and show how to correct them by using (12).

3. The Cauchy problem

To set the stage for the well-posedness study of the initial boundary value problems we start with a few observations.

a. Observation 1: symmetry properties

A necessary requirement for the symmetrization of a matrix $\mathbf{A}$ with a non-zero element a_{ij} is that also the element a_{ji} is non-zero, see Abarbanel and Gottlieb (1981) for more details. This basic requirement is satisfied for all matrices in (3),(7) and (12) except for $\bar{\mathbf{C}}$, $\bar{\mathbf{D}}$ and $\bar{\mathbf{E}}$ in (7). Consequently, (3) and (12) are symmetrizable, but (7) is not. This exposes a first potential weakness with the SWE formulation (7) compared to (3) and (12).

Remark 3. *With symmetrization of a matrix in this paper we also include the construction or preservation of skew-symmetric matrices.*

b. Observation 2: eigenvalues in Fourier-space

By Fourier-transformation of the formulations (3),(7) and (12) in x, y we obtain three initial value problems of the general form

$$\hat{\mathbf{w}}_t + \mathbf{H}\hat{\mathbf{w}} = 0. \quad (13)$$

For the SWE in terms of velocities and height, the Fourier coefficients $\hat{\mathbf{w}}$ and matrix $\mathbf{H}$ are

$$\hat{\mathbf{w}} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{h} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} i\widetilde{W} & -F & i\omega_x G \\ F & i\widetilde{W} & i\omega_y G \\ i\omega_x H & i\omega_x H & i\widetilde{W} \end{pmatrix}. \quad (14)$$

In (14) we have slightly abused standard notation by indicating the Fourier coefficients related to u, v, h with $\hat{u}, \hat{v}, \hat{h}$. This convenient notation is used in a similar way below. We have also used $\widetilde{W} = \omega_x U + \omega_y V$ where ω_x, ω_y are the wave numbers in the x and y directions respectively.

For the SWE in terms of vorticity-divergence and height, and vorticity-divergence and gradients of height, we have respectively

$$\hat{\mathbf{w}} = \begin{pmatrix} \hat{\zeta} \\ \hat{\delta} \\ \hat{h} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} i\widetilde{W} & F & 0 \\ -F & i\widetilde{W} & -(\omega_x^2 + \omega_y^2)G \\ 0 & H & i\widetilde{W} \end{pmatrix}, \quad (15)$$

$$\hat{\mathbf{w}} = \begin{pmatrix} \hat{\zeta} \\ \hat{\delta} \\ \hat{h}_x \\ \hat{h}_y \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} i\widetilde{W} & F & 0 & 0 \\ -F & i\widetilde{W} & i\omega_x G & i\omega_y G \\ 0 & i\omega_x H & i\widetilde{W} & 0 \\ 0 & i\omega_y H & 0 & i\widetilde{W} \end{pmatrix}. \quad (16)$$

It is straightforward to compute the eigenvalues of the $\mathbf{H}$ matrices above. One finds that

$$\lambda_1 = i\widetilde{W}, \quad \lambda_{2,3} = i(\widetilde{W} \pm \sqrt{F^2 + GH(\omega_x^2 + \omega_y^2)}), \quad (17)$$

where $\lambda_j, j = 1, 2, 3$ are the eigenvalues of $\mathbf{H}$ in (14) and (15). For the new formulation (16) we get one more eigenvalue $\lambda_4 = i\widetilde{W}$, which correspond to the larger system. Note that all eigenvalues are imaginary (and similar) which indicates zero time-growth for all formulations.

c. Observation 3: energy growth in Fourier-space

By multiplying (13) from the left with $\hat{\mathbf{w}}^*$ we arrive at the energy-growth rate for the Fourier coefficients

$$|\hat{\mathbf{w}}|_t^2 + \hat{\mathbf{w}}^*(\mathbf{H} + \mathbf{H}^*)\hat{\mathbf{w}} = 0, \quad (18)$$

where the star superscript indicates a conjugated transposed vector or matrix. For both the Fourier-transformed versions of (3) and (12), the term $\hat{\mathbf{w}}^*(\mathbf{H} + \mathbf{H}^*)\hat{\mathbf{w}}$ is identically zero which guarantees zero energy growth. However, for the Fourier-transformed version of (7), a mixed term with undetermined sign remain, and zero energy growth is not guaranteed. This exposes a second potential weakness of the vorticity-divergence and height formulation.

