On Production Planning and Activity Periods

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Published by Linköping University Electronic Press, 2015
Series: Linköping Studies in Economics, No. 2
ISSN: 1652-8166
URL: http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-113298

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Abstract

Consider a company which produces and sells a certain product on a market with highly variable demand. Since the demand is very high during some periods, the company will produce and create a stock in advance before these periods. On the other hand it costs money to hold a big stock, so that some balance is needed for optimum. The demand is assumed to be known in advance with sufficient accuracy. We use a technique from optimal control theory for the analysis, which leads to so-called activity periods. During such a period the stock is positive and the production is maximal, provided that the problem starts with zero stock, which is the usual case. Over a period of one or more years, there will be a few activity periods. Outside these periods the stock is zero and the policy is to choose production = the smaller of [demand, maximal production]. The "intrinsic time length" is a central concept. It is simply the maximal time a unit of the product can be stored before selling without creating a loss.

Remarks: The author realizes that this is a simplified model, since for instance the production cost and the price are both assumed constant. We hope nevertheless that the structure theorem (Th.2) and the numerical procedure for the seasonal problem can be of some interest.

Acknowledgement: The author was introduced to the problem by Professor Ou Tang and Dr. Shuoguo Wei, Linköping. Thanks for that and for many interesting discussions!

1. General Overview and Basic Ingredients of the Problem

The company produces and sells its product on a market with variable demand over a fixed time interval $0 \le t \le T$. The demand d(t) is supposed to be known in advance. The problem is to determine the production u=u(t) over the time interval so that the result $J =$ (income – costs) becomes maximal. The following parameters and variables will be used:

t = time;

u(t) = production per time unit, $0 \le u \le U$, where the maximal production rate U is constant; u is the control variable;

$\beta > 0$ = the production cost for one unit of product, also constant;

d(t) = demand for product, $0 \le d(t) \le D$, is the maximal product quantity that can be sold per unit time

x = x(t) ≥ 0 is the stock of product available at time t;

h > 0 is the storage cost per time unit and unit of stock

c_1 = price per unit of sold product

S(x,d,u) = selling function, which means that S(x(t),d(t),u(t)) units are sold per time unit at time t. Various choices for S are possible.

All this leads to the "state equation" $\dot{x} = u(t) - S(x(t), d(t), u(t))$;

i.e.: increase of stock per time unit = production minus selling.

The objective functional is therefore given by

$$J = \int_0^T [c_1\, S(x(t), d(t), u(t)) - \beta\, \mathrm{u}(t) - \mathrm{h}\, \mathrm{x(t)}]dt + c_0 x(T).$$

It is to be maximized by clever choice of the control function u(t), when x(0) is given and x(T) is free <u>or</u> prescribed. In the case that x(T) is not prescribed there is a rest value c_0 ascribed to the product. It is understood from the beginning that $c_0 \le c_1$ for obvious reasons.

The demand function d(t), $0 \le t \le T$, is assumed to be known and assumed to be piece-wise constant, having a finite number of jump discontinuities.

It is also clear that there can be no selling if there is no demand, d(t) = 0. As a basic approximation we assume that the selling is proportional to d(t) for any given x > 0.

The dependence on x can certainly be modeled in various ways.

The given problem can be discussed under different mathematical ambitions:

A. The initial problem

The concrete optimization problem for the company considered is called the <u>initial problem</u>. It normally runs over a period of one or more years. The demand, in particular, may vary considerably for a "seasonal" product during such a period, for example being considered constant each month, or week. It is understood that the control u(t) should be piece-wise continuous, and the stock x(t) should have a piece-wise continuous derivative, allowing for a finite number of discontinuities of the derivative. Clearly, x(t) is supposed to be non-negative. The selling may simply be the demand d(t) times some "suitable" function of $x > 0$, or some more complicated function of d and u, when x=0. In this context we are not aiming at complete mathematical rigour.

B. A mathematical solution, satisfying the side condition

Here, a mathematically correct solution $\{x(\cdot), u(\cdot)\}$ is wanted, satisfying the side condition $x(t) \geq 0$, plus restrictions stated below. It will be called a proper solution.

Remark: The fact that the function d(t) in our problem has a finite number of jump discontinuities means no severe difficulty.

In this context the control function u(t) must be Lebesgue measurable and satisfy $0 \leq u(t) \leq U$. The stock x(t) must be absolutely continuous and satisfy the state equation a.e. The side condition $x(t) \geq 0$ must be satisfied in the whole domain of definition. The stock $x(t)$ must satisfy initial and final conditions.

It remains to specify the selling function and the state equation.

It is assumed, for $x > 0$ only, that the selling function can be written

$S(x, d) = d\ \varphi(x)$, where the function $\varphi(x)$ is continuously differentiable and non-decreasing for $x \geq 0$. Further, $\varphi(0) = 1$.

Also, for $x > 0$, the state equation is simply $\dot{x} = u(t) - S\big(x(t), d(t)\big)$, i.e.

$\dot{x} = u(t) - d(t)\ \varphi(x(t))$.

Define $E = \{t: x(t) = 0\}$. Since x(t) is differentiable a.e., and since almost all points of E are points of accumulation, it follows that $\dot{x}(t) = 0$ a.e. on E.

Thus, a specific formula for $\dot{x}$ on E is simply not needed.

It is fundamental, however, that $\dot{x} = u(t) -$ selling function $= 0$ on E.

Thus selling = u(t) $\leq U$, and, by definition, selling $\leq d(t)$.

Consequently, u(t) = selling $\leq \min\{U, d(t)\}$. Clearly, $\min\{U, d(t)\}$ is the biggest possible value for production and selling at t that does not increase the stock. It is therefore "locally" optimal management.

The continued analysis will be based on the understanding, or condition that

u(t) = selling = $\min\{U, d(t)\}$ for almost all $t \in E$.

The results obtained in this paper refer to case B. They can, however, easily be interpreted in the context of case A.

The question of the existence of a proper solution will not be solved here. It is left for future study.

In the following analysis it is simply assumed that we have an optimal element.

2. A first look at and adaption of the control maximum principle

In this situation the usual maximum principle (MP) by Boltyanski and Pontryagin in optimal control can be applied, paying due attention to the condition x(t) > 0. We will adhere to the presentation by Evans [4].

Rigorous versions of the maximum principle are found in [L-M], pp. 318-321, or [M-S], pp. 126–127, but the most useful version for the present text is found in [4] by L. C. Evans, pp. 110–118. We will refer to [4] and use the same notation, as far as possible.

