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Energy of Taut Strings Accompanying Wiener Process

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Abstract

Let $W$ be a Wiener process. For $r > 0$ and $T > 0$ let $I_W(T,r)^2$ denote the minimal value of the energy $\int_0^T h'(t)^2 dt$ taken among all absolutely continuous functions $h(\cdot)$ defined on $[0,T]$, starting at zero and satisfying

$$W(t) - r \leq h(t) \leq W(t) + r, \quad 0 \leq t \leq T.$$ 

The function minimizing energy is a taut string, a classical object well known in Variational Calculus, in Mathematical Statistics, and in a broad range of applications. We show that there exists a constant $C \in (0,\infty)$ such that for any $q > 0$

$$\frac{r}{T^{1/2}} I_W(T,r) \xrightarrow{L^q} C, \quad \text{as} \quad \frac{r}{T^{1/2}} \to 0,$$

and for any fixed $r > 0$,

$$\frac{r}{T^{1/2}} I_W(T,r) \xrightarrow{a.s.} C, \quad \text{as} \quad T \to \infty.$$ 

Although precise value of $C$ remains unknown, we give various theoretical bounds for it, as well as rather precise results of computer simulation.

While the taut string clearly depends on entire trajectory of $W$, we also consider an adaptive version of the problem by giving a construction (called Markovian pursuit) of a random function $h(t)$ based only on the values $W(s), s \leq t$, and having minimal asymptotic energy. The solution, i.e. an optimal pursuit strategy, turns out to be related with a classical minimization problem for Fisher information on the bounded interval.

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Introduction

Given a time interval \([0, T]\) and two functional boundaries \(g_1(t) < g_2(t), 0 \leq t \leq T\), the taut string is a function \(h^*\) that for any (!) convex function \(\varphi\) provides minimum for the functional

\[
F_\varphi(h) := \int_0^T \varphi(h'(t)) \, dt
\]

among all absolutely continuous functions \(h\) with given starting and final values and satisfying

\(g_1(t) \leq h(t) \leq g_2(t), \quad 0 \leq t \leq T\).

The list of simultaneously optimized functionals includes energy \(\int_0^T h'(t)^2 \, dt\), variation \(\int_0^T |h'(t)| \, dt\), graph length \(\int_0^T \sqrt{1 + h'(t)^2} \, dt\), etc.

The first instance of taut strings that we have found in the literature is in G. Dantzig’s paper [6]. Dantzig notes there that the problem under study and its solution was discussed in R. Bellman’s seminar at RAND Corporation in 1952. The taut strings were later used in Statistics, see [18] and [7]. In the book [22, Chapter 4, Subsection 4.4], taut strings are considered in connection with problems in image processing. Quite recently, taut strings were applied to a buffer management problem in communication theory, see [23].

In this article, we study the energy of the taut string going through the tube of constant width constructed around sample path of a Wiener process \(W\), i.e. for some \(r > 0\) we let \(g_1(t) := W(t) - r, g_2(t) := W(t) + r\), see Fig. 1.

![Figure 1: A fragment of taut string accompanying Wiener process.](image)

We focus attention on the behavior in a long run: we show that when \(T \to \infty\), the taut string spends asymptotically constant amount of energy \(C^2\) per unit of time. Precise assertions are given in Theorems 1.1 and 1.2 below. The constant \(C\) shows how much energy an absolutely continuous function must spend if it is bounded to stay within a certain distance from the non-differentiable trajectory of \(W\).

Although precise value of \(C\) remains unknown, we give various theoretical bounds for it in Section 4, as well as the results of computer simulation in Section 6. The latter suggest \(C \approx 0.63\).

If we take the pursuit point of view, considering \(h(\cdot)\) as a trajectory of a particle moving with finite speed and trying to stay close to a Brownian particle, then it is much more natural to consider constructions that define \(h(t)\) in adaptive way, i.e. on the base of the known \(W(s), s \leq t\). Recall that the taut string depends on the entire trajectory \(W(s), s \leq T\), hence it does not fit the adaptive setting. In view of Markov property of \(W\), the reasonable pursuit strategy for \(h(t)\) is to move towards \(W(t)\) with the speed depending on the distance \(|h(t) - W(t)|\). In this
class of algorithms we find an optimal one in Section 5. The corresponding function spends in
average $\approx 1.57$ units of energy per unit of time. Comparing of two constants shows that we
have to pay more than double price for not knowing the future of the trajectory of $W$. To our
great surprise, the search of optimal pursuit strategy boils down to the well known variational
problem: minimize Fisher information on the class of distributions supported on a fixed bounded
interval.

One may prove that the provided algorithm is the optimal one in the entire class of adaptive
algorithms but this fact is beyond the scope of the present article.

In Section 7 we establish some connections with other well known settings and problems.

First, we recall that the famous Strassen’s functional law of the iterated logarithm (FLIL)
and its extensions handling convergence rates in FLIL actually deal exactly with the energy of
taut strings. Not surprisingly, we borrowed some techniques for evaluation of this energy from
FLIL research. Yet it should be noticed that FLIL requires very different range of parameters $r$
and $T$ than those emerging in our case. The FLIL tubes are much wider, hence the taut string
energy is much lower than ours. This is why Strassen law with its super-slow loglog rates is
so hard to reproduce in simulations, while our results handling the same type of quantities are
easily observable in computer experiment.

Second, we briefly look at the taut string as a minimizer of variation

$$V(h) := \int_0^T |h'(t)| dt.$$ 

Since $| \cdot |$ is not a strictly convex function, the corresponding variational problem typically has
many solutions. In [15, 19] another minimizer of $V(h)$ is described in detail, a so called lazy
function. As E. Schertzer pointed to us, the relations between the taut strings and lazy functions
are yet to be clarified.

Finally, we briefly describe a discrete analogue of our problem thus giving flavor of eventual
applications.

As a conclusion, Section 8 traces some forthcoming or possible developments of the treated
subject.

1 Notation and main results

Throughout the paper, we consider uniform norms

$$||h||_T := \sup_{0 \leq t \leq T} |h(t)|, \quad h \in \mathbb{C}[0, T],$$

and Sobolev-type norms

$$|h|^2_T := \int_0^T |h'(t)|^2 dt, \quad h \in AC[0, T],$$

where $AC[0, T]$ denotes the space of absolutely continuous functions on $[0, T]$. It is natural to
call $|h|^2_T$ energy.

Let $W$ be a Wiener process. We are mostly interested in its approximation characteristics

$$I_W(T, r) := \inf \{ ||h||_T; h \in AC[0, T], ||h - W||_T \leq r, h(0) = 0 \}$$

3
and
\[ I^0_W(T, r) := \inf \{ |h|_T; h \in AC[0, T], |h - W|_T \leq r, h(0) = 0, h(T) = W(T) \}. \]

The unique functions at which the infima are attained are called \textit{taut string}, resp. \textit{taut string with fixed end}.

Our main results are as follows.

\textbf{Theorem 1.1} There exists a constant \( C \in (0, \infty) \) such that if \( \frac{r}{\sqrt{T}} \to 0 \), then
\[
\frac{r}{T^{1/2}} I_W(T, r) \xrightarrow{L_q} C, \tag{1.1}
\]
\[
\frac{r}{T^{1/2}} I^0_W(T, r) \xrightarrow{L_q} C, \tag{1.2}
\]
for any \( q > 0 \).

We may complete the mean convergence with almost sure convergence to \( C \).

\textbf{Theorem 1.2} For any fixed \( r > 0 \), when \( T \to \infty \), we have
\[
\frac{r}{T^{1/2}} I_W(T, r) \xrightarrow{a.s.} C,
\]
\[
\frac{r}{T^{1/2}} I^0_W(T, r) \xrightarrow{a.s.} C.
\]

\section{Basic properties of \( I_W \) and \( I^0_W \)}

We prepare the proofs of the main results given below in Subsections 3.2 and 3.3 by exploring scaling and concentration properties of the taut string’s energy.

\subsection{Scaling}

Given two functions \( W(t) \) and \( h(t) \) on \([0, T]\), let us rescale them onto the time interval \([0, 1]\) by letting
\[
X(s) := \frac{W(sT)}{\sqrt{T}}, \quad g(s) := \frac{h(sT)}{\sqrt{T}}, \quad 0 \leq s \leq 1.
\]

Then
\[
||g - X||_1 = \frac{||h - W||_T}{\sqrt{T}}
\]
and
\[
|g|^2_1 = \int_0^1 |g'(s)|^2 ds = \int_0^1 \left( \frac{h'(sT)}{\sqrt{T}} \right)^2 ds = T \int_0^1 h'(sT)^2 ds = \int_0^T h'(t)^2 dt = |h|^2_T.
\]

The boundary conditions are also transformed properly: namely, \( h(0) = W(0) \) is equivalent to \( g(0) = X(0) \), while \( h(T) = W(T) \) is equivalent to \( g(1) = X(1) \). Therefore, \( h \) belongs to the set \( \{ h : h \in AC[0, T], ||h - W||_T \leq r, h(0) = 0 \} \) iff \( g \) belongs to the analogous set \( \{ g : g \in AC[0, 1], ||g - X||_1 \leq \frac{r}{\sqrt{T}}, g(0) = 0 \} \).

