Remarkable curves in the Euclidean plane

Jonas Granholm

LiTH-MAT-EX–2014/06–SE
Remarkable curves in the Euclidean plane

Department of Mathematics, Linköping University

Jonas Granholm

LiTH-MAT-EX–2014/06–SE

Thesis: 16 hp

Level: G2

Supervisor: Vitalij Tjatyrko,
Department of Mathematics, Linköping University

 Examiner: Vitalij Tjatyrko,
Department of Mathematics, Linköping University

Linköping: December 2014
Abstract

An important part of mathematics is the construction of good definitions. Some things, like planar graphs, are trivial to define, and other concepts, like compact sets, arise from putting a name on often used requirements (although the notion of compactness has changed over time to be more general). In other cases, such as in set theory, the natural definitions may yield undesired and even contradictory results, and it can be necessary to use a more complicated formalization.

The notion of a curve falls in the latter category. While it is intuitively clear what a curve is – line segments, empty geometric shapes, and squiggles like this: \( \bigcirc \) – it is not immediately clear how to make a general definition of curves. Their most obvious characteristic is that they have no width, so one idea may be to view curves as what can be drawn with a thin pen. This definition, however, has the weakness that even such a line has the ability to completely fill a square, making it a bad definition of curves. Today curves are generally defined by the condition of having no width, that is, being one-dimensional, together with the conditions of being compact and connected, to avoid strange cases.

In this thesis we investigate this definition and a few examples of curves.

**Keywords:**
- Curves, Cantor curves, Peano curves, Sierpiński carpet, one-dimensional, Menger curve

**URL for electronic version:**
Sammanfattning

En viktig del av matematiken är skapandet av bra definitioner. Vissa saker, som planāra grafer, är triviala att definiera, och andra koncept, som kompakt mängder, uppkommer genom att man sätter ett namn på ofta använda villkor (även om begreppet kompakthet har ändrats med tiden och blivit mer generellt). I andra fall, som i mängdlära, kan de naturliga definitionerna ge oönskade och till och med självmotsägande resultat, och det kan krävas mer komplicerade formaliseringar.


I denna uppsats undersöker vi denna definition och några exempel på kurvor.

Nyckelord:
Kurvor, Cantorkurvor, Peanokurvor, Sierpińskimattan, endimensionell, Mengerkurvan
Acknowledgements

I would like to thank my supervisor Vitalij Tjatyrko for his support and help when I have gotten stuck, and my opponent Emil Karlsson for valuable comments. I would also like to thank my classmates for friendship and wonderful discussions. Finally I want to thank my family and especially my fiancée for their unending support.
Nomenclature

Most of the recurring letters and symbols are described here.

**Letters**

- $x, y, z \in \mathbb{R}$: coordinates
- $p, q \in \mathbb{R}^n$: points
- $X, Y, \ldots \subset \mathbb{R}^n$: sets
- $f: \mathbb{R} \to \mathbb{R}$: real functions
- $F, G: \mathbb{R}^n \to \mathbb{R}^n$: mappings

**Symbols**

- $\bar{X}$: the closure of $X$
- $F \circ G(p)$: the composition $F(G(p))$
- $\|p - q\|$: the Euclidean distance between $p$ and $q$
- $\subset$: subset (not necessarily proper)
- $(a, b)$: an open interval
- $[a, b]$: a closed interval
- $I$: the closed unit interval $[0, 1]$
- $\mathbb{N}$: the set of positive integers
- $\mathbb{R}$: the set of real numbers
- $\mathbb{R}^n$: $n$-dimensional Euclidean space
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{I}$: the set of irrational numbers

Granholm, 2014.
## Contents

1 Prerequisites 1  
  1.1 Basic properties of sets in the Euclidean plane 1  
  1.2 Mappings and embeddings 2  
  1.3 Dimension theory 4  

2 Curves 7  
  2.1 Definition of a curve 7  
  2.2 Some simple curves 8  
  2.3 The sin(\(\frac{1}{x}\))-curve 9  
  2.4 The Sierpiński carpet 9  
  2.5 Other examples of curves 11  

3 Peano curves 13  
  3.1 Definition of Peano curves 13  
  3.2 A characterization of Peano curves 13  
  3.3 Explicit mappings to Peano curves 14  

4 Generalization to higher dimensions 17  
  4.1 A general definition of curves 17  
  4.2 The Kuratowski graph theorem 17  
  4.3 Three-dimensional embeddings 18  

Granholm, 2014. xiii
Chapter 1

Prerequisites

We will start by presenting some basic notions and theorems that will be used in the thesis. For simplicity some of the definitions will not be in the standard form, but adjusted to our setting.

