High-order compact finite difference schemes for the vorticity-divergence representation of the spherical shallow water equations

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SUMMARY

This work is devoted to the application of the super compact finite difference (SCFDM) and the combined compact finite difference (CCFDM) methods for spatial differencing of the spherical shallow water equations in terms of vorticity, divergence and height. The fourth-order compact, the sixth-order and eighth-order SCFDM and the sixth-order and eighth-order CCFDM schemes are used for the spatial differencing. To advance the solution in time, a semi-implicit Runge-Kutta method is used. In addition, to control the non-linear instability an eighth-order compact spatial filter is employed. For the numerical solution of the elliptic equations in the problem, a direct hybrid method which consists of a high-order compact scheme for spatial differencing in the latitude coordinate and a fast Fourier transform in longitude coordinate is utilized. The accuracy and convergence rate for all methods are verified against exact analytical solutions. Qualitative and quantitative assessment of the results for an unstable barotropic mid-latitude zonal jet employed as an initial condition, is addressed. It is revealed that the sixth-order and eighth-order CCFDM and SCFDM methods lead to a remarkable improvement of the solution over the fourth-order compact method.

KEY WORDS: Compact finite difference; High-order methods; Spherical shallow water equations; Numerical accuracy

1. Introduction

The shallow water equations are widely used in various oceanic and atmospheric problems. This model is applied to a fluid layer of constant density in which the horizontal scale of the flow is much greater than the layer depth (e.g., [1, 2]). Even though, the dynamics of the two dimensional shallow water model is less general than the three dimensional general circulation models, it is often preferred because of its mathematical and computational simplicity. The spherical shallow water equations are usually used as the first step to examine new numerical

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algorithms to be used in more complex global climate and weather prediction models. In recent years, much research (such as [3, 4, 5, 6, 7, 8, 9]) have been devoted to the development of new efficient numerical algorithms for the spherical shallow water equations.

The present work examines the application of two families of high-order compact finite difference methods to spatial differencing of the vorticity, divergence and height (VDH) representation of the spherical shallow water equations. The compact finite-difference schemes provide simple and powerful ways to reach the objectives of high accuracy and low computational cost (e.g., [10, 11, 12]). Compared with the traditional explicit finite-difference schemes of the same-order, compact schemes are significantly more accurate along with the benefit of using smaller stencil sizes. The compact finite difference methods can not be formulated on so called summation-by-parts (SBP) form [13, 14, 15, 16, 17] which is a serious drawback for finite domain problems, since the stability can not be proved. However, in this work we only consider periodic domains and this drawback is not important.

One of the main problems with the spherical harmonic method [18], which is the dominant global modelling method, is its rapid increase of computational cost with resolution. Therefore, examination of other high-order schemes, such as compact finite difference methods, as alternatives is important. Previous applications of compact schemes to idealized models of the atmosphere and oceans have shown that they are a promising alternative (e.g., [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]). In addition, there are recent advancement in parallel implementation of the compact schemes that facilitates efficient application of these methods for large scale computations on distributed memory machines (for example see Ref. [30]).

The super compact finite difference (SCFDM) and the combined compact finite difference (CCFDM) methods [36, 37, 31, 32, 33, 34, 35] are considered in this study. To our knowledge, the application of higher orders of the SCFDM and CCFDM methods for spatial differencing of the VDH representation of the spherical shallow water equations has not been studied yet.

The remainder of this paper is organized as follows. The spherical shallow water equations are presented in section 2. Details of the semi-implicit Runge-Kutta time advancing method, formulation of the high-order compact methods used for spatial differencing of the spherical shallow water equations and the spatial filter formulation are given in sections 3 and 4, respectively. The procedure used for the numerical solution of the elliptic equations is described in section 5. The accuracy and the order of convergence for the high-order compact methods used in this study is presented in section 6. Details of the test case used for the numerical solution, numerical results and discussions are given in section 7. Finally, section 8 presents the conclusions.

2. The shallow water equations

The momentum and mass continuity representation of the spherical shallow water equations in vector form can be written as (e.g., [38])

$$\frac{D\mathbf{v}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{v} + c^2\nabla\tilde{h} = 0 \quad (1)$$

$$\frac{\partial\tilde{h}}{\partial t} + \nabla \cdot [(1 + \tilde{h})\mathbf{v}] = 0. \quad (2)$$

In (1) and (2), $\mathbf{v} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}}$ is the horizontal velocity vector with u and v being the velocity components in longitudinal (λ) and latitudinal (ϕ) directions, respectively. $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors in longitudinal and latitudinal directions, respectively. $D()/Dt = \partial()/\partial t + (\mathbf{v} \cdot \nabla)()$ is the substantial derivative and $f = 2\Omega \sin \phi$ is the Coriolis parameter and Ω is the rotation rate of the earth. $\tilde{h} = (h - H)/H$ is the dimensionless perturbation depth where h is the actual depth and H is the domain area average depth. $c = \sqrt{gH}$ is the gravity wave speed and g denotes the acceleration due to gravity. The unit vector in vertical direction is denoted $\hat{\mathbf{z}}$.

Equations (1) and (2) can be rewritten in terms of relative vorticity, horizontal divergence and height as

$$\frac{\partial \zeta}{\partial t} = S_\zeta \quad (3)$$

$$\frac{\partial \delta}{\partial t} + \mathcal{H}\tilde{h} = S_\delta \quad (4)$$

$$\frac{\partial \tilde{h}}{\partial t} + \delta = S_h. \quad (5)$$

In the VDH equations (3)-(5),

$$S_\zeta = -\nabla \cdot [(\zeta + f)\mathbf{v}], \quad S_\delta = f(\zeta - f\tilde{h}) - \nabla^2\left(\frac{|\mathbf{v}|^2}{2}\right) - \nabla \cdot (-\zeta v\hat{\mathbf{i}} + \zeta u\hat{\mathbf{j}}) - u\hat{\beta}, \quad S_h = -\nabla \cdot (\tilde{h}\mathbf{v})$$

and $\mathcal{H} = c^2\nabla^2 - f^2$ is the modified Helmholtz operator, $\hat{\beta} = (df/d\phi)/a = (2\Omega/a)\cos \phi$ and a is the radius of the earth. The relative vorticity and divergence are defined as

$$\zeta = \frac{1}{a \cos \phi} \left(\frac{\partial v}{\partial \lambda} - \frac{\partial u \cos \phi}{\partial \phi} \right), \quad \delta = \frac{1}{a \cos \phi} \left(\frac{\partial u}{\partial \lambda} + \frac{\partial v \cos \phi}{\partial \phi} \right). \quad (6)$$

In addition, to find the velocity components and to close the system of equations the Helmholtz decomposition $\mathbf{v} = \hat{\mathbf{z}} \times \nabla \psi + \nabla \chi$ is employed where ψ and χ denote the stream function and velocity potential, respectively. To find the stream function and velocity potential, the following Poisson equations must be solved:

$$\nabla^2 \psi = \zeta, \quad \nabla^2 \chi = \delta. \quad (7)$$

3. The semi-implicit Runge-Kutta method

In this work the semi-implicit Runge-Kutta (hereafter, SIRK) time advancing method developed in [39] for the flux convergence form of the shallow water equations in Cartesian coordinates, is extended to the VDH representation of the spherical shallow water equations. The semi-implicit formulation of the shallow water equations takes a much simpler form when the vorticity and divergence are used as prognostic variables compared to formulations where the velocity components are used.

For the VDH form of the shallow water equations, the SIRK formulation is applied to equations (4) and (5) [40]. The Runge-Kutta formulation used in the present work is a third-order explicit three step method [41].

Equations (4) and (5) can be rewritten in the following vector form:

$$\frac{\partial \mathbf{q}}{\partial t} = L(\mathbf{q}) + N(\mathbf{q}) \quad (8)$$

where $\mathbf{q} = (\delta, \tilde{h})^T$, $L(\mathbf{q}) = (-\mathcal{H}\tilde{h}, -\delta)^T$ and $N(\mathbf{q}) = (S_\delta, S_h)^T$. The superscript T denotes transpose. The semi-implicit approach which employs the third-order Runge-Kutta method for time advancement of equation (8) leads to

$$\frac{\mathbf{q}^{n+\alpha_k} - \mathbf{q}^n}{\alpha_k \Delta t} = \frac{L(\mathbf{q}^{n+\alpha_k}) + L(\mathbf{q}^n)}{2} + N(\mathbf{q}^{n+\alpha_{k-1}}) \quad (9)$$

where the superscript n denotes the time level. The three steps of the Runge-Kutta method are written in a compact form where subscript k ($k = 1, 2, 3$) denotes the three steps of the Runge-Kutta method and the values of α_k are $\alpha_0 = 0$, $\alpha_1 = 1/3$, $\alpha_2 = 1/2$ and $\alpha_3 = 1$.

The following auxiliary variables are defined

$$\bar{\delta}^{n+\alpha_k} = \frac{\delta^{n+\alpha_k} + \delta^n}{2}, \quad \bar{h}^{n+\alpha_k} = \frac{\tilde{h}^{n+\alpha_k} + \tilde{h}^n}{2}. \quad (10)$$

By substituting the auxiliary variables (10) into equation (9), the following equations for the auxiliary variables can be found

$$(\mathcal{H} - \frac{4}{\alpha_k^2 \Delta t^2}) \bar{h}^{n+\alpha_k} = \frac{2}{\alpha_k \Delta t} (\delta^n - S_h^{n+\alpha_{k-1}}) + S_\delta^{n+\alpha_{k-1}} - \frac{4}{\alpha_k^2 \Delta t^2} \tilde{h}^n \quad (11)$$

$$\bar{\delta}^{n+\alpha_k} = \frac{2}{\alpha_k \Delta t} (\tilde{h}^n - \bar{h}^{n+\alpha_k}) + S_h^{n+\alpha_{k-1}}. \quad (12)$$

Equation (11) is a modified Helmholtz equation. The unknown variables $\delta^{n+\alpha_k}$ and $\tilde{h}^{n+\alpha_k}$ are found by substitution of the auxiliary variables into equation (10).

The time discretization of the vorticity equation (3) by the third-order Runge-Kutta method leads to:

$$\zeta^{n+\alpha_k} = \zeta^n + \alpha_k \Delta t S_\zeta^{n+\alpha_{k-1}} \quad (13)$$

where the definitions of n and α_k are given above.

4. Spatial differencing

For spatial differencing of the first and second derivatives in equations (11), (12), (13) and the diagnostic equations in (7), two families of compact finite difference schemes, i.e., the super compact finite difference method (SCFDM) and the combined compact finite difference method (CCFDM) are used.