A brief background comment:

The standard method for handling side conditions like x(t) ≥ 0 is to introduce multipliers for the side conditions, multiply and add to the so-called Hamiltonian function, which gives the Lagrangian. Then a modified maximum principle is supposed to hold for the Lagrangian. Further, complementary slackness conditions enter the picture and make it more complicated. For this more traditional approach, see [S-T], pp. 98 ff. All this is avoided here, thanks to our "activity period" approach.

We are now facing a problem in optimal control of Bolza´s type, over a given time interval. The problem is non-autonomous, since the demand depends on time. It is shown in [4] how the Bolza problem can be rewritten as a problem of Mayer´s type, which is convenient.

It was above assumed that the selling function is written $S(x, d) = d \cdot \varphi(x)$, where the function $\varphi(x)$ is continuously differentiable and non-decreasing for $x \geq 0$. Further, $\varphi(0)$ = 1. In a while it will be assumed that $\varphi(x) \equiv 1$, but

$\varphi(x)$ will be kept until further for reference purposes.

Consider for a while an element (x(t),u(t)), optimal on some interval [0,T], such that $x(t) > 0$ on the whole interval. Some notation must be changed in order to adapt to [4].

First, the notation x for the stock is changed to x_1. Next, x_2 will be the integral found in the definition of J, but now taken from 0 to t. In other words, we have

$\dot{x_1} = u - \varphi(x_1)d(t) \equiv f_1(t, x_1, u)$,

$\dot{x_2} = c_1\, \varphi(x_1)d(t) - \beta u - hx_1 \equiv f_2(t, x_1, u)$.

Introduce the Jacobian matrix A(t) as in [4], p. 115:

A(t) $= (\frac{\partial f_i}{\partial x_k}) = \begin{bmatrix} -\varphi'(x_1)d(t) & 0 \\ c_1\varphi'(x_1)d(t) - h & 0 \end{bmatrix}$. Consider the adjoint system

$\dot{\eta} = -(\eta_1, \eta_2)$ A(t), i.e.

$\dot{\eta_1} = \eta_1 \cdot \varphi'(x_1)\, d(t) - \eta_2 \cdot [c_1 \cdot \varphi'(x_1)d(t) - h]$,

$\dot{\eta_2} = 0$.

Here, $\eta = (\eta_1, \eta_2)$ *is* the so-called adjoint state variable.

In the case that the end-point is free, then a so-called transversality condition is available, giving $\eta_1(T) = c_0$ *and* $\eta_2(T) = 1$. (See [4], p. 117.) This may in some cases simplify the continued analysis . It will not be needed in this work.

In the case that the end-point is prescribed, much less information is obtained. One only gets the information that $\eta_2(T) \geq 0$. (See [4], p. 123.)

According to the maximum principle (MP), [4] p.116 -118, we form the Hamiltonian function H from the state equation and the objective functional J as follows:

$$H = \big(u - d(t)\varphi(x_1)\big) \cdot \eta_1 + (c_1 \cdot d(t)\varphi(x_1) - \beta \cdot u - h \cdot x_1) \cdot \eta_2 \,.$$

Observe here that x satisfies the dual equation $\dot{x} = \frac{\partial H}{\partial \eta}$. In the lucky case that $\eta_2 > 0$, we can replace η_2 by 1 without loosing generality, and then the only terms in H which contain u are $u\,\eta_1 - \beta u = u(\eta_1 - \beta)$. According to MP, this expression is maximized by u(t) along an optimal trajectory, for almost all t. Since the control variable u is restricted by $0 \leq u \ \leq U$, it follows that (except for a null set)

$$u(t)\ is \begin{cases} U & if\ \eta_1 > \beta \\ 0 & if\ \eta_1 < \beta \\ unspecified & if\ \eta_1 = \beta \end{cases}.$$

All this is in agreement with the “standard” deterministic maximum principle.

For obvious reasons, we make the following general assumption: $c_1 > \beta > 0$.

The interpretation of the adjoint variable η_1 as a “shadow value” for the state variable x_1 is well known. It works best for the case that the end-point is free, in which it follows from the derivation of the maximum principle from the solution of a Mayer problem. It requires, however, some regularity of the so-called value function $V(t, x_1)$, which is not always satisfied. See [4], p.85. This interpretation is nevertheless very helpful for intuitive understanding of the result. The above rule says that if this “shadow value” is higher than the cost of production, then produce at maximal rate; otherwise do not produce at all.

The concept of a switch must be explained. Let $\eta_1(t) < \beta\ for\ t < t'$, and $\eta_1(t) > \beta\ for\ t > t'$. Then, by the above rule for u(t), it follows that $u(t) = 0\ for\ t < t', and\ u(t) = U\ for\ t > t'$. This is an off-to-on switch. The meaning of an on-to-off switch is now obvious. The possibility of different behavior of η_1 at some t-value, other than a simple sign change of the crucial quantity $(\eta_1 - \beta)$ is not a priori excluded.

We want to obtain a better understanding of an optimal control. An important step will be to understand the number and position of switches.

Definitions: A demand period is an open interval on the t-axis, where d(t) is constant. The interval is always assumed maximal. Other parameters like U, h, β, c_1 are also assumed constant during a demand period.

The global problem is the just the previous optimization problem, now considered over an interval consisting of a finite number of demand periods. Other parameters than d(t) do not differ between the demand periods, unless otherwise is said.

3. Introduction of Business Periods. Guiding Function

Clearly, the side condition $x_1 \geq 0$ is an important feature of our problem. Now consider a "candidate" solution element $x_0(t)$, not necessarily optimal for the initial problem. Suppose that $x_0(t) > 0$ on an interval L, and $x_0(t) = 0$ at the endpoints of L. Assume that $x_0(t)$ is optimal, considered on $L = [T_0, T_1]$, with prescribed boundary values zero, and under the restriction $x_1 > 0$ (except for endpoints). We then say that $x_0(t)$, considered on L, defines a true active period. This will turn out to be a very useful concept. We may later also consider active periods, starting by a positive stock, or active periods ending by a positive stock, or both. But for the moment all active periods considered will be "true".

The Boltyanski-Pontryagin maximum principle is applicable to $x_0(t)$, as was briefly explained in §2. For more details, we refer to [4], Chapter 4, and in particular the Appendix. The control system here has the form (n = 1)

$\dot{x_1} = u - \varphi(x_1)d(t) \equiv f_1(t, x_1, u),$

$\dot{x_2} = c_1\, \varphi(x_1)d(t) - \beta u - hx_1 \equiv f_2(t, x_1, u).$

For technical reasons, first consider $x_0(t)$ on $[T_0, T']$ for some $T' < T_1$. Clearly, $x_0(t)$ is optimal over this interval too, and the end-point is not on the boundary of the allowed domain. The maximum principle is clearly applicable.