The mathematics in this thesis will mainly take place in the Euclidean plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, with the usual Euclidean distance function. The first two sections in this chapter is a short introduction to the topology of the Euclidean plane. It should be deducible from any introduction to topology, such as [5], and a lot will be familiar from calculus in multiple variables.

1.1 Basic properties of sets in the Euclidean plane

We begin by defining the important concepts of openness and closedness.

**Definition 1.** An open disc of radius $r$ is a set $D = \{p : \|p - p_0\| < r\}$ for some fixed point $p_0$, i.e., the set of all points closer than $r$ to the center.

**Definition 2.** A set $X \subset \mathbb{R}^2$ is open if every point in $X$ is the center of an open disc that is contained in $X$. A set $X \subset \mathbb{R}^2$ is closed if its complement $\mathbb{R}^2 \setminus X$ is open.

**Remark 1.** It is easy to see that the whole plane $\mathbb{R}^2$ is open, and thus that the empty set is closed. Furthermore, the empty set is also open, which means the whole plane is also closed. No other subset of the plane has both of these properties.

**Example 1.** One can easily show that an open disc is open. Let $p_1$ be a point in the open disc $D = \{p : \|p - p_0\| < r\}$. Then the number $r_1 = r - \|p_1 - p_0\|$ is positive, so we can define the open disc $D_1 = \{p : \|p - p_1\| < r_1\}$. According to the triangle inequality, we have that for any point $p_2 \in D_1$, the distance $\|p_2 - p_0\| \leq \|p_2 - p_1\| + \|p_1 - p_0\| < r_1 + \|p_1 - p_0\| = r$, so $p_2 \in D$.

**Lemma 1.** The union of any number of open sets is an open set. The intersection of finitely many open sets is open. The union of finitely many closed sets is closed. The intersection of any number of closed sets is closed.

**Proof.** For any point $x$ in the union of open sets, the point lies in at least one of those sets. We will call this set $V$. Since $V$ is open, there is a disc around $x$ that lies in $V$, and that disc will also belong to the union. Thus the union is open.
For any point in the intersection of finitely many open sets, all these sets contain a disc around the point. The smallest of those discs will lie in the intersection, so the intersection is open.

The properties for closed sets now follow from the above by looking at the complements.

**Definition 3.** The interior of a set $X$ is the largest open set inside $X$. The closure of a set $X$, denoted $\overline{X}$, is the smallest closed set that covers $X$.

**Example 2.** The interior of an open set is the same open set, and the closure of a closed set is the same closed set. The closure of an open disc $D = \{p : \|p - p_0\| < r\}$ is the closed disc $D = \{p : \|p - p_0\| \leq r\}$. The interior of a line is the empty set.

**Definition 4.** A set $X \subset \mathbb{R}^2$ is bounded if there is some finite distance $M$ such that $\|p_1 - p_2\| \leq M$ for any two points $p_1, p_2 \in X$.

**Definition 5.** A set $X \subset \mathbb{R}^2$ is compact if it is closed and bounded.

**Definition 6.** A set $X \subset \mathbb{R}^2$ is connected if it cannot be covered by two open sets such that $X$ has points in both of these sets, and every point in $X$ lies in exactly one of these sets. A component of a set is a connected subset that cannot be enlarged without becoming disconnected.

**Theorem 1.** The union of two intersecting connected sets is connected.

*Proof. See Theorem 23.3 in [5].*

**Theorem 2.** Let $X_1 \supset X_2 \supset \ldots$ be a sequence of nonempty compact sets. Then $X = \bigcap X_i$ is compact and nonempty. Furthermore, if each set $X_i$ is connected, then so is $X$.

*Proof. See Proposition 1.7 and Theorem 1.8 in [6].*

**Definition 7.** A set $X \subset \mathbb{R}^2$ is nondegenerate if it contains more than one point.

### 1.2 Mappings and embeddings

**Definition 8.** A mapping from a set $X \subset \mathbb{R}^2$ to a set $Y \subset \mathbb{R}^2$ is a rule assigning to every point $p \in X$ a single point $F(p) \in Y$, called the image of $p$. We will also use the notation $F(X) = \{q \in Y : q = F(p) \text{ for some } p \in X\}$.

If every point in $Y$ is the image of some point in $X$, so $F(X) = Y$, the mapping is called surjective. If no two points in $X$ have the same image, the mapping is called injective. A mapping that is both surjective and injective is called bijective.

