Efficient Recovery of Sub-Nyquist Sampled Sparse Multi-Band Signals Using Reconfigurable Multi-Channel Analysis and Modulated Synthesis Filter Banks

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Abstract—Sub-Nyquist cyclic nonuniform sampling (CNUS) of a sparse multi-band signal generates a nonuniformly sampled signal. Assuming that the corresponding uniformly sampled signal satisfies the Nyquist sampling criterion, the sequence obtained via CNUS can be passed through a reconstructor to recover the missing uniform-grid samples. At present, these reconstructors have very high design and implementation complexity that offsets the gains obtained due to sub-Nyquist sampling. In this paper, we propose a scheme that reduces the design and implementation complexity of the reconstructor. In contrast to the existing reconstructors which use only a multi-channel synthesis filter bank (FB), the proposed reconstructor utilizes both analysis and synthesis FBs which makes it feasible to achieve an order-of-magnitude reduction of the complexity. The analysis filters are implemented using polyphase networks whose branches are allpass filters with distinct fractional delays and phase shifts. In order to reduce both the design and the implementation complexity of the synthesis FB, the synthesis filters are implemented using a cosine-modulated FB. In addition to the reduced complexity of the reconstructor, the proposed multi-channel recovery scheme also supports online reconfigurability which is required in flexible (multi-mode) systems where the user subband locations vary with time.

Index Terms—Sub-Nyquist sampling, sparse multi-band signals, reconstruction, nonuniform sampling, time-interleaved analog-to-digital converters, filter banks.

I. INTRODUCTION

It is well recognized that data acquisition (analog-to-digital conversion) constitutes one of the bottlenecks in signal processing and communication systems [1]. In particular, with the increasing demands for high data rates and resolution, the power consumption of the data acquisition is becoming intolerably high, especially in battery-powered wideband communication systems. An emerging research focus is therefore to utilize structures (sparsities) in the analog signals in order to reduce the average acquisition rate and thereby reduce the cost [2]–[5]. This is referred to as sub-Nyquist sampling of sparse signals which has the potential to dramatically reduce the power consumption. Typically, in uniform sampling, a signal that is bandlimited to \( f < f_0 \) is sampled at a rate of \( f_s \geq 2f_0 \). In sub-Nyquist sampling, the average sampling rate is lower than \( 2f_0 \) but still large enough to capture the information content in the signal. There are essentially two paradigms within this area. The first covers multi-band (or multi-coset) sampling where the use of cyclic nonuniform sampling (CNUS) enables a reduction of the average sampling rate to (in principle) the Landau minimal sampling rate which is determined by the frequency occupancy [4], [6]. The other paradigm is compressive sampling (or compressed sensing) [4], [5] which in practice (so far) utilizes modulation with a (pseudo) random signal, integration, and low-rate uniform sampling. Both of these approaches have their own unique advantages and drawbacks and it is likely that both of them in the future will be employed but in different contexts depending on the application. In this paper, we are primarily interested in the CNUS approach.

For CNUS, the sub-Nyquist sampled signal is passed through a digital reconstructor to recover the uniformly spaced samples. Thus, assuming that the corresponding uniformly sampled signal satisfies the Nyquist sampling criterion, the sampling problem to be considered in this paper corresponds to the recovery of uniform-grid samples given a subset of those samples. Given \( K \) samples in each block (period) of \( M \) samples, \( K < M \), the problem is to recover the \( M - K \) missing samples. For the CNUS approach, it is known that the reconstruction can be done, in principle, via a set of ideal multi-level synthesis filters, given the sampling pattern [7]–[9]. The related problem of selecting the optimal sampling patterns has also been addressed [9]–[11]. However, the straightforward CNUS recovery scheme has very high design and implementation complexities. Also, in frequency-hopping communication systems where the active user time frames are different, the reconstruction scheme should support online reconfigurability with low complexity. Further, it is noted that here, like in [7]–[9], we only consider the recovery of the uniform-grid samples corresponding to the entire sparse multi-band signal. In order to extract the uniform-grid signal corresponding to the frequency band of each active user, regular filtering can be used at the output of the reconstructor. Also, we assume that the location of the active subbands are known and available beforehand as in [7]–[9].

A. Contributions and Outline of the Paper

In this paper, we will introduce the efficient recovery scheme shown in Fig. 1, which is derived by first expressing the reconstructor design problem in terms of multi-channel analysis and synthesis filter banks (FBs). In this scheme, the
time-interleaved analog-to-digital converters (TI-ADCs) [12]–

There exist efficient reconstruction techniques for other

magnitudes of the residual aliasing terms. It is also noted

design method in that paper offers no direct control on the

Fig. 2. Reconstruction using a set of multi-level synthesis filters [7].

scheme in Section V. Using complexity expressions for the

proposed reconstructor and the polyphase implementation of

the straightforward scheme in [7], we show that order-of-

magnitude reduction of the complexity is achievable using

the proposed reconstructor. Furthermore, in [18], the filters in

the analysis FB were designed using numerical optimization

which can be time-consuming especially for higher filter

orders and/or larger K. In Section VI of this paper, we

propose a least-squares approach for designing these filters so

that their filter coefficients can be obtained via a closed-form

solution. In addition to reducing the design effort, the closed-

form solution enables us to redetermine the filter coefficients

online, if required. Also, in Section VII, we use detailed design

examples to show that the proposed method offers significant

complexity savings, particularly for larger M. In order to

provide the necessary background for the above mentioned

sections, in Section III we review the concept of sub-Nyquist

CNUS of sparse multi-band signals. Immediately following

this introduction, in Section II, we define the notations used

in this paper as well as briefly review some of the signal

processing concepts that will be used in later sections.