\footnote{In order to justify unicity, notice that the quadratic function \( | \cdot |^2_1 \) is \textit{strictly} convex on the hyperplane \( \{ h \in AC[0, T]; h(0) = 0 \} \); any strictly convex function attains its minimum on a convex set at most at one point.}
Recall that if $W$ is a Wiener process on $[0, T]$, then $X(s) := \frac{W(sT)}{\sqrt{T}}$ is a Wiener process on $[0, 1]$. We conclude that

$$I_W(T, r) \overset{d}{=} I_W\left(1, \frac{r}{\sqrt{T}}\right).$$

Similarly,

$$I_W^0(T, r) \overset{d}{=} I_W^0\left(1, \frac{r}{\sqrt{T}}\right). \quad (2.1)$$

Therefore, assertions (1.1) and (1.2) may be rewritten in a one-parameter form

$$\varepsilon I_W(1, \varepsilon) \overset{L_q}{\to} C, \quad \text{as } \varepsilon \to 0, \quad (2.2)$$

$$\varepsilon I_W^0(T, \varepsilon) \overset{L_q}{\to} C, \quad \text{as } \varepsilon \to 0.$$ 

### 2.2 Finite moments

We will show now that both $I_W(T, r)$ and $I_W^0(T, r)$ have finite exponential moments. Yet in the following we only need that

$$D(T, r) := \mathbb{E} I_W(T, r)^2 < \infty, \quad (2.3)$$

$$D^0(T, r) := \mathbb{E} I_W^0(T, r)^2 < \infty. \quad (2.4)$$

Let $v$ be an even integer. Then $\delta := \frac{2}{v}$ is inverse to an integer, and we may cut the time interval $[0, 1]$ into $\delta^{-1}$ intervals of length $\delta$. Let $W_\delta$ be the linear interpolation of $W$ based on the knots $(j\delta, W(j\delta)), 0 \leq j \leq \delta^{-1}$. Clearly, we have either $||W_\delta - W||_1 > r$ or $I_W^0(1, r) \leq ||W_\delta||_1$. It follows that

$$P\left(I_W^0(1, r)^2 > v\right) \leq P\left(||W_\delta - W||_1 > r\right) + P\left(||W_\delta||_1^2 > v\right). \quad (2.5)$$

Notice that

$$||W_\delta - W||_1 = \max_{0 \leq t \leq 1} |W_\delta(t) - W(t)|$$

$$= \max_{0 \leq t < \delta^{-1}} \max_{0 \leq \delta \leq (j+1) \delta} |W_\delta(t) - W(t)|$$

$$= \max_{0 \leq t < \delta^{-1}} \sqrt{\delta} \max_{0 \leq t \leq 1} |B_j(t)|, \quad (2.6)$$

where $(B_j)$ are independent Brownian bridges, and

$$||W_\delta||_1^2 = \int_0^1 W_\delta(t)^2 dt$$

$$= \delta^{-1} \sum_{0 \leq j < \delta^{-1}} (W((j+1)\delta) - W(j\delta))^2 = \sum_{0 \leq j < \delta^{-1}} \eta_j^2, \quad (2.7)$$

where $(\eta_j)$ are i.i.d. standard normal random variables.

Now we may evaluate the probabilities in (2.5). By using (2.6), we obtain

$$P\left(||W_\delta - W||_1 > r\right) \leq \delta^{-1} P\left(||B||_1 > \frac{r}{\delta}\right) \leq \delta^{-1} P\left(||W||_1 > \frac{r}{\delta}\right)$$

$$\leq \frac{v}{2} \cdot 2 \cdot \exp\left(-\frac{r^2}{2\delta}\right) = v \exp(-r^2v/4).$$
On the other hand, by using Cramér–Chernoff theorem and (2.7),

\[ P \left( |W_2|_1^2 > v \right) = P \left( \sum_{0 \leq j < v/2} \eta_j^2 > v \right) \leq \exp \{-c_1 v\} \]

for all \( v \) and some universal constant \( c_1 \). It follows that

\[ P \left( I_W^0(1, r)^2 > v \right) \leq v \exp(-r^2 v/4) + \exp \{-c_1 v\}. \]

Hence,

\[ E \exp(c I_W(T, r)^2) < \infty, \]

whenever \( 0 < c < \min\{\frac{r^2}{4}, c_1\} \). By scaling we also have

\[ E \exp(c I_W^0(T, r)^2) < \infty, \]

for any \( r, T > 0 \) and sufficiently small positive \( c \). It follows from the definitions that

\[ I_W(T, r) \leq I_W^0(T, r) \quad \forall \ T, r > 0. \quad (2.8) \]

Hence, the exponential moment of \( I_W(T, r) \) is finite, too.

### 2.3 Relations between \( I_W \) and \( I_W^0 \)

We already noticed in (2.8) that \( I_W(T, r) \leq I_W^0(T, r) \). We will show now that a kind of converse estimate is also true.

**Proposition 2.1** For all positive \( T, r, \delta \) it is true that

\[ E I_W(T, r)^2 \geq E I_W^0(T + 1, r + \delta)^2 - E I_W^0(1, \delta)^2 - r^2. \quad (2.9) \]

**Proof:** Let us fix for a while the time interval \([0, 1]\) and let us approximate the trajectory of Wiener process \( W \) by functions starting from some arbitrary point \( \rho \in \mathbb{R} \). Let \( \delta > 0 \) and let \( h(\cdot) \) be the taut string with fixed end at which \( I_W^0(1, \delta) \) is attained. Then we have \( h(0) = 0, h(1) = W(1), ||h - W||_1 \leq \delta, |h|_1 = I_W^0(1, \delta) \). Let

\[ H(t) := \rho + h(t) - \rho t, \quad 0 \leq t \leq 1. \]

Then \( H(0) = \rho + h(0) = \rho, H(1) = \rho + h(1) - \rho = h(1) = W(1), ||H - W||_1 \leq ||h - W||_1 + |\rho| \max_{0 \leq t \leq 1} |1 - t| \leq \delta + |\rho|, \quad (2.10) \]

and

\[ |H|_1^2 = \int_0^1 H'(t)^2 dt = \int_0^1 (h'(t) - \rho)^2 dt \]
\[ = \int_0^1 h'(t)^2 dt + \rho^2 - 2\rho \int_0^1 h'(t) dt \]
\[ = |h|_1^2 + \rho^2 - 2\rho(h(1) - h(0)) = I_W^0(1, \delta)^2 + \rho^2 - 2\rho W(1). \quad (2.11) \]

Now we pass to the lower bound for \( I_W(T, r) \). Let us fix \( r, \delta, T \) and produce an approximation for \( W \) on \([0, T + 1]\) with the fixed end. First, let \( \hat{h}(t), 0 \leq t \leq T, \) be the taut string at which
\(I_W(T, r)\) is attained. The end point is not fixed, thus \(\rho := \hat{h}(T) - W(T)\) need not vanish. Nevertheless we still have
\[
|\rho| \leq |\hat{h} - W|_T \leq r.
\]
Now we approximate the auxiliary Wiener process
\[
\hat{W}_T(s) := W(T + s) - W(T), \quad 0 \leq s \leq 1,
\]
by the function \(H(\cdot)\) defined above and let
\[
\hat{h}(T + s) := W(T) + H(s), \quad 0 \leq s \leq 1.
\]
At the boundary point \(T\) the first definition yields the value \(\hat{h}(T) = W(T) + \rho\), the second definition yields \(\hat{h}(T) := W(T) + H(0)\); the two values coincide by the definition of function \(H\). Moreover,
\[
\hat{h}(T + 1) = W(T) + H(1) = W(T) + \hat{W}_T(1) = W(T) + W(T + 1) - W(T) = W(T + 1).
\]
Therefore, the extended function \(\hat{h}(\cdot)\) provides an absolutely continuous approximation with fixed end to \(W\) on \([0, T + 1]\). Furthermore, by (2.10) for \(0 \leq s \leq 1\) we have
\[
|W(T + s) - \hat{h}(T + s)| = |\hat{W}_T(s) + W(T) - W(T) - H(s)| = |\hat{W}_T(s) - H(s)| \leq \delta + |\rho| \leq \delta + r.
\]
Finally, by (2.11),
\[
\int_{T}^{T+1} \hat{h}'(t)^2 dt = \int_{0}^{1} \hat{H}'(s)^2 ds = |H|^2 = I_{W_T}(1, \delta)^2 + \rho^2 - 2\rho\hat{W}_T(1).
\]
We conclude that
\[
I_W^0(T + 1, r + \delta)^2 \leq \hat{h}|_{T+1}^2
\]
\[
= \hat{h}|_T^2 + \int_{T}^{T+1} \hat{h}'(t)^2 dt \leq I_W(T, r)^2 + I_{W_T}^0(1, \delta)^2 + r^2 - 2\rho\hat{W}_T(1)
\]  
and turn this relation into the desired bound
\[
I_W(T, r)^2 \geq I_W^0(T + 1, r + \delta)^2 - I_W^0(1, \delta)^2 - r^2 + 2\rho\hat{W}_T(1).
\]
Notice that \(\rho\) and \(\hat{W}_T(1)\) are independent and \(E\hat{W}_T(1) = 0\). By taking expectations we get the desired relation (2.9).  