In the SCFDM, approximations of the first (second) derivatives depends on all the nodal values and their odd (even) derivatives, but the CCFDM approximates the first and second derivatives simultaneously. See references [36, 37, 25] and [31, 32, 33, 34, 35] for more information on details of the derivation and alternative forms of the SCFDM and CCFDM, respectively.

4.1. The super compact method (SCFDM)

The SCFDM relations for approximation of the first derivative of an arbitrary function Φ in a uniform grid are

$$\mathbf{Q}_f \mathbf{F}_j = \mathbf{E}_j^F \quad (14)$$

where

$$\mathbf{F}_j = \begin{pmatrix} \Phi_j^{<1>} \\ \Phi_j^{<3>} \\ \vdots \\ \Phi_j^{<2M-1>} \end{pmatrix}, \quad \mathbf{E}_j^F = \begin{pmatrix} D^0 \Phi_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{Q}_f = \frac{1}{d} \begin{pmatrix} \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \cdots & \frac{1}{(2M-1)!} \\ -\frac{d^2}{2} D^2 & \frac{1}{2!} & \frac{1}{4!} & \cdots & \frac{1}{[2(M-1)]!} \\ 0 & -\frac{d^2}{2} D^2 & \frac{1}{2!} & \cdots & \frac{1}{[2(M-2)]!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\frac{d^2}{2} D^2 & \frac{1}{2!} \end{pmatrix}$$

In (14), the difference operators are defined as

$$D^0 \Phi_j = (\Phi_{j+1} - \Phi_{j-1})/2d, \quad D^2 \Phi_j = (\Phi_{j+1} - 2\Phi_j + \Phi_{j-1})/d^2$$

in which d denotes the grid spacing and $\Phi_j^{<l>} = d^l \left(\frac{\partial^l \Phi}{\partial x^l} \right)_j$. $\mathbf{Q}_f$ is a $M \times M$ matrix, $\mathbf{F}$ and $\mathbf{E}^F$ are M dimensional vectors. The expression $\Phi_j^{<2l-1>}/d^{2l-1}$ ($l = 1, 2, \dots, M$) approximates $\frac{\partial^{2l-1} \Phi}{\partial x^{2l-1}}$ with an accuracy of order $2(M-l+1)$ [36, 37].

In a similar way, the SCFDM relations for approximation of the second derivative on a uniform grid are

$$\mathbf{Q}_s \mathbf{S}_j = \mathbf{E}_j^S \quad (15)$$

in which

$$\mathbf{S}_j = \begin{pmatrix} \Phi_j^{<2>} \\ \Phi_j^{<4>} \\ \vdots \\ \Phi_j^{<2M>} \end{pmatrix}, \quad \mathbf{E}_j^S = \begin{pmatrix} \frac{1}{2} D^2 \Phi_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{Q}_s = \frac{1}{d^2} \begin{pmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \cdots & \frac{1}{(2M)!} \\ -\frac{d^2}{2} D^2 & \frac{1}{2!} & \frac{1}{4!} & \cdots & \frac{1}{[2(M-1)]!} \\ 0 & -\frac{d^2}{2} D^2 & \frac{1}{2!} & \cdots & \frac{1}{[2(M-2)]!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\frac{d^2}{2} D^2 & \frac{1}{2!} \end{pmatrix}$$

Here, $\mathbf{Q}_s$ is a $M \times M$ matrix, $\mathbf{S}$ and $\mathbf{E}^S$ are M dimensional vector. The expression $\Phi_j^{<2l>}/d^{2l}$ approximates $\frac{\partial^{2l} \phi}{\partial x^{2l}}$ with an accuracy of order $2(M-l+1)$ [36, 37].

Using different values of M ($M = 1, 2, 3, \dots$) in equations (14) and (15) leads to different orders of the SCFDM relations for approximation of the first and second derivatives. For example, $M = 1$ leads to the conventional second-order central finite difference formulation. The fourth-order Pade type compact finite difference are found by using $M = 2$. Higher values of M leads to the higher order SCFDM relations.

4.1.1. The SCFDM relations Using $M = 2, 3, 4$ in equation (14) leads to the fourth-order, sixth-order and eighth-order SCFDM relations for the approximation of the first derivative. The fourth-order relation reads

$$\frac{1}{6}\Phi_{j-1}^{(1)} + \frac{2}{3}\Phi_j^{(1)} + \frac{1}{6}\Phi_{j+1}^{(1)} - \frac{1}{2d}(\Phi_{j+1} - \Phi_{j-1}) = 0, \quad (16)$$

the sixth-order relations are

$$\begin{cases} \Phi_{j+1} - \Phi_{j-1} - 2d\Phi_j^{(1)} - \frac{d^3}{60}\Phi_{j-1}^{(3)} - \frac{3d^3}{10}\Phi_j^{(3)} - \frac{d^3}{60}\Phi_{j+1}^{(3)} = 0 \\ \Phi_{j+1}^{(1)} - 2\Phi_j^{(1)} + \Phi_{j-1}^{(1)} - \frac{d^2}{12}\Phi_{j-1}^{(3)} - \frac{5d^2}{6}\Phi_j^{(3)} - \frac{d^2}{12}\Phi_{j+1}^{(3)} = 0, \end{cases} \quad (17)$$

and the eighth-order relations are found as

$$\begin{cases} \Phi_{j+1} - \Phi_{j-1} - 2d\Phi_j^{(1)} - \frac{d^3}{3}\Phi_j^{(3)} - \frac{2d^5}{5040}\Phi_{j-1}^{(5)} - \frac{2d^5}{126}\Phi_j^{(5)} - \frac{2d^5}{5040}\Phi_{j+1}^{(5)} = 0 \\ \Phi_{j+1}^{(1)} - 2\Phi_j^{(1)} + \Phi_{j-1}^{(1)} - d^2\Phi_j^{(3)} - \frac{d^4}{360}\Phi_{j-1}^{(5)} - \frac{7d^4}{90}\Phi_j^{(5)} - \frac{d^4}{360}\Phi_{j+1}^{(5)} = 0 \\ \Phi_{j+1}^{(3)} - 2\Phi_j^{(3)} + \Phi_{j-1}^{(3)} - \frac{d^2}{12}\Phi_{j-1}^{(5)} - \frac{5d^2}{6}\Phi_j^{(5)} - \frac{d^2}{12}\Phi_{j+1}^{(5)} = 0. \end{cases} \quad (18)$$

Similarly, using $M = 2, 3, 4$ in equation (15) leads to the fourth-order, sixth-order and eighth-order SCFDM relations for the approximation of the second derivative. The fourth-order relation is

$$\frac{1}{12}\Phi_{j-1}^{(2)} + \frac{5}{6}\Phi_j^{(2)} + \frac{1}{12}\Phi_{j+1}^{(2)} - \frac{1}{d^2}(\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}) = 0, \quad (19)$$

the sixth-order relations are

$$\begin{cases} \Phi_{j+1} - 2\Phi_j + \Phi_{j-1} - d^2\Phi_j^{(2)} - \frac{d^4}{360}\Phi_{j-1}^{(4)} - \frac{7d^4}{90}\Phi_j^{(4)} - \frac{d^4}{360}\Phi_{j+1}^{(4)} = 0 \\ \Phi_{j+1}^{(2)} - 2\Phi_j^{(2)} + \Phi_{j-1}^{(2)} - \frac{d^2}{12}\Phi_{j-1}^{(4)} - \frac{5d^2}{6}\Phi_j^{(4)} - \frac{d^2}{12}\Phi_{j+1}^{(4)} = 0, \end{cases} \quad (20)$$

and the eighth-order relations are found as

$$\begin{cases} \Phi_{j+1} - 2\Phi_j + \Phi_{j-1} - d^2\Phi_j^{(2)} - \frac{d^4}{12}\Phi_j^{(4)} - \frac{d^6}{20160}\Phi_{j-1}^{(6)} - \frac{3d^6}{1120}\Phi_j^{(6)} - \frac{d^6}{20160}\Phi_{j+1}^{(6)} = 0 \\ \Phi_{j+1}^{(2)} - 2\Phi_j^{(2)} + \Phi_{j-1}^{(2)} - d^2\Phi_j^{(4)} - \frac{d^4}{360}\Phi_{j-1}^{(6)} - \frac{7d^4}{90}\Phi_j^{(6)} - \frac{d^4}{360}\Phi_{j+1}^{(6)} = 0 \\ \Phi_{j+1}^{(4)} - 2\Phi_j^{(4)} + \Phi_{j-1}^{(4)} - \frac{d^2}{12}\Phi_{j-1}^{(6)} - \frac{5d^2}{6}\Phi_j^{(6)} - \frac{d^2}{12}\Phi_{j+1}^{(6)} = 0. \end{cases} \quad (21)$$

In these equations, the superscript (k) denotes k th spatial derivative.

4.2. The combined compact method (CCFDM)

The CCFDM relations for the first and second derivatives on a uniform grid can be written as

$$\mathbf{Q}_c \mathbf{T}_j = \mathbf{G}_j \quad (22)$$

where

$$\mathbf{T}_j = \begin{pmatrix} \Phi_j^{<1>} \\ \Phi_j^{<2>} \\ \vdots \\ \Phi_j^{<2N-1>} \\ \Phi_j^{<2N>} \end{pmatrix}, \quad \mathbf{G}_j = \begin{pmatrix} D^0\Phi_j \\ D^2\Phi_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\mathbf{Q}_c = \begin{pmatrix} \frac{a}{1!} & 0 & \frac{a}{3!} & 0 & \cdots & \frac{a}{(2N-1)!} & 0 \\ 0 & \frac{2b}{2!} & 0 & \frac{2b}{4!} & \cdots & 0 & \frac{2b}{(2N)!} \\ -D^0 & \frac{a}{1!} & 0 & \frac{a}{3!} & \cdots & 0 & \frac{a}{(2N-1)!} \\ -D^2 & 0 & \frac{2b}{2!} & 0 & \cdots & \frac{2b}{(2N)!} & 0 \\ 0 & -D^0 & \frac{a}{1!} & 0 & \cdots & 0 & \frac{a}{(2N-3)!} \\ 0 & -D^2 & 0 & \frac{2b}{2!} & \cdots & \frac{2b}{(2N-2)!} & 0 \\ 0 & 0 & -D^0 & \frac{a}{1!} & \cdots & 0 & \frac{a}{(2N-5)!} \\ 0 & 0 & -D^2 & 0 & \cdots & \frac{2b}{(2N-4)!} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \end{pmatrix}$$

and $a = 1/d$, $b = 1/d^2$. For the CCFDM method, $\mathbf{Q}_c$ is a $2N \times 2N$ matrix and $\mathbf{T}$ and G are $2N$ dimensional vectors. Equation (22) approximates both the first and second derivatives of Φ with an accuracy of order $2N$.