The basic statement of the principle is found in [4], p.123. The question as to whether the situation is "abnormal" or not will be resolved below. The adjoint variable is here written $\eta(t) = (\eta_1(t), \eta_2(t))$, instead of p*(t) as in [4].

The perturbation cone K(T′) (in our case just a sector in the plane) is well defined and plays a central role. The unit vector e_2 cannot be interior to the cone K (see [4], p.122), because that would quickly lead to a contradiction, namely a better element could then be constructed, i.e. an element with the same value for x_1, and a bigger value for x_2. Hence, there exists a separating vector $w = (w_1, w_2)$, such that $w \cdot z \leq 0$ for all $z \in K(T')$, and $w \cdot e_2 = w_2 \geq 0$. The terminal condition for the adjoint vector is $\eta(T') = w$, and so $\eta_2(T') = w_2$. The maximization statement is the same as in [4], p.123-124, although in slightly different notation. This argument is much the same as in [1], pp. 247-254 (much more in detail). See also [2], pp. 108-109, or [7], pp. 92-107.

Two different cases must be considered: $w_2 > 0$, or $w_2 = 0$ (abnormal case).

1. Let $w_2 > 0$. As explained in §2, the adjoint variable $\eta(t)$ will satisfy the system $\dot{\eta_1} = \eta_1 \cdot \varphi'(x_1)\, d(t) - \eta_2 \cdot [c_1 \cdot \varphi'(x_1)d(t) - h], \dot{\eta_2} = 0.$

From now on, we specialize on the case $\boldsymbol{\varphi(x) \equiv 1}$.

Divide $\eta\,(t)$ by w_2 to get $\eta_2(t) \equiv 1$, and simplify the first equation into $\dot{\eta_1} =$ h, after observing that $\varphi'(x_1) \equiv 0$.

Therefore, only a switch "off to on" is a priori possible. But $x_0(t)$ starts from zero and immediately becomes positive, so the only possibility is that $u(t) = U$ on the whole $[T_0, T']$. Further, U > d(t) must hold on L near the left end-point.

2. Let $w_2 = 0$. Also here, $\dot{\eta_2} = 0$, and so $\eta_2(t) \equiv 0$. Clearly, $\eta_1(t) \equiv const. \neq 0$. The Hamiltonian reduces to $= \eta_1(t) \cdot (u - d(t))$, to be maximized a.e. along the optimal

$x_0(t)$. This implies $u(t) \equiv 0$ or $u(t) \equiv U$ on $[T_0, T']$. The case $u(t) \equiv 0$ is clearly impossible, and thus $u(t) \equiv U$.

But T´ was arbitrary, except that $T_0 < T' < T_1$.

Consequently $u(t) \equiv U$ on L in any case. To summarize:

Theorem 1

An optimal solution $x_0(t)$ must satisfy $u(t) \equiv U$ a.e. during a true active period.

The question is now: does the same result hold for any active period, which starts from zero? Consider a problem, where the end-point is free, and a term $c_0 \cdot x(T_1)$ is added to J. Consider an optimal solution $x_0(t)$, starting from zero, and ending with $x_0(T_1) > 0$ (otherwise we are back in the previous case). Then the previous argument is applicable to $x_0(t)$, and so $u(t) \equiv U$ a.e. for $T_0 \leq t \leq T_1$. But in this case more information is available, namely certain end-point conditions. These will be considered in §5.

Theorem 1'

An optimal solution $x_0(t)$, starting from zero, must satisfy $u(t) \equiv U$ a.e. during a final active period, where the endpoint is free.

Also in this case, U > d(t) must hold on L near the left end-point.

The equation governing the development of the stock $x_1(t)$ on L has the form

$\dot{x} = U - d(t)$, where $d(t)$ is non-negative and constant on each demand period, i.e. piece-wise constant. The value of d(t) at an endpoint of a demand period has no importance. The form of the equation implies that all possible solutions during an active period will be restrictions or "vertically" translated restrictions of one and the same piece-wise linear solution X(t), as long as

$x_0(t) > 0$. The function X(t) will be called the guiding function of the problem.

Thus $X(t) = \int_0^t (U - d(s))ds$, for $0 \leq t \leq T$.

Corollary of Theorem 1

An optimal solution $x_0(t)$ is a restriction (+ a constant) of the guiding function during a true active period.

This will be the key to a numerical solution.

Clearly, an active period for $x_0(t)$ consists of a finite number of demand periods, or parts thereof, such that $x_0(t)$ is linear on each demand period. It starts from zero and ends non-negative. In the situation that $x_0(t)$ ends by a positive value further information can be obtained, as seen in §5. Clearly, $\dot{x_0}(t)$ will have jump discontinuities only at endpoints of demand periods. Further, $x_0(t)$ may start from zero at an interior point of a demand period, or from an end-point, and similarly for the ending.

Before summarizing the results so far: assume that $x(t) = 0$ on some interval L´. As was observed in §1, the best our company can do during that time is to choose $u(t) =$

$\min[d(t), U]$, i.e. produce and sell what can be sold without creating a stock. We can now summarize:

Main structure theorem (Theorem 2)

If x(t) is optimal on the "global" interval L, then L is decomposed into a finite number of activity periods: positive stock, u(t) = U; and a finite number of "passive" periods: zero stock,

u(t) = min[d(t), U].

It can certainly occur that one of these categories is empty.

Corollary

If x(t) is optimal, then x(t) is piece-wise linear on [L] and its derivative has a finite number of discontinuities. These discontinuities occur at points, where d(t) has a discontinuity, and at points where an activity period begins or ends.

4. Useful General Observations. Formulas for Some Derivatives

We begin here by observing that the whole problem is trivial if $d(t) \geq U$ always, or if $d(t) \leq U$ always. In any of these two cases the choice

u*(t) = min (U, d(t)) is clearly optimal. Thus, assume from now on that

$\max(\mathrm{d(t)}) > U > \min\big(d(t)\big)$.

Consider again the initial problem with arbitrary boundary data. The following simple fact will be useful.

Lemma 1

Let $x^*(t) \geq 0$ be a proper solution to our basic problem. Let

d(t) < U immediately to the left of some point t_0 and d(t) > U immediately to the right of t_0. Then $x^*(t_0) > 0$.