**2.4 Concentration**

We first notice an almost obvious Lipschitz property of the functionals under consideration.

**Proposition 2.2** For any \(T, r > 0\), any \(w \in C[0, T]\), \(g \in AC[0, T]\) we have
\[
|I_{w+g}^0(T, r) - I_{w}^0(T, r)| \leq |g|_T \tag{2.13}
\]
and
\[
|I_{w+g}(T, r) - I_{w}(T, r)| \leq |g|_T. \tag{2.14}
\]
It is remarkable that the Lipschitz constant in the right hand side does not depend on $r$ and $T$.

**Proof:** Let $h$ be the taut string at which $I^0_W(T, r)$ is attained. Then the function $\hat{h} := h + g$ satisfies the boundary conditions $\hat{h}(0) = (w + g)(0)$, $\hat{h}(T) = (w + g)(T)$ as well as

$$||\hat{h} - (w + g)||_T = ||(h + g) - (w + g)||_T = ||h - w||_T \leq r.$$

Therefore,

$$I^0_{w+g}(T, r) \leq \hat{h}|T| \leq |h|_T + |g|_T = I^0_w(T, r) + |g|_T.$$

By applying the latter inequality to $\tilde{w} := w + g$ and $\tilde{g} := -g$ in place of $w$ and $g$ we obtain

$$I^0_w(T, r) \leq I^0_{w+g}(T, r) + |g|_T,$$

and (2.13) follows. The proof of (2.14) is exactly the same. \qed

In the rest of the subsection parameters $T$ and $r$ are fixed, and we drop them from our notation, thus writing $I^0_W$ instead of $I^0_W(T, r)$, etc. Let $m^0$ be a median for the random variable $I^0_W$. The famous concentration inequality for Lipschitz functionals of Gaussian random vectors (see [17, Theorem 6.2 and Example 4.4]) asserts that for any $\rho > 0$

$$\mathbb{P}(I^0_W \geq m^0 + \rho) \leq \mathbb{P}(N \geq \rho),$$

$$\mathbb{P}(I^0_W \leq m^0 - \rho) \leq \mathbb{P}(N \geq \rho),$$

where $N$ is a standard normal random variable.

It follows that

$$VarI^0_W = \inf_y \mathbb{E}((I^0_W - y)^2) \leq \mathbb{E}(I^0_W - m^0)^2 = 2 \int_0^\infty \rho \mathbb{P}(|I^0_W - m^0| \geq \rho) \, d\rho$$

$$\leq 2 \int_0^\infty \rho \mathbb{P}(|N| \geq \rho) \, d\rho = \mathbb{E}|N|^2 = 1.$$ 

Moreover,

$$|\mathbb{E} I^0_W - m^0| \leq \mathbb{E}|I^0_W - m^0| \leq \sqrt{\mathbb{E}(I^0_W - m^0)^2} \leq 1$$

and

$$\mathbb{E} I^0_W \leq \sqrt{\mathbb{E}[(I^0_W)^2]} = \sqrt{[\mathbb{E} I^0_W]^2 + VarI^0_W} \leq \sqrt{[\mathbb{E} I^0_W]^2 + 1} \leq I^0_W + 1.$$ 

Finally, we infer

$$m^0 - 1 \leq \sqrt{\mathbb{E}[(I^0_W)^2]} \leq m^0 + 2. \quad (2.15)$$

We will also need that for any $q > 0$

$$\mathbb{E}|I^0_W - m^0|^q = q \int_0^\infty \rho^{q-1} \mathbb{P}(|I^0_W - m^0| \geq \rho) \, d\rho$$

$$\leq q \int_0^\infty \rho^{q-1} \mathbb{P}(|N| \geq \rho) \, d\rho = \mathbb{E}|N|^q. \quad (2.16)$$

Similarly, for the median $m$ of $I_W$ we obtain

$$m - 1 \leq \sqrt{\mathbb{E}[(I_W)^2]} \leq m + 2$$

and

$$\mathbb{E}|I_W - m|^q \leq \mathbb{E}|N|^q. \quad (2.17)$$

8
3 Asymptotics

3.1 Asymptotics of the second moments and medians

Recall that $D(T, r) := \mathbb{E} I_W(T, r)^2$ and $D^0(T, r) := \mathbb{E} I^0_W(T, r)^2$ are the second moments. We prove the following.

**Proposition 3.1** There exists a constant $C \in [0, \infty)$ such that if $r \sqrt{T} \to 0$, then

$$\frac{r^2}{T} D(T, r) \to C^2, \quad (3.1)$$

$$\frac{r^2}{T} D^0(T, r) \to C^2. \quad (3.2)$$

**Proof:** In proving (3.1), the following sub-additivity property plays the key role. For any $r, T_1, T_2 > 0$ we have

$$I_W(T_1 + T_2, r)^2 \leq I^0_W(T_1, r)^2 + I^0_{W_T}(T_2, r)^2, \quad (3.3)$$

where $\tilde{W}_{T_1}(s) := W(T_1 + s) - W(T_1)$ is a Wiener process. This means that we may approximate $W$ by taut strings with fixed ends separately on the intervals $[0, T_1]$ and $[T_1, T_1 + T_2]$ by gluing them at $T_1$ due to the fixed end condition imposed on the first string.

Notice that $I_W(\cdot, r)$ does not possess such a nice subadditivity property.

By taking expectations in (3.3), we obtain

$$D^0(T_1 + T_2, r) \leq D^0(T_1, r) + D^0(T_2, r). \quad (3.4)$$

Since $I^0_W(1, \varepsilon)$ is a decreasing random function w.r.t. argument $\varepsilon$, the function $D^0(1, \varepsilon) = \mathbb{E} I^0_W(1, \varepsilon)^2$ is also decreasing in $\varepsilon$. By the scaling argument (2.1) we observe that for any fixed $r > 0$

$$D^0(T, r) = D^0(1, \frac{r}{\sqrt{T}}) \quad (3.5)$$

is an increasing function w.r.t. the argument $T$.

Fix any $T_0 > 0$. By using monotonicity of $D^0(T, r)$ in $T$ and iterating subadditivity (3.4) we obtain

$$\limsup_{T \to \infty} \frac{D^0(T, r)}{T} = \limsup_{T \to \infty} \max_{0 \leq \tau \leq T_0} \frac{D^0(kT_0 + \tau, r)}{kT_0 + \tau} \leq \limsup_{k \to \infty} \frac{D^0((k + 1)T_0, r)}{kT_0} \leq \lim_{k \to \infty} \frac{(k + 1)D^0(T_0, r)}{kT_0} = \frac{D^0(T_0, r)}{T_0}.$$

By optimizing over $T_0$ we find

$$\limsup_{T \to \infty} \frac{D^0(T, r)}{T} \leq \inf_{T > 0} \frac{D^0(T, r)}{T} \leq \liminf_{T \to \infty} \frac{D^0(T, r)}{T}.$$

It follows that there exists a finite limit

$$\lim_{T \to \infty} \frac{D^0(T, r)}{T} = \inf_{T > 0} \frac{D^0(T, r)}{T} := C_r.$$
By using the scaling (3.5) we find the limit
\[
C^2 := \lim_{\varepsilon \to 0} \varepsilon^2 D^0(1, \varepsilon) = r^2 \lim_{T \to \infty} \frac{D^0(T, r)}{T} = C_r r^2.
\]

Now the relation (3.2) with varying \( r \) follows by another application of the scaling argument (3.5).

Now we pass to (3.1). For fixed \( r > 0 \) it follows from (2.8) and (3.2) that
\[
\limsup_{T \to \infty} \frac{D(T, r)}{T} \leq \lim_{T \to \infty} \frac{D^0(T, r)}{T} = C_r = C^2 r^{-2}.
\]

Conversely, for any fixed \( \delta > 0 \) it follows from (2.9) that
\[
\liminf_{T \to \infty} \frac{D(T, r)}{T} \geq \lim_{T \to \infty} \frac{D^0(T, r + \delta)}{T} = C^2 (r + \delta)^{-2}.
\]

By letting \( \delta \to 0 \) we infer
\[
\lim_{T \to \infty} \frac{D(T, r)}{T} = C^2 r^{-2},
\]
which is (3.1) for fixed \( r \). The case of varying \( r \) in (3.1) follows by the same scaling arguments as above. \( \square \)

Remark: We will show later in Subsection 4.3 that \( C > 0 \).