Remark 4. *The weakness of formulation (7) in the energy-growth sense above is not really problematic for the Cauchy problem since the eigenvalues are purely imaginary. However, it will show up as a definite weakness for the related initial boundary value problem.*

4. The initial boundary value problem

As mentioned above, well-posedness for the SWE (3) in terms of horizontal velocities is already known and is therefore excluded in the following. We focus on the vorticity-divergence formulations (7) and (12).

a. Preliminaries

Roughly speaking, an initial boundary value problem is well-posed if a unique solution that depends continuously on the initial and boundary data exist. We will define these concepts more precisely below. Consider the following general initial boundary value problem

$$\begin{aligned}\frac{\partial \mathbf{q}}{\partial t} + \mathbf{P}\mathbf{q} &= \mathbf{F}, \quad \mathbf{x} \in \Omega, \quad t \geq 0 \\ \mathbf{L}\mathbf{q} &= \mathbf{g}, \quad \mathbf{x} \in \partial\Omega, \quad t \geq 0 \\ \mathbf{q} &= \mathbf{f}, \quad \mathbf{x} \in \Omega, \quad t = 0\end{aligned}\tag{19}$$

where $\mathbf{q}$ is the solution, $\mathbf{P}$ is the spatial differential operator and $\mathbf{L}$ is the boundary operator. In this paper $\mathbf{P}$ and $\mathbf{L}$ are linear operators. $\mathbf{F}$ is a forcing function, and $\mathbf{g}$ and $\mathbf{f}$ are boundary and initial functions respectively. $\mathbf{F}$, $\mathbf{g}$ and $\mathbf{f}$ are the known data of the problem.

Definition 1. *The initial boundary value problem (19) with $\mathbf{F} = \mathbf{g} = 0$ is well-posed, if for every $\mathbf{f} \in C^\infty$ that vanishes in a neighborhood of $\partial\Omega$, a unique smooth solution exist that satisfies the estimate*

$$\|\mathbf{q}(\cdot, t)\|_\Omega^2 \leq K_1(t) \|\mathbf{f}\|_\Omega^2\tag{20}$$

where the function $K_1(t)$ is bounded independently of $\mathbf{f}$.

Remark 5. *Well-posedness of (19) require that an appropriate number of boundary conditions (number of linearly independent rows in $\mathbf{L}$) with the correct form of $\mathbf{L}$ (the rows in $\mathbf{L}$ have appropriate elements) is used. Too many boundary conditions means that no existence is possible, and too few that neither the estimate (20) nor uniqueness can be obtained.*

Definition 2. *The initial boundary value problem (19) is strongly well-posed, if it is well-posed and*

$$\|\mathbf{q}\|_{\Omega}^2 \leq K_2(t) \left(\|\mathbf{f}\|_{\Omega}^2 + \int_0^t (\|\mathbf{F}(\cdot, \tau)\|_{\Omega}^2 + \|\mathbf{g}(\tau)\|_{\partial\Omega}^2) d\tau \right) \quad (21)$$

holds wher the function $K_2(t)$ is bounded independently of $\mathbf{f}, \mathbf{F}$ and $\mathbf{g}$.

Remark 6. *The boundary and initial data in (19) must be compatible for a smooth solution.*

More details on the definitions above, are given in Gustafsson et al. (1995). In this paper we analyse the linearized constant coefficient problem. This is not a severe limitation since by using the linearization and localization principles (see Kreiss and Lorenz (1989) and Strang (1964)) it can be shown that if the constant coefficient and linearized form of an initial boundary value system is well-posed then the associated smooth nonlinear problem is also well-posed.

b. Non-existing energy estimates for the standard SWE in vorticity-divergence form

As mentioned above, the standard form (7) of the vorticity, divergence and height formulation cannot be symmetrized. Hence, we treat each equation separately, multiply with the dependent variable and integrate over the domain Ω with boundary $\partial\Omega$ and outward pointing normal n .

The first two rows in (7) must be weighted equally when added to cancel the skew-symmetric terms. We find

$$(\|\zeta\|^2 + \|\delta\|^2)_t + \oint_{\partial\Omega} W(\zeta^2 + \delta^2) + 2C^2\delta(\nabla h \cdot n)ds = 2C^2 \int_{\Omega} (\nabla h \cdot \nabla \delta) dxdy, \quad (22)$$

where we have introduced the outward pointing velocity $W = (U, V) \cdot n$, the gravity wave speed $C = \sqrt{GH}$ and the norm $\|w\|^2 = \int_{\Omega} w^T w ds$.

