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Model Reduction using a Frequency-Limited $\mathcal{H}_2$ -Cost

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Abstract

We propose a method for model reduction on a given frequency range, without the use of input and output filter weights. The method uses a nonlinear optimization approach to minimize a frequency limited $\mathcal{H}_2$ like cost function. An important contribution of the paper is the derivation of the gradient of the proposed cost function. The fact that we have a closed form expression for the gradient and that considerations have been taken to make the gradient computationally efficient to compute enables us to efficiently use off-the-shelf optimization software to solve the optimization problem.

Keywords: model reduction, $\mathcal{H}_2$ -norm, optimization

1. Introduction

Given a linear time-invariant (LTI) dynamical model,

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)\end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$ and $\mathbf{D} \in \mathbb{R}^{p \times m}$, the model reduction problem is to find a reduced order model

$$\begin{aligned}\dot{\hat{x}}(t) &= \hat{\mathbf{A}}\hat{x}(t) + \hat{\mathbf{B}}u(t), \\ \hat{y}(t) &= \hat{\mathbf{C}}\hat{x}(t) + \hat{\mathbf{D}}u(t),\end{aligned}$$

with $\hat{\mathbf{A}} \in \mathbb{R}^{\hat{n} \times \hat{n}}$, $\hat{\mathbf{B}} \in \mathbb{R}^{\hat{n} \times m}$, $\hat{\mathbf{C}} \in \mathbb{R}^{p \times \hat{n}}$ and $\hat{\mathbf{D}} \in \mathbb{R}^{p \times m}$ with $\hat{n} < n$, where this reduced order model describes the original model well in some metric. In this paper we are interested in a reduced order model that describes the model well on a given frequency range. This is motivated by situations where the given model is valid only for a certain frequency range, for example as in Varga

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et al. (2012) where models coming from aerodynamical and structural mechanics computations describing a flexible structure are only valid up to a certain frequency.

For a review of balanced truncation model reduction approaches, both ordinary and frequency-weighted, see e.g. Gugercin and Antoulas (2004) and Ghafoor and Sreeram (2008). Some of the most commonly used frequency-weighted methods, according to Gugercin and Antoulas (2004), are Wang et al. (1999), Enns (1984) and Lin and Chiu (1992), which all use different balanced truncation approaches. In many of the frequency-weighted methods one has to specify input and output filter weights. In Gawronski and Juang (1990) they introduce a method which does not need these weighting functions, by introducing frequency-limited Gramians. This method can be interpreted as using ideal low-, band- or high-pass filters as weights. However, this method has the drawback of not always producing stable models. One approach to remedy this has been presented in Gugercin and Antoulas (2004), where they introduce a modification of the method in Gawronski and Juang (1990), and additionally derive an $\mathcal{H}_\infty$ bound for the error. Sahlan et al. (2012) also presents a modification of the method from Gawronski and Juang (1990), however this method is only applicable to SISO models. Note that these methods, whilst having an upper bound on the $\mathcal{H}_\infty$ error, do not minimize an explicit measure.

In this paper a method that uses a variant of the $\mathcal{H}_2$ -norm as the measure of performance will be presented. The problem of finding a reduced order model that, in $\mathcal{H}_2$ sense, resembles the original model well has been a goal in many investigations. Especially since the work of Meier and Luenberger (1967), and especially Wilson (1970), in which they derive first order optimality conditions for minimization of the $\mathcal{H}_2$ -norm, see for example Lepschy et al. (1991), Beattie and Gugercin (2007), Gugercin et al. (2008), Fulcheri and Olivi (1998), Yan and Lam (1999) and references therein. One reason for this could be the fact that $\mathcal{H}_2$ criterion provides a meaningful characterization of the error, both in deterministic and stochastic contexts.

Finding global minimizers for the $\mathcal{H}_2$ -approximation problem is very difficult, it is in fact a non-linear non-convex optimization problem, the existing methods for $\mathcal{H}_2$ -approximation have the more modest goal of finding local minimizers and can be categorized into two categories; methods using tangential interpolation techniques or methods using gradient-flow techniques.

The gradient-flow algorithms use the gradients for the state space matrices derived in Wilson (1974) and let these evolve in time to find a local approximation of the given system. The different algorithms use different techniques to assure that the reduced model is stable, to speed up the process and to guarantee convergence. Examples of these algorithms are Yan and Lam (1999), Fulcheri and Olivi (1998) and Huang et al. (2001).

The interpolation based $\mathcal{H}_2$ model reduction techniques try to find a model whose transfer function interpolates the transfer function of the full-order system (and its derivative) at selected interpolation points. These methods often use computationally effective Krylov-based algorithms which makes these techniques suitable for large-scale problems. Examples of these algorithms are Xu

and Zeng (2011), Beattie and Gugercin (2007) and Poussot-Vassal (2011). Recently two variants of these methods have been published. The first one is a frequency-weighted version that is applicable on SISO systems, see Anić et al. (2013), and the second one is a frequency-limited version with the same goal as the method presented in this article, but using a different technique, see Vuillemin et al. (2013).

One of the main contributions in this paper is a method which uses optimization, and not truncation, to find an $\mathcal{H}_2$ -optimal reduced order model, and, where this optimization is *only* performed over a *limited* frequency interval. Another important contribution is the derivation of the gradient for the cost function. The fact that we have a closed form expression for the gradient and that considerations have been taken to make the gradient computationally efficient to compute enables us to efficiently use off-the-shelf optimization software to solve the optimization problem.

2. Frequency Limited Gramians

The method proposed in this paper uses the idea presented in Gawronski and Juang (1990), in which they introduce frequency-limited Gramians. Before presenting the frequency-limited Gramians we define the standard Gramians, see, e.g., Zhou et al. (1996), in time and frequency domain, for reference. For a system G , that is stable and described by

$$G : \begin{cases} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t) \end{cases} \quad (1)$$

and denoted as $G : \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right]$, the observability and controllability Gramians are defined as

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}\tau} \mathbf{B} \mathbf{B}^\top e^{\mathbf{A}^\top \tau} d\tau = \frac{1}{2\pi} \int_{-\infty}^\infty \mathbf{H}(\nu) \mathbf{B} \mathbf{B}^\top \mathbf{H}^*(\nu) d\nu, \quad (2a)$$

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^\top \tau} \mathbf{C}^\top \mathbf{C} e^{\mathbf{A}\tau} d\tau = \frac{1}{2\pi} \int_{-\infty}^\infty \mathbf{H}^*(\nu) \mathbf{C}^\top \mathbf{C} \mathbf{H}(\nu) d\nu, \quad (2b)$$

where $\mathbf{H}(\nu) = (i\nu\mathbb{I} - \mathbf{A})^{-1}$ and $\mathbf{H}^*(\nu)$ denotes the conjugate transpose of $\mathbf{H}(\nu)$. The controllability and observability Gramians satisfy, respectively, the Lyapunov equations

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{B}\mathbf{B}^\top = \mathbf{0}, \quad (3a)$$

$$\mathbf{A}^\top \mathbf{Q} + \mathbf{Q}\mathbf{A} + \mathbf{C}^\top \mathbf{C} = \mathbf{0}. \quad (3b)$$