Proof: This goes by contradiction. Assume that $x^*(t_0) = 0$. Then a better element can be constructed. First, it is clear that $x^*(t) = 0$ must hold on some interval to the right of t_0, because of the dominating demand. Let s > 0 be a parameter at our disposal. Let y(t) be an admissible candidate for our optimization problem, obtained by replacing u*(t) by U on the interval $t_0 - s \leq t \leq t_0$. Put $\lambda = y(t_0) > 0$. Clearly, $y(t) = x^*(t)$ will hold for $t > t_0 + a \cdot \lambda$, for some positive constant a. Now, if s goes to 0, then so does λ.

Compare the merits of $x^*(t)$ and y(t). Clearly, the difference in storage cost will be $o(\lambda)$. The difference in selling returns will be $c_1 \cdot \lambda$ in favour of y(t). The difference in production cost will be $\beta \cdot \lambda$ in favour of $x^*(t)$. Now, since $c_1 > \beta$, it follows that y(t) is better than $x^*(t)$, for s small enough, contradicting the optimality of $x^*(t)$. This completes the proof.

Terminology. Let t´ be a point where $d(t) - U$ changes sign from strictly negative to strictly positive. Then t´ is called a check-point. (Just to have a convenient name.) So an optimal element is positive at each check-point.

The concept of intrinsic time length for our problem will be useful later. It is defined as $l_0 = \frac{1}{h}(c_1 - \beta)$. Clearly, it is a measure of the possible profit of one unit of product versus the storage cost; not very surprising! It has indeed the dimension of time.

Again, let $x^*(t) \geq 0$ be a proper solution, such that $x^*(t) = 0$ at the endpoints of the basic interval. Then, provided that there is at least one check- point, then there must be at least one true active period. Clearly, there may be several active periods. Some of these may be isolated, and others may be adjacent. Each active period must be internally optimal, and so an analysis for these is valid, whereas a little more can be proved for "isolated" active periods. Certainly, there may occur a "non-true" initial active period or a "non-true" final active period.

In order to find a way of computing a proper solution, it is clearly needed to look at the functional J, evaluated for a suitable family of restrictions, or "vertically" translated restrictions, of the guiding function X(t). The idea of the construction is that the wanted function element $x_0(t)$ is imbedded and "trapped" in a one-parameter family of restrictions of X(t); then to be identified from its optimizing property by using an appropriate derivative of J.

Some simple technical preparations are needed.

Let X(t) be increasing and linear on some interval $t' \le t \le t''$;

decreasing and linear on some interval $t^{**} \le t \le t^{*}$.

Assume further that $t'' < t^{**}, X(t'') = X(t^{**})$ and $X(t') = X(t^{*})$.

Assume finally that $X(t) > X(t'')$ for $t'' < t < t^{**}$.

We now define a one-to-one correspondence $t_1 \leftrightarrow t_2$ between the intervals [t′,t′′] and $[t^{**}, t^{*}]$ by requiring that $X(t_1) = X(t_2)$. Clearly, this correspondence is linear. Write $d(t) = D_1$ for $t' < t < t''$, and $d(t) = D_2$ for $t^{**} < t < t^{*}$. Clearly, $D_1 < U < D_2$.

Now put $x(t) = X(t) - X(t_1)$. This defines a true active period for $t_1 < t < t_2(t_1)$, not necessarily optimal.

Lemma 2

Consider the above situation. A basic formula, describing the result of an infinitesimal shift of the active period is needed; in other words, a derivative of J with respect to $t_1 \in$ [t′, t′′] . The result is:

$\frac{dJ}{dt_1^i} = (U - D_1)[h(t_2 - t_1) - (c_1 - \beta)]$.

This is called an “inner” derivative for reasons which will become clear.

Proof: To begin with, it is enough to consider the contribution to the overall functional J from the interval $[t', t^{*}]$, since t_1 has no influence outside it. But it is not enough to look at $\int_{t_1}^{t_2(t_1)} \dots$. It is necessary to specify the situation outside $[t_1, t_2]$, when t_1 and t_2 are slightly perturbed. As mentioned,

$x(t) = X(t) - X(t_1)$ holds for $t_1 \le t \le t_2$. Further, the following **convention** will be used:

for $t' < t < t_1$ and for $t_2 < t < t^{*}$, it is assumed that x(t) = 0, and

$u(t) = selling\ function = \min[U, d(t)]$. Clearly, the company wants to produce and sell, also outside the activity period without creating a stock there.

We look for a derivative of $J = \int_0^T \big(c_1 d(t) - \beta u(t) - hx(t)\big)dt$. To find $\frac{dJ}{dt_1}$ it is sufficient to consider $\int_{t_1}^{t_2(t_1)} \dots$ and to remember the convention near t_1 and t_2. This will be clearly seen below.

Observe that there are no restrictions on x(t) outside the interval $[t', t^{*}]$. In the following derivation nothing happens outside this interval.

Let t_1 be an arbitrary point on (t', t''). The corresponding x-trajectory $(x(t) = X(t) - X(t_1))$ reaches zero at $t_2(t_1)$, but not earlier. The demand near t_1 is $D_1 < U$, and the demand near t_2 is $D_2 > U$. For comparison, let another trajectory start at $t_1 + \delta$, where $\delta > 0$ will soon be sent to zero. The perturbed trajectory reaches zero at time $t_2 - \delta'$. Because of volume conservation these quantities are linked by the simple relation $\delta(U - D_1) = \delta'(D_2 - U) =$ volume perturbation. The “merits” of the trajectories must be compared:

Change of production cost $= \delta \cdot (U - D_1) \cdot \beta > 0$; in favour of perturbed curve.

Change of selling income: $c_1 \cdot \delta' \cdot (D_2 - U) > 0$; in favour of unperturbed curve.

Change of storage cost: $h \cdot \delta \cdot (U - D_1)(t_2 - t_1) + o(\delta) > 0$; in favour of perturbed curve.

It is understood here that during the interval $[t_1, t_1 + \delta]$ it holds $u(t) = D_1$ for the perturbed curve, and $u(t) = U$ for the unperturbed curve. The selling is D_1 for both curves.

During the interval $(t_2 - \delta', t_2)$ the selling is D_2 for the unperturbed curve, and U for the perturbed curve. There is no change of production cost here.

The change of storage cost should be obvious, so the quantities are clear.

Adding things together, we get

$\Delta J = h \cdot \delta \cdot (U - D_1)(t_2 - t_1) + o(\delta) + \delta \cdot (U - D_1) \cdot \beta - c_1 \cdot \delta' \cdot (D_2 - U)$.