II. PRELIMINARIES

A. Notations

Bold lowercase letters are used to denote vectors while

bold uppercase letters are used to denote matrices. Transpose

and conjugate-transpose are represented using \( (\cdot)^T \)

and \( (\cdot)^\dagger \), respectively. For a filter with impulse response coefficients

\( h(n) \), we use \( H(z) \) to denote its transfer function which is

defined as \( H(z) = \sum_n h(n)z^{-n} \). The frequency response of the filter is denoted by \( H(e^{j\omega}) \) and is obtained from the

transfer function by replacing \( z \) with \( e^{j\omega} \).

B. Polyphase Decomposition

Any filter \( H(z) \) can generally be expressed in terms of its

polyphase components \( H_m(z) \), \( m = 0, 1, \ldots, M-1 \), as [19],

[20]

\[
H(z) = \sum_{m=0}^{M-1} z^{-m}H_m(z^M). \tag{1}
\]

Polyphase decomposition as in (1) along with the noble

identities shown in Fig. 3 [20], can be used to derive efficient

structures for decimation and interpolation. For example, con-

sider the decimator shown in Fig. 4(a). Expressing \( H(z) \) in

Fig. 4(a) as in (1) and then propagating the downsampler to

the left using the noble identity shown in Fig. 3(a), we get the
Fig. 3. Noble identities.

(a) $x(n) \rightarrow H(z) \rightarrow \downarrow M \rightarrow y(n)$

(b) $x(n) \rightarrow \downarrow M \rightarrow H_0(z) \rightarrow \oplus \rightarrow y(n)$

$z^{-1} \cdot \omega \rightarrow \downarrow M \rightarrow H_{M-1}(z)$

Fig. 4. (a) Decimator. (b) Equivalent representation of (a) using the $M$ polyphase branches of the filter $H(z)$.

polyphase structure in Fig. 4(b). It can be seen that, unlike in Fig. 4(a), in the polyphase structure the filtering takes place at the lower rate. It is noted that the corresponding polyphase structure for the interpolator can be obtained by transposing the structure in Fig. 4(b) and replacing each downsampler with an upsampler [20].

C. Generalized Fractional-Delay Filter

A generalized fractional-delay (FD) filter has a phase shift in addition to the fractional delay [21] and its frequency response can be expressed as

$$H(e^{j\omega}) = e^{j(\omega d + \alpha \text{sgn}(\omega))}, \quad \omega \in [-\pi, \pi]$$

with $d, \alpha \in \mathbb{R}$. Here, $d$ represents the fractional delay, $\alpha$ is the additional phase shift, and $\text{sgn}(\omega)$ denotes the sign of $\omega$.

III. SUB-NYQUIST CYCLIC NONUNIFORM SAMPLING OF SPARSE MULTI-BAND SIGNALS

Assume that $x_a(t)$ is a real-valued continuous-time signal that carries information within the frequency band $\omega \in (-2\pi f_0, 2\pi f_0), \ f_0 < 1/(2T)$. Uniform sampling of $x_a(t)$ at a sampling frequency of $f_s = 1/T$ results in a discrete-time sequence $x(n) = x_a(nT)$. Below, for the sake of simplicity, we assume that $T = 1$. Now it is assumed that the band $\omega \in [0, \pi]$ is divided into $M$ granularity bands of equal width $\pi/M$. In sparse multi-band signals, at any given time frame, only $K$ of the $M$ granularity bands ($K < M$) are allocated to users. In this paper, $r_i \in [0, 1, \ldots, M - 1], \ i = 1, 2, \ldots, K$, denote the active granularity bands. Figure 5 shows the principle spectrum of a sparse multi-band signal when $M = 16, \ K = 3$, and with active granularity bands $r_{1,2,3} = [1, 4, 10]$. A user can occupy one or several consecutive granularity bands. Further, to be able to design practical filters, we assume a certain amount of redundancy (oversampling) which corresponds to transition bands between user bands. In case of such sparse multi-band signals, uniform sampling will generate more samples than what is required to prevent information loss. The number of samples generated during the sampling process can be reduced by using CNUS which only uses a subset of the uniform samples $x(n)$, i.e., $x(Mn - m_\ell)$, $\ell = 1, 2, \ldots, K$ with $m_\ell \in [0, 1, \ldots, M - 1]$. It can be viewed as if the available input samples $x_\ell(n) = x(M\nu - m_\ell)$, $\ell = 1, 2, \ldots, K, \ \nu \in \mathbb{Z}$, are obtained from the uniform-grid samples $x(n)$ as shown in Fig. 2. A practical implementation of the CNUS is an $M$-channel TI-ADC [22] where only a subset of the channels are used. A reconstructor can then be used to recover the uniformly sampled sequence $x(n)$ from $x_\ell(n)$, $\ell = 1, 2, \ldots, K$, for a given set of $K$ granularity bands, provided the sampling instants $m_\ell$ are selected properly [11].