Now we narrow the frequency band, from $(-\infty, \infty)$ to $(-\omega, \omega)$ where $\omega < \infty$. We define the frequency-limited Gramians, see Gawronski and Juang (1990), as

$$\mathbf{P}_\omega = \frac{1}{2\pi} \int_{-\omega}^{\omega} \mathbf{H}(\nu) \mathbf{B} \mathbf{B}^\top \mathbf{H}^*(\nu) d\nu, \quad (4a)$$

$$\mathbf{Q}_\omega = \frac{1}{2\pi} \int_{-\omega}^{\omega} \mathbf{H}^*(\nu) \mathbf{C}^\top \mathbf{C} \mathbf{H}(\nu) d\nu. \quad (4b)$$

These Gramians can be shown to satisfy the following Lyapunov equations, see Gawronski and Juang (1990),

$$\mathbf{A} \mathbf{P}_\omega + \mathbf{P}_\omega \mathbf{A}^\top + \mathbf{S}_\omega \mathbf{B} \mathbf{B}^\top + \mathbf{B} \mathbf{B}^\top \mathbf{S}_\omega^* = \mathbf{0}, \quad (5a)$$

$$\mathbf{A}^\top \mathbf{Q}_\omega + \mathbf{Q}_\omega \mathbf{A} + \mathbf{S}_\omega^* \mathbf{C}^\top \mathbf{C} + \mathbf{C}^\top \mathbf{C} \mathbf{S}_\omega = \mathbf{0}, \quad (5b)$$

with

$$\mathbf{S}_\omega = \frac{i}{2\pi} \ln ((\mathbf{A} + i\omega \mathbb{I})(\mathbf{A} - i\omega \mathbb{I})^{-1}) \quad (6)$$

In Gawronski and Juang (1990) they continue by creating a balanced system and performing a balanced truncation using the newly defined frequency-limited Gramians. A drawback of this method is that since the terms $\mathbf{S}_\omega \mathbf{B} \mathbf{B}^\top + \mathbf{B} \mathbf{B}^\top \mathbf{S}_\omega^*$ and $\mathbf{S}_\omega^* \mathbf{C}^\top \mathbf{C} + \mathbf{C}^\top \mathbf{C} \mathbf{S}_\omega$ are not guaranteed to be positive semi-definite, stability of the reduced order model cannot be guaranteed. There exists a modification to this in Gugercin and Antoulas (2004) where they propose a remedy to this.

Remark 1. By using addition/subtraction of two or more different frequency-limited Gramians it is possible to focus on one or more arbitrary frequency ranges, e.g., you can construct the frequency-limited controllability Gramian, $\mathbf{P}_\Omega$, for the interval $\omega \in \Omega = [\omega_1, \omega_2] \cup [\omega_3, \omega_4]$ as

$$\mathbf{A} \mathbf{P}_\Omega + \mathbf{P}_\Omega \mathbf{A}^\top + \mathbf{S}_\Omega \mathbf{B} \mathbf{B}^\top + \mathbf{B} \mathbf{B}^\top \mathbf{S}_\Omega^* = \mathbf{0}, \quad (7)$$

with $\mathbf{S}_\Omega = \mathbf{S}_{\omega_2} - \mathbf{S}_{\omega_1} + \mathbf{S}_{\omega_4} - \mathbf{S}_{\omega_3}$.

3. Frequency Limited Model Reduction using Optimization

The $\mathcal{H}_2$ -norm of G , in (1), assuming that $\mathbf{D} = \mathbf{0}$ so that the system becomes strictly proper, can be expressed as

$$\|G\|_{\mathcal{H}_2}^2 = \text{tr} \int_0^\infty \mathbf{C} \mathbf{e}^{\mathbf{A}\tau} \mathbf{B} \mathbf{B}^\top \mathbf{e}^{\mathbf{A}^\top \tau} \mathbf{C}^\top d\tau \quad (8a)$$

$$= \frac{1}{2\pi} \text{tr} \int_{-\infty}^{\infty} G(i\nu) G^*(i\nu) d\nu \quad (8b)$$

$$= \frac{1}{2\pi} \text{tr} \int_{-\infty}^{\infty} \mathbf{C} \mathbf{H}(\nu) \mathbf{B} \mathbf{B}^\top \mathbf{H}^*(\nu) \mathbf{C}^\top d\nu = \text{tr} \mathbf{C} \mathbf{P} \mathbf{C}^\top \quad (8c)$$

$$= \text{tr} \int_0^\infty \mathbf{B}^\top \mathbf{e}^{\mathbf{A}^\top \tau} \mathbf{C}^\top \mathbf{C} \mathbf{e}^{\mathbf{A}\tau} \mathbf{B} d\tau \quad (8d)$$

$$= \frac{1}{2\pi} \text{tr} \int_{-\infty}^{\infty} \mathbf{B}^\top \mathbf{H}^*(\nu) \mathbf{C}^\top \mathbf{C} \mathbf{H}(\nu) \mathbf{B} d\nu = \text{tr} \mathbf{B}^\top \mathbf{Q} \mathbf{B}. \quad (8e)$$

In this paper we introduce a new frequency-limited $\mathcal{H}_2$ -like norm that uses the frequency-limited Gramians presented in the previous section, and we denote the new measure by $\|G\|_{\mathcal{H}_{2,\omega}}$, with

$$\|G\|_{\mathcal{H}_{2,\omega}}^2 = \frac{1}{2\pi} \operatorname{tr} \int_{-\omega}^{\omega} G(i\nu) G^*(i\nu) d\nu \quad (9)$$

$$= \frac{1}{2\pi} \operatorname{tr} \int_{-\omega}^{\omega} (\mathbf{C}\mathbf{H}(\nu)\mathbf{B} + \mathbf{D}) (\mathbf{B}^\top \mathbf{H}^*(\nu) \mathbf{C}^\top + \mathbf{D}^\top) d\nu \quad (10)$$

$$= \operatorname{tr} \mathbf{C} \mathbf{P}_\omega \mathbf{C}^\top + 2 \operatorname{tr} \left[\left(\mathbf{C} \mathbf{S}_\omega \mathbf{B} + \mathbf{D} \frac{\omega}{2\pi} \right) \mathbf{D}^\top \right]. \quad (11)$$

One thing that differs from the ordinary $\mathcal{H}_2$ -norm is that, if we do not include an infinite interval in Ω , i.e., include $\omega = \infty$ as the end frequency, then the system does not need to be strictly proper. This means that we can, in this case, have $\mathbf{D} \neq \mathbf{0}$.