**Definition 9.** The inverse of a bijective mapping $F : X \to Y$ is the mapping $F^{-1} : Y \to X$ that assigns to every point in $Y$ the unique point in $X$ that is mapped to it by $F$, that is, $F^{-1}(q) = p \iff F(p) = q$. 
Definition 10. Let \( p \in X \). Then the mapping \( F : X \to Y \) is called \textit{continuous in} \( p \) if for every open disc around \( F(p) \) there is an open disc around \( p \) such that the images of all points in the disc around \( p \) lies in the disc around \( F(p) \). If a mapping is continuous in all points where it is defined, it is simply called \textit{continuous}.

Example 3. The mapping \( (x, 0) \to (x, f(x)) \) is continuous if and only if the function \( f \) is continuous as a real function.

Definition 11. A bijective mapping is called a \textit{homeomorphism} if both it and its inverse are continuous. Two sets in \( \mathbb{R}^2 \) are called \textit{homeomorphic} if there is a homeomorphism between them.

Example 4. Some simple examples of homeomorphisms are scalings, translations and rotations.

Homeomorphism can be seen as mappings that stretch and twist sets without changing their structure. Sets that are homeomorphic are often seen as different realizations of the same topological spaces.

Example 5. A closed disc is homeomorphic to a closed square, but not to an open disc. An open disc is homeomorphic to an open square and to the whole plane. None of these are homeomorphic to a line segment.

Theorem 3. The properties of being open, compact, connected or nondegenerate are preserved by homeomorphisms, i.e., if \( X \) and \( Y \) are homeomorphic and \( X \) has one of these properties, then so does \( Y \).

Proof. For openness, see Theorem 7.9 in [7]. For compactness and connectedness, see Theorem 1 of section §41 – III and Theorem 3 of section §46 – I in [4]. The fact that nondegenerate sets are preserved follows trivially from the bijectiveness of homeomorphisms.

Remark 2. Even though compactness is preserved by homeomorphisms, the properties of just being a closed or bounded subset of the plane are not, since the closed but unbounded real line \( \{(x, 0) : x \in \mathbb{R} \} \) is homeomorphic to the interval \( \{(x, 0) : 0 < x < 1 \} \), which is bounded but not closed. If the mapping can be extended to a homeomorphism from the whole plane to itself, however, then these properties are preserved as well.

Theorem 4. Any continuous bijection between compact sets in \( \mathbb{R}^2 \) is a homeomorphism.

Proof. See Theorem 26.6 in [5].

Definition 12. An \textit{embedding} of a set \( X \) into a set \( Y \) is a homeomorphism from \( X \) to a subset of \( Y \).

Example 6. A line segment can be embedded in a circle, and both of these can be embedded in a disc. The converse is not true.

Theorem 5. All properties defined in this section are preserved by finite composition, i.e., if \( F \) and \( G \) both have one of these properties, then the composition \( F \circ G \) has the same property.

Proof. This is easy to verify by looking at the images of points and sets involved.
Definition 13. Let $F_n$ be a sequence of mappings on a set $X$. We say that the sequence $(F_n)$ converges uniformly to the mapping $F$ if for every $\varepsilon > 0$ there is an integer $N$ such that $\|F_n(p) - F(p)\| < \varepsilon$ for all $n > N$ and all $p \in X$.

Theorem 6. If $(F_n)$ is a sequence of continuous mappings on a set $X$ that converges uniformly to a mapping $F$, then $F$ is continuous.

Proof. See Theorem 21.6 in [5].

1.3 Dimension theory

There are three main ways to define dimension in topology. These definitions sometimes give different values, but they coincide in the class of separable metrizable spaces, which includes all subsets of $\mathbb{R}^n$. For a more complete presentation of dimension theory, see [1]. We will use a definition called the covering dimension, which was formalized by Čech in 1933, based on previous work by Lebesgue.

To define the covering dimension we need a few notions.

Definition 14. A cover of a set $X \subset \mathbb{R}^n$ is a collection $\{A_\lambda : \lambda \in \Lambda\}$ of subsets of $\mathbb{R}^n$, where $\Lambda$ is an arbitrary index set, such that $X \subset \bigcup_{\lambda \in \Lambda} A_\lambda$. An open cover is a cover consisting of open sets.

Definition 15. A cover $A$ is a refinement of a cover $B$ if they cover the same set and for every $A \in A$ there is a set $B \in B$ such that $A \subset B$. An open refinement is a refinement that is an open cover.