The energy method applied to the evolution equation for the height gives

$$\|h\|_t^2 + \oint_{\partial\Omega} W h^2 ds + 2C^2 \int_{\Omega} h \delta dxdy = 0. \quad (23)$$

By inspection, it is immediately clear that no linear combination of (22) and (23) can lead to an energy estimate due to different character of the remaining volume terms, and in particular the existence of the nonlinear gradient term on the righthand side of (22). Hence boundary conditions that could potentially bound the line integral are not of interest.

A close look at the remaining volume terms in (22) and (23) reveal that the terms in (23) appear differentiated in (22). This suggest that it might be worthwhile to differentiate the height equation, and see if an estimate can be obtained. This observation lead to the new formulation (12), which is the main topic of this paper and what will be considered next.

5. The new SWE in vorticity-divergence form

As mentioned above, the formulation (12) can be symmetrized. We multiply each term with a diagonal symmetrizing matrix $\mathbf{S}$ to obtain

$$\mathbf{v}_t + \mathbf{A}^s \mathbf{v}_x + \mathbf{B}^s \mathbf{v}_y + \mathbf{C}^s \mathbf{v} = 0, \quad (24)$$

where $\mathbf{v} = \mathbf{S}\mathbf{u}$, $\mathbf{A}^s = \mathbf{S}\mathbf{A}\mathbf{S}^{-1}$, $\mathbf{B}^s = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$ and $\mathbf{C}^s = \mathbf{S}\mathbf{C}\mathbf{S}^{-1}$. The specific choice $\mathbf{S} = \text{diag}(1, 1, \sqrt{G/H}, \sqrt{G/H})$ leads to

$$\mathbf{A}^s = \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & U & C & 0 \\ 0 & C & U & 0 \\ 0 & 0 & 0 & U \end{pmatrix}, \quad \mathbf{B}^s = \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & V & 0 & C \\ 0 & 0 & V & 0 \\ 0 & C & 0 & V \end{pmatrix}, \quad \mathbf{C}^s = \begin{pmatrix} 0 & F & 0 & 0 \\ -F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system (24) is now symmetric, and the energy method is applicable.

a. The energy method

We multiply (24) with $\mathbf{v}^T$ and integrate over the domain Ω with boundary $\partial\Omega$ and outward pointing normal $n = (\alpha, \beta)$. This yields

$$\|\mathbf{v}\|_t^2 + \oint_{\partial\Omega} \mathbf{v}^T (\alpha \mathbf{A}^s + \beta \mathbf{B}^s) \mathbf{v} \, ds = 0. \quad (25)$$

The symmetric boundary matrix $\mathbf{Q} = \alpha \mathbf{A}^s + \beta \mathbf{B}^s$ has the eigen-decomposition $\mathbf{Q} = \mathbf{X} \mathbf{\Lambda} \mathbf{X}^T$ where $\mathbf{\Lambda} = \text{diag}(W - C, W, W, W + C)$, $W = (U, V) \cdot n = \alpha U + \beta V$ and

$$\mathbf{Q} = \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & W & \alpha C & \beta C \\ 0 & \alpha C & W & 0 \\ 0 & \beta C & 0 & W \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ -\alpha/\sqrt{2} & 0 & -\beta & \alpha/\sqrt{2} \\ -\beta/\sqrt{2} & 0 & \alpha & \beta/\sqrt{2} \end{pmatrix}.$$

By using these definitions, the estimate (25) can be written on characteristic form as

$$\|\mathbf{v}\|_t^2 + \oint_{\partial\Omega} (\mathbf{X}^T \mathbf{v})^T \mathbf{\Lambda} (\mathbf{X}^T \mathbf{v}) \, ds = 0, \quad (26)$$

which provides us with information about the number of and type of boundary conditions.

b. The number of boundary conditions

The number of necessary boundary conditions to obtain a bound is equal to the number of negative eigenvalues in $\mathbf{\Lambda}$ (which is the number of growth terms in (26)). The different cases are

- Supercritical inflow: $W < 0$, $|W| > C \Rightarrow$ four boundary conditions.
- Subcritical inflow: $W < 0$, $|W| \leq C \Rightarrow$ three boundary conditions.
- Subcritical outflow: $W \geq 0$, $|W| \leq C \Rightarrow$ one boundary conditions.
- Supercritical outflow: $W > 0$, $|W| > C \Rightarrow$ zero boundary conditions.