We may complete the convergence of second moments with convergence of medians.

**Corollary 3.2** Let \( m^0(T, r) \), resp. \( m(T, r) \), be a median of \( I^0_W(T, r) \), resp. \( I_W(T, r) \). If \( \frac{r}{\sqrt{T}} \to 0 \), then
\[
\frac{r}{\sqrt{T}} m(T, r) \to C, \quad \text{(3.6)}
\]
\[
\frac{r}{\sqrt{T}} m^0(T, r) \to C. \quad \text{(3.7)}
\]

**Proof:** Indeed (2.15) writes as
\[
m^0(T, r) - 1 \leq \sqrt{D^0(T, r)} \leq m^0(T, r) + 2.
\]

Therefore, (3.7) follows immediately from (3.2). Relation (3.6) follows from (3.1) in the same way. \( \square \)

### 3.2 \( L_q \)-convergence

**Proof of Theorem 1.1.** Let \( q > 0 \). We have to prove that if \( \frac{r}{\sqrt{T}} \to 0 \), then
\[
\frac{r}{\sqrt{T}} I_W(T, r) \overset{L_q}{\to} C, \quad \text{(3.8)}
\]
\[
\frac{r}{\sqrt{T}} I^0_W(T, r) \overset{L_q}{\to} C. \quad \text{(3.9)}
\]
In view of (3.7) the proof of (3.9) reduces to
\[ \frac{r}{\sqrt{T}} (I^0_W(T, r) - m^0(T, r)) \to_L 0. \]
Indeed by (2.16) we have
\[ \left( \frac{r}{\sqrt{T}} \right)^q \mathbb{E} |I^0_W(T, r) - m^0(T, r)|^q \leq \left( \frac{r^2}{T} \right)^{q/2} \mathbb{E} |N|^q \to 0 \]
and (3.9) follows.
Relation (3.8) follows from (3.6) and (2.17) in the same way. □

3.3 Almost sure convergence

**Proof of Theorem 1.2.** For any fixed \( r > 0 \), when \( T \to \infty \), we must prove
\[ \frac{r}{T^{1/2}} I_W(T, r) \xrightarrow{a.s.} C, \quad (3.10) \]
\[ \frac{r}{T^{1/2}} I^0_W(T, r) \xrightarrow{a.s.} C. \quad (3.11) \]

Consider first an exponential subsequence \( T_k := a^k \) with arbitrary fixed \( a > 1 \). By moment estimate (2.17) and Chebyshev inequality, for any \( \varepsilon > 0 \) we have
\[
\sum_{k=1}^{\infty} \mathbb{P} \left( T_k^{-1/2} |I_W(T_k, r) - m(T_k, r)| > \varepsilon \right) \leq \varepsilon^{-q} \sum_{k=1}^{\infty} T_k^{-q/2} \mathbb{E} |I_W(T_k, r) - m(T_k, r)|^q \\
\leq \varepsilon^{-q} \mathbb{E} |N|^q \sum_{k=1}^{\infty} T_k^{-q/2} < \infty.
\]
Borel–Cantelli lemma yields
\[ \lim_{k \to \infty} T_k^{-1/2} (I_W(T_k, r) - m(T_k, r)) = 0 \quad \text{a.s.} \]
Taking the convergence of medians (3.6) into account, we obtain
\[ \lim_{k \to \infty} \frac{r}{T_k^{1/2}} I_W(T_k, r) = C \quad \text{a.s.} \]
Since the function \( I_W(\cdot, r) \) is non-decreasing, for any \( T \in [T_k, T_{k+1}] \) we have the chain
\[ \frac{r I_W(T_k, r)}{(a T_k)^{1/2}} = \frac{r I_W(T_k, r)}{T_k^{1/2}} \leq \frac{r I_W(T, r)}{T_k^{1/2}} \leq \frac{r I_W(T_{k+1}, r)}{T_{k+1}^{1/2}} = \frac{r I_W(T_{k+1}, r)}{(T_{k+1}/a)^{1/2}}. \]
It follows that
\[ a^{-1/2} C \leq \liminf_{T \to \infty} \frac{r}{T^{1/2}} I_W(T, r) \leq \limsup_{T \to \infty} \frac{r}{T^{1/2}} I_W(T, r) \leq a^{1/2} C \quad \text{a.s.} \]
By letting \( a \downarrow 1 \) we obtain (3.10).

The inequality (2.8) yields now the lower bound in (3.11), namely,
\[ \lim_{T \to \infty} \frac{r}{T^{1/2}} I^0_W(T, r) \geq \lim_{T \to \infty} \frac{r}{T^{1/2}} I_W(T, r) = C \quad \text{a.s.} \]
The proof of the upper bound in (3.11) requires more efforts because the monotonicity of $I^0_W(\cdot, r)$ is missing. By using (2.12), for any $r > 0, \delta > 0$ we have
\[
\limsup_{T \to \infty} \frac{r^2}{T} I^0_W(T + 1, r + \delta)^2 \leq \limsup_{T \to \infty} \frac{r^2}{T} \left( I_W(T, r)^2 + I^0_{W_T}(1, \delta)^2 + 2r|\tilde{W}_T(1)| \right)
\leq C^2 + \limsup_{T \to \infty} \frac{r^2}{T} I^0_{W_T}(1, \delta)^2 + 2r \limsup_{T \to \infty} \frac{r^2}{T} |\tilde{W}_T(1)|.
\]
Now we show that both remaining limits vanish. Indeed, It is well known that
\[
\limsup_{T \to \infty} \frac{|\tilde{W}_T(1)|}{\sqrt{2 \ln T}} = \limsup_{T \to \infty} \frac{|W(T + 1) - W(T)|}{\sqrt{2 \ln T}} = 1,
\]
hence,
\[
\limsup_{T \to \infty} \frac{|\tilde{W}_T(1)|}{T} = 0.
\]
Now write
\[
\limsup_{T \to \infty} \frac{I^0_{W_T}(1, \delta)^2}{T} = \limsup_{k \to \infty} \sup_{k \leq T \leq k+1} \frac{I^0_{W_T}(1, \delta)^2}{k} := \limsup_{k \to \infty} V_k(W)^2/k,
\]
where $V_k(W)$ are identically distributed random variables satisfying Lipschitz condition due to (2.13). Let $m$ be the common median of $V_k$. By concentration inequality it follows that for any $x > 0$
\[
P\{V_k(W) > m + x\} \leq P(N > x) \leq \exp\{-x^2/2\}.
\]
Borel–Cantelli lemma yields now that
\[
\limsup_{k \to \infty} \frac{V_k(W)}{\sqrt{2 \ln k}} \leq 1,
\]
Hence,
\[
\limsup_{k \to \infty} \frac{V_k(W)^2}{k} = 0.
\]
We conclude that
\[
\limsup_{T \to \infty} \frac{r^2}{T} I^0_W(T + 1, r + \delta)^2 \leq C^2
\]
and by letting $\delta \to 0$ we are done with proving upper bound in (3.11). □

4 Quantitative estimates and algorithms

In this section we provide several theoretical lower and upper bounds for $C$.

4.1 Isoperimetric and small deviation bounds

This subsection closely follows the ideas of Griffin and Kuelbs [9]. Let $c > 0$. Then for any $\varepsilon > 0$ we have
\[
P(\varepsilon I_W(1, \varepsilon) \geq c) = P(I_W(1, \varepsilon) \geq c \varepsilon^{-1}) = P(W \notin \varepsilon U + c \varepsilon^{-1} K)
\]
where $U := \{ x : \| x \|_1 \leq 1 \}$ and $K := \{ h : |h|_1 \leq 1 \}$. According to the Gaussian isoperimetric inequality (cf. [1, 25], or e.g. [16, Section 11]),

$$
\mathbb{P}\left( W \notin \varepsilon U + c \varepsilon^{-1} K \right) \leq 1 - \Phi\left( c \varepsilon^{-1} + \Phi^{-1}\left( \mathbb{P}(W \in \varepsilon U) \right) \right),
$$

where $\Phi(\cdot)$ is the distribution function of the standard normal law. It is well known that $\Phi^{-1}(p) \sim -\sqrt{2\ln p}$, as $p \to 0$. On the other hand, by the classical small deviation estimate, following from the Petrovskii formula of the distribution of $\| W \|_1$ (cf. [20] or e.g. [16, Section 18])

$$
\ln \mathbb{P}(W \in \varepsilon U) = \ln \mathbb{P}(\| W \|_1 \leq \varepsilon) \sim -\frac{\pi^2}{8} \varepsilon^{-2}, \quad \text{as } \varepsilon \to 0.
$$