The method proposed in this paper is a model reduction method that, given a model G , finds a reduced order model, $\hat{G}$, that is a good approximation on a given frequency interval, e.g. $[0, \omega]$. The objective is to minimize the error between the given model and the sought reduced order model in a frequency-limited $\mathcal{H}_2$ -norm, using the frequency-limited Gramians. We formulate the optimization problem

$$\underset{\hat{G}}{\text{minimize}} \left\| G - \hat{G} \right\|_{\mathcal{H}_{2,\omega}}^2 = \underset{\hat{G}}{\text{minimize}} \|E\|_{\mathcal{H}_{2,\omega}}^2, \quad (12)$$

where

$$\|E\|_{\mathcal{H}_{2,\omega}}^2 = \frac{1}{2\pi} \operatorname{tr} \int_{-\omega}^{\omega} E(i\nu) E^*(i\nu) d\nu. \quad (13)$$

Assume that the system E is stable and described by

$$E : \left[\begin{array}{c|c} \mathbf{A}_E & \mathbf{B}_E \\ \hline \mathbf{C}_E & \mathbf{D}_E \end{array} \right]. \quad (14)$$

Given G and $\hat{G}$, represented as

$$G : \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right], \hat{G} : \left[\begin{array}{c|c} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hline \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{array} \right], \quad (15)$$

the error system can be realized, in state space form, as

$$E : \left[\begin{array}{c|c} \mathbf{A}_E & \mathbf{B}_E \\ \hline \mathbf{C}_E & \mathbf{D}_E \end{array} \right] = \left[\begin{array}{c|c} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{A}} \end{pmatrix} & \begin{pmatrix} \mathbf{B} \\ \hat{\mathbf{B}} \end{pmatrix} \\ \hline \begin{pmatrix} \mathbf{C} & -\hat{\mathbf{C}} \end{pmatrix} & \mathbf{D} - \hat{\mathbf{D}} \end{array} \right]. \quad (16)$$

This realization of the error system will later prove beneficial when rewriting the optimization problem. Throughout the paper we will assume that the given model is stable.

The cost function for the optimization problem (12) can be written as

$$\|E\|_{\mathcal{H}_2, \omega}^2 = \text{tr } \mathbf{C}_E \mathbf{P}_{E, \omega} \mathbf{C}_E^\top + 2 \text{tr} \left[\left(\mathbf{C}_E \mathbf{S}_{E, \omega} \mathbf{B}_E + \mathbf{D}_E \frac{\omega}{2\pi} \right) \mathbf{D}_E^\top \right] \quad (17a)$$

$$= \text{tr } \mathbf{B}_E^\top \mathbf{Q}_{E, \omega} \mathbf{B}_E + 2 \text{tr} \left[\left(\mathbf{C}_E \mathbf{S}_{E, \omega} \mathbf{B}_E + \mathbf{D}_E \frac{\omega}{2\pi} \right) \mathbf{D}_E^\top \right]. \quad (17b)$$

where

$$\mathbf{A}_E \mathbf{P}_{E, \omega} + \mathbf{P}_{E, \omega} \mathbf{A}_E^\top + \mathbf{S}_{E, \omega} \mathbf{B}_E \mathbf{B}_E^\top + \mathbf{B}_E \mathbf{B}_E^\top \mathbf{S}_{E, \omega}^* = \mathbf{0}, \quad (18a)$$

$$\mathbf{A}_E^\top \mathbf{Q}_{E, \omega} + \mathbf{Q}_{E, \omega} \mathbf{A}_E + \mathbf{S}_{E, \omega}^* \mathbf{C}_E^\top \mathbf{C}_E + \mathbf{C}_E^\top \mathbf{C}_E \mathbf{S}_{E, \omega} = \mathbf{0}, \quad (18b)$$

with

$$\mathbf{S}_{E, \omega} = \frac{i}{2\pi} \ln \left((\mathbf{A}_E + i\omega \mathbb{I})(\mathbf{A}_E - i\omega \mathbb{I})^{-1} \right). \quad (19)$$

In this paper we have also derived a simpler expression for the matrix $\mathbf{S}_{E, \omega}$, compared to what is presented in Gawronski and Juang (1990).

Lemma 1. *For a matrix $\mathbf{A}$ that is Hurwitz it holds that*

$$\mathbf{S}_\omega = \frac{i}{2\pi} \ln \left((\mathbf{A} + i\omega \mathbb{I})(\mathbf{A} - i\omega \mathbb{I})^{-1} \right) = \text{Re} \left[\frac{i}{\pi} \ln (-\mathbf{A} - i\omega \mathbb{I}) \right], \quad (20)$$

where $\ln \mathbf{A}$ denotes the matrix complex logarithm using the principal branch.

Proof. $\mathbf{S}_\omega$ can be written as

$$\mathbf{S}_\omega = \frac{i}{2\pi} \ln \left((-\mathbf{A} - i\omega \mathbb{I})(-\mathbf{A} + i\omega \mathbb{I})^{-1} \right). \quad (21)$$

The matrices $(-\mathbf{A} - i\omega \mathbb{I})$ and $(-\mathbf{A} + i\omega \mathbb{I})$ commute and since $\mathbf{A}$ is Hurwitz, both $(-\mathbf{A} - i\omega \mathbb{I})$ and $(-\mathbf{A} + i\omega \mathbb{I})$ will have their eigenvalues in the right half plane. Now using Theorems 11.2 and 11.3 in Higham (2008), yields

$$\mathbf{S}_\omega = \frac{i}{2\pi} \ln \left((-\mathbf{A} - i\omega \mathbb{I})(-\mathbf{A} + i\omega \mathbb{I})^{-1} \right) = \frac{i}{2\pi} \ln(-\mathbf{A} - i\omega \mathbb{I}) - \frac{i}{2\pi} \ln(-\mathbf{A} + i\omega \mathbb{I}).$$

Since the principal branch of the logarithm is used, Theorem 1.18 in Higham (2008) can be used. Which for this case means that given a matrix $\mathbf{C} \in \mathbb{C}^{n \times n}$ it holds that $\ln \overline{\mathbf{C}} = \overline{\ln \mathbf{C}}$. $\mathbf{S}_\omega$ becomes

$$\mathbf{S}_\omega = \frac{i}{2\pi} \ln(-\mathbf{A} - i\omega \mathbb{I}) + \overline{\frac{i}{2\pi} \ln(-\mathbf{A} - i\omega \mathbb{I})} = \text{Re} \left[\frac{i}{\pi} \ln(-\mathbf{A} - i\omega \mathbb{I}) \right].$$

□

Now we want to rewrite the cost function (17) to a more computationally tractable form. This is done by using the realization given in (16) and by partitioning the Gramians $\mathbf{P}_{E, \omega}$ and $\mathbf{Q}_{E, \omega}$ as

$$\mathbf{P}_{E, \omega} = \begin{pmatrix} \mathbf{P}_\omega & \mathbf{X}_\omega \\ \mathbf{X}_\omega^\top & \hat{\mathbf{P}}_\omega \end{pmatrix}, \quad \mathbf{Q}_{E, \omega} = \begin{pmatrix} \mathbf{Q}_\omega & \mathbf{Y}_\omega \\ \mathbf{Y}_\omega^\top & \hat{\mathbf{Q}}_\omega \end{pmatrix}, \quad (22)$$

and $\mathbf{S}_{E,\omega}$ as

$$\mathbf{S}_{E,\omega} = \begin{pmatrix} \mathbf{S}_\omega & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{S}}_\omega \end{pmatrix}. \quad (23)$$