Once the number of boundary conditions allowed are known, we can choose the type of boundary conditions that limits the energy.

c. General form of the boundary conditions

The most straightforward choice of boundary conditions is to specify the characteristic variables that correspond to negative eigenvalues in terms of given data. This is often referred to as characteristic boundary conditions which are used both for inviscid (Oliger and Sundström (1978)) and viscous (Nordström (1995)) flow problems. A general formulation is

$$(\mathbf{X}^T \mathbf{v}^T)_- = \mathbf{R}(\mathbf{X}^T \mathbf{v})_+ + \mathbf{g}, \quad (27)$$

where the subscript minus denote ingoing characteristic variables and the subscript plus, outgoing ones. In (27) we specify the ingoing characteristic variable in term of the outgoing ones and data. The matrix $\mathbf{R}$ has the same number of rows as the number of negative eigenvalues in $\mathbf{\Lambda}$ and $\mathbf{g}$ is given data. With $\mathbf{R} = 0$ we have the characteristic boundary conditions. We keep the type of boundary conditions general at this point but will come back later to more explicit and specific forms.

Remark 7. *Note that the number and form of boundary conditions vary with the normal velocity W along the boundary $\partial\Omega$.*

d. Well-posedness of the new SWE formulation

The quadratic boundary terms in (26) can be rewritten as

$$(\mathbf{X}^T \mathbf{v})^T \mathbf{\Lambda} (\mathbf{X}^T \mathbf{v}) = \begin{pmatrix} (\mathbf{X}^T \mathbf{v})_- \\ (\mathbf{X}^T \mathbf{v})_+ \end{pmatrix}^T \begin{pmatrix} \mathbf{\Lambda}_- & 0 \\ 0 & \mathbf{\Lambda}_+ \end{pmatrix} \begin{pmatrix} (\mathbf{X}^T \mathbf{v})_- \\ (\mathbf{X}^T \mathbf{v})_+ \end{pmatrix}. \quad (28)$$

By inserting (27) with zero data in (28) we get

$$(\mathbf{X}^T \mathbf{v})^T \mathbf{\Lambda} (\mathbf{X}^T \mathbf{v}) = (\mathbf{X}^T \mathbf{v})_+^T (\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R}) (\mathbf{X}^T \mathbf{v})_+, \quad (29)$$

and we can now prove

Theorem 1. *The initial boundary value problem (12) with homogeneous boundary conditions of the form (27) and with the matrix $\mathbf{R}$ chosen such that*

$$\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R} \geq 0, \quad (30)$$

is well-posed.

Condition (30) means that $\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R}$ must be a positive semi-definite matrix.

Proof. Time-integration of (26) in combination with (27) for zero data and $\mathbf{R}$ such that (30) holds leads to

$$\|\mathbf{v}\|^2 \leq \|\mathbf{f}\|^2, \quad (31)$$

where $\mathbf{f}$ is the initial data. The estimate (31) has the form (20) for $\mathbf{v}$ and hence also for $\mathbf{u}$.

The estimate (31) guarantees uniqueness by considering another possible solution with the same data. The difference between the solutions satisfies the estimate (31) with zero right hand side. Existence is guaranteed by the minimal number of boundary conditions used. $\square$

e. Strong well-posedness of the new SWE formulation

Next we consider the case with non-zero data in (27) which leads to

$$\|\mathbf{v}\|_t^2 = - \oint_{\delta\Omega} \begin{bmatrix} (\mathbf{X}^T \mathbf{v})_+ \\ \mathbf{g} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R} & (\mathbf{\Lambda}_- \mathbf{R})^T \\ (\mathbf{\Lambda}_- \mathbf{R}) & \mathbf{\Lambda}_- \end{bmatrix} \begin{bmatrix} (\mathbf{X}^T \mathbf{v})_+ \\ \mathbf{g} \end{bmatrix} ds. \quad (32)$$