Definition 16. The order of a cover is the maximal number $n$ such that there is a point of the covered set that lies in $n + 1$ of the sets in the cover.

Now we are ready to define the dimension of a set.

Definition 17. To every set $X \subset \mathbb{R}^n$ we assign the dimension of $X$, denoted $\dim X$, according to the following rules:

- $\dim X \leq n$, where $n = -1, 0, 1, \ldots$, if every finite open cover of $X$ has a finite open refinement of order $\leq n$
- $\dim X = n$ if $\dim X \leq n$ and $\dim X \not\leq n - 1$

Remark 3. It is easy to see that only the empty set will have dimension $-1$.

Remark 4. In more general spaces it is necessary to define infinite-dimensional sets, if the first condition is never satisfied. All sets in $\mathbb{R}^n$ are however finite-dimensional, which follows from Theorems 8 and 9.

Remark 5. It follows from the definition that a set is one-dimensional if any finite open cover of it can be openly refined so that at any point in the set no more than two elements of the cover overlap, see Figure 1.1.

The following theorems indicate that this definition of dimension is consistent with our intuitive definition of dimension.

Theorem 7. The dimension of a set is preserved by homeomorphisms, i.e., if $X$ and $Y$ are homeomorphic, then $\dim X = \dim Y$.

Proof. This is clear since open covers, subsets, and intersections are preserved by homeomorphisms.
Theorem 8. Let $A \subset B \subset \mathbb{R}^n$. Then $\dim A \leq \dim B$.

Proof. This is obvious, since a cover of $B$ is also a cover of $A$. \hfill \Box

Theorem 9. $\dim \mathbb{R}^n = n$

Proof. See Theorem 1.8.2 in [1]. \hfill \Box

Theorem 10. Any open disc in the plane with positive radius has dimension 2.

Proof. Since an open disc is homeomorphic to the plane, Theorem 3 gives that this is equivalent to the case $n = 2$ of Theorem 9. \hfill \Box

We shall finish this section by characterizing some one-dimensional sets in the plane.

Lemma 2. Let $X \subset \mathbb{R}^2$. Then $\dim X \leq 1$ if and only if $X$ does not contain an open disc.

Proof. See Theorem 20 of Chapter 2 in [2]. \hfill \Box

Lemma 3. Any connected, nondegenerate set in $\mathbb{R}^n$ has at least dimension 1.

Proof. Assume $X \subset \mathbb{R}^n$ is connected and nondegenerate. Then $\dim X > -1$, since only the empty set can have dimension $-1$. Let $p$ and $q$ be two distinct points in $X$. Then $\{\mathbb{R}^2 \setminus \{p\}, \mathbb{R}^2 \setminus \{q\}\}$ is an open cover of $X$. If $\dim X = 0$, then that cover has an open refinement of open sets of order zero, where $p$ and $q$ lies in different sets. But that is a contradiction, since $X$ is connected. Hence $\dim X \geq 1$. \hfill \Box

These last two lemmas give us the following useful theorem:

Theorem 11. Any connected, nondegenerate set in $\mathbb{R}^2$ that does not contain an open disc is one-dimensional.
Chapter 2

Curves

2.1 Definition of a curve

So how should we define a curve in the plane? Our intuitive picture of a curve was something like this $\mathcal{C}$. A simple way to create such curves is of course to draw them with a pencil. We will see in Chapter 3, however, that this approach cannot be used to create a consistent definition of what we intuitively mean by curves. We will instead set up a few conditions so that anything that satisfies our conditions is sufficiently nice to be considered curves.

The most important characteristic of a curve is of course that it is one-dimensional. To only consider one curve at a time, and avoid constructions with isolated points we will require curves to be connected. Finally we will require curves to be compact, for reasons that will be apparent shortly.

**Definition 18.** A curve is a compact and connected one-dimensional subset of the plane $\mathbb{R}^2$.

This definition is adapted from the first chapter of [2], on which much of this thesis is based.

We will start with two simple theorems about curves, the first of which follows immediately from the fact that all properties defining curves are preserved by homeomorphisms.