Hence,

$$
\Phi^{-1}(\mathbb{P}(W \in \varepsilon U)) \sim -\frac{\pi}{2} \varepsilon^{-1}, \quad \text{as } \varepsilon \to 0.
$$

It follows that

$$
\mathbb{P}\left( W \notin \varepsilon U + c \varepsilon^{-1} K \right) \leq 1 - \Phi\left( c \varepsilon^{-1} - \frac{\pi}{2} \varepsilon^{-1}(1 + o(1)) \right) \to 0, \quad \text{as } \varepsilon \to 0,
$$

whenever $c > \frac{\pi}{2}$. Since $\varepsilon I_W(1, \varepsilon) \xrightarrow{P} C$ by (2.2), we end up with the bound

$$
C \leq \frac{\pi}{2}.
$$

### 4.2 Free knot approximation: constructive approach

Here we provide a more constructive approach to building strings having the right order of energy and properly approximating Wiener process. Let $\varepsilon > 0$ and let $W(s), 0 \leq s \leq 1$, be a Wiener process. Consider a sequence of stopping times $\tau_j$ defined by $\tau_0 := 0$ and

$$
\tau_j := \inf\{ t \geq \tau_{j-1} : |W(t) - W(\tau_{j-1})| \geq \varepsilon/2 \}, \quad j \geq 1.
$$

By continuity of $W$ we clearly have

$$
|W(\tau_j) - W(\tau_{j-1})| = \frac{\varepsilon}{2}.
$$

Let $g(\cdot)$ be the linear interpolation of $W(\cdot)$ built upon the knots $(\tau_j, W(\tau_j))$. We stress that the knots are random, since they depend on the process trajectory $W(\cdot)$. This randomness is typical for free knot approximation, cf. [4, 5].

We have a good approximation of $W$ by $g$ in the uniform norm, since for any $t \in [\tau_{j-1}, \tau_j]$ it is true that

$$
|W(t) - W(\tau_{j-1})| \leq \frac{\varepsilon}{2},
$$

$$
|g(t) - W(\tau_{j-1})| \leq |W(\tau_j) - W(\tau_{j-1})| = \frac{\varepsilon}{2},
$$

hence

$$
||g - W||_1 \leq \varepsilon. \quad (4.1)
$$

Let us now evaluate Sobolev norm $|g|_1$. First, we determine the required number of knots $N_\varepsilon$ from the condition

$$
\tau_{N_\varepsilon-1} < 1 \leq \tau_{N_\varepsilon}.
$$
Then
\[ |g|^2_1 = \int_0^1 g'(s)^2 ds \leq \sum_{j=1}^{N_\varepsilon} \frac{(W(\tau_j) - W(\tau_{j-1}))^2}{\tau_j - \tau_{j-1}} = \sum_{j=1}^{N_\varepsilon} \frac{(\varepsilon/2)^2}{\Delta_j} \]

where \( \Delta_j := \tau_j - \tau_{j-1} \) are independent random variables identically distributed with \((\varepsilon/2)^2 \theta\) and
\[ \theta := \inf\{t > 0 : |W(t)| = 1\}. \]

Therefore,
\[ |g|^2_1 \leq \sum_{j=1}^{N_\varepsilon} \theta_j^{-1} \quad (4.2) \]

where \( \theta_j \) are independent copies of \( \theta \). Recall that \( E_1 := \mathbb{E} \theta < \infty \) and \( E_2 := \mathbb{E}(\theta^{-1}) < \infty \). By applying the law of large numbers we show that \( N_\varepsilon \) has order of growth \( \varepsilon^{-2} = n \), and that, by the same argument, the sum in the right hand side of (4.2) also has the same order. Indeed, let \( c > 0 \). Then
\[ P\left(N_\varepsilon > c \varepsilon^{-2}\right) = P\left(\sum_{j=1}^{N_\varepsilon} \Delta_j < 1\right) = P\left(\frac{\varepsilon^2}{4} \sum_{j=1}^{N_\varepsilon} \theta_j < 1\right) \]
\[ = P\left(\frac{1}{c \varepsilon^{-2}} \sum_{j=1}^{N_\varepsilon} \theta_j < \frac{4}{c}\right) \rightarrow 0, \]
whenever \( \frac{4}{c} < E_1 \). Furthermore, for any \( v > 0 \)
\[ P\left(\sum_{j=1}^{N_\varepsilon} \theta_j^{-1} \geq \frac{v^2}{\varepsilon^2}\right) = P\left(\frac{1}{c \varepsilon^{-2}} \sum_{j=1}^{N_\varepsilon} \theta_j^{-1} \geq \frac{v^2}{c}\right) \rightarrow 0, \]
whenever \( v^2 > c E_2 \).

By (4.1), we have \( I_{W}(1, \varepsilon) \leq |g|_1 \). Therefore, by using (4.2) and subsequent estimates, we have
\[ P\left(\varepsilon I_{W}(1, \varepsilon) \geq v\right) \leq P\left(\varepsilon |g|_1 \geq v\right) = P\left(|g|^2_1 \geq v^2 \varepsilon^{-2}\right) \]
\[ \leq P\left(\sum_{j=1}^{N_\varepsilon} \theta_j^{-1} \geq v^2 \varepsilon^{-2}\right) \]
\[ \leq P\left(N_\varepsilon > c \varepsilon^{-2}\right) + P\left(\sum_{j=1}^{N_\varepsilon} \theta_j^{-1} \geq \frac{v^2}{\varepsilon^2}\right). \]

It follows that
\[ P\left(\varepsilon I_{W}(1, \varepsilon) \geq v\right) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \]
whenever \( \frac{4}{c} < E_1 \) and \( v^2 > c E_2 \). By letting \( v^2 \searrow c E_2, c \searrow \frac{4}{E_1} \), we obtain
\[ P\left(\varepsilon I_{W}(1, \varepsilon) \geq x\right) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \]
whenever \( x > 2\sqrt{E_2/E_1} \). Since \( \varepsilon I_W(1, \varepsilon) \xrightarrow{P} C \) by (2.2), it follows that
\[
C \leq 2\sqrt{E_2/E_1}. \tag{4.3}
\]

It is of interest to calculate the constants \( E_1 \) and \( E_2 \) in this bound. By Wald identity,
\[
1 = \mathbb{E} W(\theta)^2 = \mathbb{E} \theta = E_1.
\]

Next, let \( M_t := \sup_{0 \leq s \leq t} |W(s)| \). Then for any \( r > 0 \)
\[
\mathbb{P}(\theta^{-1} \geq r) = \mathbb{P}(\theta \leq r^{-1}) = \mathbb{P}(M_{r^{-1}} \geq 1) = \mathbb{P}(r^{-1/2}M_1 \geq 1) = \mathbb{P}(M_1^2 \geq r).
\]

Therefore, \( \theta^{-1} \) and \( M_1^2 \) are equidistributed. The distribution of \( M_1 \) is still inconvenient for calculations. However, it is convenient to work with \( M_\tau \), where \( \tau \) is a standard exponential random variable independent of \( W \). Indeed, by [3, Formula 1.15.2] we have for any \( a > 0 \)
\[
\mathbb{P}(M_\tau \geq a) = [\cosh(\sqrt{2a})]^{-1},
\]
hence, by using [8, Formula 860.531],
\[
\mathbb{E} M_\tau^2 = \int_0^\infty 2a \mathbb{P}(M_\tau \geq a) \, da = \int_0^\infty \frac{2a}{\cosh(\sqrt{2a})} \, da = \int_0^\infty \frac{x}{\cosh(x)} \, dx \approx 1.832.
\]

On the other hand,
\[
\mathbb{E} M_\tau^2 = \int_0^\infty \mathbb{E} M_t^2 e^{-t} \, dt = \int_0^\infty \mathbb{E} M_t^2 t e^{-t} \, dt = \mathbb{E} M_t^2.
\]

We conclude that
\[
E_2 = \mathbb{E} \theta^{-1} = \mathbb{E} M_1^2 = \mathbb{E} M_\tau^2 \approx 1.832.
\]

Thus numerical bound from (4.3) becomes \( C \leq 2\sqrt{1.832} \approx 2.7 \).