A reconstruction scheme using multi-level synthesis filters $A_\ell(z), \ \ell = 1, 2, \ldots, K$, as shown in Fig. 2, was proposed in [7]. It was shown that perfect reconstruction, i.e., $\hat{x}(\nu) = x(\nu)$, can be achieved in principle using ideal multi-level synthesis filters $A_\ell(z)$. Perfect reconstruction (PR) is generally not feasible with realizable filters. However, in practice, it is sufficient to determine $A_\ell(z)$ such that PR is approximated within a given tolerance. This can be carried out by designing $A_\ell(z)$ straightforwardly, assuming no a priori relations between the filters. However, the reconstructor thus designed may become intolerably costly in real-time applications as the computational complexity of this approach is roughly $N_AK/M$ multiplications per corrected output sample, where $N_A$ is the filter order of $A_\ell(z)$. Also, at a later time frame, if the location of the $K$ bands change, then all $A_\ell(z)$ need redesign. The design complexity of $A_\ell(z)$ is high as regular filter design with many unknowns is too computationally intensive and time consuming to be carried out online.

IV. PROPOSED RECONSTRUCTION USING ANALYSIS AND SYNTHESIS FBs

In this paper, to reduce the complexity, we describe the reconstruction in terms of both analysis and synthesis filters as shown in Fig. 6. Expressing the reconstruction in terms of analysis and synthesis filters as shown in Fig. 6 enables efficient implementation of the overall reconstructor (to be considered in Section V). The complexity reduction is due to the fact that the synthesis filters $C_\ell(z)$ can be efficiently realized using a cosine-modulated FB whereas a common set of fixed subfilters can be utilized to implement all the filters $B_k(z)$ in the analysis FB as shown in the proposed reconstructor in Fig. 1. It will be shown below that the

2Like in [7], the proposed reconstructor can be extended to use noninteger values for $m_\ell$. However, since practical implementations of CNUS schemes make use of TI-ADCs, we assume $m_\ell$ to be an integer as this appears to be the preferred choice.
components of conventional bandpass filters can be considered as a special case of the unconventional bandpass filter when all the samples in $x(n)$ are available. We will now state the expression for the non-zero polyphase components in the following theorem.

**Theorem 1:** In the proposed reconstructor in Fig. 6, the non-zero polyphase components $B_{km}(z)e^{j\omega m}$, $m \in [0, 1, \ldots, M - 1]$, $k = 1, 2, \ldots, K$, of the unconventional bandpass filter $B_k(e^{j\omega})$, $k = 1, 2, \ldots, K$, in (4) are generalized FD filters given by

$$
B_{km}(e^{j\omega}) = \frac{\beta_{km}}{M} e^{j(\omega m + \alpha_{km} \text{sgn}(\omega))}, \quad \omega \in [-\pi, \pi]
$$

with $\beta_{km}, \alpha_{km} \in \mathbb{R}$.

**Proof.** In order to prove Theorem 1 we show that, with $B_{km}(e^{j\omega})$ as in (5), the reconstructor in Fig. 2 [7] is equivalent to the proposed reconstructor in Fig. 6. In the following derivation, we assume as in [7] that the reconstruction of a sub-Nyquist sampled signal with $K$ active bands is performed using ideal synthesis filters $A_k(z)$, $k = 1, 2, \ldots, K$. As can be seen from Fig. 8, the frequency response of each synthesis filter $A_k(z)$, $k = 1, 2, \ldots, K$, has non-zero levels in the occupied granularity bands $r_i \in [0, 1, \ldots, M - 1]$, $i = 1, 2, \ldots, L$, and is zero elsewhere. In the granularity band $r_k$, the frequency response of the synthesis filter $A_k(z)$ is given by

$$
A_k(e^{j\omega}) = \frac{1}{M} \beta_{km} e^{j\omega m + \alpha_{km} \text{sgn}(\omega)} C_k(e^{j\omega})
$$

where $\omega \in \{[-(r_k + 1)\pi/M, -r_k\pi/M] \cup [r_k\pi/M, (r_k + 1)\pi/M]\}$, $C_k(e^{j\omega})$ is a bandpass filter with passband at the granularity band $r_k$ so that

$$
C_k(e^{j\omega}) = \left\{
\begin{array}{ll}
M, & \omega \in \{[-(r_k + 1)\pi/M, -r_k\pi/M] \cup [r_k\pi/M, (r_k + 1)\pi/M]\} \\
0, & \text{elsewhere}
\end{array}
\right.
$$

(7)

and $\beta_{km}, \alpha_{km}$ are the modulus and angle, respectively, of the complex constant $u_{km}$ that correspond to the level of $A_k(e^{j\omega})$ in the band $r_k$. Considering the contributions from all the synthesis filters $A_k(e^{j\omega})$, $k = 1, 2, \ldots, K$, to the overall frequency response in the granularity band $r_k$, the structure in Fig. 2 can be redrawn for the band $r_k$ as shown in Fig. 9(a) to the left through the upsample
Theorem 2: Consider the bandpass filters $B_k(z)$, $k = 1, 2, \ldots, K$, which extract the active subbands $r_k \in [0, 1, \ldots, M - 1]$, $k = 1, 2, \ldots, K$, respectively. Let $v_k$ be a vector $(K \times 1)$ containing all the $K$ complex constants $v_{km} e^{j \omega m}$ corresponding to the non-zero polyphase components of $B_k(z)$, $k \in [1, 2, \ldots, K]$. Then, $v_k$ can be determined using matrix inversion as