**Theorem 12.** Any set in $\mathbb{R}^2$ that is homeomorphic to a curve is a curve.

**Theorem 13.** The union of two intersecting curves is a curve.

**Proof.** Let $X_1$ and $X_2$ be two intersecting curves. It is easy to see that $X_1 \cup X_2$ is compact and connected. Now recall [Theorem 11]. To see that $X_1 \cup X_2$ is one-dimensional we assume the contrary and imagine a disc inside it. We know that $X_1$ is one-dimensional, so there must be at least one point in the disc that belongs to $X_2 \setminus X_1$. Since $X_1$ is a closed set, there must then be a smaller disc around that point that does not intersect $X_1$. But the smaller disc must by our assumption lie in $X_1 \cup X_2$, so we can conclude that it is a subset of $X_2$, which is a contradiction as $X_2$ is also one-dimensional. Thus $X_1 \cup X_2$ is one-dimensional, so it is a curve. \qed
Chapter 2. Curves

Figure 2.1: Some simple curves

This last theorem is an important reason to require compactness, as the
closedness of the curves is used in the proof. The necessity closedness is made
clear in the following example.

Example 7. Let \( C_Q = \{(x, y) \in I^2 : x \in \mathbb{Q}\} \cup \{(x, 1) : x \in I\} \) be the unit
square filled with vertical teeth for each rational number, connected by the line
from \((0, 1)\) to \((1, 1)\). We shall call this space the rational comb.

In the same way, let \( C_I = \{(x, y) \in I^2 : x \in I\} \cup \{(x, 1) : x \in I\} \) be a similar
space where the rational numbers are replaced with the irrational. We shall call
this space the irrational comb.

Both these spaces satisfy all conditions of being a curve except for compact-
ness, as they are bounded but not closed. The union \( C_Q \cup C_I = I^2 \), however,
is two-dimensional, demonstrating the necessity of closedness in the definition.

Remark 6. In Example 7 we could see why the definition requires curves to be
closed, but not why boundedness is necessary. That comes from the fact that
closedness and boundedness are tightly connected, as we can see from the fact
that the open unit interval \([0, 1[\) and the real line \(\mathbb{R}\) are homeomorphic. Thus
Theorem 13 could not hold if curves are not closed.

2.2 Some simple curves

Some simple examples of curves can be seen in Figure 2.1. A simple way of
producing curves is through graphs of continuous functions on closed intervals,
as in Figure 2.2.

Theorem 14. The set \( X = \{(x, f(x)) : a \leq x \leq b, f \text{ continuous}\} \) in \(\mathbb{R}^2\) is a
curve.

These are examples of the most basic curves: arcs.

Definition 19. An arc is a set homeomorphic to the closed unit interval \([0, 1]\).

The unit interval is clearly compact, connected, and one-dimensional, so it is
easy to see that all arcs are curves. A more intricate example is the \(\sin \left(\frac{1}{x}\right)\)-curve.

Figure 2.2: The curve \( y = x^3 - \frac{x}{2}, -1 \leq x \leq 1 \)
2.3 The \( \sin \left( \frac{1}{x} \right) \)-curve

The function \( f(x) = \sin \frac{1}{x} \) oscillates with increasing frequency as \( x \) approaches zero. The function cannot be continuously extended to \( x = 0 \), since any value \(-1 \leq y \leq 1\) can be found as the limit \( f(x_n) \) for some sequence \( x_n \to 0 \). Nevertheless, the function can still be used to construct an interesting curve. Let \( S = \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \). Then the closure \( \bar{S} = S \cup \{(0, y) : -1 \leq y \leq 1\} \) is a curve called the \( \sin \left( \frac{1}{x} \right) \)-curve (see Figure 2.3).

**Theorem 15.** The \( \sin \left( \frac{1}{x} \right) \)-curve

\[ \bar{S} = \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\} \]

is a curve.

2.4 The Sierpiński carpet

Let \( C_0 \) denote the unit square \( \{(x,y) : x,y \in I\} \). Divide it into nine equal subsquares, and let \( C_1 \) denote the set obtained by removing the interior of the middle square, as in Figure 2.4. Divide each of the eight remaining subsquares in the same way and continue the process to create \( C_2, C_3, \) etc. The remainder \( C = \bigcap C_i \) is called the Sierpiński carpet.

**Theorem 16.** The Sierpiński carpet is a curve.

**Proof.** Each set \( C_i \) is clearly compact and connected, so \( C \) is compact and connected as well, according to Theorem 2. We will now use Theorem 11 to prove that \( C \) is one-dimensional. \( C \) is clearly nondegenerate, since the border of the unit square remains in every \( C_i \), and thus in \( C \).