$\mathbf{P}_\omega, \mathbf{Q}_\omega, \hat{\mathbf{P}}_\omega, \hat{\mathbf{Q}}_\omega, \mathbf{X}_\omega$ and $\mathbf{Y}_\omega$ satisfy, due to (18), the Sylvester and Lyapunov equations

$$\mathbf{A}\mathbf{P}_\omega + \mathbf{P}_\omega\mathbf{A}^\top + \mathbf{S}_\omega\mathbf{B}\mathbf{B}^\top + \mathbf{B}\mathbf{B}^\top\mathbf{S}_\omega^* = \mathbf{0}, \quad (24a)$$

$$\mathbf{A}\mathbf{X}_\omega + \mathbf{X}_\omega\hat{\mathbf{A}}^\top + \mathbf{S}_\omega\mathbf{B}\hat{\mathbf{B}}^\top + \mathbf{B}\hat{\mathbf{B}}^\top\mathbf{S}_\omega^* = \mathbf{0}, \quad (24b)$$

$$\hat{\mathbf{A}}\hat{\mathbf{P}}_\omega + \hat{\mathbf{P}}_\omega\hat{\mathbf{A}}^\top + \hat{\mathbf{S}}_\omega\hat{\mathbf{B}}\hat{\mathbf{B}}^\top + \hat{\mathbf{B}}\hat{\mathbf{B}}^\top\hat{\mathbf{S}}_\omega^* = \mathbf{0}, \quad (24c)$$

$$\mathbf{A}^\top\mathbf{Q}_\omega + \mathbf{Q}_\omega\mathbf{A} + \mathbf{S}_\omega^*\mathbf{C}^\top\mathbf{C} + \mathbf{C}^\top\mathbf{C}\mathbf{S}_\omega = \mathbf{0}, \quad (24d)$$

$$\mathbf{A}^\top\mathbf{Y}_\omega + \mathbf{Y}_\omega\hat{\mathbf{A}} - \mathbf{S}_\omega^*\mathbf{C}^\top\hat{\mathbf{C}} - \mathbf{C}^\top\hat{\mathbf{C}}\mathbf{S}_\omega = \mathbf{0}, \quad (24e)$$

$$\hat{\mathbf{A}}^\top\hat{\mathbf{Q}}_\omega + \hat{\mathbf{Q}}_\omega\hat{\mathbf{A}} + \hat{\mathbf{S}}_\omega^*\hat{\mathbf{C}}^\top\hat{\mathbf{C}} + \hat{\mathbf{C}}^\top\hat{\mathbf{C}}\hat{\mathbf{S}}_\omega = \mathbf{0}, \quad (24f)$$

with

$$\mathbf{S}_\omega = \text{Re} \left[\frac{i}{\pi} \ln(-\mathbf{A} - i\omega\mathbb{I}) \right], \quad \hat{\mathbf{S}}_\omega = \text{Re} \left[\frac{i}{\pi} \ln(-\hat{\mathbf{A}} - i\omega\mathbb{I}) \right]. \quad (25)$$

Note that $\mathbf{P}_\omega$ and $\mathbf{Q}_\omega$ satisfy the Lyapunov equations for the frequency-limited controllability and observability Gramians for the given model, and $\hat{\mathbf{P}}_\omega$ and $\hat{\mathbf{Q}}_\omega$ satisfy the Lyapunov equations for the frequency-limited controllability and observability Gramians for the sought model.

With the partitioning of $\mathbf{P}_{E,\omega}$ and $\mathbf{Q}_{E,\omega}$ it is possible to rewrite (17) in two alternative forms

$$\begin{aligned} \|E\|_{\mathcal{H}_2,\omega}^2 = & \text{tr} \left(\mathbf{B}^\top\mathbf{Q}_\omega\mathbf{B} + 2\mathbf{B}^\top\mathbf{Y}_\omega\hat{\mathbf{B}} + \hat{\mathbf{B}}^\top\hat{\mathbf{Q}}_\omega\hat{\mathbf{B}} \right) \\ & + 2 \text{tr} \left[\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi} - \left(\hat{\mathbf{C}}\hat{\mathbf{S}}_\omega\hat{\mathbf{B}} + \hat{\mathbf{D}}\frac{\omega}{2\pi} \right) \right] \left(\mathbf{D}^\top - \hat{\mathbf{D}}^\top \right), \end{aligned} \quad (26a)$$

$$\begin{aligned} \|E\|_{\mathcal{H}_2,\omega}^2 = & \text{tr} \left(\mathbf{C}\mathbf{P}_\omega\mathbf{C}^\top - 2\mathbf{C}\mathbf{X}_\omega\hat{\mathbf{C}}^\top + \hat{\mathbf{C}}\hat{\mathbf{P}}_\omega\hat{\mathbf{C}}^\top \right) \\ & + 2 \text{tr} \left[\mathbf{C}\mathbf{S}_\omega\mathbf{B} + \mathbf{D}\frac{\omega}{2\pi} - \left(\hat{\mathbf{C}}\hat{\mathbf{S}}_\omega\hat{\mathbf{B}} + \hat{\mathbf{D}}\frac{\omega}{2\pi} \right) \right] \left(\mathbf{D}^\top - \hat{\mathbf{D}}^\top \right). \end{aligned} \quad (26b)$$

Remark 2. Note that neither the term $\mathbf{B}^\top\mathbf{Q}_\omega\mathbf{B}$ nor the term $\mathbf{C}\mathbf{P}_\omega\mathbf{C}^\top$, which are included in the cost function (26), depend on the optimization variables, $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$. Hence, these terms can be excluded from the optimization. These are the only terms including $\mathbf{P}_\omega$ and $\mathbf{Q}_\omega$ which are the most costly to compute.

When optimizing the frequency-limited $\mathcal{H}_2$ -norm using the system matrices as optimization variables we have the freedom to choose which elements we optimize over, i.e., we can introduce structure in $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$, as long as we

can find an $\hat{\mathbf{A}}$ that is Hurwitz. Let us introduce the matrices $\mathbf{S}_{\hat{\mathbf{A}}}, \mathbf{S}_{\hat{\mathbf{B}}}, \mathbf{S}_{\hat{\mathbf{C}}}$ and $\mathbf{S}_{\hat{\mathbf{D}}}$ which hold the structure of the sought matrices, i.e.,

$$[\mathbf{S}_{\hat{\mathbf{A}}}]_{ij} = \begin{cases} 1, & \text{if } [\hat{\mathbf{A}}]_{ij} \text{ is a free variable;} \\ 0, & \text{otherwise.} \end{cases} \quad (27)$$

We will see in the next section, that due to the element-wise differentiation, this structure will be inherited in the gradient.

The parametrization of the sought system using the full system matrices is of course redundant, which leads to a non-unique minimum of the cost function in the parameter space. This leads, in theory, to a singular Hessian matrix. However, this is taken care of in most quasi-newton solvers to ensure that the minimum is reached in a numerically stable way. The over-parametrization can also help the algorithm numerically, see McKelvey and Helmersson (1997).

3.1. Gradient of the Cost Function

An appealing feature of the proposed nonlinear optimization approach, using our proposed $\mathcal{H}_2$ -like measure to solve the problem, is that the equations (26) are differentiable in the system matrices, $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$. In addition, the closed form expression obtained when differentiating the cost function is expressed in the given data ($\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$), the optimization variables ($\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$) and solutions to the equations in (24).