Dividing by δ and sending it to zero gives

$\frac{dJ}{dt_1^i} = h \cdot (U - D_1)(t_2 - t_1) + \beta(U - D_1) - c_1 \cdot \frac{\delta'}{\delta} \cdot (D_2 - U)$,

where the i sign (inner) indicates that this is a one-sided derivative. From above we have the relation $\frac{\delta'}{\delta} = \frac{U - D_1}{D_2 - U}$. Inserting this into the above formula, we find, as was claimed

$\frac{dJ}{dt_1^i} = (U - D_1)[h(t_2 - t_1) - (c_1 - \beta)]$.

Observe that $\frac{dJ}{dt_1^i} > 0$ if $(t_2 - t_1) >$ the intrinsic length l_0; it is expected.

Corollary 1: It is now clear that a corresponding "outer" derivative can be derived in a completely analogous way. Let $t_1 \in (t', t'']$. The result is:

$\frac{dJ}{dt_1^o} = (U - D_1)[h(t_2 - t_1) - (c_1 - \beta)]$.

Corollary 2: Assume that $t' < t_1 < t''$. Then the derivatives are clearly equal and continuous in a neighbourhood of t_1.

Let t_1 vary over (t', t''). Then t_2 is a linear function of t_1, and $\frac{dJ}{dt_1} = (U - D_1)[h(t_2 - t_1) - (c_1 - \beta)]$. Thus, $J(t_1) \in C^1(t', t'')$, and a second derivative is easily computed. We have already $\frac{dt_2}{dt_1} = \frac{D_1 - U}{D_2 - U} < 0$, and a simple calculation gives

$\frac{d^2J}{dt_1{}^2} = -(U - D_1) \cdot h \cdot \frac{D_2 - D_1}{D_2 - U} = \text{const.} < 0$.

Thus, J is just a polynomial of degree 2, with negative leading coefficient, as long as $t_1 \in (t', t'')$.

It seems suitable to finish this section by two simple results:

Theorem 3

A true activity period cannot last longer than the intrinsic length l_0.

Proof: The inner derivative is always well defined and given by

$\frac{dJ}{dt_1^i} = (U - D_1)[h(t_2 - t_1) - (c_1 - \beta)]$. This is enough.

Theorem 4

Let (t_1, t_2) be any true activity period. Assume that $\mathrm{d(t)} < \mathrm{U}$ on some interval immediately to the left of t_1 and $\mathrm{d(t)} > \mathrm{U}$ on some interval immediately to the right of t_2.

Then $t_2 - t_1 = l_0 =$ the intrinsic length.

Proof: Look at the outer derivative! Because of the assumptions concerning d(t), the endpoints are free to move (no adjacent activity periods!) so that the interval can be expanded. If $t_2 - t_1$ were less than l_0 , then this would imply a negative outer derivative. Then a slight expansion would give an improvement, contradicting optimality. Together with Theorem 3, this completes the proof.

Remark: This argument is correct also if d(t) should happen to be discontinuous at $t = t_1$. Just approximate t_1 from the left and perform a simple limiting procedure.

5. Solving Problems with a Seasonal Demand

For problems where the demand function has a simple seasonal structure it is possible to use our preceding results to design a rather simple computational method to find an optimal solution. The idea is to exploit some helpful monotonicity properties of the guiding function. It is assumed that the demand for the product is low in the first part of the basic interval [0,T], then higher in the middle part and then low again.

To be concrete, let the basic interval [0,T] be divided into open sub-intervals $I_k, k = 1,2, \ldots, M + N + P$, such that d(t) is constant and non-negative on each sub-interval. It is further assumed that d(t) < U on I_k for $k = 1, \ldots, M$; d(t) > U on the following N intervals and finally $d(t) < U$ on the last P intervals. Write $I_k = (t_k^-, t_k)$ for all k. Clearly, X(t) is strictly increasing on $[0, t_M]$, strictly decreasing on $[t_M, t_{M+N}]$, and strictly increasing on $[t_{M+N}, t_{M+N+P}]$. Consider X(t) on the interval $[0, t_{M+N}]$. It has a strict maximum at $t = t_M$. Further, t_M is a check-point and $x_0(t_M) > 0$ for any optimal $x_0(\cdot)$, as proved above. Obviously, there exists an activity period, starting at some t_0 on the interval $[0, t_M)$. Let it be represented by $x_0(\cdot)$. Write $T = t_{M+N+P}$.

Consider now the seasonal problem under the end condition x(T) = 0.

In this case the activity period must be a "true" one, and so Theorem 1 is applicable, according to which the active trajectory $x_0(\cdot)$ must be a restriction of the guiding function (+ a constant), as long as $x_0(t) > 0$. But X(t) is strictly increasing on $[t_{M+N}, t_{M+N+P}]$, which means that $x_0(t') = 0$ must hold for some $t' \leq t_{M+N}$. Thus, $x_0(t) > 0$ on the interval $t_0 \leq t \leq t'$ and $x_0(t) = 0$ for $t \geq t'$. The problem is that t_0 and t′ are both unknown.

1. Assume to begin with that $X(t_{M+N}) \leq 0 = X(0)$. (The notationally simpler case.)

Put $t^* = \min \{t; t > 0,\ X(t) \leq 0\}$. Thus, $t_{M+N} \geq t^* > t_M$ and

$X(t^*) = 0$. Because of the nice continuity and monotonicity properties of X(t) on the interval $[0, t^*]$, the relation $X(t_1) = X(t_2)$ defines a piece-wise linear strictly monotone mapping $t_1 \to t_2$, for $t_1 \in [0, t_M]$, suitably stored by the computer. <u>The basic interval for optimization and identification of t_0 is now</u> $[0, t_M]$. The idea of the construction is evidently that the wanted function element $x_0(\cdot)$, on $[t_0, t']$ is imbedded and "trapped" in a one-parameter family of restrictions of X(t); then to be identified from its optimizing property. It will be carried out in some detail below, after a short digression on case 2. Note that the one-to-one correspondence $t_1 \leftrightarrow t_2$ here is the same as in §4, but now expanded considerably.

<u>Definition</u>: a <u>break point</u> is a value for t_1, where d(t) has a discontinuity, or such that d(t) has a discontinuity at $t_2(t_1)$.

There will be a finite number of break points on $[0, t_M]$, easily identified. The number t* and all break points are easily recorded by the computer. The correspondence $t_1 \leftrightarrow t_2$ is linear between break points.