![Figure 2.4: Construction of the Sierpiński carpet](image-url)
To show that \( C \) does not contain any open disc, let \( p \) be the center of a hypothetical open disc \( D \) that lies in \( C \). Then obviously \( p \) must lie in \( C \), so for each \( C_i \), the point \( p \) lies in one of the \( 8^i \) subsquares of \( C_i \). Each of these subsquares have width \( \frac{1}{2^i} \), so for a big enough value of \( i \), the subsquare is smaller than the disc, so the whole subsquare is contained in \( D \). But the center of the subsquare does not belong to \( C \), so some points of the open disc \( D \) do not lie in \( C \), which is a clear contradiction. This shows that the Sierpiński carpet is one-dimensional, so we can conclude that it is indeed a curve.

The Sierpiński carpet has a very interesting property, called universality. This means that every curve in the plane can be embedded in the Sierpiński carpet, so in a sense it is the biggest or most complex plane curve there is. For this reason it is sometimes called the Sierpiński universal plane curve.

**Theorem 17.** Every curve in the Euclidean plane is homeomorphic to a subset of the Sierpiński carpet.

**Proof.** Let \( X \) be an arbitrary plane curve. We shall construct a function that maps \( X \) to a subset of the Sierpiński carpet \( C \), and show that it is a homeomorphism. First, since \( X \) is bounded, we can let \( F_0 \) be a homeomorphism that maps \( X \) into the unit square \( C_0 \) simply by scaling and translating.

We will now use a homeomorphism \( G \), seen in Figure 2.5, that maps all points in \( C_0 \) except the center \( \left( \frac{1}{2}, \frac{1}{2} \right) \) into \( C_1 \) by simply moving each point outwards along a straight line originating in the center. The distance from the edge of \( C_0 \) to the center is reduced by one third, so we can make two observations that will be useful shortly: the distance between two points after the transformation is at least two thirds of the distance before, and the points on the edge of \( C_0 \) are not moved at all.

If \( \left( \frac{1}{2}, \frac{1}{2} \right) \notin F_0(X) \) we let \( F_1 = G \). Then the homeomorphism \( F_1 \circ F_0 \) maps \( X \) into \( C_1 \). If \( \left( \frac{1}{2}, \frac{1}{2} \right) \in F_0(X) \) we cannot simply use \( G \), but that is not a big problem, since the one-dimensionality lets us pick another point arbitrarily close to the center to expand from. Now divide \( C_1 \) into \( 8 \) subsquares as in the definition of the Sierpiński carpet, and let \( F_2 \) be the mapping created by using \( G \) on each subsquare. Since the edge of each subsquare is kept still, this mapping is also a homeomorphism, so \( F_2 \circ F_1 \circ F_0 \) is a homeomorphism that maps \( X \) into \( C_2 \). Note that distance between two points is now at least \( \left( \frac{2}{3} \right)^2 \) times the distance between them after \( F_0 \) had been applied. Continue in this manner to create \( F_3, F_4, \ldots \), and the limiting function \( F = \ldots \circ F_2 \circ F_1 \circ F_0 \) that maps \( X \) into the Sierpiński carpet \( C \).

\[ \text{Figure 2.5: The help function } G \]
2.5. Other examples of curves

The Sierpiński triangle

The Hawaiian earring

The Cantor brush

Figure 2.6: Other examples of curves

It is easy to see that the sequence of functions $F_0, F_1 \circ F_0, \ldots$ converges uniformly to $F$, so the function $F$ is continuous by Theorem 6. The distance between two points will after each iteration be at least $(\frac{2}{3})^i$ times the distance between them after $F_0$ has been applied. The width of the subsquares, however, is $(\frac{1}{3})^i$, so any pair of points in $X$ will eventually end up in two different subsquares. This means that the function $F$ is injective, so according to Theorem 4 $F$ is a homeomorphism between $X$ and $F(X) \subset C$. Thus we can conclude that any curve $X$ is embeddable in the Sierpiński carpet.

2.5 Other examples of curves

Other examples of well-known topological structures that are curves include the Sierpiński triangle, the Hawaiian earring, and the Cantor brush. These can all be seen in Figure 2.6.

The Sierpiński triangle is constructed in a manner similar to the Sierpiński carpet, with central triangles removed in each iteration. The Hawaiian earring is made of a countable number of circles with diminishing radii and a common tangent. The Cantor brush is created by connecting a single vertex to all points of the Cantor middle third set, which is constructed by iteratively removing the middle third of a line segment, similar to the construction of the Sierpiński carpet.

Theorem 18. The Sierpiński triangle, the Hawaiian earring, and the Cantor brush are all curves.