To show this we start by differentiating with respect to $\hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$. First we note that neither $\mathbf{Q}_\omega, \mathbf{Y}_\omega$ nor $\hat{\mathbf{Q}}_\omega$ in equation (26a) depends on $\hat{\mathbf{B}}$ which means that the equation is quadratic in $\hat{\mathbf{B}}$. Analogous observations can be made with equation (26b) and the variable $\hat{\mathbf{C}}$ and similarly with $\hat{\mathbf{D}}$. Hence, the derivative of the cost function with respect $\hat{\mathbf{B}}, \hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ becomes

$$\frac{\partial \|\mathbf{E}\|_{\mathcal{H}_2, \omega}^2}{\partial \hat{\mathbf{B}}} = 2 \left(\hat{\mathbf{Q}}_\omega \hat{\mathbf{B}} + \mathbf{Y}_\omega^\top \mathbf{B} - \hat{\mathbf{S}}_\omega^\top \hat{\mathbf{C}}^\top (\mathbf{D} - \hat{\mathbf{D}}) \right) \odot \mathbf{S}_{\hat{\mathbf{B}}}, \quad (28a)$$

$$\frac{\partial \|\mathbf{E}\|_{\mathcal{H}_2, \omega}^2}{\partial \hat{\mathbf{C}}} = 2 \left(\hat{\mathbf{C}} \hat{\mathbf{P}}_\omega - \mathbf{C} \mathbf{X}_\omega - (\mathbf{D} - \hat{\mathbf{D}}) \hat{\mathbf{B}}^\top \hat{\mathbf{S}}_\omega^\top \right) \odot \mathbf{S}_{\hat{\mathbf{C}}}, \quad (28b)$$

$$\frac{\partial \|\mathbf{E}\|_{\mathcal{H}_2, \omega}^2}{\partial \hat{\mathbf{D}}} = -2 \left(\mathbf{C} \mathbf{S}_\omega \mathbf{B} + \mathbf{D} \frac{\omega}{2\pi} - \hat{\mathbf{C}} \hat{\mathbf{S}}_\omega \hat{\mathbf{B}} - \hat{\mathbf{D}} \frac{\omega}{2\pi} \right) \odot \mathbf{S}_{\hat{\mathbf{D}}}, \quad (28c)$$

where $\odot$ represents the Hadamard product of matrices, i.e., element-wise multiplication.

For the more complicated case of differentiating with respect to $\hat{\mathbf{A}}$ we observe that $\hat{\mathbf{Q}}_\omega$ and $\mathbf{Y}_\omega$ do depend on $\hat{\mathbf{A}}$, see the equations in (24). The calculations of this part of the gradient are lengthy and can be found in Appendix A.

The complete gradient becomes

$$\frac{\partial \|E\|_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{A}}} = 2 \left(\mathbf{Y}_\omega^\top \mathbf{X} + \hat{\mathbf{Q}}_\omega \hat{\mathbf{P}} \right) \odot \mathbf{S}_{\hat{\mathbf{A}}} - 2\mathbf{W} \odot \mathbf{S}_{\hat{\mathbf{A}}}, \quad (29a)$$

$$\frac{\partial \|E\|_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{B}}} = 2 \left(\hat{\mathbf{Q}}_\omega \hat{\mathbf{B}} + \mathbf{Y}_\omega^\top \mathbf{B} - \hat{\mathbf{S}}_\omega^\top \hat{\mathbf{C}}^\top (\mathbf{D} - \hat{\mathbf{D}}) \right) \odot \mathbf{S}_{\hat{\mathbf{B}}}, \quad (29b)$$

$$\frac{\partial \|E\|_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{C}}} = 2 \left(\hat{\mathbf{C}} \hat{\mathbf{P}}_\omega - \mathbf{C} \mathbf{X}_\omega - (\mathbf{D} - \hat{\mathbf{D}}) \hat{\mathbf{B}}^\top \hat{\mathbf{S}}_\omega^\top \right) \odot \mathbf{S}_{\hat{\mathbf{C}}}, \quad (29c)$$

$$\frac{\partial \|E\|_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{D}}} = -2 \left(\mathbf{C} \mathbf{S}_\omega \mathbf{B} + \mathbf{D} \frac{\omega}{2\pi} - \hat{\mathbf{C}} \hat{\mathbf{S}}_\omega \hat{\mathbf{B}} - \hat{\mathbf{D}} \frac{\omega}{2\pi} \right) \odot \mathbf{S}_{\hat{\mathbf{D}}}, \quad (29d)$$

where

$$\mathbf{W} = \text{Re} \left[\frac{i}{\pi} L \left(-\hat{\mathbf{A}} - i\omega \mathbb{I}, \mathbf{V} \right) \right]^\top, \quad (30a)$$

$$\mathbf{V} = \hat{\mathbf{C}}^\top \hat{\mathbf{C}} \hat{\mathbf{P}} - \hat{\mathbf{C}}^\top \mathbf{C} \mathbf{X} - \hat{\mathbf{C}}^\top (\mathbf{D} - \hat{\mathbf{D}}) \hat{\mathbf{B}}^\top \quad (30b)$$

$$\mathbf{0} = \hat{\mathbf{A}} \hat{\mathbf{P}} + \hat{\mathbf{P}} \hat{\mathbf{A}}^\top + \hat{\mathbf{B}} \hat{\mathbf{B}}^\top \quad (30c)$$

$$\mathbf{0} = \mathbf{A} \mathbf{X} + \mathbf{X} \mathbf{A}^\top + \mathbf{B} \mathbf{B}^\top \quad (30d)$$

with the function $L(\cdot, \cdot)$ being the Frechét derivative of the matrix logarithm, see Higham (2008).

Remark 3. Remember that if $\omega = \infty$ is included as the end frequency we need to have $\mathbf{D}_E = \mathbf{D} - \hat{\mathbf{D}} = \mathbf{0}$, i.e., $\hat{\mathbf{D}}$ constant and with $\mathbf{D} = \hat{\mathbf{D}}$.

3.2. Numerical Remarks

The most computationally heavy parts in the cost function are solving the Lyapunov equations in (24a) and (24d) and computing $\mathbf{S}_\omega$ which cost $\mathcal{O}(n^3)$ to compute. However, these entities do not depend on the optimization variables and can be computed ones, before the optimization starts and during the iterations in the optimization the cost is $\mathcal{O}(n^2 n_r)$ to compute the cost function. To compute the gradient we also need to compute the Frechét derivative of a matrix of size n_r by n_r which costs about $\mathcal{O}(n_r^3)$ to compute, see Al-Mohy et al. (2012) for more details. This means that the computational cost to compute the cost function and its gradient is, first an overhead of $\mathcal{O}(n^3)$ and then $\mathcal{O}(n_r n^2)$ per iteration.

Looking at the matrices used in this article to evaluate the matrix logarithm and its Frechét derivative one realizes that complex arithmetic is needed to compute these entities. However, the resulting matrices used, i.e., $\mathbf{S}_\omega$, $\hat{\mathbf{S}}_\omega$ and $\mathbf{W}$, are always real since we are only interested in the real part of the computations. This means that complex arithmetic is needed to compute parts of the cost function and its gradient. However, it is not needed in the optimization solver to perform a step. By supplying the cost function and its gradient in

this computationally efficient forms the presented method can be used in any off-the-shelf Quasi-Newton solver.

The optimization problem described in this article is a non-convex problem and we can not guarantee to find any global minima, hence, it is very important to have a good starting point when performing the optimization. Additionally, to keep the cost function bounded, the initial point need to be a stable model. The method described in this article can, hence, be thought of as a two step method, first use your favorite model reduction method to find a stable model of right order, e.g. balanced truncation, and then use this model to initialize the optimization problem described herein.

In the optimization problem described in this article we try to optimize over Hurwitz matrices of a specific order, which is a well known hard problem. As can be expected we do not have any solution to this, however, by starting with a stable model ($\hat{\mathbf{A}}$ Hurwitz) we can check the eigenvalues of the new $\hat{\mathbf{A}}$ when a step is proposed to assure that the next step keeps the stability of the model.