Substitution of different values of N ($N = 1, 2, 3, \dots$) in equation (22) leads to various orders of the CCFDM relations for approximation of the first and second derivatives simultaneously. Similar to the SCFDM, $N = 1$ and $N = 2$ lead to the second-order central finite difference and fourth-order Pade type compact finite difference relations. Higher values of N leads to the higher order CCFDM relations.

4.2.1. The CCFDM relations For the CCFDM method by using $N = 3, 4$ in equation (22), the sixth-order and eighth-order relations for the approximation of the first and second derivatives are obtained. The sixth-order relations are

$$\begin{cases} \frac{7}{16}(\Phi_{j+1}^{(1)} + \Phi_{j-1}^{(1)}) + \Phi_j^{(1)} - \frac{d}{16}(\Phi_{j+1}^{(2)} - \Phi_{j-1}^{(2)}) - \frac{15}{16d}(\Phi_{j+1} - \Phi_{j-1}) = 0 \\ \frac{9}{8d}(\Phi_{j+1}^{(1)} - \Phi_{j-1}^{(1)}) - \frac{1}{8}(\Phi_{j+1}^{(2)} + \Phi_{j-1}^{(2)}) + \Phi_j^{(2)} - \frac{3}{d^2}(\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}) = 0, \end{cases} \quad (23)$$

and the eighth-order relations are found as

$$\begin{cases} \frac{19}{32}(\Phi_{j+1}^{(1)} + \Phi_{j-1}^{(1)}) + \Phi_j^{(1)} - \frac{d}{8}(\Phi_{j+1}^{(2)} - \Phi_{j-1}^{(2)}) + \frac{d^2}{96}(\Phi_{j+1}^{(3)} + \Phi_{j-1}^{(3)}) - \frac{35}{32d}(\Phi_{j+1} - \Phi_{j-1}) = 0 \\ \frac{29}{16d}(\Phi_{j+1}^{(1)} - \Phi_{j-1}^{(1)}) - \frac{5}{16}(\Phi_{j+1}^{(2)} + \Phi_{j-1}^{(2)}) + \Phi_j^{(2)} + \frac{d}{48}(\Phi_{j+1}^{(3)} - \Phi_{j-1}^{(3)}) - \frac{4}{d^2}(\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}) = 0 \\ \frac{-105}{16d^2}(\Phi_{j+1}^{(1)} + \Phi_{j-1}^{(1)}) + \frac{15}{8d}(\Phi_{j+1}^{(2)} - \Phi_{j-1}^{(2)}) - \frac{3}{16}(\Phi_{j+1}^{(3)} + \Phi_{j-1}^{(3)}) + \Phi_j^{(3)} + \frac{105}{16d^3}(\Phi_{j+1} - \Phi_{j-1}) = 0. \end{cases} \quad (24)$$

4.3. The numerical grid

The VDH form of the spherical shallow water equations are solved on the doubly periodic longitude-latitude grid system ($0 \leq \lambda \leq 2\pi$, $-\pi/2 \leq \phi \leq \pi/2$) of reference [42]. The grid points in this system are arranged such that no grid points are located at the poles. This grid system has the advantage of avoiding the singularities at the poles and enabling the periodic

boundary conditions to be imposed in both longitude and latitude directions. It should be noted that the sign of a quantity must be adjusted properly when using the periodic boundary condition in the latitude direction, depending on whether the quantity is a scalar or a vector. The grid points are defined as $(\lambda_i, \phi_j) = (i\Delta\lambda, -\pi/2 + (j - 1/2)\Delta\phi)$ where $\Delta\lambda = 2\pi/N_\lambda$ and $\Delta\phi = \pi/N_\phi$. N_λ and N_ϕ are the number of grid points in latitude and longitude directions, respectively.

4.4. The spatial filter

The generation of fine-scale vortical structures by the nonlinear advection terms is a dominant feature of the shallow water equations. A problem facing any grid-based numerical method developed for the shallow water equations is misinterpretation of the fine-scale structures due to insufficient resolution and the potential for non-linear numerical instability. In this work, an eighth-order compact filter [47] has been selected to control the non-linear instability. For a discrete field Υ with values Υ_j in either λ or ϕ direction, the stencil of the compact filter reads

$$\sum_{i=0}^{i=2} \alpha_i (\tilde{\Upsilon}_{j-i} + \tilde{\Upsilon}_{j+i}) = \sum_{i=0}^{i=4} a_i (\Upsilon_{j-i} + \Upsilon_{j+i}) \quad (25)$$

where $\tilde{\Upsilon}_j$ is the filtered Υ at point j and

$$\begin{aligned} \alpha_0 &= 0.5, \alpha_1 = 0.66624, \alpha_2 = 0.16688, \\ 2a_0 &= 0.99965, a_1 = 0.66652, a_2 = 0.16674, a_3 = 4 \times 10^{-5}, a_4 = -5 \times 10^{-6}. \end{aligned}$$

The spatial filter is sequentially applied in latitude and longitude directions at each time step of the time integration.

In addition, on a regular latitude-longitude system the lines of constant longitude converge at the poles which leads to a high concentration of grid points near poles. The cluster of grid points forces a short time step to satisfy the CFL stability condition and this is often referred to as the polar problem [48]. A simple spatial filter of the following form

$$\tilde{\Upsilon}_j = \Upsilon_j + \sigma_p (\Delta\phi^2)^p \left(\frac{\partial^2 \Upsilon}{\partial \phi^2} \right)_j^p, \quad p = 1, 2, 3, \dots \quad (26)$$

is used as a smoother to reduce the polar problem. In (26), $\tilde{\Upsilon}_j$ denotes the filtered values. The unknown coefficient σ_p is determined by setting the location of cutoff in the transfer function of the filter (e.g., [49]). The spatial filter coefficient can be found as

$$\sigma_p = \left(\frac{-1}{T_2(\kappa_{\max})} \right)^p \quad (27)$$

where T_2 is the transfer function of the second derivative. Table I presents values of σ_p for the various schemes used in this study.

The same spatial differencing method that is applied to discretize the governing equations is used to estimate the even derivative in the spatial filter (26).

We apply the spatial filter (26) only in longitude direction and near the poles, i.e., $\frac{0.9\pi}{2} \leq \phi \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq \phi \leq -\frac{0.9\pi}{2}$, to reduce the polar problem. We have used $p = 2$ for the filter power.

Table I. values of the spatial filter coefficient, σ_p , for different methods.

Method	σ_p
Fourth-order compact	$\left(\frac{-1}{6}\right)^p$
Sixth-order SCFDM	$\left(\frac{-1}{7.05882353}\right)^p$
Sixth-order CCFDM	$\left(\frac{-1}{9.6}\right)^p$
Eighth-order SCFDM	$\left(\frac{-1}{7.67741935}\right)^p$
Eighth-order CCFDM	$\left(\frac{-1}{9.84615385}\right)^p$

5. The elliptic equations

The SIRC formulation used to advance the shallow water equations in time leads to a set of modified Helmholtz equations (11) at each successive step of the Runge-Kutta method. To find the velocity components, the two Poisson equations given in (7) must be solved at each step.

To limit the computational cost, an efficient algorithm for numerical solution of the Helmholtz and Poisson equations must be used (e.g., [43]). Therefore, the separable elliptic equations (11) and (7) are solved using a direct method. In this approach, the spatial discretization of the derivatives in longitudinal direction are performed by discrete Fourier series and the FFT (Fast Fourier Transform) technique. In the latitudinal direction we discretize using different orders of the SCFDM and CCFDM schemes. The procedure is similar to the one developed in [43, 44, 45]. However, some changes are applied to make the procedure easier and avoiding the interpolation used in [45].

It is worth to note that in this work the FFT technique is only used to spatial discretization of the derivatives in longitude direction for numerical solution of the Helmholtz and Poisson equations. In other parts of the algorithm, all spatial differencings are performed by the compact schemes in both latitude and longitude directions.

5.1. The Helmholtz equation

The modified Helmholtz equation (11) can be written as

$$\nabla^2 \bar{h} - \beta \bar{h} = R \quad (28)$$

where the Laplacian operator is

$$\nabla^2 = \frac{1}{a^2 \cos^2 \phi} \frac{\partial^2}{\partial \lambda^2} + \frac{1}{a^2 \cos \phi} \left(\frac{\partial}{\partial \phi} \cos \phi \frac{\partial}{\partial \phi} \right).$$

In (28), β is a function of ϕ and R denotes the known right hand side of the elliptic equation.

The longitudinal parts of $\bar{h}$ and R in (28) are approximated by truncated Fourier series. By substitution of the Fourier components $\bar{h} = \hat{h}_m e^{im\lambda}$ and $R = \hat{R}_m e^{im\lambda}$ with wave number m into equation (28) we have

$$\left(\frac{-m^2}{a^2 \cos^2 \phi} - \beta \right) \hat{h} + \frac{1}{a^2} \frac{d^2 \hat{h}}{d\phi^2} - \frac{\tan \phi}{a^2} \frac{d\hat{h}}{d\phi} = \hat{R}. \quad (29)$$

Now, the fourth-order compact, the sixth- and eighth-order SCFDM and CCFDM methods are used to discretize the first and second latitudinal derivatives in equation (29). We arrive at the following block tridiagonal system of equations

$$\mathbf{A}_j^S \mathbf{U}_{j-1}^S + \mathbf{B}_j^S \mathbf{U}_j^S + \mathbf{C}_j^S \mathbf{U}_{j+1}^S = \mathbf{D}_j^S \quad (30)$$

where the superscript S is used to denote the spatial differencing method. The solution vector $\mathbf{U}$ for the different methods are

$$\mathbf{U}_j^{\text{C4}} = \begin{pmatrix} \hat{h}_j \\ \hat{h}_j^{(1)} \\ \hat{h}_j^{(2)} \end{pmatrix}, \mathbf{U}_j^{\text{SC6}} = \begin{pmatrix} \hat{h}_j \\ \hat{h}_j^{(1)} \\ \hat{h}_j^{(2)} \\ \hat{h}_j^{(3)} \\ \hat{h}_j^{(4)} \end{pmatrix}, \mathbf{U}_j^{\text{SC8}} = \begin{pmatrix} \hat{h}_j \\ \hat{h}_j^{(1)} \\ \hat{h}_j^{(2)} \\ \hat{h}_j^{(3)} \\ \hat{h}_j^{(4)} \\ \hat{h}_j^{(5)} \\ \hat{h}_j^{(6)} \end{pmatrix}, \mathbf{U}_j^{\text{CC6}} = \begin{pmatrix} \hat{h}_j \\ \hat{h}_j^{(1)} \\ \hat{h}_j^{(2)} \end{pmatrix}, \mathbf{U}_j^{\text{CC8}} = \begin{pmatrix} \hat{h}_j \\ \hat{h}_j^{(1)} \\ \hat{h}_j^{(2)} \\ \hat{h}_j^{(3)} \end{pmatrix}. \quad (31)$$

The superscripts C4, SC6, SC8, CC6, CC8 denote the fourth-order compact, the sixth-order SCFDM, the eighth-order SCFDM, the sixth-order CCFDM and the eighth-order CCFDM methods, respectively. Equation (30) is a three-point implicit formulation which solution leads to a linear block tridiagonal system of equations. The block size for the fourth-order compact is 3×3 , for the sixth-order SCFDM is 5×5 , for the eighth-order SCFDM is 7×7 , for the sixth-order CCFDM is 3×3 and for the eighth-order CCFDM is 4×4 .