Proof. We will start by noting that they are all nondegenerate. To show that they are compact and connected we will use Theorem 2. The Sierpiński triangle is constructed by triangle with open inner sets removed, so each iteration is clearly compact and connected. The Hawaiian earring can be constructed a sequence of sets where all but finitely many of the circles are replaced by a closed disc, and these sets are of course all compact and connected as well. The Cantor brush can be constructed by starting with a solid triangle and iteratively removing the open middle third, in each step leaving a compact and connected set. We can conclude that all three sets are compact and connected. Finally, like for the Sierpiński carpet, we can use Theorem 11 to show that they are one-dimensional.
Chapter 3

Peano curves

3.1 Definition of Peano curves

It may seem a bit odd to define curves not through the stroke of a pencil, but with a somewhat arbitrary set of conditions and a complicated notion of dimension. The stroke of a pencil, which can be formalized with a continuous mapping $t \mapsto (x, y)$, was actually the definition mathematicians used for curves for a long time. We will call these objects Peano curves.

While the two definitions overlap to some extent, there are curves that do not satisfy the definition of Peano curves. We will see, however, that the main problem that made mathematicians abandon Peano curves is not what the definition excludes, but what it does not exclude.

Definition 20. A Peano curve is the image of the closed interval $I$ under a continuous mapping.

Many curves, especially simple ones, are Peano curves. Trivially, all arcs satisfy the definition, as do all curves in Figure 2.1. An example of a curve that is not a Peano curve is the $\sin\left(\frac{1}{x}\right)$-curve. No continuous line inside it that starts in the point $(1, \sin 1)$ can ever reach the vertical line $(0, y)$, since if such a line existed, then the point where it first met the vertical line would be the limit $\lim_{x \to 0} \sin \frac{1}{x}$, but that limit does not exist.

3.2 A characterization of Peano curves

So how can we know whether or not a certain object is a Peano curve? Finding a continuous mapping that fits the need can be tricky, and proving that no such mapping exists could be even harder. Fortunately there is a simple condition for when it is possible. The central part of this condition is the property of being locally connected.

Definition 21. A set $X \subset \mathbb{R}^2$ is locally connected if every component of the intersection between $X$ and an open set in $\mathbb{R}^2$ is the intersection between $X$ and some open set in $\mathbb{R}^2$.

They can also be called Peano spaces or Jordan curves. The terminology is a bit ambiguous as all three names can refer to other things as well.

Granholm, 2014.
This is a quite technical definition, but hopefully an example will make it clearer.

**Example 8.** The \( \sin \left( \frac{1}{x} \right) \)-curve is not locally connected. The intersection between the \( \sin \left( \frac{1}{x} \right) \)-curve and a small open disc around \((0,0)\) can be seen in Figure 3.1. The components of this intersection are a bunch of disjoint segments of the function \( \sin \frac{1}{x} \) and a part of the vertical line \((0,y)\). The part of the vertical line is however not the intersection between the \( \sin \left( \frac{1}{x} \right) \)-curve and any open set, since any open set intersecting the vertical line will also intersect some of the other segments.

**Theorem 19.** As set \( X \subset \mathbb{R}^2 \) is a Peano curve if and only if it is compact, connected, and locally connected.

**Proof.** See Theorem 5.9 of Chapter 5 in [3].

From this we can again see that the \( \sin \left( \frac{1}{x} \right) \)-curve is not a Peano curve, and with the same argument as in Example 8 we can see that the Cantor brush is not a Peano curve either. On the other hand the Hawaiian earring and the Sierpiński triangle, as well as the important Sierpiński carpet, satisfy all of these conditions, so they are Peano curves. Somewhat surprising, considering the definition of Peano curves, is the fact that the unit square also fulfills the conditions and thus is a Peano curve. It becomes less surprising, though, once we realize that this characterization of Peano curves is just our original definition of curves, with the crucial condition of one-dimensionality replaced by local connectedness, a condition not really related to curves.

We can now see that the class of Peano curves excludes objects that can easily be considered curves, while it includes objects that are nothing like curves. It is clearly not a good way to define curves in the plane.

### 3.3 Explicit mappings to Peano curves

In the last section of this chapter we will sketch explicit mappings from the unit interval to the Hawaiian earring, the Sierpiński triangle and carpet, and the unit square.

The easiest of these is the Hawaiian earring. Simply map the points \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \) to the point where all circles intersect, and the intervals between these points to each of the circles. Finally map 0 to the point of intersection to make the mapping continuous.