4. Numerical Examples

In this section we will present three examples that will show the applicability of the proposed method, and compare with other relevant methods. To be able to measure how well the different methods perform the relative error of the norm is used, i.e.

$$\frac{\|G - \hat{G}\|_{\mathcal{H}_2, \omega}}{\|G\|_{\mathcal{H}_2, \omega}}. \quad (31)$$

To shorten the names and make the figures more readable we will denote our proposed method as FLH₂NL. The methods that will be used for comparison, in the different examples, are

- BT– ordinary balanced truncation, the implementation used is the function **schurm** in Robust Control Toolbox in MATLAB
- WBT– weighted balanced truncation, an implementation of the method in Enns (1984)
- FLBT– frequency limited balanced truncation, an implementation of the method in Gawronski and Juang (1990)
- MFLBT– modified frequency limited balanced truncation, an implementation of the method in Gugercin and Antoulas (2004)
- FLISTIA– frequency limited iterative tangential interpolation algorithm (see Vuillemin et al. (2013)), the implementation in the MORE-toolbox is used
- SPANOS– weighted $\mathcal{H}_2$ -model reduction, an implementation of the method in Spanos et al. (1992)

When solving the examples in this section we use the function `fminunc` in MATLAB, that uses a quasi-Newton algorithm, to perform the optimization. Since the optimization problem is a non-convex problem the initialization of the optimization is important. In the presented examples we have used the model from balanced truncation to initialize the optimization.

We will start with a small example to illustrate how the frequency limited methods behave and show some bode plots. We will then continue with two larger examples to show the applicability for more realistic systems.

Example 1 (Small illustrative example). *This example addresses a small model with four states. The model is composed of two second order models in series, one with a resonance frequency at $\omega = 1$ and the other at $\omega = 3$. We will limit the frequency range to $\omega \in [0, 2]$ and find a model with only two states to try to only capture the first model.*

$$G = G_1 G_2 = \frac{1}{s^2 + 0.2s + 1} \frac{9}{s^2 + 0.003s + 9}. \quad (32)$$

We compare the methods FLH₂NL, FLISTIA, FLBT and MFLBT, we also compare these methods with the methods SPANOS and WBT using a low pass Butterworth filter of order 10 with a cut off frequency of 2. The results from the different methods can be seen in Figures 1 and 2 and Table 1. As can be seen in the result we are successful in finding a good model for the relevant frequencies with FLH₂NL, FLISTIA, FLBT, SPANOS and WBT. However, MFLBT captures the wrong resonance mode (from our perspective) and fails completely in the lower frequency region. Interesting to note is how the models, that does a good job, sacrifices the model fit at higher frequencies for the lower.

Table 1: Numerical results for Example 1

	$\frac{\ G - \hat{G}\ _{\mathcal{H}_2, \omega}}{\ G\ _{\mathcal{H}_2, \omega}}$	$\frac{\ G - \hat{G}\ _{\mathcal{H}_\infty, \omega}}{\ G\ _{\mathcal{H}_\infty, \omega}}$
WBT	3.01e-01	2.91e-01
FLISTIA	6.38e-02	3.96e-02
MFLBT	1.00e+00	1.00e+00
FLBT	6.31e-02	4.00e-02
SPANOS	5.97e-02	3.95e-02
FLH ₂ NL	1.02e-02	1.15e-02

Example 2 (CD player). *This example uses a slightly larger model, a model of a compact disc player with 120 states and two inputs and two outputs, see Leibfritz and Lipinski (2003). In this example we will only look at the transfer function from second input to the first output. Here we will focus on a banded frequency interval, $\omega \in [10, 1000]$ where the main peak gain is and reduce the model to a model with 12 states. We will compare the frequency limited methods FLBT, MFLBT and FLH₂NL and the weighted method WBT with a tenth order Butterworth band pass filter with cutoff frequencies equal to ω . Looking at the*

Figure 1: The true and reduced order models for Example 1. The dashed vertical line denotes $\omega = 2$. The reader is referred to the digital version of this paper, for a color plot.

Figure 2: The error models for the different methods for Example 1. The dashed vertical line denotes $\omega = 2$. The reader is referred to the digital version of this paper, for a color plot.

Figure 3: The true and reduced order models for Example 2. The dashed vertical lines denotes $\omega = 10$ and $\omega = 10000$. The reader is referred to the digital version of this paper, for a color plot.

results in Figure 3, Figure 4 and Table 2 we see that in this case all the methods does a good job, and again FLH₂NL finds the best model.

Table 2: Numerical results for Example 2

	$\frac{\ G - \hat{G}\ _{\mathcal{H}_2, \omega}}{\ G\ _{\mathcal{H}_2, \omega}}$	$\frac{\ G - \hat{G}\ _{\mathcal{H}_\infty, \omega}}{\ G\ _{\mathcal{H}_\infty, \omega}}$
WBT	1.24e-03	9.50e-04
MFLBT	1.25e-03	9.43e-04
FLBT	1.24e-03	9.41e-04
FLH ₂ NL	6.95e-04	6.83e-04

Example 3 (Clamped Beam Model, varying frequency interval). *In this example we will use the model of a clamped beam, a SISO model with 348 states which can be found in Leibfritz and Lipinski (2003). In this example we will try to focus on certain frequency intervals of the model. We will focus on different frequency intervals, $[0, \omega]$, $\omega \in [2, 40]$ and fix the reduced order model to have 12 states, $n_r = 12$. We will use the method FLH₂NL and compare with the frequency limited methods FLISTIA, FLBT and MFLBT. Additionally, we will use the method WBT with a tenth order Butterworth low-pass filter, with the cutoff frequency equal to ω . Looking at the left plot of Figure 5 we see that for small ω , the $\mathcal{H}_2$ optimal methods do very well. However, for $\omega > 7$ we see that FLH₂NL, even though optimizing the same cost function, does better than FLISTIA. As in the previous example, we see that the relative $\mathcal{H}_\infty$ -norm still keeps low. Al-*

Figure 4: The error models for the different methods for Example 2. The dashed vertical lines denotes $\omega = 10$ and $\omega = 10000$. The reader is referred to the digital version of this paper, for a color plot.

most for all ω , the $\mathcal{H}_2$ optimal methods have better relative $\mathcal{H}_\infty$ -error than the methods using balanced truncation.

In all three of the above examples we observe that the proposed method finds an at least as good model as the other methods, but with the proposed method we can guarantee that the reduced order model is stable and with the possibility to impose structure in the system matrices.