2. Assume now that $X(t_{M+N}) > 0$. Also here, the active period will necessarily be of the form $x_0(t) = X(t) - C > 0$ on some open interval containing t_M. Choosing $C = X(t_{M+N})$ defines an interval which must contain the sought active period. Therefore, put $t^* = \min \{t; t > 0, X(t) = X(t_{M+N})\}$. It is easily computed. Clearly, $0 < t^* \leq t_0 < t_M$, and $X(t^*) = X(t_{M+N})$. Now consider Y(t) := X(t) – X(t*) on the

interval $[t^*, t_{M+N}]$. Then $Y(t^*) = Y(t_{M+N}) = 0$. Further, $Y(t) \geq x_0(t)$ for $t^* \leq t \leq t_{M+N}$. Let $l_0 < t_{M+N} - t^*$

The basic interval for identification of t_0 in this case is $[t^*, t_M]$. The method is the same as in case 1; only notations differ. It will not be carried out in detail here.

We return to the situation and notation in case 1. Clearly, $J(t_1) = \int_0^{t^*} \ldots dt$ is continuous for $t_1 \in [0, t_M]$, and, according to §4, reduces to a polynomial of order two on each of the intervals, written $A_j, j = 1,2, \ldots, J$, between the break points. For each interval A_j there exists a "twin interval" B_j to the right of t_M, defined by the one-to-one correspondence $t_1 \leftrightarrow t_2$. The demand on A_j is denoted $D_{1,j}$ and the demand on B_j is written $D_{2,j}$. Further, $t_2 - t_1$ is a continuous, strictly decreasing function of t_1, and it is readily seen from the formula

$$\frac{dJ}{dt_1} = \left(U - D_{1,j}\right)[h(t_2 - t_1) - (c_1 - \beta)]$$

that this derivative can change sign only once; from + to -. The factor $\left(U - D_{1,j}\right)$ can be discontinuous, but does not change sign (positive) at a break point. Therefore, $J(t_1)$ has a unique maximum on $[0, t_M]$, for identifying the point t_0.

Observe first that $t_1 = t_M$ is an impossible maximum point, since the above derivative would then be negative.

Then observe that for $t_1 = 0$, we get $\frac{dJ}{dt_1} = \left(U - D_{1,1}\right)[ht^* - (c_1 - \beta)]$. Thus, if $t^* > l_0$, then $t_1 = 0$ cannot be optimal, because of $\frac{dJ}{dt_1} > 0$.

If $t^* \leq l_0$ (cheap storing!), then $t_1 = 0$ is optimal, because of $\frac{dJ}{dt_1} \leq 0$ on the whole interval.

So assume now $t^* > l_0$; the more interesting case.

Since all constants involved are known, we can easily sketch a suitable numerical procedure for finding t_0, which also gives the wanted activity period. It seems that the simplest way would be to find t_1 such that $t_2(t_1) - t_1 = l_0$; now a routine matter.

The procedure will be as follows:

1. Identify and list all break points on $[0, t_M]$.

2. For each break point, find $t_2 - t_1$

3. On some sub-interval I* between break points, $(t_2 - t_1) - l_0 =$ will change sign, unless we already have found a zero for this quantity.

4. For this particular sub-interval I*, find the value of

$$\frac{dt_2}{dt_1} - 1 = \frac{D_{1,j} - D_{2,j}}{D_{2,j} - U} = const. < 0.$$ Observe that $D_{1,j} < U$ and $D_{2,j} > U$.

5. Find t_1 on I*, such that $t_2 - t_1 = l_0$. This value of t_1 defines the optimal activity period. So this value of t_1 is the wanted starting point t_0.

6. General Observations for the Case of a Free End-Point

For the moment, we make no assumptions concerning seasonal demand. Consider an optimal trajectory $x_0(\cdot)$, starting from zero at time $t = t_1$. Let $x_0(\cdot)$ be the last active period. No restriction yet concerning the end-point. Theorem 1´ is applicable, according to which the trajectory must be a restriction of the guiding function. But here, in contrast to §5, a final activity period can occur close to t = T , depending on the final “payment” c_0.

Lemma 3

(“Closing” lemma). Let $c_0 \leq \beta$. Then $x_0(T) = 0$.

Proof: Assume that $x_0(T) > 0$. We will look at a derivative of J with respect to the starting point of the trajectory $x_0(\cdot)$, much as in §4. As in §4, the trajectory is imbedded in a family of restrictions (+ constant) of X(t). Let $x_0(t)$ start from zero at $t = t_1 < T$. The demand immediately to the right of t_1 is denoted by $D_1 < U$. Find an inner derivative, as in §4.

For comparison, let another trajectory start at $t_1 + \delta$, where $\delta > 0$ will soon be sent to zero. In other words, the higher production starts δ time units later. Clearly, $x_0(t_1 + \delta) - x_0{}'(t_1 + \delta) = \delta(U - D_1)$, where $x_0{}'(\cdot)$ is the perturbed trajectory. Then, by volume conservation, we also have

$x_0(T) - x_0{}'(T) = \delta(U - D_1)$.

The merits of the trajectories are now compared, using the convention in §4:

Change of production cost: $\cdot\, \delta \cdot (U - D_1) > 0$; in favour of perturbed curve.

Change of storage cost: $h \cdot \delta \cdot (U - D_1)(T - t_1) + o(\delta) > 0$; in favour of perturbed curve.

Change of final payment: $c_0 \cdot \delta \cdot (U - D_1) \geq 0$; in favour of unperturbed curve.

No change of selling near t_1!

Adding things together, we get

$\Delta J = \cdot\, (U - D_1)[h(T - t_1) + \beta - c_0] + o(\delta)$; in favour of perturbed curve.

Thus, $\frac{dJ}{dt_1^i} = (U - D_1)[h(T - t_1) + \beta - c_0]$, where “i” again stands for “inner”. Consequently, if $x_0(\cdot)$ is an optimal trajectory, then $h(T - t_1) + \beta - c_0 \leq 0$, which immediately gives $T - t_1 \leq \frac{1}{h}(c_0 - \beta) \leq 0$. The contradiction proves the lemma.

Observe that this is correct also if d(t) should happen to be discontinuous at t_1.

Assume now that $c_0 > \beta$. Various situations can still occur, but the following is clear:

Lemma 4

Let $c_0 > \beta$ and let $d(t) < U$ near t=T. Then $x_0(T) > 0$ at optimum.

Proof: This is very similar, though not identical, to the proof of Lemma 1. We leave the details to the reader.

Consider again an optimal trajectory $x_0(\cdot)$, starting at $t = t_1$, and such that $x_0(T) > 0$. The argument in the proof of Lemma 3 is still valid and gives $T - t_1 \leq \frac{1}{h}(c_0 - \beta)$. In order to apply an outer derivative, we must assume:

1. $t_1 >$ the left endpoint of the basic interval

2. the demand immediately to the left of t_1 is $D'_1 < U$. ($D'_1 = D_1$ is not needed).