To close the system at the south pole ($j = 1$) and the north pole ($j = N_\phi$), additional conditions are needed. The boundary values are obtained by using the symmetry constraint of the Fourier coefficients (e.g., [44]). For the grid system used in the present work, the symmetry constraints imply that at $j = 1$ we have $\hat{h}_0^{(k)} = (-1)^k (-1)^m \hat{h}_1^{(k)}$ and $\hat{h}_{N_\phi+1}^{(k)} = (-1)^k (-1)^m \hat{h}_{N_\phi}^{(k)}$ for $j = N_\phi$. The superscript (k) denotes as before the k th derivative ($k = 0, 1, 3, \dots$) and m denotes the wave number. There is one sign change due to odd and even wave numbers and another sign change related to the spatial derivative.

By inserting these values into equation (30) at the boundaries and solving the resulting block tridiagonal system of equations, the values of $\hat{h}$ are obtained and then by using FFT, the values of $\bar{h}_{i,j}$ are found. The details of matrix coefficients $\mathbf{A}_j^S$, $\mathbf{B}_j^S$, $\mathbf{C}_j^S$ and right hand side vector $\mathbf{D}_j^S$ are given in appendix I.

5.2. The Poisson equation

The Poisson equations in (7) have the following form

$$\nabla^2 \psi = \zeta. \quad (32)$$

The procedure for the numerical solution of the Poisson equations is similar to the procedure for the modified Helmholtz equation above and can be reached by letting $\beta = 0$ in equation (29). For nonzero wave numbers ($m \neq 0$) equation (32) can be solved in a similar manner as the modified Helmholtz equation.

The procedure for the zeroth Fourier mode ($m = 0$) is different. By using $\beta = 0$ and $m = 0$ in (29), $\hat{h}$ vanishes and the Poisson equation for wave number $m = 0$ is singular, since the

solution is unique within a constant. In fact, the Poisson equation on a sphere must satisfy the following compatibility condition (e.g., [46, 44])

$$\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \zeta \cos \phi d\phi d\lambda = 0 \quad (33)$$

to guarantee the uniqueness of solution.

By substitution of the Fourier components in equation (32), and considering $m = 0$ we have

$$\frac{1}{a^2} \frac{d^2 \bar{\psi}}{d\phi^2} - \frac{\tan \phi}{a^2} \frac{d\bar{\psi}}{d\phi} = \bar{\zeta} \quad (34)$$

in which $\bar{\psi} = \hat{\psi}(m = 0)$ and $\bar{\zeta} = \hat{\zeta}(m = 0)$. After application of the Fourier method in longitude direction the equivalent form of the compatibility condition (33) for the latitudinal direction is

$$\int_{-\pi/2}^{\pi/2} \bar{\zeta} \cos \phi d\phi = 0. \quad (35)$$

To impose the compatibility condition, first the fourth-order compact, the sixth- and eighth-order SCFDM and the sixth- and eighth-order CCFDM methods are used to discretize the first and second latitudinal derivatives in equation (34) in the same way as for (29), and then the right hand side of equation (34) is replaced by $\bar{\zeta}_j - P$ where P is a constant. P can be found by using the discrete version of the compatibility condition (35) as

$$P = \sum_{j=1}^{j=N_\phi} \bar{\zeta}_j \cos \phi_j. \quad (36)$$

Now, by finding the values of $\bar{\psi}_j$ for the zeroth Fourier mode and having the values of $\hat{\psi}_j$ for other wave numbers we employ the FFT to find the solution of the Poisson equation. The matrix coefficients and right hand side vector for different compact methods used to discretize equation (34) are also described in appendix I.

6. The accuracy and order of convergence

The accuracy and the spatial order of convergence of high-order compact schemes used in this work is assessed by using the standard test cases proposed by Williamson *et al.* [38]. We use test cases 1 and 2 which have analytical solutions.

Test case 1 studies the advection of a cosine bell around a sphere. The parameter α is used to set the angle of solid body rotation. We have used different values of α as suggested by Williamson *et al.* [38] to perform the simulations.

Test case 2 is a steady state analytical solution to the full nonlinear spherical shallow water equations. In this study we set $\alpha = 0$ which means that the Coriolis parameter is only a function of latitude (i.e., for other choices the Coriolis parameter will be a function of both latitude and longitude). The direct approach used to solve the elliptic equations on sphere is limited to separable elliptic equations (see section 5). Furthermore, the coefficients of separable elliptic equations on the sphere can be at most a function of latitude (e.g., [43]). Therefore,

we limit our calculation to the case $\alpha = 0$. Note that in most practical and operational global atmospheric models the Coriolis parameter is only a function of latitude.

The following normalized global errors are used for the error measurements [38]:

$$l_1() = \frac{I(|\Psi_n(\lambda, \phi) - \Psi_e(\lambda, \phi)|)}{I(|\Psi_e(\lambda, \phi)|)} \quad (37)$$

$$l_2(\Psi) = \frac{\{I(|\Psi_n(\lambda, \phi) - \Psi_e(\lambda, \phi)|^2)\}^{1/2}}{\{I(|\Psi_e(\lambda, \phi)|^2)\}^{1/2}} \quad (38)$$

$$l_\infty(\Psi) = \frac{\max_{\text{all } \lambda, \phi} (|\Psi_n(\lambda, \phi) - \Psi_e(\lambda, \phi)|)}{\max_{\text{all } \lambda, \phi} (|\Psi_e(\lambda, \phi)|)} \quad (39)$$

where Ψ_n and Ψ_e are the numerical and exact solutions, respectively. The function I in (37)-(39) is defined as

$$I(\Psi) = \frac{1}{4\pi} \sum_{k=1}^{N_\lambda} \sum_{l=1}^{N_\phi} \Psi_{k,l} \cos \phi \Delta \phi \Delta \lambda \quad (40)$$

in which N_λ and N_ϕ are the number of grid points in longitude and latitude directions and $\Psi_{k,l}$ is the argument at grid point (λ_k, ϕ_l) .

Tables II and III present values of normalized global errors $l_1(h)$, $l_2(h)$ and $l_\infty(h)$, where $h = H(1 + \tilde{h})$, for test case 1 after one rotation (i.e., at 12 days). The grid resolution is $N_\lambda \times N_\phi = 128 \times 64$ and a time step of $\Delta t = 240$ s is used. Table II shows the $l_2(h)$ error for different compact methods when different values of parameter α are used. In addition, to be able to compare our results with others, table III presents normalized global errors for different methods when the parameter for the angle of solid body rotation is set to $\alpha = \pi/2 - 0.05$. The results using other methods are included in the table for comparison. In particular, we have included results using spectral transforms [22] and double Fourier series [50].

Table II. Normalized global error $l_2(h)$ for test case 1 for different methods and different values of α after one rotation. The grid resolution is $N_\lambda \times N_\phi = 128 \times 64$.

Method	$\alpha = 0$	$\alpha = 0.05$	$\alpha = \frac{\pi}{2} - 0.05$	$\alpha = \frac{\pi}{2}$
Fourth-order compact	0.043	0.043	0.045	0.047
Sixth-order SCFDM	0.012	0.012	0.023	0.021
Sixth-order CCFDM	0.010	0.011	0.019	0.019
Eighth-order SCFDM	0.007	0.007	0.016	0.016
Eighth-order CCFDM	0.006	0.007	0.017	0.015

The 6th-order and 8th-order SCFDM and CCFDM methods generate, as expected, less errors than the 4th-order compact method. The results of our 4th-order compact is very close to the results obtained by the fourth-order compact with spherical harmonic filter in [22]. Note the similarity of the SCFDM and CCFDM methods to the results generated by the spectral transform and the double Fourier series methods. It can also be seen that the SCFDM and CCFDM schemes are more accurate than the explicit 6th-order and 8th-order central finite difference methods.

Table III. Normalized global errors for test case 1 for different methods after one rotation. The grid resolution is $N_\lambda \times N_\phi = 128 \times 64$ and $\alpha = \pi/2 - 0.05$.

Method	$l_1(h)$	$l_2(h)$	$l_\infty(h)$
Fourth-order compact	0.177	0.045	0.036
Sixth-order SCFDM	0.136	0.023	0.009
Sixth-order CCFDM	0.119	0.019	0.009
Eighth-order SCFDM	0.103	0.016	0.007
Eighth-order CCFDM	0.109	0.017	0.008
Sixth-order central finite difference [24]	0.154	0.041	0.034
Eighth-order central finite difference [24]	0.090	0.022	0.013
Fourth-order compact with spherical harmonic filter [22]	NA	0.042	NA
Spectral Transform Method [22]	NA	0.011	NA
Double Fourier Series [50]	0.047	0.022	NA

To assess the spatial order of convergence of high-order compact schemes, we start by measuring the spatial order of convergence of the elliptic equations. First a Helmholtz equation with a known analytical solution is considered. The following elliptic equation

$$\nabla^2 \Psi - a\Psi = R_\Psi \quad (41)$$

with the Rossby-Haurwitz wave [38],

$$\Psi = -a^2 \nu \sin \phi + a^2 K \cos^4 \phi \sin \phi \cos 4\lambda \quad (42)$$

as the analytical solution, is solved. In equation (41), $R_\Psi = 2\nu \sin \phi - 30K \sin \phi \cos^4 \phi \cos 4\lambda + a^3 \nu \sin \phi - a^3 K \cos^4 \phi \sin \phi \cos 4\lambda$ and a is the radius of the Earth. The constants ν and K are $\nu = K = 7.848 \times 10^{-6} \text{ s}^{-1}$. To investigate the order of convergence, the numerical solution of equation (41) is compared with the analytical solution (42).