$$v_k = D^{-1}b_k$$

where $D$ is a $K \times K$ generalized Vandermonde matrix given by

$$D = \frac{1}{M} \begin{bmatrix} e^{j2\pi q_1 m_1 / M} & e^{j2\pi q_1 m_2 / M} & \cdots & e^{j2\pi q_1 m_K / M} \\ e^{j2\pi q_2 m_1 / M} & e^{j2\pi q_2 m_2 / M} & \cdots & e^{j2\pi q_2 m_K / M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j2\pi q_K m_1 / M} & e^{j2\pi q_K m_2 / M} & \cdots & e^{j2\pi q_K m_K / M} \end{bmatrix}$$

and $b_k$ is a vector $(K \times 1)$ matrix containing $K - 1$ zeros and unity for the position $k$. In (12), $q_i \in [0, 1, \ldots, M - 1]$, $i = 1, 2, \ldots, K$, depend on the corresponding active subband locations $r_i \in [0, 1, \ldots, M - 1]$ and is given by

$$q_i = \begin{cases} r_i + \frac{1}{2}, & \text{odd } r_i \\ M - \frac{r_i}{2}, & \text{even } r_i \neq 0 \end{cases} (13)$$

Proof. We divide the frequency range $[-\pi / M, 2\pi - \pi / M]$ into $M$ adjacent regions of equal width $2\pi / M$ as shown in Fig. 10(a). Thus, region $\ell$, $\ell \in [0, 1, \ldots, M - 1]$, covers the frequencies in $[-\pi / M + 2\pi p / M, -\pi / M + 2\pi (p + 1) / M]$. The passband of the desired bandpass filter $B_k(e^{j\omega})$ covers the band $\omega \in [r_k \pi / M, (r_k + 1) \pi / M]$ and thus also $\omega \in [2\pi - (r_k + 1) \pi / M, 2\pi - r_k \pi / M]$ as shown in Fig. 10(b). Further, comparing Figs. 10(a) and 10(b), we can see that if an active subband $r_i$, $i \in [1, 2, \ldots, K]$, occupies the left (right) half of a region $p$, it will also occupy the right (left) half of the region $M - p$.

Next, we make use of the fact that the non-zero polyphase components $B_{km} e^{j\omega}$ in (5) are $2\pi$-periodic with respect to $\omega$. This implies that $B_{km} e^{j\omega} = B_{km} e^{j(\omega - 2\pi p)}$. Thus, $B_{km} e^{j\omega}$ for $\omega \in [-\pi + 2\pi p, -\pi + 2\pi (p + 1)]$, $p \in Z$. It is further noted that $B_{km} e^{j\omega} = B_{km} e^{j(\omega + 2\pi p)}$ for $\omega \in [-\pi + 2\pi p, -\pi + 2\pi (p + 1)]$. The above discussion is not limited to the FB representations. It can be seen that (14) and (15) also correspond to the right and left half, respectively, of region $M - p$. Thus, if $q_i \in [0, 1, \ldots, M - 1]$, represent the region whose left half is occupied by the active subband $r_i$, then we have

$$B_k(e^{-j\omega}) = \frac{1}{M} \sum_{\ell=1}^{K} \beta_{km} e^{j2\pi pm / M}$$

and for $\omega \in [2\pi p / M, -\pi / M + 2\pi (p + 1) / M]$ (right part of region $p$), we obtain

$$B_k(e^{j\omega}) = \frac{1}{M} \sum_{\ell=1}^{K} \beta_{km} e^{j2\pi pm / M}.$$
the requirement on $B_k(e^{j\omega})$ in $q_i$ is equal to the requirement in the right half of the region $M - q_i$. Consequently, for the bandpass filter $B_k(e^{j\omega})$ it suffices to solve a system of $K$ equations corresponding to the left half of the $K$ regions $q_i$, $i = 1, 2, \ldots, K$. More precisely, the right hand side of (14) should equal unity in the region $q_k$ and zero in the $K - 1$ regions $q_i$, $i \in [1, 2, \ldots, K]$, $i \neq k$. Thus, using $v_{km\ell} = \beta_{km\ell}e^{j\omega_{km\ell}}$ in (14), we obtain the system of equations

$$\mathbf{D}v_k = b_k \quad (16)$$

where

$$v_k = [v_{km1}, v_{km2}, \ldots, v_{kmK}]^T. \quad (17)$$

The vector $v_k$ corresponding to the bandpass filter $B_k(e^{j\omega})$ can then be determined using (11).

Theorem 2 shows that the vectors $v_k$ corresponding to all the $K$ bandpass filters $B_k(e^{j\omega})$, $k = 1, 2, \ldots, K$, can be determined by inverting a single $K \times K$ matrix. Also, consistent with the results in [7], it can be seen from (12) that there is always at least one set of sampling instants that corresponds to an invertible matrix, namely $m_\ell = 0, 1, \ldots, K$, since for these sampling points the generalized Vandermonde matrix $\mathbf{D}$ reduces to a Vandermonde matrix. However, these sampling instants may not guarantee that the matrix $\mathbf{D}$ is well conditioned. In order to ensure that $\mathbf{D}$ is well conditioned, optimal sampling instants can be selected depending on the active subband locations as outlined in [9], [11].