### 4.3 Oscillation lower bound

Fix an arbitrary \( x > 0 \) (to be optimized later on). Let \( n \) be a positive integer and let \( \varepsilon := xn^{-1/2} \). Let us split the interval \([0, 1]\) into \( n \) intervals \( \Delta_j := [j/n, (j + 1)/n] \) of length \( n^{-1} \). Let
\[
Y_j := \left( \max_{s \in \Delta_j} W(s) - \min_{t \in \Delta_j} W(t) - 2\varepsilon \right)_+ = (W(t_j) - W(s_j) - 2\varepsilon)_+, \quad j = 0, 1, \ldots, n-1,
\]
where \( s_j, t_j \) are the points where the maximum and the minimum of \( W \) are attained. Notice that by the standard properties of Wiener process (self-similarity, independence and stationarity of increments) the variables \( Y_j \) are independent and identically distributed with \( n^{-1/2} Y_x \), where
\[
Y_x := \left( \max_{0 \leq s \leq 1} W(s) - \min_{0 \leq t \leq 1} W(t) - 2x \right)_+ \geq 0.
\]

Take any function \( h \in \mathbb{C}[0, 1] \) such that \( ||h - W||_1 \leq \varepsilon \). We have
\[
h(s_j) - h(t_j) \geq W(s_j) - \varepsilon - (W(t_j) + \varepsilon) = W(s_j) - (W(t_j) - 2\varepsilon); \quad |h(s_j) - h(t_j)| \geq (W(s_j) - W(t_j) - 2\varepsilon)_+ = Y_j.
\]
Furthermore, by Hölder inequality,

$$|h|^2_1 = \int_0^1 h'(t)^2 \, dt = \sum_{j=0}^{n-1} \int_{s_j}^{t_j} h'(t)^2 \, dt$$

$$\geq \sum_{j=0}^{n-1} \int_{s_j}^{t_j} h'(t)^2 \, dt \geq \sum_{j=0}^{n-1} \left( \frac{f_{s_j} |h'(t)| \, dt}{|s_j - t_j|} \right)^2$$

$$\geq \sum_{j=0}^{n-1} \frac{|h(s_j) - h(t_j)|^2}{|s_j - t_j|} \geq n \sum_{j=0}^{n-1} Y_j^2 = \sum_{j=0}^{n-1} (Y(j))^2,$$

where $Y(j)$ are i.i.d. copies of $Y_x$. It follows that

$$I_W(1, \varepsilon)^2 \geq \sum_{j=0}^{n-1} (Y(j))^2,$$

thus

$$\varepsilon^2 I_W(1, \varepsilon)^2 \geq \varepsilon^2 \sum_{j=0}^{n-1} (Y(j))^2 = x^2 n^{-1} \sum_{j=0}^{n-1} (Y(j))^2.$$

Since $\varepsilon^2 I_W(1, \varepsilon)^2 \xrightarrow{P} C^2$ by (2.2), and by the law of large numbers $n^{-1} \sum_{j=0}^{n-1} (Y(j))^2 \xrightarrow{P} \mathbb{E} Y_x^2$, we infer that

$$C^2 \geq x^2 \mathbb{E} Y_x^2 > 0.$$

Let us explore what does it mean numerically. Recall that $Y_x = (R - 2x)_+$ where $R$ is the range of Wiener process on the unit interval of time. According to [3, Formula 1.15.4(1)], $R$ has the following distribution function,

$$\mathbb{P}(R \leq y) = 1 + 4 \sum_{k=1}^{\infty} (-1)^k \text{Erfc} \left( \frac{ky}{\sqrt{2}} \right),$$

where $\text{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} \, du$. It follows that $R$ has a density

$$p_R(y) = 4 \sqrt{2/\pi} \sum_{k=1}^{\infty} (-1)^{k+1} k^2 \exp \left\{ -k^2 y^2 / 2 \right\}.$$

Then

$$\mathbb{E} Y_x^2 = \int_{2x}^{\infty} (y - 2x)^2 p_R(y) \, dy = 4 \sqrt{2/\pi} \sum_{k=1}^{\infty} (-1)^{k+1} k^2 \int_{2x}^{\infty} (y - 2x)^2 \exp \left\{ -k^2 y^2 / 2 \right\} \, dy$$

$$= \frac{4 \sqrt{2/\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \left[ (4kx^2 + \frac{1}{x})\sqrt{2\pi} \widehat{\Phi}(2kx) - 2x \exp \{-2k^2 x^2 \} \right]}{\int_{2x}^{\infty} (y - 2x)^2 \exp \left\{ -k^2 y^2 / 2 \right\} \, dy},$$

where $\widehat{\Phi}(\cdot)$ stands for the tail of standard normal distribution. The series is so rapidly decreasing in $k$ that mostly the first term $k = 1$ is relevant. By making numerical optimization in $x$ we find the best values near $x \approx 0.5$ where we obtain $x^2 \mathbb{E} Y_x^2 \approx 0.145$, thus

$$C \geq x \sqrt{\mathbb{E} Y_x^2} \approx 0.381.$$
5 Markovian pursuit

In practice, it is often necessary to build an approximation to the process adaptively, i.e. in real time, because as parameter (viewed as time) advances, we may only know the trajectory of approximated process (Wiener process in our case) up to the current instant.

In this setting, approximation problem becomes a pursuit problem. One may think of a person walking with a dog along one-dimensional path. Wiener process represents the disordered dog’s walk, while the person tries to keep the dog on a leash of given length by moving with finite speed and expending minimal energy per unit of time.

We will construct an absolutely continuous approximating process \( x(t) \) such that
\[
|x(t) - W(t)| \leq 1, \quad t \in \mathbb{R}
\]
In view of Markov property of Wiener process, the reasonable strategy is to determine the derivative \( x'(t) \) as a function of the distance to the target,
\[
x'(t) := b(x(t) - W(t))
\]
where the odd function \( b(\cdot) \) defined on \([-1, 1]\) explodes to \(-\infty\) at 1 and to \(+\infty\) at \(-1\), thus preventing the exit of \( x(t) - W(t) \) from the corridor \([-1, 1]\).

In this section we will find an optimal speed function \( b(\cdot) \). As a by-product, this will give us another upper bound on \( C \).

Let \( X(t) := x(t) - W(t) \). In our setting, \( X(\cdot) \) is a diffusion process satisfying a simple SDE
\[
dX = b(X)dt - dW(t).
\]
Recall some important facts about the univariate diffusion, cf. [2, Chapter IV.11], [3, Chapter 2].

Let
\[
B(x) := 2 \int_{-\infty}^{x} b(s)ds.
\]
We stress that \( B \) is defined as an indefinite integral, i.e. up to an additive constant. Let
\[
p_0(x) := e^{B(x)}.
\]
If
\[
\int_{-1}^{1} \frac{dx}{p_0(x)} = \int_{-1}^{1} \frac{dx}{p_0(x)} = \infty,
\]
then the boundaries \( \pm 1 \) belong to the entrance type and do not belong to the exit type in Feller classification. This means that diffusion remains inside the corridor \([-1, 1]\) all along its infinite horizon of life. Moreover, the normalized density
\[
p(x) := Q^{-1}p_0(x),
\]
where \( Q := \int_{-1}^{1} p_0(x)dx \) is the normalizing factor, is the density of the stationary distribution of diffusion \( X \) considered as a mixing Markov process. We conclude that at large intervals of time \([0, T]\)
\[
\int_{0}^{T} x'(t)^2 dt = \int_{0}^{T} b(X(t))^2 dt \sim T \int_{-1}^{1} b(x)^2 p(x) dx
\]
\[
= T \int_{-1}^{1} \left( \frac{np}{2} \right)'(x)^2 p(x) dx = \frac{T}{4} \int_{-1}^{1} \frac{p'(x)^2}{p(x)} dx := \frac{I(p)}{4} T, \quad \text{as } T \to \infty,
\]
where, quite unexpectedly, Fisher information $I(p)$ shows up in the asymptotics.

The next step is to solve the variational problem

$$\min \left\{ I(p) \mid \int_{-1}^{1} p(x)dx = 1 \right\}$$

over the set of even densities concentrated on $[-1,1]$ and satisfying (5.1).