Table IV presents the normalized global error l_2 for the Ψ field of different compact methods at different resolutions. It can be seen that the solution is calculated to machine precision. In fact the accuracy of Ψ is determined by the accuracy of the methods used to discretize the elliptic equation (i.e., using FFT to spatial discretization of the derivatives in longitude and using a high-order compact method to spatial differencing of derivatives in latitude direction). The values of global error l_2 for Ψ indicate that the solution of the elliptic equation has converged to the exact solution.

The compact finite difference methods are only applied for discretization in the latitudinal direction of the elliptic equations (see section 5). Therefore, to find the order of convergence we use $\partial\Psi/\partial\phi$ instead of Ψ . Figure 1 presents the normalized global error l_2 for $\partial\Psi/\partial\phi$ at different uniform horizontal resolutions in longitude and latitude directions for different schemes. Figure 1 shows that the different convergence rates are in agreement with theoretical order of convergence. In addition, it is seen that the 6th-order CCFDM method produces less error than the 6th-order SCFDM method and the 8th-order CCFDM generates less error than the 8th-order SCFDM method, respectively. Furthermore, a comparison of results presented in table IV with those reported in figure 1 reveals that the accuracy of solution of elliptic

Table IV. Normalized l_2 error of the Ψ field for different grid points obtained using 4th-order compact method, 6th-order SCFDM, 6th-order CCFDM, 8th-order SCFDM and 8th-order CCFDM.

$N_\lambda \times N_\phi$	4th-order Compact	6th-order SCFDM	6th-order CCFDM	8th-order SCFDM	8th-order CCFDM
64×32	1.437×10^{-16}	1.437×10^{-16}	1.437×10^{-16}	1.437×10^{-16}	1.437×10^{-16}
128×64	2.389×10^{-16}	2.389×10^{-16}	2.389×10^{-16}	2.389×10^{-16}	2.389×10^{-16}
256×128	1.904×10^{-16}	1.904×10^{-16}	1.904×10^{-16}	1.904×10^{-16}	1.904×10^{-16}
512×256	2.417×10^{-16}	2.417×10^{-16}	3.851×10^{-16}	2.417×10^{-16}	3.908×10^{-16}

Figure 1. Normalized l_2 error of $\partial\Psi/\partial\phi$ field versus number of grid points obtained using 4th-order compact method, 6th-order SCFDM, 6th-order CCFDM, 8th-order SCFDM and 8th-order CCFDM. Lines with slopes 4, 6 and 8 are also shown. A uniform grid resolution is used in both λ and ϕ directions.

equation (Ψ) is higher than the accuracy of $\partial\Psi/\partial\phi$ which is calculated by compact methods used in this study.

The results in figure 1 present the spatial order of convergence for the elliptic equations involved in the algorithm. The order of convergence is in agreement with the design order of the schemes.

Next, we employ test case number 2 proposed by Williamson *et al.* [38] to measure the order of convergence for the full nonlinear equations. The following relation is used to calculate the

convergence rate

$$\mathbf{q} = \frac{\log_{10} \left(\frac{l_2(\Phi)^{\Delta\lambda_1}}{l_2(\Phi)^{\Delta\lambda_2}} \right)}{\log_{10} \left(\frac{\Delta\lambda_1}{\Delta\lambda_2} \right)} \quad (43)$$

where $l_2(\Phi)^{\Delta\lambda_1}$ denotes the l_2 -error of numerical solution corresponding with spatial grid space $\Delta\lambda_1$ ($= \Delta\phi_1$). Table V presents the convergence order for the height field at different grid resolutions. To study the accuracy in space we minimize the temporal error by using a very small time step, $\Delta t = 0.1$ s. We calculate the convergence rate at time $t = 500$ s. The coarsest grid resolution used to calculate the convergence rate in equation (43) is $N_\lambda \times N_\phi = 32 \times 16$. It can be seen that, the convergence orders are in agreement with theoretical order of convergence.

Table V. The convergence rate, $\mathbf{q}$, for the height field of test case 2 [38] at $t = 500$ s (with $\Delta t = 0.1$ s).

$N_\lambda \times N_\phi$	4th-order Compact	6th-order SCFDM	6th-order CCFDM	8th-order SCFDM	8th-order CCFDM
48×24	3.96991	6.00077	6.00201	8.00065	8.00160
64×32	3.99340	6.00058	6.00105	7.99967	8.00265
80×40	3.99793	6.00036	6.00063	7.99980	8.00797
96×48	3.99923	6.00024	6.00044	7.99979	8.04020
128×64	3.99978	6.00007	6.00017	7.99984	8.00301

In addition, the normalized global error l_2 for the height field in test case 2 after 1 day (86400 s) integration of the spherical shallow water equations at different uniform horizontal resolutions for various methods is presented in figure 2. A time step of 30 s is used for all resolutions. The results of figure 2 reveal that by using larger time steps and longer integration times, the order of convergence is reduced to second order. This finding is in agreement with the results reported by others (for example see references [22, 50, 26]). The reduced order of accuracy for long time integration is a result of the inevitable error growth present in hyperbolic problems. For cases involving boundary conditions, the error growth can be avoided, see [51] for an example.

7. Numerical results

7.1. Initial condition

The test case proposed in reference [52] which is an unstable barotropic mid-latitude zonal jet is used as the starting point for the numerical solution and evaluation of the spherical shallow water equations. The initial vorticity field of the unstable barotropic zonal jet is

$$\zeta = \frac{1}{a \cos \phi} \left(u \sin \phi - \cos \phi \frac{\partial u}{\partial \phi} \right). \quad (44)$$

Figure 2. Normalized l_2 error of h field versus number of grid points generated by different methods for test case 2 with $\alpha = 0$ at day 1. Line with slop 2 is also shown. A uniform grid resolution is used in both λ and ϕ directions.

For this test case $v = 0$ and $\delta = 0$ and

$$u = \begin{cases} 0 & \text{for } \phi \leq \phi_0 \\ \frac{u_{\max}}{e_n} \left(\frac{1}{(\phi - \phi_0)(\phi - \phi_1)} \right) & \text{for } \phi_0 < \phi < \phi_1 \\ 0 & \text{for } \phi \geq \phi_1 \end{cases} \quad (45)$$

is used in which the constants are

$$u_{\max} = 80 \text{ ms}^{-1}, \quad \phi_0 = \frac{\pi}{7}, \quad \phi_1 = \frac{\pi}{2} - \phi_0, \quad e_n = \exp\left[\frac{-4}{(\phi_1 - \phi_0)^2}\right].$$

The following relation is used to find the initial height

$$h = \frac{1}{g} \left\{ gh_0 - \int^{\phi} au(\phi') \left[f + \frac{\tan(\phi')}{a} u(\phi') \right] d\phi' \right\} \quad (46)$$

where the value of the constant h_0 needs to be found such that the domain area average of depth is $H = 10000$ m. Here, we have used the trapezoid rule to evaluate the integral in equation (46) numerically and find h to machine precision.

Next, the following perturbation is added to the basic height in equation (46) to trigger the barotropic instability

$$h'(\lambda, \phi) = h_p \cos(\phi) e^{-(\lambda/\gamma)^2} e^{-[(\phi_2 - \phi)/\bar{\beta}]^2}, \quad \text{for } -\pi < \lambda < \pi \quad (47)$$

where λ is longitude and the constants are $h_p = 120$ m, $\gamma = 1/3$, $\phi_2 = \pi/4$ and $\bar{\beta} = 1/15$.

7.2. Grid resolution and time step

The grid resolutions used in the numerical experiments reported here are $N_\lambda \times N_\phi = 128 \times 64, 256 \times 128, 512 \times 256$. To estimate the time steps a fixed Courant number of less than unity based on gravity-wave speed is used. The time steps $\Delta t = 120, 30, 6$ seconds have been used for the successive grids.

7.3. Results and discussions

We begin by providing some qualitative results for the vorticity field. Figure 3 presents the time evolution of the vorticity field for $N_\lambda \times N_\phi = 256 \times 128$ resolution at $t = 0, 2, 4, 6$ days (1 day = 86400 s) calculated by the eighth-order CCFDM. A qualitative comparison of these results with those presented in [52, 26, 53, 54, 8] indicates the validity of computations. In addition, figures 4 and 5 provide a qualitative comparison of the vorticity field calculated by the compact schemes used in this study at time $t = 6$ for $N_\lambda \times N_\phi = 256 \times 128$ and $N_\lambda \times N_\phi = 512 \times 256$ resolutions. It can be seen that the solutions generated by various schemes are qualitatively similar and a quantitative assessment of the solutions is needed to better inspect and understand the properties of different methods.

Figure 6 presents the time evolution of the maximum vorticity gradient for different methods. As sharp gradients of vorticity field are a feature of the unstable barotropic mid-latitude zonal jet test case [52, 26], it can be seen that all methods represent the vorticity field in a similar way. Note that the results of figure 6 are in good agreement with those presented in [52, 54] for different resolutions of the spectral method.

By inspecting the time evolution of potential enstrophy C_2 and potential palinstrophy P defined by

$$C_2 = \frac{1}{2} \langle (1 + \tilde{h}) Q^2 \rangle, \quad P = \frac{1}{2} \langle (1 + \tilde{h}) |\nabla Q| \rangle \quad (48)$$

we can compare the global spatial distribution of potential vorticity ($Q = (\zeta + f)/(1 + \tilde{h})$) in the solutions. In equation (48) the domain area average is denoted by $\langle \cdot \rangle$. The time variations of the percentage changes in C_2' and P , i.e. $(C_2'(t) - C_2'(0))/C_2'(0)$ and $(P(t) - P(0))/P(0)$ multiplied by 100, where $C_2' = C_2 - \langle f^2 \rangle / 2$ are shown in figures 7 and 8 for different methods and different grid resolutions. We have subtracted the time-independent contribution of $\langle f^2 \rangle / 2$ from C_2 to have a better quantification of the effects of damping/discretization error in destroying the global conservation of potential enstrophy.

Figure 7 shows that the sixth-order and eighth-order SCFDM and CCFDM methods have better global conservation properties of the potential enstrophy than the 4th-order compact method. For instance, at $N_\lambda \times N_\phi = 256 \times 128$ resolution, a less than 0.2% reduction in C_2' during the first 3 days is followed by a much stronger loss of potential enstrophy to near 2% at time $t = 6$. It can also be seen that a slightly better conservation of potential enstrophy is obtained when the order of spatial differencing of each scheme is increased.