By adding and subtracting $\mathbf{g}^T \mathbf{U} \mathbf{g}$, where $\mathbf{U}$ is a bounded positive semi-definite matrix we get,

$$\begin{aligned} \|\mathbf{v}\|_t^2 &= - \oint_{\delta\Omega} \begin{bmatrix} (\mathbf{X}^T \mathbf{v})_+ \\ \mathbf{g} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R} & (\mathbf{\Lambda}_- \mathbf{R})^T \\ (\mathbf{\Lambda}_- \mathbf{R}) & \mathbf{U} \end{bmatrix} \begin{bmatrix} (\mathbf{X}^T \mathbf{v})_+ \\ \mathbf{g} \end{bmatrix} ds \\ &+ \oint_{\delta\Omega} \mathbf{g}^T (|\mathbf{\Lambda}_-| + \mathbf{U}) \mathbf{g} ds. \end{aligned} \quad (33)$$

To bound the right hand side of (33) in terms of data, we must show that the matrix in the first line integral is positive semi-definite. We multiply from the left and right with another non-singular matrix such that,

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R} & (\mathbf{\Lambda}_- \mathbf{R})^T \\ (\mathbf{\Lambda}_- \mathbf{R}) & \mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \quad (34)$$

Note that the multiplication with the matrix above does not change the definiteness.

By the choice $\mathbf{N} = -(\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R})^{-1}(\mathbf{\Lambda}_- \mathbf{R})^T$, which require

$$(\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R}) > 0, \quad (35)$$

we obtain the following block diagonal matrix,

$$\mathbf{M} = \begin{bmatrix} \mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R} & \mathbf{0} \\ \mathbf{0} & -(\mathbf{\Lambda}_- \mathbf{R})^T (\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R})^{-1} (\mathbf{\Lambda}_- \mathbf{R}) + \mathbf{U} \end{bmatrix}. \quad (36)$$

A positive semi-definite *Schur complement* $-(\mathbf{\Lambda}_- \mathbf{R})^T (\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R})^{-1} (\mathbf{\Lambda}_- \mathbf{R}) + \mathbf{U}$ lead to a positive semi-definite $\mathbf{M}$ since (35) holds. Hence we need

$$\mathbf{U} \geq (\mathbf{\Lambda}_- \mathbf{R})^T (\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R})^{-1} (\mathbf{\Lambda}_- \mathbf{R}). \quad (37)$$

We can prove the main result of this paper.

Theorem 2. *The problem (12) with boundary condition (27) subject to (35) and the choice (37) is strongly well-posed.*

Proof. Integrating (33) in time with the conditions (35) and (37) leads to the estimate

$$\|\mathbf{v}\|^2 \leq \|\mathbf{f}\|^2 + \int_0^t \oint \mathbf{g}^T (|\mathbf{\Lambda}_-| + \mathbf{U}) \mathbf{g} \, ds \, dt. \quad (38)$$

By including a forcing function $\mathbf{F}$ in (12), the estimate (38) will be augmented with a volume term and have the form (21).

The estimate(38) guarantees uniqueness by considering another possible solution with the same data. The difference between the solutions satisfies the estimate (31) with zero

right hand side. Existence is guaranteed by the minimal number of boundary conditions used. $\square$

Remark 8. *Note that condition (35) selects the possible matrices $\mathbf{R}$ that can be used and that the choice (37) can always be made if (35) holds.*

f. Characteristic variables and well-posed boundary conditions

The general form of the boundary condition (27) can be explicitly written out by calculating the characteristic variables in terms of the original variables. We have

$$\mathbf{X}^T \mathbf{v} = \begin{pmatrix} (\delta - (\partial h / \partial n)(C/H)) / \sqrt{2} \\ \zeta \\ \partial h / \partial \tau \\ (\delta + (\partial h / \partial n)(C/H)) / \sqrt{2} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}. \quad (39)$$

where $\partial h / \partial n = \alpha h_x + \beta h_y = \nabla h \cdot n$ and $\partial h / \partial \tau = -\beta h_x + \alpha h_y = \nabla h \cdot \tau$ are the normal and tangential derivatives respectively. The characteristic variables C_{1-4} in (39) correspond to the eigenvalue organization in $\mathbf{\Lambda} = \text{diag}(W - C, W, W, W + C)$.

This implies that a subcritical inflow boundary one should specify the top three variables in terms of the last one which leads to an $\mathbf{R}$ matrix with three rows and one column. At a subcritical outflow boundary one should specify the first variable in terms of the last three which yields an $\mathbf{R}$ matrix with one row and three columns.