Although the solution is well known (see the references below), we recall it here for completeness. For Lagrange variation (with one indefinite multiplier) we have, for any smooth function $\delta(\cdot)$ supported by $(-1,1)$

$$I(p + \delta) - \lambda^2 \int_{-1}^{1} (p + \delta)(x)dx - \left( I(p) - \lambda^2 \int_{-1}^{1} p(x)dx \right)$$

$$= \int_{-1}^{1} \left[ \frac{(p'(x) + \delta'(x))^2}{p(x) + \delta(x)} - \frac{p'(x)^2}{p(x)} - \lambda^2 \delta(x) \right] dx$$

$$\sim \int_{-1}^{1} \left[ \frac{2p'(x)\delta'(x)}{p(x)} - \frac{p'(x)^2\delta(x)}{p(x)^2} - \lambda^2 \delta(x) \right] dx$$

$$= \int_{-1}^{1} \left[ -2 \left( \frac{p'}{p} \right)'(x) - \frac{p'(x)^2}{p(x)^2} - \lambda^2 \right] \delta(x)dx, \quad \text{as } \delta \to 0.$$

We obtain variational equation,

$$2 \left( \frac{p'}{p} \right)'(x) + \frac{p'(x)^2}{p(x)^2} + \lambda^2 = 0.$$

By letting $\beta(x) := (\ln p)'(x) = \frac{p'}{p}(x)$, we have $2\beta' + \beta^2 + \lambda^2 = 0$ which yields \( \frac{d\beta}{dx} = -\frac{d\beta}{x^2} \) and

$$\frac{1}{\lambda} \arctan(\beta/\lambda) = c - \frac{x}{2}.$$

Since by symmetry $p'(0) = 0$, we have $\beta(0) = 0$, thus $c = 0$ and

$$\frac{1}{\lambda} \arctan(\beta/\lambda) = -\frac{x}{2},$$

or, equivalently,

$$\beta(x) = -\lambda \tan(\lambda x/2).$$

Furthermore, since $p(\cdot)$ should vanish on the boundary $\pm 1$, $\beta$ should explode, i.e. $\beta(\pm1) = \mp\infty$, we obtain $\lambda = \pi$. Hence,

$$\beta(x) = -\pi \tan(\pi x/2).$$

Next,

$$\ln p(x) = \int (\ln p)'(x)dx = \int \beta(x)dx$$

$$= -\pi \int \tan(\pi x/2)dx = c + 2 \ln \cos(\pi x/2).$$

Therefore, the density of the optimal invariant measure is

$$p(x) = c_1 \cos^2(\pi x/2).$$
Since
\[ \int_{-1}^{1} \cos^2(\pi x/2) \, dx = \frac{1}{2} \int_{-1}^{1} (1 + \cos(\pi x)) \, dx = 1, \]
we have \( c_1 = 1 \), thus
\[ p(x) = \cos^2(\pi x/2). \]
This distribution, as a minimizer of Fisher information on an interval, can be also found in [12, 14], [21, p.63]. Luckily for us, this \( p(\cdot) \) satisfies
\[ \int_{-1}^{1} \frac{dx}{p(x)} = \int_{-1}^{1} \frac{dx}{p(x)} = \infty, \]
so that (5.1) is satisfied, and we really have entrance boards for the optimal regime.

It remains now to calculate the optimal Fisher information,
\[
I(p) = \int_{-1}^{1} \frac{p'(x)^2}{p(x)} \, dx = \int_{-1}^{1} b(x)^2 p(x) \, dx
= \pi^2 \int_{-1}^{1} \tan^2(\pi x/2) \cos^2(\pi x/2) dx = \pi^2 \int_{-1}^{1} \sin^2(\pi x/2) dx
= \pi^2.
\]
The optimum is attained at the speed strategy
\[ b(x) = \frac{\beta(x)}{2} = -\frac{\pi}{2} \tan(\pi x/2); \]
see an example of its implementation in Fig. 2.

The specialists, including the anonymous referee, recognized in the optimal process \( X \) the Brownian motion conditioned to stay in the unit strip, which also is a diffusion having the unit diffusion coefficient and drift \( b(x) \) as above. The nature of this coincidence remains a full mystery to the authors.

One may prove that the provided algorithm is the optimal one in the entire class of adaptive algorithms but this fact is beyond the scope of the present article.

Finally, notice that the problem of Markovian pursuit that we considered here, is very close to the settings of stochastic control theory, see e.g. Karatzas [13].

![Figure 2: Optimal Markovian pursuit accompanying Wiener process](image)

As a by-product we get a bound for non-Markovian asymptotic bound,
\[ C \leq \frac{I(p)^{1/2}}{2} = \frac{\pi}{2}, \]
surprisingly the same as the bound obtained in Subsection 4.1 by completely different method.
6 Some simulation results

We simulate paths of Wiener process \( W(t) \) on the interval \([0, T]\) with \( N + 1 \) equally distant knots, i.e. \( (iT/N, W(iT/N)) \), \( i = 0, 1, ..., N \). For each simulated path we compute the discrete taut string with fixed end and the discrete Markovian pursuit trajectory within the tube of radius 1. By "discrete" we mean that the function values \( h(iT/N) \) are computed and thereafter consecutive pairs of the computed knots

\[
(iT/N, h(iT/N)), \quad i = 0, 1, ..., N,
\]

are joined by line segments so that the resulting function \( h \) on \([0, T]\) is a piecewise linear function. The normalized square root of energy of a discrete function can then be computed according to

\[
|h|_T := \frac{1}{T^{1/2}} \left( \int_0^T h'(t)^2 \, dt \right)^{1/2} = \frac{N^{1/2}}{T} \left( \sum_{i=1}^{N} (h(iT/N) - h((i-1)T/N))^2 \right)^{1/2}.
\]

Notice that closeness at the knots is equivalent to the uniform closeness for the discrete functions.

Let us briefly describe the algorithm we use for computing the discrete taut string \( h \) with fixed end. This algorithm is a particular case of an unpublished algorithm due to N. Kruglyak and E. Setterqvist for more general discrete taut string problems. We would like to emphasize that the algorithm constructs \( h \) in a finite number of steps.

Fix \( h(0) = 0 \) and \( h(T) = W(T) \). Let

\[
\ell(i/N) := \frac{N-i}{N} h(0) + \frac{i}{N} h(T), \quad i = 0, ..., N,
\]

be the linear function interpolating the known ends of the string, and let

\[
\rho_i := \min_{x \in [W(iT/N) - 1, W(iT/N) + 1]} |x - \ell(i/N)|, \quad i = 1, ..., N - 1.
\]

Then the following is done:

1) If \( \max_{i \in \{1, ..., N-1\}} \rho_i = 0 \), we fix

\[
h(iT/N) := \ell(i/N), \quad i = 1, ..., N - 1.
\]
The resulting \( h \) is the discrete taut string and we are done.

2) If \( \max_{i \in \{1, \ldots, N-1\}} \rho_i > 0 \), we choose an index \( k \in \{1, \ldots, N-1\} \) that satisfies \( \rho_k = \max_{i \in \{1, \ldots, N-1\}} \rho_i \) and fix \( h(kT/N) \) to be equal to the endpoint of the interval \( [W(kT/N) - 1, W(kT/N) + 1] \) that is nearest to \( \ell(k/N) \).

Let us explain why this is the correct value of the discrete taut string \( h \). Without loss of generality we may assume that

\[
W(kT/N) - 1 > \ell(k/N).
\]

By the choice of \( k \) we have

\[
W(iT/N) - 1 \leq \ell(i/N) + \rho_k, \quad i = 1, \ldots, N-1.
\]

Therefore the linearly trimmed taut string

\[
\hat{h}(i/N) := \min \{ h(i/N), \ell(i/N) + \rho_k \}
\]

also goes through our unit tube. Since the energy of a trimmed function does not exceed the energy of the initial function, and since the taut string uniquely delivers the minimal energy among all functions going through the tube, we conclude that \( \hat{h} = h \). In particular,

\[
h(k/N) = \hat{h}(k/N) \leq \ell(k/N) + \rho_k = W(kT/N) - 1,
\]

On the other hand, since \( h \) goes through the tube, we have \( h(k/N) \geq W(kT/N) - 1 \). Hence \( h(k/N) = W(kT/N) - 1 \), as claimed.

The procedure 1) and 2) is then applied for the resulting subsets \( \{0, T/N, \ldots, kT/N\} \) and \( \{kT/N, (k+1)T/N, \ldots, T\} \) of \( \{0, T/N, \ldots, T\} \), then applied for their resulting subsets and so on. One may say that iterations form a subset of a binary tree. After at most \( N - 1 \) iterations, the discrete taut string \( h \) has been computed at each \( iT/N, i = 1, \ldots, N - 1 \).

Next we describe the computation of the discrete Markovian pursuit. The stochastic differential equation

\[
h'(t) := -\frac{\pi}{2} \tan \left(\frac{\pi}{2} h(t) - W(t)\right)
\]

is discretized with a backward finite difference method which result in the equations

\[
\frac{h \left(\frac{iT}{N}\right) - h \left(\frac{(i-1)T}{N}\right)}{\frac{T}{N}} = \frac{\pi}{2} \tan \left(\frac{\pi}{2} \left( h \left(\frac{(i-1)T}{N}\right) - W \left(\frac{(i-1)T}{N}\right) \right) \right)
\]

for \( i = 1, \ldots, N \) together with the initial condition \( h(0) = 0 \). If \( h(iT/N) \) is close to the boundaries \( W(iT/N) \pm 1 \) we might experience numerical instability at subsequent time points due to \( \tan \left(\frac{\pi x}{2}\right) \rightarrow \pm \infty \) when \( x \rightarrow \pm 1^\pm \). To avoid this, we constrain \( h(iT/N) \) to the interval

\[
\left[ W \left(\frac{iT}{N}\right) - 0.99, W \left(\frac{iT}{N}\right) + 0.99 \right].
\]

If the computed value of \( h(iT/N) \), given by (6.1), is outside this interval we set \( h(iT/N) \) equal to the nearest endpoint of the interval.