5. Conclusions

In this paper we have proposed a method, based on nonlinear optimization, that uses the frequency-limited Gramians introduced in Gawronski and Juang (1990), and the method does not have the drawback of finding unstable models. We use these Gramians to construct a frequency-limited $\mathcal{H}_2$ -norm which describes the cost function of the optimization problem. We derive a gradient of the proposed cost function which enables us to use off-the-shelf optimization software to solve the problem efficiently. The derivation of the method also enables us to impose structural constraints, e.g., upper triangular $\mathbf{A}$ -matrix, in the system matrices. The derivation follows closely the technique in Petersson (2010) and it is easy to extend the method to identify LPV-models. We have also presented three examples of different sizes and characteristics to show the applicability of the method. The method uses a nonlinear optimization approach on a nonconvex problem, hence the unavoidable problem of local minima is present. However, we consider this as a two step approach where you use your

Figure 5: Reduction of a clamped beam model to 12 states with focus on the frequency interval $[0, \omega]$, $\omega \in [2, 40]$ using FLH₂NL, FLISTIA, FLBT, MFLBT and WBT. The filter used for the weighted method is a tenth order Butterworth low-pass filter with cutoff frequency ω . To the left we see the relative $\mathcal{H}_2$ error and to the right the relative $\mathcal{H}_\infty$ error. The reader is referred to the digital version of this paper, for a color plot.

favorite reduction method as an initial point to this method which will refine the solution, as we have shown in the examples.

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Appendix A. Derivation of the Gradient w.r.t. $\hat{\mathbf{A}}$

In this appendix we present the differentiation of the cost function with respect to $\hat{\mathbf{A}}$.

The cost function is

$$\begin{aligned} \|E\|_{\mathcal{H}_2, \omega}^2 = & \text{tr} \left(\mathbf{B}^\top \mathbf{Q}_\omega \mathbf{B} + 2\mathbf{B}^\top \mathbf{Y}_\omega \hat{\mathbf{B}} + \hat{\mathbf{B}}^\top \hat{\mathbf{Q}}_\omega \hat{\mathbf{B}} \right) \\ & + 2 \text{tr} \left[\mathbf{C} \mathbf{S}_\omega \mathbf{B} + \mathbf{D} \frac{\omega}{2\pi} - \left(\hat{\mathbf{C}} \hat{\mathbf{S}}_\omega \hat{\mathbf{B}} + \hat{\mathbf{D}} \frac{\omega}{2\pi} \right) \right] \left(\mathbf{D}^\top - \hat{\mathbf{D}}^\top \right) \end{aligned} \quad (\text{A.1})$$

and by looking at the equations

$$\mathbf{A}^\top \mathbf{Y}_\omega + \mathbf{Y}_\omega \hat{\mathbf{A}} - \mathbf{S}_\omega^* \mathbf{C}^\top \hat{\mathbf{C}} - \mathbf{C}^\top \hat{\mathbf{C}} \hat{\mathbf{S}}_\omega = \mathbf{0}, \quad (\text{A.2a})$$

$$\hat{\mathbf{A}}^\top \hat{\mathbf{Q}}_\omega + \hat{\mathbf{Q}}_\omega \hat{\mathbf{A}} + \hat{\mathbf{S}}_\omega^* \hat{\mathbf{C}}^\top \hat{\mathbf{C}} + \hat{\mathbf{C}}^\top \hat{\mathbf{C}} \hat{\mathbf{S}}_\omega = \mathbf{0}, \quad (\text{A.2b})$$

we observe that $\hat{\mathbf{Q}}_\omega$ and $\mathbf{Y}_\omega$ depend on $\hat{\mathbf{A}}$ which we need to keep in mind when differentiating (A.1) with respect to $\hat{\mathbf{A}}$. Hence, $\left[\frac{\partial ||E||_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{A}}}\right]_{ij}$ becomes

$$\left[\frac{\partial ||E||_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{A}}}\right]_{ij} = \text{tr} \left(2\hat{\mathbf{B}}\hat{\mathbf{B}}^\top \frac{\partial \mathbf{Y}_\omega}{\partial \hat{a}_{ij}} + \hat{\mathbf{B}}\hat{\mathbf{B}}^\top \frac{\partial \hat{\mathbf{Q}}_\omega}{\partial \hat{a}_{ij}} - 2\hat{\mathbf{C}} \frac{\partial \hat{\mathbf{S}}_\omega}{\partial \hat{a}_{ij}} \hat{\mathbf{B}} (\mathbf{D}^\top - \hat{\mathbf{D}}^\top) \right), \quad (\text{A.3})$$

where $\frac{\partial \mathbf{Y}_\omega}{\partial \hat{a}_{ij}}$ and $\frac{\partial \hat{\mathbf{Q}}_\omega}{\partial \hat{a}_{ij}}$ depend on $\hat{\mathbf{A}}$ via the differentiated versions of the equations in (A.2),

$$\hat{\mathbf{A}}^\top \frac{\partial \mathbf{Y}_\omega^\top}{\partial \hat{a}_{ij}} + \frac{\partial \mathbf{Y}_\omega^\top}{\partial \hat{a}_{ij}} \hat{\mathbf{A}} + \frac{\partial \hat{\mathbf{A}}^\top}{\partial \hat{a}_{ij}} \mathbf{Y}_\omega^\top - \frac{\partial \hat{\mathbf{S}}_\omega^*}{\partial \hat{a}_{ij}} \hat{\mathbf{C}}^\top \mathbf{C} = \mathbf{0}, \quad (\text{A.4a})$$

$$\hat{\mathbf{A}}^\top \frac{\partial \hat{\mathbf{Q}}_\omega}{\partial \hat{a}_{ij}} + \frac{\partial \hat{\mathbf{Q}}_\omega}{\partial \hat{a}_{ij}} \hat{\mathbf{A}} + \frac{\partial \hat{\mathbf{A}}^\top}{\partial \hat{a}_{ij}} \hat{\mathbf{Q}}_\omega + \hat{\mathbf{Q}}_\omega \frac{\partial \hat{\mathbf{A}}}{\partial \hat{a}_{ij}} + \frac{\partial \hat{\mathbf{S}}_\omega^*}{\partial \hat{a}_{ij}} \hat{\mathbf{C}}^\top \hat{\mathbf{C}} + \hat{\mathbf{C}}^\top \hat{\mathbf{C}} \frac{\partial \hat{\mathbf{S}}_\omega}{\partial \hat{a}_{ij}} = \mathbf{0}. \quad (\text{A.4b})$$

To be able to substitute $\frac{\partial \hat{\mathbf{Q}}_\omega}{\partial \hat{a}_{ij}}$ and $\frac{\partial \mathbf{Y}_\omega}{\partial \hat{a}_{ij}}$ to something more computationally tractable we use the following lemma.

Lemma 2. *If $\mathbf{M}$ and $\mathbf{N}$ satisfy the Sylvester equations*

$$\mathbf{A}\mathbf{M} + \mathbf{M}\mathbf{B} + \mathbf{C} = \mathbf{0}, \quad \mathbf{N}\mathbf{A} + \mathbf{B}\mathbf{N} + \mathbf{D} = \mathbf{0}, \quad (\text{A.5})$$

then $\text{tr } \mathbf{C}\mathbf{N} = \text{tr } \mathbf{D}\mathbf{M}$.