Now an outer derivative can be involved, as before. The result is $\frac{dJ}{dt_1^o} = (U - D_1{'}) \cdot [h(T - t_1) + \beta - c_0]$. The optimality clearly implies that $T - t_1 \geq \frac{1}{h}(c_0 - \beta)$. Combine with the previous result and summarize:

Theorem 5

Consider an optimal trajectory $x_0(\cdot)$, starting from zero at $t = t_1$, and ending with $x_0(T) > 0$.

Then $T - t_1 \leq \frac{1}{h}(c_0 - \beta)$.

If, furthermore, $d(t) < U$ on some interval immediately to the left of t_1, then $T - t_1 = \frac{1}{h}(c_0 - \beta)$.

This result should be compared to Theorem 4.

Observe that no assumptions were made above (in case B) concerning seasonal demand.

7. Seasonal Problem and Free End-Point

Now make the same assumptions on seasonal demand as in the beginning of §5, and free end-point. We are not aiming at a solution of the general case, but rather a case suitable for numerical solution:

Assume first that $\int_0^{t_{M+N}} \big(U - d(s)\big) ds \leq 0$, i.e. $X(t_{M+N}) \leq 0$. In this case, as we already know, $x_0(t_{M+N}) = 0$any optimal $x_0(\cdot)$. (Compare the arguments in the beginning of §5.) This means that there exists a true activity period, which can be determined using the method from §5. Now assume that $c_0 \leq \beta$. Then, as seen from Lemma 3, there is no more activity period, and so the solution is complete. Finally, assume that $c_0 > \beta$. Further, $d(t) < U$ for $t > t_{M+N}$, because of the seasonality assumptions. It now follows from Lemma 4´ that there exists a final a.p. such that $x_0(T) > 0$. For convenience, put

$l_0{}' = \frac{1}{h}(c_0 - \beta)$.

We still have the derivative

$\frac{dJ}{dt_1^i} = (U - D_1)[h(T - t_1) + \beta - c_0]$, and this expression is clearly positive for $t_1 < T - l_0{}'$ and negative for $t_1 > T - l_0{}'$. Clearly, t_1 must be chosen as close as possible to $T - l_0{}'$.

Thus, if $t_{M+N} \leq T - l_0{}'$, then $t_1 = T - l_0{}'$ is optimal.

If $t_{M+N} > T - l_0{}'$, then clearly $t_1 = t_{M+N}$ is optimal. (Note that t_{M+N} is the smallest possible value for t_1.)

Thus, the problem is solved, at least in principle.

It remains to consider the case $X(t_{M+N}) > 0$, seasonal demand, and free end-point.

For this to be interesting, assume that $c_0 > \beta$. Further, $d(t) < U$ for $t > t_{M+N}$, and it follows from Lemma 3 that $x_0(T) > 0$ for any optimal $x_0(\cdot)$. As before, $x_0(t_M) > 0$ is already known. The question is now: does the set

$A := \{t: x_0(t) > 0\}$ consist of one or two components? It is easy to see from examples with suitable values for the parameters that both cases can occur. We are not aiming at a complete analysis here, just a few easy results.

First observation: Assume that $l_0{}' \leq T - t_M$. Then the set A has two separate components, i.e. two activity periods.

Proof: Assume the contrary. Then there is just one activity period, starting at some $t_1 < t_M$. From above we have then the inner derivative

$\frac{dJ}{dt_1^i} = (U - D_1)[h(T - t_1) + \beta - c_0]$, also to be written as

$\frac{dJ}{dt_1^i} = (U - D_1)\, h\, [(T - t_1) - l_0{}']$. But at optimum necessarily

$\frac{dJ}{dt_1^i} \leq 0$, so $(T - t_1) \leq l_0{}'$, i.e. $t_1 \geq T - l_0{}' \geq t_M$. But this contradicts $t_1 < t_M$. In other words: $l_0{}' \leq T - t_M$ simply means that the payment $c_0 - \beta$ is so bad that one should not store stuff from t up to T.

Second observation: Assume that $l_0{}' > T$. Then the set A has only one component, i.e. just one activity period.

Proof: Simply disprove the possibility that $x_0(t_{M+N}) = 0$. This will be enough. So assume that this is the case. It follows from the earlier analysis in §7 that $x_0(t) > 0$ holds for $t_{M+N} < t \leq T$. Put $t_1 = \min\{t: X(t) = X(t_{M+N})\}$.

Clearly, $0 < t_1 < t_M$. Consider $Y(t) := X(t) - X(T_{M+N})$, which is a possible activity period on $t_1 \leq t \leq t_{M+N}$. Clearly, $Y(t) = x_0(t)$ for $t_1 \leq t \leq t_{M+N}$.

Now make a shift of $x_0(t)$ by decreasing the starting point t_1, and consider the outer derivative of J with respect to t_1. According to the analysis in §§ 4 and 5 it will be $\frac{dJ}{dt_1^o} = (U - D_1') \cdot h[(T - t_1) - l_0'] < 0$. This means that slightly decreasing t_1 will give a better value for J and a positive value for $x_0(t_{M+N})$. Contradiction!

In other words: $l_0' > T$ simply means that the payment $c_0 - \beta$ is, after all, so good that it makes sense to store stuff over the basic interval. Compare above!

8. Examples

1. A very simple example with seasonal demand

Using the notation of §5, put M = N = P = 1. This means low season, demand $D_1 < U$; then high season, demand $D_0 > U$; finally low season, demand $D_2 < U$. The length of these periods need not be equal. Keep the notation t_M, t_{M+N}, T. Clearly, the guiding function $X(t)$ is linear and increasing on

$[0, t_M]$, linear and decreasing on $[t_M, t_{M+N}]$, finally linear and increasing on $[t_{M+N}, T]$. As before, t_M is a check-point and $x_0(t_M) > 0$ if $x_0(\cdot)$ is optimal. The sign of $X(t_{M+N})$ depends on the parameters, but let us assume that $X(t_{M+N}) \leq 0$, for simplicity of presentation.

The end condition is $x(T) = 0$. Thus, only one activity period for $x_0(\cdot)$.

As before (§5), we have $t^* = \min\ \{t; t > 0,\ X(t) \leq 0\}$. Thus, $t_{M+N} \geq t^* > t_M$ and $X(t^*) = 0$. Further, the relation $X(t_1) = X(t_2)$ defines a linear strictly monotone mapping $t_1 \to t_2$, for $t_1 \in [0, t_M]$, and where $t_2 \in [t_M, t^*]$.