We simulated 100000 independent paths of the Wiener process on the interval \([0, T]\) with \( T = 1000 \) and \( N = 1000000 \). For this sample, the mean of \( |h|_T \) for the taut string with fixed end was approximately 0.63, see Fig. 3 for the corresponding histogram. Using the same sample of paths, the mean of \( |h|_T \) for the Markovian pursuit was approximately 1.62, see Fig. 4, which is reasonably close to the theoretical constant \( \frac{\pi}{2} \approx 1.57 \) when \( T \rightarrow \infty \).
7 Some related problems

By different reasons, taut strings and similar objects already appeared, sometimes implicitly, in probabilistic problems. Therefore, it seems reasonable to place our results into a historical context.

7.1 Strassen’s functional law of the iterated logarithm

Strassen’s functional law of the iterated logarithm [24, 16] claims that

$$\limsup_{T \to \infty} \inf_{|h|_1 \leq 1} \left\| \frac{W(T)}{\sqrt{2T \ln \ln T}} - h \right\|_1 = 0 \quad \text{a.s.}$$

Grill and Talagrand [11, 26] independently established the optimal convergence rate in this law by proving that for some finite positive constants $c_1$, $c_2$ it is true that

$$c_1 < \limsup_{T \to \infty} (\ln \ln T)^{2/3} \inf_{|h|_1 \leq 1} \left\| \frac{W(T)}{\sqrt{2T \ln \ln T}} - h \right\|_1 < c_2 \quad \text{a.s.}$$

Due to scaling properties of the function $I_W(\cdot, \cdot)$, the latter statement just means that

$$\limsup_{T \to \infty} \frac{I_W(T, c_1(2T)^{1/2}(\ln \ln T)^{-1/6})}{(2 \ln \ln T)^{1/2}} > 1,$$

$$\limsup_{T \to \infty} \frac{I_W(T, c_2(2T)^{1/2}(\ln \ln T)^{-1/6})}{(2 \ln \ln T)^{1/2}} < 1.$$

Grill [10] and Griffin and Kuelbs [9] showed a similar lim inf result asserting that for some $c_3 > 0$ and any $c_4 > \frac{7}{8}$ it is true that

$$c_3 < \liminf_{T \to \infty} (\ln \ln T) \inf_{|h|_1 \leq 1} \left\| \frac{W(T)}{\sqrt{2T \ln \ln T}} - h \right\|_1 < c_4 \quad \text{a.s.}$$

which means, in our notations, that

$$\liminf_{T \to \infty} \frac{I_W(T, c_3(2T)^{1/2}(\ln \ln T)^{-1/2})}{(2 \ln \ln T)^{1/2}} > 1,$$

$$\liminf_{T \to \infty} \frac{I_W(T, c_4(2T)^{1/2}(\ln \ln T)^{-1/2})}{(2 \ln \ln T)^{1/2}} < 1.$$
We may conclude that the tubes relevant to the Strassen law are much larger than ours. Accordingly, the respective minimal energy is much lower.

### 7.2 \( L_1 \)-optimizers: lazy functions

By its definition, the taut string is the unique minimizer of \( \int_0^T h'(t)^2 dt \) among the functions whose graphs pass through the corresponding tube. It is well known, however, that when both endpoints are fixed, the taut string also is a minimizer for any functional \( \int_0^T \varphi(h'(t))dt \) whenever \( \varphi(\cdot) \) is a convex function. In the recent literature, much attention was paid to the case \( \varphi(x) = |x| \), i.e. to the minimization of variation

\[
\mathcal{V}(h) := \int_0^T |h'(t)| dt,
\]

see [15, 19]. Notice that since \(|\cdot|\) is not a strictly convex function, the corresponding variational problem typically has many solutions. Moreover, since the variation is well defined not only on absolutely continuous functions, the natural functional domain for optimization becomes wider. In [19] another minimizer of \( \mathcal{V}(h) \) is described in detail, a so called “lazy function”. When possible, this function remains constant; otherwise, it follows the boundary of the tube. Notice that lazy function need not be absolutely continuous; it only has a bounded variation.

For the case when the tube is constructed around a sample path of Wiener process, [19] suggests a description of lazy function as an inverse to appropriate subordinator. Although the taut string and lazy function both solve the same variational problem, the relations between them are yet to be clarified.

### 7.3 A related discrete applied problem

We describe in this section an interesting discrete applied problem coming from information transmission that turns out to be related with discrete taut string construction. This problem actually was an initial motivation for our research.

Consider the following information transmission unit represented on Fig. 5.

![Information transmission unit diagram](image)

**Figure 5:** Information transmission unit.

We have the discrete time count: \( j = 1, 2, 3, \ldots \). At each time \( j \) an amount of information \( S_j \) enters the system and should be transmitted through a channel. The channel’s transmission capacity \( C_j \) varies upon the time (for example, the channel may be shared with other tasks external to our information flow). We are interested in the situation when the channel capacity is insufficient for transmission, i.e. \( S_j \geq C_j \). We may place a part of the excessive information into a buffer of given size \( B \) and drop (loose) the remaining part. Let \( L_j \) denote the loss size. This variable remains under our partial control, yet within buffer size limitations. Let \( B_j \) denote the amount of information stocked in the buffer. One necessarily has

\[
0 \leq B_j \leq B.
\]

(7.1)
Given $\varphi : [0, 1] \to \mathbb{R}_+$ – an increasing convex penalty function, define the penalty functional by the formula

$$F := \sum_{j=1}^{n} \varphi \left( \frac{L_j}{S_j} \right) S_j.$$ 

Given $(S_j), (C_j),$ and $B$, we are interested to minimize $F$ by controlling $L_j$. It is important to notice that eventual non-linearity of $\varphi(\cdot)$ is a natural feature because a small loss of information, e.g. of a graphical one, is more likely to be repaired by interpolation methods than a large loss.

The process of system work may be analyzed through the buffer balance equation. We clearly have

$$B_j = B_{j-1} + (S_j - C_j - L_j).$$

Therefore,

$$B_k = \sum_{j=1}^{k} (S_j - C_j) - \sum_{j=1}^{k} L_j.$$

Now the buffer bounds (7.1) mean that

$$\sum_{j=1}^{k} (S_j - C_j) - B \leq \sum_{j=1}^{k} L_j \leq \sum_{j=1}^{k} (S_j - C_j).$$

In other words, the accumulated loss curve $\sum_{j=1}^{k} L_j$ must go within a (random) band of fixed width $B$, see Fig. 6. Note that on the picture we use the operational time, i.e. the accumulated entrance flow $\sum_{j=1}^{k} S_j$ instead of the usual time $j$.

Therefore the minimum

$$F = F(L) = \sum_{j=1}^{n} \varphi \left( \frac{L_j}{S_j} \right) S_j = \int_{S} \varphi(L'(s))ds \xrightarrow{\text{min}}$$

where $S := \sum_{j=1}^{n} S_j$, is attained at the corresponding taut string.

The greedy FIFO strategy (“first in, first out”) which consists in keeping the buffer full all the time corresponds to the accumulated loss graph going along the lower border of the admissible corridor. It is usually non-optimal at all.

Assuming that information excess $S_j - C_j$ is a sequence of identically distributed random variables, we arrive to the problem of construction of taut string accompanying sums of i.i.d. random variables with positive drift.

8 Final remarks

After energy evaluation for the taut strings accompanying Wiener process, many similar questions arise.

Within the same framework, it would be natural to study more general functionals of taut string by replacing energy with the functionals $\int_{0}^{T} \varphi(h'(t)) dt$ with more or less general convex function $\varphi$. Since in the long run the derivative of accompanying taut string seems to be close to an ergodic stationary process, characterized by its invariant distribution, say $\mu$, it is natural to expect that an ergodic theorem holds in the form

$$\frac{1}{T} \int_{0}^{T} \varphi(h'(t)) dt \xrightarrow{\text{a.s.}} \int_{\mathbb{R}} \varphi(x) \mu(dx)$$
thus extending our Theorem 1.2.

It is natural to explore the energy and similar characteristics of the taut strings accompanying other processes. The fractional Brownian motion is the first obvious candidate, but in general, the class of processes with stationary increments including non-Gaussian Lévy processes seems to be a natural framework for this extension. Notice that the energy we handled here has a special relation to Wiener process, because it coincides with the squared norm of the corresponding reproducing kernel. This makes our proofs easier but we hope that handling energy for other processes is still possible.

One can also modify the form of the tube that defines required closeness between the string and the process. For example if we measure the distance between the string and the process in $L_2$-norm instead of the uniform one, then all calculations become explicit, and the analogue of constant $C$ may be calculated precisely. This will be a subject of forthcoming publication.

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