Figure 3. The time evolution of the vorticity field at 2-day time intervals for unstable barotropic jet generated by the eighth-order CCFDM. The grid resolution is $N_\lambda \times N_\phi = 256 \times 128$ and contour interval is $1 \times 10^{-5} \text{ s}^{-1}$. Solid and dashed lines are for positive and negative contours, respectively. The zero contour is not shown and only the Northern Hemisphere is shown in figure.

For the potential palinstrophy presented in figure 8, as the grid resolution increases and the solution converges, a rapid growth of disturbances in the unstable jet which reaches a maximum at time $t = 6$ days can be observed. This is in agreement with the results presented in figure 6.

It is also meaningful to look at the long term time evolution of the globally averaged total energy defined by

$$E = \left\langle \frac{1}{2}h(u^2 + v^2) + \frac{1}{2}gh^2 \right\rangle \quad (49)$$

$h = H(1 + \tilde{h})$ and $\langle \rangle$ denotes the domain area average. Figure 9 shows a 20 days time evolution of the percentage changes in E' defined by $(E'(t) - E'(0))/E'(0)$ multiplied by 100, where $E' = E - \langle c^2 H \rangle / 2$. The time-independent part $\langle c^2 H \rangle / 2$ has been subtracted from E to have a better representation of changes in total energy. It is seen that the eighth-order SCFDM and CCFDM methods at time $t = 20$ days show approximately 0.6 percent reduction in total energy, the sixth-order SCFDM and CCFDM methods exhibit approximately 0.75 percent reduction in total energy and the fourth-order compact method shows approximately 1 percent reduction in total energy. Therefore, the worst results are generated by the fourth-order compact method.

In addition, Figure 9 presents the cumulative effect of the spatial filter, used to control

Figure 4. The vorticity field at time $t = 6$ days, for (a) 4th-order compact method, (b) 6th-order SCFDM, (c) 6th-order CCFDM, (d) 8th-order SCFDM and (e) 8th-order CCFDM. The grid resolution is $N_\lambda \times N_\phi = 256 \times 128$ and the contour interval is $1 \times 10^{-5} \text{ s}^{-1}$. Solid and dashed lines are for positive and negative contours, respectively.

the nonlinear instability, on the conservation of total energy and other invariants for different methods. It should be noted that the type and accuracy of the spatial filter plays a crucial role on the conservation properties of the numerical scheme and using an unsuitable spatial filter degrades it.

In addition, to compare the computational cost of different schemes used in this study, table VI presents the CPU times. For a meaningful comparison, the CPU times have been normalized by the CPU time of the fourth-order compact scheme at $N_\lambda \times N_\phi = 128 \times 64$ resolution. It can be seen the CCFDM methods are less expensive than the SCFDM methods.

Figure 5. The vorticity field at time $t = 6$ days, for (a) 4th-order compact method, (b) 6th-order SCFDM, (c) 6th-order CCFDM, (d) 8th-order SCFDM and (e) 8th-order CCFDM. The grid resolution is $N_\lambda \times N_\phi = 512 \times 256$ and the contour interval is $1 \times 10^{-5} \text{ s}^{-1}$. Solid and dashed lines are for positive and negative contours, respectively.

8. Conclusions

The focus of this work was on the application and qualitative and quantitative comparison of two families of high-order compact finite difference methods, namely, the super compact finite difference and the combined compact finite difference methods, to spatial differencing of the spherical shallow water equations in terms of vorticity, divergence and height on a regular latitude-longitude grid. They were compared with the standard fourth-order compact scheme. In addition, a semi-implicit Runge-Kutta method was developed for time advancing of the vorticity, divergence and height representation of the spherical shallow water equations. For

Figure 6. Time evolution of maximum vorticity gradient for different methods at grid resolutions (a) $N_\lambda \times N_\phi = 128 \times 64$, (b) $N_\lambda \times N_\phi = 256 \times 128$ and (c) $N_\lambda \times N_\phi = 512 \times 256$

numerical solution of the elliptic equations appeared in formulation of the problem the direct procedure proposed in references [44, 45] with minor changes was extended to the high-order compact finite difference schemes.

A study of the convergence rate, using an analytical test case, verifies the order of convergence for the full nonlinear equations. Qualitative and quantitative measurements for the test case developed in reference [52], shows that using the sixth-order and the eighth-order CCFDM and SCFDM for spatial differencing of the spherical shallow water equations lead to a noticeable improvement of the solution over the fourth-order compact method. However, the performance of sixth-order and eighth-order CCFDM methods is better than the sixth-order and eighth-order SCFDM methods.

Figure 7. Time evolution of the percentage of relative change in potential enstrophy for different methods at (a) $N_\lambda \times N_\phi = 128 \times 64$ and (b) $N_\lambda \times N_\phi = 256 \times 128$ resolutions.

Figure 8. Time evolution of the percentage of relative change in potential palinstrophy for different methods at (a) $N_\lambda \times N_\phi = 128 \times 64$ and (b) $N_\lambda \times N_\phi = 256 \times 128$ resolutions.

Figure 9. Time evolution of the percentage of relative change in total energy for different methods at $N_\lambda \times N_\phi = 256 \times 128$ resolution.

Table VI. The CPU times for different methods at $t = 1$ day. The unit of CPU time is taken to be that of the fourth-order compact method at $N_\lambda \times N_\phi = 128 \times 64$ resolution.

Method	$N_\lambda \times N_\phi = 128 \times 64$	$N_\lambda \times N_\phi = 256 \times 128$	$N_\lambda \times N_\phi = 512 \times 256$
Fourth-order Compact	1.00	15.73	315.23
Sixth-order SCFDM	3.44	56.23	1180.23
Sixth-order CCFDM	2.42	39.05	778.41
Eighth-order SCFDM	8.61	135.93	2723.65
Eighth-order CCFDM	5.42	88.64	1773.65

Furthermore, the methods in terms of computational cost from the lowest to highest are: the fourth-order compact, the sixth-order CCFDM, the sixth-order SCFDM, the eighth-order CCFDM, and the eighth-order SCFDM. Therefore, in terms of qualitative and quantitative measurements including computational cost we conclude that the sixth-order and eighth-order CCFDM methods are superior to the sixth-order and eighth-order SCFDM methods for spatial differencing of the spherical shallow water equations.

In addition, to avoid the polar problem and using a polar filter it is worth to replace the regular latitude-longitude grid system used in the present work by a Yin-Yang grid [55, 7] to examine the performance of the compact finite difference method on it. In addition, using this grid system enables us to use larger time steps that will decrease the computational cost of the algorithms. We will report the results of this approach in a future work.

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APPENDIX

I. Details of block tridiagonal system for numerical solution of elliptic equations

The matrix coefficients $\mathbf{A}_j^S$, $\mathbf{B}_j^S$, $\mathbf{C}_j^S$ and right hand side vector $\mathbf{D}_j^S$ of the block tridiagonal system obtained from different compact method used in this study are presented here.

I.1. The Helmholtz equation

The matrix coefficients for the fourth-order compact method for $2 < j < N_\phi - 1$ are

$$\mathbf{B}_j^{C4} = \begin{pmatrix} \alpha_j & -\tan \phi_j & 1 \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{d^2} & 0 & \frac{5}{6} \end{pmatrix}, \mathbf{A}_j^{C4} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2d} & \frac{1}{6} & 0 \\ \frac{1}{d^2} & 0 & \frac{1}{12} \end{pmatrix}, \mathbf{C}_j^{C4} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{-1}{2d} & \frac{1}{6} & 0 \\ \frac{-1}{d^2} & 0 & \frac{1}{12} \end{pmatrix}$$

where d is grid space and

$$\alpha_j = \frac{-m^2}{\cos^2 \phi_j} - a^2 \beta_j \quad (50)$$

and coefficients for $j = 1$ and $j = N_\phi$ are

$$\mathbf{B}_1^{C4} = \begin{pmatrix} \alpha_1 & -\tan \phi_1 & 1 \\ \frac{(-1)^m}{\frac{2d}{d^2}} & \frac{2}{3} + \frac{(-1)^{m+1}}{6} & 0 \\ -\frac{(-1)^m}{d^2} + \frac{2}{d^2} & 0 & \frac{(-1)^m}{12} + \frac{5}{6} \end{pmatrix}, \mathbf{C}_1^{C4} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{-1}{2d} & \frac{1}{6} & 0 \\ \frac{-1}{d^2} & 0 & \frac{1}{12} \end{pmatrix}$$

$$\mathbf{B}_{N_\phi}^{C4} = \begin{pmatrix} \alpha_{N_\phi} & -\tan \phi_{N_\phi} & 1 \\ \frac{-(-1)^m}{\frac{2d}{d^2}} & \frac{2}{3} + \frac{(-1)^{m+1}}{6} & 0 \\ -\frac{(-1)^m}{d^2} + \frac{2}{d^2} & 0 & \frac{-(-1)^m}{12} + \frac{5}{6} \end{pmatrix}, \mathbf{A}_{N_\phi}^{C4} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2d} & \frac{1}{6} & 0 \\ \frac{1}{d^2} & 0 & \frac{1}{12} \end{pmatrix}$$

and the right hand side vector $\mathbf{D}_j$ for $1 < j < N_\phi$ is

$$\mathbf{D}_j^{C4} = \begin{pmatrix} a^2 \hat{R}_j \\ 0 \\ 0 \end{pmatrix}$$

The matrix coefficients for the sixth-order SCFDM method for $2 < j < N_\phi - 1$ are

$$\mathbf{B}_j^{\text{SC6}} = \begin{pmatrix} \alpha_j & -\tan \phi_j & 1 & 0 & 0 \\ 0 & -2d & 0 & -\frac{3}{10}d^3 & 0 \\ 0 & -2 & 0 & -\frac{5}{6}d^2 & 0 \\ -2 & 0 & -d^2 & 0 & -\frac{7}{90}d^4 \\ 0 & 0 & -2 & 0 & -\frac{5}{6}d^2 \end{pmatrix}, \mathbf{A}_j^{\text{SC6}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -\frac{d^3}{60} & 0 \\ 0 & 1 & 0 & -\frac{d^2}{12} & 0 \\ 1 & 0 & 0 & 0 & -\frac{d^4}{360} \\ 0 & 0 & 1 & 0 & -\frac{d^2}{12} \end{pmatrix}$$

$$\mathbf{C}_j^{\text{SC6}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{d^3}{60} & 0 \\ 0 & 1 & 0 & -\frac{d^2}{12} & 0 \\ 1 & 0 & 0 & 0 & -\frac{d^4}{360} \\ 0 & 0 & 1 & 0 & -\frac{d^2}{12} \end{pmatrix}$$