Proof. Multiplying the left equation in (A.5) with $\mathbf{N}$ from the left, the right equation with $\mathbf{M}$ from the right and taking the trace of both equations entails

$$\text{tr } \mathbf{N}\mathbf{A}\mathbf{M} + \text{tr } \mathbf{M}\mathbf{B}\mathbf{N} = -\text{tr } \mathbf{N}\mathbf{C}, \quad \text{tr } \mathbf{N}\mathbf{A}\mathbf{M} + \text{tr } \mathbf{M}\mathbf{B}\mathbf{N} = -\text{tr } \mathbf{D}\mathbf{M}.$$

Hence, $\text{tr } \mathbf{C}\mathbf{N} = \text{tr } \mathbf{D}\mathbf{M}$. $\square$

Studying Lemma 2, the two factors in front of $\frac{\partial \mathbf{Y}_\omega}{\partial \hat{a}_{ij}}$ and $\frac{\partial \hat{\mathbf{Q}}_\omega}{\partial \hat{a}_{ij}}$ in (A.3) and the structure of the Lyapunov/Sylvester equations in (A.4), $\hat{\mathbf{A}}^\top \cdot + \cdot \hat{\mathbf{A}} + \star = 0$ and $\hat{\mathbf{A}}^\top \cdot + \cdot \hat{\mathbf{A}} + \star = 0$, brings us to the conclusion that, to do the substitution, we need to solve two additional Lyapunov/Sylvester equations, namely

$$\mathbf{A}\mathbf{X} + \mathbf{X}\hat{\mathbf{A}}^\top + \hat{\mathbf{B}}\hat{\mathbf{B}}^\top = \mathbf{0}, \quad (\text{A.6a})$$

$$\hat{\mathbf{A}}\hat{\mathbf{P}} + \hat{\mathbf{P}}\hat{\mathbf{A}}^\top + \hat{\mathbf{B}}\hat{\mathbf{B}}^\top = \mathbf{0}. \quad (\text{A.6b})$$

Note that $\hat{\mathbf{P}}$ in (A.6b) is the controllability Gramian for the reduced order model.

Rewriting (A.3) using Lemma 2, (A.4) and (A.6) leads to

$$\begin{aligned} \left[\frac{\partial ||E||_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{A}}}\right]_{ij} &= 2 \text{tr} \left[\frac{\partial \hat{\mathbf{A}}^\top}{\partial \hat{a}_{ij}} \left(\mathbf{Y}_\omega^\top \mathbf{X} + \hat{\mathbf{Q}}_\omega \hat{\mathbf{P}} \right) + \frac{\partial \hat{\mathbf{S}}_\omega^*}{\partial \hat{a}_{ij}} \left(\hat{\mathbf{C}}^\top \hat{\mathbf{C}} \hat{\mathbf{P}} - \hat{\mathbf{C}}^\top \mathbf{C} \mathbf{X} \right) \right] \\ &\quad - 2 \text{tr} \left[\frac{\partial \hat{\mathbf{S}}_\omega}{\partial \hat{a}_{ij}} \left[\hat{\mathbf{B}} (\mathbf{D}^\top - \hat{\mathbf{D}}^\top) \hat{\mathbf{C}} \right] \right]. \end{aligned} \quad (\text{A.7})$$

What remains is to rewrite the two last terms in (A.7), which includes $\frac{\partial \hat{\mathbf{S}}_\omega}{\partial \hat{a}_{ij}}$ and $\frac{\partial \hat{\mathbf{S}}_\omega^*}{\partial \hat{a}_{ij}}$. Recall the definition of $\hat{\mathbf{S}}_\omega$,

$$\hat{\mathbf{S}}_\omega = \text{Re} \left[\frac{i}{\pi} \ln \left(-\hat{\mathbf{A}} - i\omega \mathbb{I} \right) \right] = \text{Re} \left[\frac{i}{\pi} \ln \left(r(\hat{\mathbf{A}}) \right) \right] \quad (\text{A.8})$$

and differentiate with respect to an element in $\hat{\mathbf{A}}$, i.e., a_{ij} . This yields

$$\frac{\partial \hat{\mathbf{S}}_\omega}{\partial a_{ij}} = \text{Re} \left[\frac{i}{2\pi} L \left(r(\hat{\mathbf{A}}), \frac{\partial r(\hat{\mathbf{A}})}{\partial a_{ij}} \right) \right] = \text{Re} \left[\frac{i}{2\pi} L \left(r(\hat{\mathbf{A}}), -\frac{\partial \hat{\mathbf{A}}}{\partial a_{ij}} \right) \right] \quad (\text{A.9})$$

where $L(\mathbf{A}, \mathbf{E})$ is the Frechét derivative of the matrix logarithm, see Higham (2008), with

$$L(\mathbf{A}, \mathbf{E}) = \int_0^1 (t(\mathbf{A} - \mathbb{I}) + \mathbb{I})^{-1} \mathbf{E} (t(\mathbf{A} - \mathbb{I}) + \mathbb{I})^{-1} dt, \quad (\text{A.10})$$

$$r(\hat{\mathbf{A}}) = -\hat{\mathbf{A}} - i\omega \mathbb{I}. \quad (\text{A.11})$$

The function $L(\mathbf{A}, \mathbf{E})$ can be efficiently evaluated using an algorithm described in Higham (2008).

By substituting (A.9) into (A.7) and using (A.10) with the fact that we can interchange the tr-operator and the integral we obtain

$$\begin{aligned} \left[\frac{\partial \|E\|_{\mathcal{H}_{2,\omega}}^2}{\partial \hat{\mathbf{A}}} \right]_{ij} &= 2 \text{tr} \left[\frac{\partial \hat{\mathbf{A}}^\top}{\partial \hat{a}_{ij}} \left(\mathbf{Y}_\omega^\top \mathbf{X} + \hat{\mathbf{Q}}_\omega \hat{\mathbf{P}} \right) + \frac{\partial \hat{\mathbf{S}}_\omega^*}{\partial \hat{a}_{ij}} \left(\hat{\mathbf{C}}^\top \hat{\mathbf{C}} \hat{\mathbf{P}} - \hat{\mathbf{C}}^\top \mathbf{C} \mathbf{X} \right) \right] \\ &\quad - 2 \text{tr} \left[\frac{\partial \hat{\mathbf{S}}_\omega}{\partial \hat{a}_{ij}} \left[\hat{\mathbf{B}} \left(\mathbf{D}^\top - \hat{\mathbf{D}}^\top \right) \hat{\mathbf{C}} \right] \right] \\ &= 2 \text{tr} \left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial \hat{a}_{ij}} \left(\mathbf{Y}_\omega^\top \mathbf{X} + \hat{\mathbf{Q}}_\omega \hat{\mathbf{P}} \right) \right) - 2 \text{tr} \left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial \hat{a}_{ij}} \text{Re} \left[\frac{i}{\pi} L \left(r(\hat{\mathbf{A}}), \mathbf{V} \right) \right]^\top \right) \\ &= \text{tr} \left(\frac{\partial \hat{\mathbf{A}}^\top}{\partial \hat{a}_{ij}} \left[2 \left(\mathbf{Y}_\omega^\top \mathbf{X} + \hat{\mathbf{Q}}_\omega \hat{\mathbf{P}} \right) - 2\mathbf{W} \right] \right), \quad (\text{A.12}) \end{aligned}$$

where

$$\mathbf{W} = \text{Re} \left[\frac{i}{\pi} L \left(-\hat{\mathbf{A}} - i\omega \mathbb{I}, \mathbf{V} \right) \right]^\top, \quad (\text{A.13})$$

$$\mathbf{V} = \hat{\mathbf{C}}^\top \hat{\mathbf{C}} \hat{\mathbf{P}} - \hat{\mathbf{C}}^\top \mathbf{C} \mathbf{X} - \hat{\mathbf{C}}^\top \left(\mathbf{D} - \hat{\mathbf{D}} \right) \hat{\mathbf{B}}^\top. \quad (\text{A.14})$$