Assume first that $l_0 \geq t^*$, i.e. it is very cheap to store. It then follows easily from Lemma 2 that the optimal solution $x_0(t)$ is given by $X(t)$ itself for

$0 \leq t \leq\ t^*$, and by zero for $t \geq t^*$. There is a similar argument in §5.

Assume then that $t^* > l_0 > 0$, i.e. not very cheap to store. Then there exists a unique couple t_1, t_2 such that $X(t_1) = X(t_2)$, $0 < t_1 < t_M$, and $t_2 - t_1 = l_0$. Now, as known from §5, the couple t_1, t_2 defines the unique optimal solution, i.e. $x_0(t) > 0$ for $t_1 < t < t_2$, and otherwise $x_0(t) = 0$. Observe that if h is decreased, then l_0 increases and the activity interval increases, as expected.

The important moments are: $0 < t_1 < t_M < t_2 <\ t^* < t_{M+N} < T$. The optimal control $u_0(t)$ and $x_0(t)$ can be specified as follows:

$u_0(t) = D_1$ for $0 < t < t_1$, passivity period; $x_0(t) = 0$;

$u_0(t) = U$ for $t_1 < t < t_2$, activity period ; $x_0(t) > 0$;

$u_0(t) = U$ for $t_2 < t < t_{M+N}$, passivity period; $x_0(t) = 0$;

$u_0(t) = D_2$ for $t_{M+N} < t < T$, passivity period; $x_0(t) = 0$.

The reader should draw a figure illustrating the situation! It is also instructive to look at the variation of the selling over the whole interval [0,T].

Observe that the onset t_1 of high production and of $x_0(t) > 0$ can not occur earlier than $t_M - l_0$. It can come close to $t_M - l_0$ if D_0 is very big. All this is natural in view of the interpretation of l_0.

2. Another example involving seasonal demand.

Consider a producing and selling company over a period of 1 year. It is assumed that the demand function d(t) is highly seasonal. During months 1-8 the demand is step-wise increasing, and during months 9-12 it is step-wise decreasing. The demand is strictly bigger than U during months 5-10, i.e. high season; and strictly less than U the other months. Let d_k denote the demand in interval number k. The process starts with vanishing stock , and must end likewise.

The values are as follows: U = 4,5. On intervals $I_k, k = 1, 2, \dots, 8$ we have $d_k = k$. On intervals $I_k, k = 9, \dots, 12$ we have $d_k = 16 - k$. (The reader should draw a simple figure and fill in the step function d(t) , plus the maximal production level U to see the situation better.)

Further notation: $I_k = (k-1, k)$. In the notation of §5 we therefore have

M = 4, N= 7 and P=1. The value of U, as well as values assigned to d(t) in the intervals are of course artificial, and chosen in order to get simple arguments.

By construction, the demand function has the following simple anti-symmetry property: $U - d_k = d_{9-k} - U \; for \; k = 1,2,3,4$. Consequently,

$$\sum_{k=1}^{4}(U - d_k) = \sum_{k=1}^{4}(d_{9-k} - U) = \sum_{j=5}^{8}(d_j - U) = -\sum_{j=5}^{8}(U - d_j).$$

Thus, $\sum_{i=1}^{8}(U - d_k) = 0$. This implies that $X(8) = 0$. In the notation of §5, this means that $t^* = 8$.

Let $x^*(t) \geq 0$ be a proper solution to the problem. Clearly, $t = 4$ is a check-point and it follows from Lemma 1 that $x^*(4) > 0$. Thus, there exists an activity period containing $t = 4$. It must start in one of the intervals 1,2,3,4.

Theoretically, it may end in any of the intervals number 5- 8, and which one is determined by the parameters of the problem. Note that the important other scalars of the problem c_1, h, β have not yet been involved. Assume until further that the end-point condition is $x^*(T) = 0$. Clearly, then there cannot be any more activity period. Let the activity period end at a point $t_0 \leq 8$. Then, $x^*(t) = 0$ for $t \geq t_0$. The starting point of the activity period must be determined from the optimization conditions, derived in §5. To do so, a value must be assigned to the quantity $l_0 = \frac{1}{h}(c_1 - \beta)$, i.e. the intrinsic length of the problem.

Assume first that $l_0 \geq 8$. In this case it follows from the formula for interior derivative in §4 that the optimal $x^*(t)$ is given by $X(t)$ for $0 \leq t \leq 8$, and $x^*(t) = 0$ for $8 \leq t \leq 12$. Note that the solution is unique.

Then assume that $0 < l_0 < 8$. In this case here exists a unique $t_1 \in (0,4)$ such that $8 - 2t_1 = l_0$. From the analysis in §4 we have

$$\frac{dJ}{dt_1^i} = (U - D_1)[h(t_2 - t_1) - (c_1 - \beta)];$$

it follows that this inner derivative is zero, and moreover t_1 defines the starting point of the optimal activity period. Also here, the solution $x^*(t)$ is unique. It can be written as $x^*(t) = X(t) - X(t_1)$ for $t_1 \leq t \leq 8 - t_1$, and otherwise $x^*(t) = 0$. In terms of the optimal control $u^*(t)$ this means that

$u^*(t) = 4{,}5 \; (= U)$ for $t_1 \leq t \leq 8 - t_1$ and $u^*(t) = \min\big(U, d(t)\big)$ otherwise.

The reader should draw a sketch of the graph of $x^*(t)$! Observe that increasing h means decreasing l_0, i.e. increasing t_1 , which means decreasing activity period, not too surprising!

Finally, consider the case of a free endpoint $x^*(12)$! In the case $c_0 \leq \beta$, we know already that $x^*(t) = 0$ for $t \geq 8$.

So, let $c_0 > \beta$ and consider the “reduced intrinsic length” $l_0' = \frac{1}{h}(c_0 - \beta) > 0$.

According to Lemma 4 there exists a final activity period starting at some t_1^* which must necessarily satisfy $11 \le t_1^* < 12$. To determine t_1^* completely recall the formula from §5: $\frac{dJ}{dt_1^i} = (U - D_1)[h(T - t_1) + \beta - c_0]$. It can now be written as $\frac{dJ}{dt_1^i} = (4{,}5 - 4)\, h\,(T - t_1 - l_0')$. Thus, this expression is positive for $t_1 < T - l_0'$ and negative for $t_1 > T - l_0'$. For optimum, t_1^* must clearly be chosen as close as possible to $T - l_0'$. Consequently, if $11 \le T - l_0'$,

then $t_1^* = T - l_0'$, and otherwise $t_1^* = 11$. The optimal solution is now completely determined and unique.

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