and coefficients for $j = 1$ and $j = N_\phi$ are

$$\mathbf{B}_1^{\text{SC6}} = \begin{pmatrix} \alpha_1 & -\tan \phi_1 & 1 & 0 & 0 \\ -(-1)^m & -2d & 0 & -\frac{3d^3}{10} - \frac{(-1)^{m+1}d^3}{60} & 0 \\ 0 & -2 + (-1)^{m+1} & 0 & -\frac{5d^2}{6} - \frac{(-1)^{m+1}d^2}{12} & 0 \\ -2 + (-1)^m & 0 & -d^2 & 0 & -\frac{7d^4}{90} - \frac{(-1)^m d^4}{360} \\ 0 & 0 & -2 + (-1)^m & 0 & -\frac{5d^2}{6} - \frac{(-1)^m d^2}{12} \end{pmatrix}$$

$$\begin{aligned}
\mathbf{C}_1^{\text{SC6}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{d^3}{60} & 0 \\ 0 & 1 & 0 & -\frac{d^2}{12} & 0 \\ 1 & 0 & 0 & 0 & -\frac{d^4}{360} \\ 0 & 0 & 1 & 0 & -\frac{d^2}{12} \end{pmatrix} \\
\mathbf{B}_{N_\phi}^{\text{SC6}} &= \begin{pmatrix} \alpha_{N_\phi} & -\tan \phi_{N_\phi} & 1 & 0 & 0 \\ (-1)^m & -2d & 0 & -\frac{3d^3}{10} - \frac{(-1)^{m+1}d^3}{60} & 0 \\ 0 & -2 + (-1)^{m+1} & 0 & -\frac{5d^2}{6} - \frac{(-1)^{m+1}d^2}{12} & 0 \\ -2 + (-1)^m & 0 & -d^2 & 0 & -\frac{7d^4}{90} - \frac{(-1)^m d^4}{360} \\ 0 & 0 & -2 + (-1)^m & 0 & -\frac{5d^2}{6} - \frac{(-1)^m d^2}{12} \end{pmatrix} \\
\mathbf{A}_{N_\phi}^{\text{SC6}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -\frac{d^3}{60} & 0 \\ 0 & 1 & 0 & -\frac{d^2}{12} & 0 \\ 1 & 0 & 0 & 0 & -\frac{d^4}{360} \\ 0 & 0 & 1 & 0 & -\frac{d^2}{12} \end{pmatrix}
\end{aligned}$$

and the right hand side vector $\mathbf{D}_j$ for $1 < j < N_\phi$ is

$$\mathbf{D}_j^{\text{SC6}} = \begin{pmatrix} a^2 \hat{R}_j \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix coefficients for the eighth-order SCFDM method for $2 < j < N_\phi - 1$ are

$$\begin{aligned}
\mathbf{B}_j^{\text{SC8}} &= \begin{pmatrix} \alpha_j & -\tan \phi_j & 1 & 0 & 0 & 0 & 0 \\ 0 & -2d & 0 & -\frac{d^3}{3} & 0 & -\frac{2d^5}{126} & 0 \\ 0 & d & 0 & \frac{d^3}{2} & 0 & \frac{7d^5}{180} & 0 \\ 0 & 0 & 0 & d^3 & 0 & \frac{5d^5}{12} & 0 \\ -2 & 0 & -d^2 & 0 & -\frac{d^4}{12} & 0 & -\frac{6d^6}{2240} \\ 0 & 0 & d^2 & 0 & \frac{d^4}{2} & 0 & \frac{7d^6}{180} \\ 0 & 0 & 0 & 0 & d^4 & 0 & \frac{5d^6}{12} \end{pmatrix} \\
\mathbf{A}_j^{\text{SC8}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -\frac{2d^5}{5040} & 0 \\ 0 & -\frac{d}{2} & 0 & 0 & 0 & \frac{d^5}{720} & 0 \\ 0 & 0 & 0 & -\frac{d^3}{2} & 0 & \frac{d^5}{24} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\frac{2d^6}{40320} \\ 0 & 0 & -\frac{d^2}{2} & 0 & 0 & 0 & \frac{d^6}{720} \\ 0 & 0 & 0 & 0 & -\frac{d^4}{2} & 0 & \frac{d^6}{24} \end{pmatrix} \\
\mathbf{C}_j^{\text{SC8}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\frac{2d^5}{5040} & 0 \\ 0 & -\frac{d}{2} & 0 & 0 & 0 & \frac{d^5}{720} & 0 \\ 0 & 0 & 0 & -\frac{d^3}{2} & 0 & \frac{d^5}{24} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\frac{2d^6}{40320} \\ 0 & 0 & -\frac{d^2}{2} & 0 & 0 & 0 & \frac{d^6}{720} \\ 0 & 0 & 0 & 0 & -\frac{d^4}{2} & 0 & \frac{d^6}{24} \end{pmatrix}
\end{aligned}$$

and coefficients for $j = 1$ and $j = N_\phi$ are

$$\mathbf{B}_1^{\text{SC8}} = \begin{pmatrix} \alpha_1 & -\tan \phi_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -(-1)^m & -2d & 0 & \frac{-d^3}{3} & 0 & \frac{-2d^5}{126} - \frac{(-1)^{m+1}2d^5}{5040} & 0 & 0 \\ 0 & d - \frac{(-1)^{m+1}d}{2} & 0 & \frac{d^3}{2} & 0 & \frac{7d^5}{180} + \frac{(-1)^{m+1}d^5}{720} & 0 & 0 \\ 0 & 0 & 0 & d^3 - \frac{(-1)^{m+1}d^3}{2} & 0 & \frac{5d^5}{12} + \frac{(-1)^{m+1}}{24} & 0 & 0 \\ -2 + (-1)^m & 0 & -d^2 & 0 & \frac{-d^4}{12} & 0 & \frac{-6d^6}{2240} - \frac{(-1)^m 2d^6}{40320} & 0 \\ 0 & 0 & d^2 - \frac{(-1)^m d^2}{2} & 0 & \frac{d^4}{2} & 0 & \frac{7d^6}{180} + \frac{(-1)^m d^6}{720} & 0 \\ 0 & 0 & 0 & 0 & d^4 - \frac{(-1)^m d^4}{2} & 0 & \frac{5d^6}{12} + \frac{(-1)^m}{24} & 0 \end{pmatrix}$$

$$\mathbf{C}_1^{\text{SC8}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \frac{-2d^5}{5040} & 0 & 0 \\ 0 & \frac{-d}{2} & 0 & 0 & 0 & \frac{d^5}{720} & 0 & 0 \\ 0 & 0 & 0 & \frac{-d^3}{2} & 0 & \frac{d^5}{24} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \frac{-2d^6}{40320} & 0 \\ 0 & 0 & \frac{-d^2}{2} & 0 & 0 & 0 & \frac{d^6}{720} & 0 \\ 0 & 0 & 0 & 0 & \frac{-d^4}{2} & 0 & \frac{d^6}{24} & 0 \end{pmatrix}$$

$$\mathbf{B}_{N_\phi}^{\text{SC8}} = \begin{pmatrix} \alpha_{N_\phi} & -\tan \phi_{N_\phi} & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^m & -2d & 0 & \frac{-d^3}{3} & 0 & \frac{-2d^5}{126} - \frac{(-1)^{m+1}2d^5}{5040} & 0 & 0 \\ 0 & d - \frac{(-1)^{m+1}d}{2} & 0 & \frac{d^3}{2} & 0 & \frac{7d^5}{180} + \frac{(-1)^{m+1}d^5}{720} & 0 & 0 \\ 0 & 0 & 0 & d^3 - \frac{(-1)^{m+1}d^3}{2} & 0 & \frac{5d^5}{12} + \frac{(-1)^{m+1}}{24} & 0 & 0 \\ -2 + (-1)^m & 0 & -d^2 & 0 & \frac{-d^4}{12} & 0 & \frac{-6d^6}{2240} - \frac{(-1)^m 2d^6}{40320} & 0 \\ 0 & 0 & d^2 - \frac{(-1)^m d^2}{2} & 0 & \frac{d^4}{2} & 0 & \frac{7d^6}{180} + \frac{(-1)^m d^6}{720} & 0 \\ 0 & 0 & 0 & 0 & d^4 - \frac{(-1)^m d^4}{2} & 0 & \frac{5d^6}{12} + \frac{(-1)^m}{24} & 0 \end{pmatrix}$$

$$\mathbf{A}_{N_\phi}^{\text{SC8}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & \frac{-2d^5}{5040} & 0 & 0 \\ 0 & \frac{-d}{2} & 0 & 0 & 0 & \frac{d^5}{720} & 0 & 0 \\ 0 & 0 & 0 & \frac{-d^3}{2} & 0 & \frac{d^5}{24} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & \frac{-2d^6}{40320} & 0 \\ 0 & 0 & \frac{-d^2}{2} & 0 & 0 & 0 & \frac{d^6}{720} & 0 \\ 0 & 0 & 0 & 0 & \frac{-d^4}{2} & 0 & \frac{d^6}{24} & 0 \end{pmatrix}$$

and the right hand side vector $\mathbf{D}_j$ for $1 < j < N_\phi$ is

$$\mathbf{D}_j^{\text{SC8}} = \begin{pmatrix} a^2 \hat{R}_j \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix coefficients for the sixth-order CCFDM method for $2 < j < N_\phi - 1$ are

$$\mathbf{B}_j^{\text{CC6}} = \begin{pmatrix} \alpha_j & -\tan \phi_j & 1 \\ 0 & 1 & 0 \\ \frac{6}{d^2} & 0 & 1 \end{pmatrix}, \mathbf{A}_j^{\text{CC6}} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{15}{16d} & \frac{7}{16} & \frac{d}{16} \\ \frac{15}{d^2} & \frac{7}{8d} & \frac{1}{8} \end{pmatrix}, \mathbf{C}_j^{\text{CC6}} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{-15}{16d} & \frac{7}{16} & \frac{-d}{16} \\ \frac{-15}{d^2} & \frac{7}{8d} & \frac{-1}{8} \end{pmatrix}$$

and coefficients for $j = 1$ and $j = N_\phi$ are

$$\mathbf{B}_1^{\text{CC6}} = \begin{pmatrix} \alpha_1 & -\tan \phi_1 & 1 \\ \frac{(-1)^m 15}{16d} & 1 + \frac{(-1)^{m+1}7}{16} & \frac{(-1)^m d}{16} \\ \frac{6}{d^2} - \frac{(-1)^m 3}{d^2} & \frac{-(-1)^{m+1}9}{8d} & 1 - \frac{(-1)^m}{8} \end{pmatrix}, \mathbf{C}_1^{\text{CC6}} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{-15}{16d} & \frac{7}{16} & \frac{-d}{16} \\ \frac{-15}{d^2} & \frac{7}{8d} & \frac{-1}{8} \end{pmatrix}$$

$$\mathbf{B}_{N_\phi}^{\text{CC6}} = \begin{pmatrix} \alpha_{N_\phi} & -\tan \phi_{N_\phi} & 1 \\ \frac{-(-1)^{m+1}15}{16d} & 1 + \frac{(-1)^{m+1}7}{8d} & \frac{-(-1)^m d}{16} \\ \frac{6}{d^2} - \frac{(-1)^m 3}{d^2} & \frac{(-1)^{m+1}9}{8d} & 1 - \frac{(-1)^m}{8} \end{pmatrix}, \mathbf{A}_{N_\phi}^{\text{CC6}} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{15}{16d} & \frac{7}{16} & \frac{d}{8} \\ \frac{15}{d^2} & \frac{7}{8d} & \frac{1}{8} \end{pmatrix}$$

and the right hand side vector for $1 < j < N_\phi$ is

$$\mathbf{D}_j^{\text{CC6}} = \begin{pmatrix} a^2 \hat{R}_j \\ 0 \\ 0 \end{pmatrix}$$