To create continuous mappings from the unit interval to the other sets is more complicated, and requires limiting processes. We will start with the Sierpiński triangle.
3.3. Explicit mappings to Peano curves

Figure 3.2: Construction of a continuous mapping to the Sierpiński triangle

Figure 3.3: Construction of a continuous mapping to the Sierpiński carpet

The iterations in the limiting process will correspond to the iterations in the creation of the Sierpiński triangle. First map the unit interval to one edge of the initial triangle. As the first subtriangle is removed, the interval is broken up into three parts, and each part is mapped to the edge of one of the remaining three subtriangles (see Figure 3.2). As the next set of subtriangles are removed, each of the three segments are broken up in three again, and the process is continued forever. A line segment that is mapped into a subtriangle in one step will always remain inside that subtriangle, so the sequence of mappings is uniformly convergent. Thus, according to Theorem 6, the limit will be a continuous mapping. Furthermore, the distance from any point in the Sierpiński triangle to the image of the mapping will shrink to zero, which means that the limit mapping will pass through all points of the triangle. Thus we have created a continuous mapping that maps the unit interval to the Sierpiński triangle.

The process is similar for the Sierpiński carpet. In this case, however, the line runs diagonally through the square and is broken up into 11 segments that are mapped into the eight subsquares, so that in three of the subsquares the lines overlap. The process is illustrated in Figure 3.3, where the lines are drawn with rounded corners for clarity.

For the unit square we do the same except that in each iteration we map the segments to all nine subsquares, and no lines have to overlap (see Figure 3.4).

Figure 3.4: Construction of a continuous mapping to the unit square
Chapter 4

Generalization to higher dimensions

As stated in Chapter 1, we have only considered curves in the plane. There is nothing that limits us to two dimensions, however – the definition is trivial to adjust to $\mathbb{R}^n$.

4.1 A general definition of curves

Definition 22. A curve is a compact and connected one-dimensional subset of the space $\mathbb{R}^n$.

Some of these general curves can of course be embedded in the plane, while others cannot. Graph theory gives some indication for when this is possible, through the Kuratowski graph theorem.

4.2 The Kuratowski graph theorem

Two objects are important in the Kuratowski graph theorem: the complete graph $K_5$ and the complete bipartite graph $K_{3,3}$. The complete graph $K_5$ consists of five points that are all connected to each other by edges. In our context edges are curves, and the edges can not intersect each other except in the five points. The bipartite graph $K_{3,3}$ consists of two sets of three points, all connected to every point in the other set. Two-dimensional representations of $K_5$ and $K_{3,3}$ can be seen in Figure 4.1.

Theorem 20. If a curve in $\mathbb{R}^n$ contains either $K_5$ or $K_{3,3}$, it is not embeddable in $\mathbb{R}^2$.

For graphs, this condition is both necessary and sufficient. For curves it is not that easy, since curves can lack a planar embedding without containing either $K_5$ or $K_{3,3}$, as seen in the following example.

Example 9. Let

$$L = \left\{ \left( \frac{i}{2}, y \right) : n \in \mathbb{N}, y \in I \right\} \cup \left\{ (0, y) : y \in I \right\} \cup \left\{ (x, y) : x \in I, y = 0, 1 \right\},$$

Granholm, 2014.
as depicted in Figure 4.2. The curve produced by merging three copies of $L$ at the leftmost edge $\{(0, y) : y \in I\}$ cannot be embedded in the plane, even though it does not contain either $K_5$ or $K_{3,3}$.

### 4.3 Three-dimensional embeddings

The curves that are not embeddable in $\mathbb{R}^2$ are still quite nice.

**Theorem 21.** *Any curve in $\mathbb{R}^n$ is embeddable in $\mathbb{R}^3$.***

**Proof.** See the more general case of Theorem 1.11.4 in [1].

Since all curves are embeddable in $\mathbb{R}^3$, we can construct a curve that is universal for all curves, in the same way as the Sierpiński carpet is universal for planar curves. This curve is called the Menger curve and is constructed as follows:

Let $M_0$ denote the unit cube $\{(x, y, z) : x, y, z \in I\}$. Divide it into 27 equal subcubes and remove the central subcubes on each of the six faces and the subcube in the middle, leaving 20 subcubes around the edges to form $M_1$, as in Figure 4.3. Repeat this for all remaining subcubes to create $M_2$, $M_3$ etc. The Menger curve is the remainder $M = \bigcap M_i$.

**Theorem 22.** *Every curve is homeomorphic to a subset of the Menger curve.*

After applying [Theorem 21] the proof of this theorem becomes virtually identical to the two-dimensional case in [Theorem 17], so it will not be repeated here.
Figure 4.3: Construction of the Menger curve
Bibliography