V. PROPOSED EFFICIENT RECONSTRUCTOR

Using the reconstruction scheme described in Section IV, we will now derive the proposed efficient reconfigurable reconstructor shown in Fig. 1.

A. Synthesis and Analysis FBs

In order to implement the cosine-modulated synthesis FB, a lowpass filter with cutoff frequency at $\pi/2M$ is used as the prototype filter $P(z)$ [20]. The coefficients of the synthesis filters $c_k(n)$ can be expressed in terms of the impulse response of the prototype filter $\varphi(n)$ as [20]

$$c_k(n) = 2M \varphi(n) \cos \left( \frac{\pi}{M} (k + 0.5)(n - \frac{N_F}{2}) - (-1)^k \frac{\pi}{4} \right). \quad (18)$$

The overall complexity of the synthesis FB correspond to that of the prototype filter plus the cost of a real or complex transform block. By using a fast-transform algorithm, the cost of such a transform block can be made small when compared to the cost of the filters.

In the analysis FB, since the polyphase components of each $B_k(z)$ are as given in (5), all the analysis filters can be expressed with a common set of fixed subfilters, $F_\ell(z)$ and $G_\ell(z)$, $\ell = 1, 2, \ldots, K$. The different analysis filters are then obtained via different pairs of values of $\beta_{km\ell}$ and $\theta_{km\ell} = \alpha_{km\ell} + \pi/4$ such that

$$B_{km\ell}(z) = \frac{\beta_{km\ell}}{M} \left[ \cos(\theta_{km\ell})F_\ell(z) + \sin(\theta_{km\ell})G_\ell(z) \right] \quad (19)$$

where

$$F_\ell(e^{j\omega}) \approx e^{j\omega m\ell/M}, \quad G_\ell(e^{j\omega}) \approx \text{sgn}(\omega) \times je^{j\omega m\ell/M}. \quad (20)$$

It is noted that the additional phase of $\pi/4$ in $\theta_{km\ell}$ is required to ensure proper matching between adjacent analysis and synthesis filters in the case of overlapping granularity bands and when cosine-modulated synthesis FB is used. This is similar to the additional constants used for matching in conventional cosine-modulated FBs [20]. However, the additional constant used in $\theta_{km\ell}$ is $\pi/4$ instead of $(-1)^{k}\pi/4$ which is used in conventional cosine-modulated FBs. This is because, in the proposed reconstructor, the additional phase constants are applied on the polyphase components of the analysis filter. In conventional cosine-modulated FBs, the additional phase constants are applied on the overall analysis and synthesis filters as in (18).

B. Computational Complexity

In this paper we consider computational complexity as the number of real multiplications required per corrected output sample (see Footnote 1). Based on the discussions above, and polyphase realizations in which all the filtering takes place at the downsampled rate, the computational complexity of the proposed reconstructor in Fig. 1 can be approximated as

$$C_{\text{prop}} \approx \frac{N_P}{M} + \log_2(M) + \frac{2N_FK}{M} + \frac{2K^2}{M}. \quad (21)$$

In (21), $N_P$ is the order of the prototype filter for the synthesis FB and $N_F$ is the order of the fixed subfilters $F_\ell(z)$ and $G_\ell(z)$. The first two terms in the expression for $C_{\text{prop}}$ in (21), correspond to the computational complexity of the cosine-modulated synthesis FB assuming that the $2M \times M$ transform block is implemented using a fast-transform algorithm [23].
The third term is the computational complexity of the $2K$ subfilters $F_{\ell}(z)$ and $G_{\ell}(z)$ whereas the fourth term corresponds to the complexity of the $2K^2$ multipliers whose coefficients are the scaled $\cos(\cdot)$ and $\sin(\cdot)$ terms in (19). Typically, $N_P$ is about an order of magnitude larger than $M$ as explained below. An approximate estimate of the order of the prototype filter, $N_P$, is given by [24]

$$N_P \approx -\frac{2}{3} \log_{10}(10\delta_c\delta_s) \frac{2\pi}{\omega_s - \omega_c} \tag{22}$$

where $\delta_c$, $\delta_s$, $\omega_c$, and $\omega_s$ denote the passband ripple, stopband ripple, passband edge, and stopband edge, respectively, of the prototype filter. Assuming that $\rho$ is the percentage occupancy of a granularity band, for a prototype filter with transition band centered at $\pi/2M$, $\omega_s - \omega_c = \varepsilon \pi/M$ where $\varepsilon = 1 - \rho/100$. For example, if $\rho$ varies between 20–60%, for a prototype filter with passband and stopband ripple of $-60$ dB, $N_P$ will be between $9M-17M$. Also, the order of the subfilters $F_{\ell}(z)$ and $G_{\ell}(z)$ is $N_F \approx N_P/M$. The complexity of the polyphase implementation of the straightforward scheme in Fig. 2 can be estimated as

$$C_{reg} \approx \frac{N_P K}{M} \tag{23}$$

As exemplified in Fig. 11, which plots the ratio $C_{reg}/C_{prop}$ for $N_P = 13M$ and $N_F = N_P/M$, order-of-magnitude savings are feasible, via proper choices of $M$ and $K$ (also see Example 1 in Section VII for a specific example).