The matrix coefficients for the eighth-order CCFDM method for $2 < j < N_\phi - 1$ are

$$\mathbf{B}_j^{\text{CC8}} = \begin{pmatrix} \alpha_j & -\tan \phi_j & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{8}{d^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{A}_j^{\text{CC8}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{35}{32d} & \frac{19}{32} & \frac{d}{8} & \frac{d^2}{96} \\ \frac{-4}{d^2} & \frac{-29}{16d} & \frac{-5}{16} & \frac{-d}{48} \\ \frac{-105}{16d^3} & \frac{-105}{16d^2} & \frac{-15}{8d} & \frac{-3}{16} \end{pmatrix}$$

$$\mathbf{C}_j^{\text{CC8}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{-35}{32d} & \frac{19}{32} & \frac{-d}{8} & \frac{d^2}{96} \\ \frac{-4}{d^2} & \frac{-29}{16d} & \frac{-5}{16} & \frac{-d}{48} \\ \frac{-105}{16d^3} & \frac{-105}{16d^2} & \frac{-15}{8d} & \frac{-3}{16} \end{pmatrix}$$

and coefficients for $j = 1$ and $j = N_\phi$ are

$$\mathbf{B}_1^{\text{CC8}} = \begin{pmatrix} \alpha_1 & -\tan \phi_1 & 1 & 0 \\ \frac{(-1)^m 35}{32d} & 1 + \frac{(-1)^{m+1}19}{8d} & \frac{(-1)^m d}{8} & \frac{(-1)^{m+1}d^2}{96} \\ \frac{8}{d^2} - \frac{(-1)^m 4}{d^2} & \frac{-(-1)^{m+1}29}{16d} & 1 - \frac{(-1)^m 5}{16} & \frac{-(-1)^{m+1}d}{48} \\ \frac{-(-1)^m 105}{16d^3} & \frac{-(-1)^{m+1}105}{16d^2} & \frac{-(-1)^m 15}{8d} & 1 - \frac{(-1)^{m+1}3}{16} \end{pmatrix}, \mathbf{C}_1^{\text{CC8}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{-35}{32d} & \frac{19}{32} & \frac{-d}{8} & \frac{d^2}{96} \\ \frac{-4}{d^2} & \frac{-29}{16d} & \frac{-5}{16} & \frac{-d}{48} \\ \frac{-105}{16d^3} & \frac{-105}{16d^2} & \frac{-15}{8d} & \frac{-3}{16} \end{pmatrix}$$

$$\mathbf{B}_{N_\phi}^{\text{CC8}} = \begin{pmatrix} \alpha_{N_\phi} & -\tan \phi_{N_\phi} & 1 & 0 \\ \frac{-(-1)^m 35}{32d} & 1 + \frac{(-1)^{m+1}19}{8d} & \frac{-(-1)^m d}{8} & \frac{(-1)^{m+1}d^2}{96} \\ \frac{8}{d^2} - \frac{(-1)^m 4}{d^2} & \frac{(-1)^{m+1}29}{16d} & 1 - \frac{(-1)^m 5}{16} & \frac{(-1)^{m+1}d}{48} \\ \frac{(-1)^m 105}{16d^3} & \frac{-(-1)^{m+1}105}{16d^2} & \frac{(-1)^m 15}{8d} & 1 - \frac{(-1)^{m+1}3}{16} \end{pmatrix}, \mathbf{A}_{N_\phi}^{\text{CC8}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{35}{32d} & \frac{19}{32} & \frac{d}{8} & \frac{d^2}{96} \\ \frac{-4}{d^2} & \frac{-29}{16d} & \frac{-5}{16} & \frac{-d}{48} \\ \frac{-105}{16d^3} & \frac{-105}{16d^2} & \frac{-15}{8d} & \frac{-3}{16} \end{pmatrix}$$

and the right hand side vector for $1 < j < N_\phi$ is

$$\mathbf{D}_j^{\text{CC8}} = \begin{pmatrix} a^2 \hat{R}_j \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

I.2. The Poisson equation

The matrix coefficients for the Poisson equation for nonzero wave numbers ($m \neq 0$) are found by setting $\beta_j = 0$ in (50). For wave number zero ($m = 0$) equation (34) is used and we need to modify the first row of $\mathbf{B}_j^{\text{S}}$ and $\mathbf{D}_j^{\text{S}}$.

II. Aliasing and phase speed errors of the compact schemes

II.1. Aliasing error

To assess the aliasing error generated by the compact finite difference schemes used in this paper we follow the path set in Ref. [18] and use a general approach to compare the aliasing error properties of finite difference methods used in this paper.

The one-dimensional advection equation with a variable wave speed is:

$$\frac{\partial \varphi}{\partial t} + C(x, t) \frac{\partial \varphi}{\partial x} = 0 \quad (51)$$

Using Fourier expansion at some instant in time, C and φ can be expanded in x . We consider the interaction of an individual pair of Fourier modes:

$$C = \tilde{c}e^{ik_1x}, \quad \varphi = \tilde{a}e^{ik_2x} \quad (52)$$

where $i = \sqrt{-1}$ and k_1 and k_2 are wave numbers.

The general form of the approximation of $C(x)\partial\varphi/\partial x$ by a finite difference method at a grid point j can be written as:

$$C(x) \frac{\partial \varphi}{\partial x} \Big|_j = \tilde{c} \tilde{a} T_1(k_2) e^{i(k_1+k_2)x_j} \quad (53)$$

in which $x_j = jd$ and T_1 is the transfer function of the first derivative. The transfer function, T , of a difference scheme, S , is defined by $S(e^{ikx}) = T(k)e^{ikx}$. We assume that a wave with wave number k_2 in φ interacts with the wave speed to force the time derivative of φ to an aliased wave number $\tilde{k}$ where $|\tilde{k}| = 2k_{\max} - (k_1 + k_2)$ and $k_{\max} = \pi/d$.

To assess the aliasing property of a finite difference method we can compute the rate at which this aliasing appears. Now, if we assume that interacting waves have a unit amplitude at a given instant, the rate at which interactions between wave numbers k_1 and k_2 force the growth at the aliased wave number, $\tilde{k}$, is calculated as below:

$$G_{k_2 \rightarrow \tilde{k}} = |T_1(k_2)| \quad (54)$$

where $T_1(k_2)$ is the transfer function of the finite difference method used to approximate $\partial\varphi/\partial x$. The transfer function of the first derivative approximation by the fourth-order compact and the sixth-order SCFDM methods is given in Ref. [25]. The transfer functions for the eighth-order SCFDM, the sixth-order CCFDM and the eighth-order CCFDM schemes can be calculated in a similar way.

Figure 10 presents contour plots of the growth rates, $G_{k_2 \rightarrow \tilde{k}}$, as a function of k_2d and $\tilde{k}d$ for different compact finite difference methods. In this figure the dashed diagonal line is contour plot of $k_1d = \pi$. Since the aliasing for the interaction of wave numbers k_2 and $\tilde{k}$ can appear for $k_1 \leq \pi/d$, the contour plots are shown only for this case (below the diagonal line).

Figure 10 shows that for a fixed $\tilde{k}d$, the growth rate for small wave numbers (long waves lengths) in the different methods is similar but for the high wave numbers (short waves lengths) the behaviour of compact methods differs. For example, for the case $k_2d = 3.0$ we can see in figure 10 that the growth rate for the fourth-order compact method is 0.42, for the 6th-order SCFDM is 0.69, for the 6th-order CCFDM is 0.74, for the 8th-order SCFDM is 0.94 and for the 8th-order CCFDM is 1.02. Therefore, the growth rate moves toward higher wave numbers for the higher order schemes which means that the aliasing error is more severe for that case.

II.2. Dispersion relation and phase speed error

The one-dimensional linear advection equation is used to address the the phase speed error of the different compact finite difference methods in this study. We consider

$$\frac{\partial \varphi}{\partial t} + c \frac{\partial \varphi}{\partial x} = 0 \quad (55)$$

where the wave speed c is constant.

By approximating the spatial derivative with a difference method and assuming a wave solution, $\varphi(t) = e^{i(kx_j - \omega t)}$, the general form of the dispersion relation is:

$$\omega = \frac{cT_1(k)}{i} \quad (56)$$

Figure 10. Plot of $G_{k_2 \rightarrow \tilde{k}}$ as a function of k_2 and $\tilde{k}$ for (a) 4th-order compact method, (b) 6th-order SCFDM, (c) 6th-order CCFDM, (d) 8th-order SCFDM and (e) 8th-order CCFDM.

Figure 11. The values of ω/c (scaled frequency) as a function of wave number for different compact methods.

where ω is the frequency, $T_1(k)$ is the transfer function of the difference scheme used to approximate the first derivative and $i = \sqrt{-1}$. Using the dispersion relation (56), the phase speed can be found:

$$\Theta = \frac{\omega}{k} = \frac{cT_1(k)}{ik}. \quad (57)$$

Figure 12. Phase speed as a function of wave number for different compact methods.

Figures 11 and 12 present the values of ω/c and phase speed, Θ , as a function of wave number for the different compact finite difference methods used in this study. The exact values of the scaled frequency and phase speed, i.e., $\omega/c = k$ and $\Theta = c$ are also shown in these figures. Figures 11 and 12 indicate that the higher order methods are more accurate, produce less phase speed errors, and

outperform the lower-order ones for all wave numbers. The phase speed of the $2d$ wave (wave number $k = \pi/d$) is of course zero for all schemes.

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