As an example we consider the subcritical outflow case with one boundary condition allowed. We specify the ingoing characteristic variable C_1 in terms of the outgoing C_4 and data, i.e. $C_1 = R_4 C_4 + g(t)$ which yields $\mathbf{R} = (0, 0, R_4)$. Condition (35) becomes

$$(\mathbf{\Lambda}_+ + \mathbf{R}^T \mathbf{\Lambda}_- \mathbf{R}) = \begin{pmatrix} W & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & W + C + R_4^2(W - C) \end{pmatrix} > 0, \quad (40)$$

and the condition for strong well posedness is $W(1 + R_4^2) + C(1 - R_4^2) > 0$. The particular choices $R_4 = \pm 1$ correspond to specifying the divergence δ and $\partial h / \partial n$ respectively.

Remark 9. *Note that both W and $\partial h / \partial n$ changes sign with a varying normal n and that the new characteristic variables are easy to interpret physically.*

g. A final observation regarding time-integration

We have shown that the problem (12) with boundary condition (27) can be made well-posed as well as strongly well-posed. The system (12) has the new dependent variables h_x, h_y and does not contain the height h itself. To find the time-evolution of the height, one must first solve the system (12) and next use (6) (which is now an ordinary differential equation) to integrate the height h forward in time.

6. Concluding remarks

A completely new vorticity-divergence formulation of the two-dimensional SWE including boundary conditions is derived. The new formulation is necessary since the standard one does not lead to a well-posed initial boundary value problem for limited area modeling.

The new vorticity-divergence formulation include four dependent variables instead of three, and require more equations and boundary conditions than the conventional formulation. On the other hand, it forms a symmetrizable hyperbolic set of equations with well defined boundary conditions that leads to a well-posed problem with a bounded energy.

The new formulation does not have the height as dependent variable, the but instead the gradients of the height. The time-evolution of the height h , is obtained by first solving the new SWE system, and in a postprocessing step, use one of the standard SWE for the height evolution to integrate h forward in time.

REFERENCES

- Abarbanel, S. and D. Gottlieb, 1981: Optimal time splitting for two- and three- dimensional Navier-Stokes equations with mixed derivatives. *Journal of Computational Physics*, **41**, 1–33.
- Boyd, J. P., D. Deng, Q. Chen, and S. Gao, 2013: Applications of bivariate fourier series for solving the poisson equation in limited-area modeling of the atmosphere: Higher accuracy with a boundary buffer strip discarded and an improved order-raising procedure. *Mon. Wea. Rev.*, **141**, 4154–4164.
- Durran, D. R., 2010: *Numerical methods for fluid dynamics with applications to geophysics*. Springer, 516 pp.
- Ghader, S. and J. Nordström, 2014: Revisiting well-posed boundary conditions for the shallow water equations. *Dynamics of Atmospheres and Oceans*, **66**, 1–9.
- Gustafsson, B., 2008: *High order difference methods for time dependent PDE*. Springer, 334 pp.
- Gustafsson, B., H. O. Kreiss, and J. Oliger, 1995: *Time dependent problems and difference methods*. John Wiley and Sons, 642 pp.
- Kreiss, H. and J. Lorenz, 1989: *Initial boundary value problems and the Navier-Stokes equations*. Academic Press.
- Nordström, J., 1995: The use of characteristic boundary conditions for the navier-stokes equations. *Computers and Fluids*, **24** (5), 609–623.
- Nordström, J., 2007: Error bounded schemes for time-dependent hyperbolic problems. *SIAM Journal of Scientific Computing*, **30**, 46–59.

- Nordström, J. and M. Svärd, 2005: Well posed boundary conditions for the Navier-Stokes equations. *SIAM J. Numer. Anal.*, **43**, 1231–1255.
- Oliger, J. and A. Sundström, 1978: Theoretical and practical aspects of some initial boundary value problems in fluid dynamics. *SIAM J. Appl. Math.*, **35**, 419–446.
- Pedlosky, J., 1987: *Geophysical fluid dynamics*. Springer-Verlag, 710 pp.
- Randall, D. A., 1994: Geostrophic adjustment and the finite-difference shallow water equations. *Mon. Wea. Rev.*, **122**, 1371–1377.
- Strang, G., 1964: Accurate partial difference methods II: Non-linear problems. *Numerische Mathematik*, **6**, 37–46.
- Trottenberg, U., C. W. Oosterlee, and A. Schüller, 2001: *Multigrid*. Academic Press.
- Vallis, G., 2006: *Atmospheric and oceanic fluid dynamics: Fundamentals and large-scale circulation*. Cambridge University Press.