C. Reconfiguration Complexity

In the proposed reconstructor, the real-time reconfiguration is simple and fast as it suffices to determine the multiplier values $\beta_{km\ell}$ and $\theta_{km\ell}$ using (11). Thus, during reconfiguration, only the coefficients of the $2K^2$ multipliers corresponding to the scaled $\cos(\cdot)$ and $\sin(\cdot)$ terms in (19) need to be updated. As explained in Section VI below, the subfilters $F_{\ell}(z)$ and $G_{\ell}(z)$, as well as the prototype filter for the cosine-modulated synthesis FB, are designed once offline and are fixed in the implementation. Due to this, all the multipliers in the cosine-modulated FB as well as in the fixed subfilters can be implemented using fixed-coefficient multipliers. This helps to reduce the overall implementation complexity since, compared to variable-coefficient multipliers, efficient techniques can be used to implement the fixed-coefficient multipliers [25], [26]. Moreover, using a common set of fixed subfilters to implement all the analysis filters $B_k(z)$, $k = 1, 2, \ldots, K$, results in fewer design variables which helps to reduce the design complexity of the analysis FB.

VI. DESIGN OF THE PROPOSED RECONSTRUCTOR

In this section, we introduce a procedure to design the proposed reconstructor. Here, we assume that the sampling instants $m_\ell$, $\ell = 1, 2, \ldots, K$, are selected such that for the given active subbands $r_k$, $k = 1, 2, \ldots, K$, $D$ in (11) is an invertible matrix. Using the analysis and the synthesis FB representation in Fig. 6 for the proposed reconstruction scheme, the Fourier transform of the reconstructed output can be written as

$$Y(e^{j\omega}) = V_0(e^{j\omega})X(e^{j\omega}) + \sum_{\xi=1}^{M-1} V_\xi(e^{j\omega})X(e^{j(\omega-2\pi\xi/M)}) \tag{24}$$

where $V_0(e^{j\omega})$ is the distortion function and $V_\xi(e^{j\omega})$, $\xi = 1, 2, \ldots, M-1$, are the aliasing functions with

$$V_\xi(e^{j\omega}) = \frac{1}{M} \sum_{k=1}^{K} B_k(e^{j(\omega-2\pi\xi/M)})C_k(e^{j\omega}) \tag{25}$$

for $\xi = 0, 1, \ldots, M-1$. As can be seen from (24) and (25), the analysis and synthesis filters should be designed such that the distortion and aliasing functions approximate unity and zero, respectively, in the active subband locations. The overall design complexity becomes very high if the subfilters $F_{\ell}(e^{j\omega})$ and $G_{\ell}(e^{j\omega})$ in (19) and the prototype filter for the cosine-modulated synthesis FB are designed together. Therefore, to reduce the overall design complexity, we propose the following design procedure. First, the prototype filter $P(e^{j\omega})$ is designed and fixed. Next, the coefficients of the $2K$ subfilters $F_{\ell}(e^{j\omega})$ and $G_{\ell}(e^{j\omega})$ are determined such that the distortion and aliasing terms are kept below a certain desired level. Due to the large number of constraints that need to be satisfied during the optimization, we use a least-squares approach so that the subfilter coefficients can be obtained via a closed-form solution. Compared to numerical optimization, such a closed-form solution significantly reduces the design time. Also, during reconfiguration, if a new set of sampling instants are selected, the closed-form solution makes it feasible to redetermine the coefficients online.

A. Prototype Filter Design

The prototype filter $P(e^{j\omega})$ is a power-symmetric lowpass filter with a passband edge at $\omega_c = (1 - \varepsilon)\pi/2M$ and a stopband edge at $\omega_s = (1 + \varepsilon)\pi/2M$ with $\varepsilon$ related to the percentage occupancy $\rho$ of the subband as $\varepsilon = 1 - \rho/100$. Due to the power-symmetry constraints as in (26) below, it is not possible to use a least-squares approach for the design.
of \(P(e^{j\omega})\). However, unlike the design of the 2\(K\) subfilters \(F_k(e^{j\omega})\) and \(G_k(e^{j\omega})\), the prototype filter can be designed using numerical optimization techniques as the optimization has fewer constraints. Also, the coefficients of \(P(e^{j\omega})\) are determined offline and only once, since the same \(P(e^{j\omega})\) can be used even if the sampling instants change. In the subsequent design example sections, we use the MATLAB minimax optimization function \(\text{fminimax}\) for the design of \(P(e^{j\omega})\). Using minimax design, the coefficients of \(P(e^{j\omega})\) are determined such that the prototype filter approximates the passband and the stopband responses with unity and zero, respectively, as well as the power-symmetry property in the transition band with tolerances \(\delta_0\), \(\delta_1\), and \(\delta_2\) according to:

\[
|P(e^{j\omega}) - 1| \leq \delta_0, \quad \omega \in [0, \omega_c]
\]

\[
|P(e^{j\omega})| \leq \delta_1, \quad \omega \in [\omega_s, \pi]
\]

\[
|1 - |P(e^{j\omega})|^2 - |P(e^{j(\omega - \pi/M)})|^2| \leq \delta_2, \quad \omega \in [\omega_c, \omega_s].
\]

The coefficients of \(P(e^{j\omega})\) can therefore be obtained by solving the minimax optimization problem:

Given the order of the prototype filter \(N_p\), determine the coefficients \(\phi(n)\) of the prototype filter \(P(e^{j\omega})\) and a parameter \(\delta\), to minimize \(\delta\) subject to:

\[
|P(e^{j\omega}) - 1| \leq \delta, \quad \omega \in [0, \omega_c]
\]

\[
|P(e^{j\omega})| \leq \delta, \quad \omega \in [\omega_s, \pi]
\]

\[
|1 - |P(e^{j\omega})|^2 - |P(e^{j(\omega - \pi/M)})|^2| \leq \delta, \quad \omega \in [\omega_c, \omega_s].
\]

The filter \(P(e^{j\omega})\) designed by solving the above optimization problem satisfies (26) if, after the optimization, \(\delta \leq \min(\delta_0, \delta_1, \delta_2)\). A good initial solution for the optimization problem can be obtained using, for example, the methods in [27], [28]. Our experiments indicate that \(\delta\) should be 6–8 dB lower than the specified amplitude of the residual aliasing terms after reconstruction.

B. Least-Squares Design of \(F_k(z)\) and \(G_k(z)\)

After determining the coefficients of the lowpass prototype filter for the synthesis FB, we use a least-squares approach to determine the coefficients of the fixed subfilters \(F_k(z)\) and \(G_k(z)\). The coefficients are determined such that they minimize an error power function \(\mathcal{P}\) defined as:

\[
\mathcal{P} = \mathcal{P}_0 + \sum_{\xi=1}^{M-1} \mathcal{P}_\xi
\]

where

\[
\mathcal{P}_0 = \frac{1}{2\pi} \int_{\Omega} |V_0(e^{j\omega}) - 1|^2 d\omega, \quad \Omega \in \Omega_{r_1,0}
\]

and

\[
\mathcal{P}_\xi = \frac{1}{2\pi} \int_{\Omega} |V_\xi(e^{j\omega})|^2 d\omega, \quad \Omega \in \Omega_{r_1,\xi}
\]

with \(\Omega_{r_1,0}, r_i \in [0, 1, \ldots, M-1], i = 1, 2, \ldots, K\), representing the active subband locations and \(\Omega_{r_1,\xi}, \xi = 1, \ldots, M - 1\)

\[\text{In this paper, to simplify derivations, we assume that all filters are noncausal. The designed filters can be easily made causal by adding suitable delays.}\]
C. Design of Reconfigurable Reconstructors

In a reconfigurable reconstructor, first, the prototype filter for the cosine-modulated synthesis FB, is designed as outlined in Section VI-A. Further, the subfilters \( F_{\ell}(z) \) and \( G_{\ell}(z) \) in the analysis FB are designed and fixed based on the sampling instants. In applications where all the \( L \) possible combinations (\( L \) modes) of the \( K \) active subbands use the same set of sampling instants, during reconfiguration, it suffices to redefine the complex constants \( v_k \) in (11). Following a least-squares approach similar to the one outlined in Section VI-B, the coefficients of the subfilters \( F_{\ell}(z) \) and \( G_{\ell}(z) \) for the reconfigurable reconstructor are then determined using

\[
E(\xi, \omega) = \begin{bmatrix}
a_{11}(\xi, \omega)e(\omega, N_F) & b_{11}(\xi, \omega)e(\omega, N_F) & \cdots & a_{1K}(\xi, \omega)e(\omega, N_F) & b_{1K}(\xi, \omega)e(\omega, N_F) \\
a_{21}(\xi, \omega)e(\omega, N_F) & b_{21}(\xi, \omega)e(\omega, N_F) & \cdots & a_{2K}(\xi, \omega)e(\omega, N_F) & b_{2K}(\xi, \omega)e(\omega, N_F) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{K1}(\xi, \omega)e(\omega, N_F) & b_{K1}(\xi, \omega)e(\omega, N_F) & \cdots & a_{KK}(\xi, \omega)e(\omega, N_F) & b_{KK}(\xi, \omega)e(\omega, N_F)
\end{bmatrix}
\]

(36)

D. Design Complexity

Splitting the reconstructor design into two parts, as discussed above, makes it feasible to design and implement a reconfigurable reconstructor, especially for larger \( M \). This is exemplified using a design example in Section VII. During reconfiguration, the proposed reconstructor can be re- configured online by inverting a single \( K \times K \) matrix if all modes use the same set of sampling instants. If each mode uses a different set of sampling instants, during reconfiguration, the reconfiguration requires only one additional \( 2K(N_F + 1) \times 2K(N_F + 1) \) matrix inversion. In contrast, for the straightforward scheme [7], the reconfiguration involves inverting several \( (N_A + 1) \times (N_A + 1) \) matrices where \( N_A \) is the order of each multi-level synthesis filter \( A_{\ell}(z) \) in Fig. 2. Typically, \( N_A > 2K(N_F + 1) \) as can be seen from the examples in Section VII.

VII. DESIGN EXAMPLES

Example 1: In this example, we assume that there are three active users with two possible combinations of active band locations. It is assumed that at any given time frame, the active frequency bands can be either \([[3.2–4.8], [7.2–7.8], [11.2–11.8]] \times \pi/16 \) or \([[3.2–3.8], [7.2–7.8], [11.2–12.8]] \times \pi/16 \). Further, it is assumed that the reconstructor should be designed such that aliasing terms are kept below \(-60 \) dB.

For a given combination of active band locations, the number of channels, \( K \), required to implement the CNUS scheme will depend on the total number of granularity bands \( M \). In this example, the number of granularity bands \( M \) is chosen so as to get the least implementation complexity for the reconstructor. In order to have practical filters, a transition band is included in each active granularity band and, depending on \( M \), the percentage occupancy \( \rho \) (see Section VI-A) of a granularity band is assumed to be within 20–60\%.

As shown in Fig. 12, for the two possible combinations of active band locations assumed in this example, the least computational complexity is obtained with \( M = 32 \). When the total bandwidth is divided into \( M = 32 \) granularity bands, with the information containing frequency bands assumed in this example, only \( K = 8 \) granularity bands are active at any given time frame. Thus, at any given time frame, the users can be allocated either the granularity bands \([6–9, 14, 15, 22, 23]\) or the bands \([6, 7, 14, 15, 22–25]\). For the above two possible combination of band locations (two modes), we used the sub-Nyquist sampling points, \( m = 0, 3, 5, 14, 16, 19, 21, 30 \), which ensures that \( D \) in (11) is an invertible matrix. The sampling instants were determined using the method in [29].

Based on the occupied frequencies and the active bands, the percentage band occupancy \( \rho \) of the lowpass prototype filter \( P(z) \) is fixed at 20\%. The prototype filter is designed to be a power-symmetric lowpass filter of order 386 with \( \omega_c = 0.2\pi/64 \) and \( \omega_s = 1.8\pi/64 \). It is found that, for the 16 subfilters, \( F_{\ell}(z) \) and \( G_{\ell}(z) \), a filter order \( N_F = 14 \) is sufficient to keep the aliasing terms below \(-60 \) dB.

In order to determine the coefficients of the multi-level synthesis filters in the straightforward scheme in [7], we used
the time-varying reconstructor design method in [30] but with some of the impulse response coefficients set to zero due to the CNU5 scheme. It is found that the straightforward scheme would require a reconstructor with eight synthesis filters of order \( N_A = 318 \).

Table I tabulates the reconstructor complexity when the specification in this example is implemented using the straightforward and the proposed reconstructor. As can be seen from Table I, the proposed reconstructor offers significant reduction in complexity due to the efficient realization in Fig. 1. It can be seen that during reconfiguration from one mode to the other, the proposed reconstructor requires significantly fewer multipliers to be updated online. The coefficients of these multipliers can be either determined offline and stored in a memory or determined online using a single \( 8 \times 8 \) matrix inversion. In contrast, the straightforward scheme would require a larger memory or eight \( 319 \times 319 \) online matrix inversions.

Figure 13 shows all the distortion and aliasing terms of the reconstructor for the two possible combinations of user band locations. It can be seen that, in the required bands, the aliasing terms are not greater than \(-60 \) dB which validates the reconfigurability between the two different combinations of user band locations. The reconfigurability of the reconstructor is illustrated in Figs. 14 and 15 by configuring it for one set of active band locations and using it to reconstruct a sub-Nyquist sampled multi-tone input with tones in the active band region. The spectrum without reconstruction in Figs. 14 and 15 corresponds to the spectrum of the sub-Nyquist sampled signal with zeros inserted into the time instants where the samples are missing.

**Example 2:** This example illustrates that, for larger \( M \), the proposed method provides even more significant savings in the design and implementation complexity of the reconstructor compared to the straightforward method that uses only synthesis FBs. This is in line with the complexity comparison in Section V-B. Here, we consider an example where the information containing frequency bands are \( \{3.21-3.82\}, \{7.21-7.82\}, \{20.21-21.82\}, \{46.01-47.99\}, \{54-55\} \times \pi/64 \) and the reconstructor should be designed to keep the aliasing terms below \(-40 \) dB. For the above frequency bands, the computational complexity of the reconstructor is least when \( M = 128 \) and \( K = 18 \). Consequently, the active granularity bands are \{6, 7, 14, 15, 40–43, 91–96, 107–110\} with \( \rho = 29\% \). Further, we use the sub-Nyquist sampling points \( m = 3 \) and \( N' \) represent the number of multiplications per corrected output sample and the number of multipliers to be updated during reconfiguration, respectively. The reconfiguration complexity is the number of online matrix inversions. For the straightforward reconstructor, since we assume a polyphase implementation, \( C \) is computed as in (23).
In this paper, we proposed a reconfigurable reduced-complexity reconstructor for sub-Nyquist sampled sparse multi-band signals. The reconstructor was derived by expressing the reconstruction problem in terms of both analysis and synthesis FBs. We showed that the nonzero polyphase components of the bandpass filters in the analysis FB are generalized FD filters. Due to this, the analysis filters can be expressed in terms of a common set of fixed subfilters and a set of multipliers, thereby reducing the complexity. Moreover, since the filters in the synthesis FB are regular bandpass filters, further reduction in complexity was achieved by implementing these filters using a cosine-modulated FB. We also showed that, compared to the straightforward reconstructor, the proposed reconstructor makes it feasible to achieve order-of-magnitude reduction in the computational complexity. In addition, the proposed reconstructor provides significant reduction in the complexity of the online reconfiguration block as only the coefficients of the set of multipliers in the analysis FB have to be redetermined.

**REFERENCES**


