Oblique derivative problem
for non-divergence parabolic equations
with discontinuous in time coefficients

Vladimir Kozlov* and Alexander Nazarov†

Abstract
We consider an oblique derivative problem for non-divergence parabolic equations with discontinuous in $t$ coefficients in a half-space. We obtain weighted coercive estimates of solutions in anisotropic Sobolev spaces. We also give an application of this result to linear parabolic equations in a bounded domain. In particular, if the boundary is of class $C^{1,\delta}$, $\delta \in (0, 1]$, then we present a coercive estimate of solutions in weighted anisotropic Sobolev spaces, where the weight is a power of the distance to the boundary.

1 Introduction
Consider the parabolic equation

\[(L_0 u)(x, t) \equiv \partial_t u(x, t) - a^{ij}(t)D_i D_j u(x, t) = f(x, t)\]  

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Here and elsewhere $D_i$ denotes the operator of differentiation with respect to $x_i$ and $\partial_t u$ is the derivative of $u$ with respect to $t$.

*Department of Mathematics, University of Linköping, SE-581 83 Linköping, Sweden
†St.-Petersburg Department of Steklov Mathematical Institute, Fontanka, 27, St.-Petersburg, 191023, Russia, and St.-Petersburg State University, Universitetskii pr. 28, St.-Petersburg, 198504, Russia
The only assumptions about the coefficients in (1) is that $a^{ij}$ are measurable real valued functions of $t$ satisfying $a^{ij} = a^{ji}$ and

$$\nu |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq \nu^{-1} |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad \nu = \text{const} > 0.$$ (2)

It was proved by Krylov [2, 3] that for $f \in L_{p,q}(\mathbb{R}^n \times \mathbb{R})$ with $1 < p, q < \infty$, equation (1) in $\mathbb{R}^n \times \mathbb{R}$ has a unique solution such that $\partial_t u$ and $D_i D_j u$ belong to $L_{p,q}(\mathbb{R}^n \times \mathbb{R})$ and

$$\|\partial_t u\|_{p,q} + \sum_{ij} \|D_i D_j u\|_{p,q} \leq C\|f\|_{p,q}. \quad (3)$$

Here $L_{p,q}(\Omega \times I) = L_q(I \to L_p(\Omega))$ is the space of functions on $\Omega \times I$ with finite norm

$$\|f\|_{p,q} = \left( \int_\Omega \left( \int_I |f(x,t)|^p dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

(with natural change in the case $p = \infty$ or $q = \infty$).

In the authors’ paper [4] estimate (3) was supplemented by a similar one in the space $\tilde{L}_{p,q}(\mathbb{R}^n \times \mathbb{R})$

$$\|\partial_t u\|_{p,q} + \sum_{ij} \|D_i D_j u\|_{p,q} \leq C\tilde{f}\|_{p,q}. \quad (3)$$

Here $\tilde{L}_{p,q}(\Omega \times I) = L_p(\Omega \to L_q(I))$ is the space of functions on $\Omega \times I$ with finite norm

$$\|	ilde{f}\|_{p,q} = \left( \int_\Omega \left( \int_I |f(x,t)|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

(with natural change in the case $p = \infty$ or $q = \infty$). This space arises naturally in the theory of quasilinear non-divergence parabolic equations (see [9]). Note that for $p = q$ we have

$$\tilde{L}_{p,p}(\Omega \times I) = L_{p,p}(\Omega \times I) = L_p(\Omega \times I); \quad \|	ilde{f}\|_{p,p} = \|f\|_{p,p} = \|f\|_p.$$

The homogeneous Dirichlet problem for (1) in $\mathbb{R}^n_+ \times \mathbb{R}$, where $\mathbb{R}^n_+$ the half-space $\{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, was considered in [2, 4]. It was proved that its solution satisfies the following weighted coercive estimate

$$\|x_n^\mu \partial_t u\|_{p,q} + \sum_{ij} \|x_n^\mu D_i D_j u\|_{p,q} \leq C\|x_n^\mu f\|_{p,q}, \quad (4)$$

2
where $1 < p, q < \infty$ and $\mu \in \left(-\frac{1}{p}, 2 - \frac{1}{p}\right)$ (in [2] this estimate was proved only for $\mu \in (1 - \frac{1}{p}, 2 - \frac{1}{p})$). An analog of estimate (4), where the norm $\| \cdot \|_{p,q}$ is replaced by $\| \cdot \|_{p,q}^*$, is also proved in [4].

In the paper [5] the homogeneous Dirichlet problem for (1) in cones and wedges was considered, and coercive estimates for solutions were obtained in the scales of weighted $L_{p,q}$ and $\tilde{L}_{p,q}$ spaces, where the weight is a power of the distance to the vertex (edge).

Let us turn to the oblique derivative problem in the half-space $\mathbb{R}^n_+$. Now equation (1) is satisfied for $x_n > 0$ and $\frac{\partial u}{\partial \gamma} = 0$ for $x_n = 0$. Here $\gamma$ is a constant vector field with $\gamma_n > 0$.

By changing the spatial variables one can reduce the boundary condition to the case

$$D_n u = 0 \quad \text{for} \quad x_n = 0. \quad (5)$$

One of the main results of this paper is the proof of estimate (4) and its analog for the norm $\| \cdot \|_{p,q}$, for solutions of the oblique derivative problem (1), (5) with arbitrary $p, q \in (1, \infty)$ and for $\mu$ satisfying

$$-\frac{1}{p} < \mu < 1 - \frac{1}{p}. \quad (6)$$

In the case of time independent coefficients such estimates for the Neumann problem were proved in [9].

We use an approach based on the study of the Green functions. In Section 2 we collect (partially known) results on the estimate of the Green function and of solutions to the Dirichlet problem for equation (1). Section 3 is devoted to the estimates of the Green function of problem (1), (5).

In Section 4 we apply the obtained estimates to the oblique derivative problem for linear non-divergence parabolic equations with discontinuous in time coefficients in cylinders $\Omega \times (0, T)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$. We prove solvability results in weighted $L_{p,q}$ and $\tilde{L}_{p,q}$ spaces, where the weight is a power of the distance to the boundary of $\Omega$. The smoothness of the boundary is characterized by smoothness of local isomorphisms in neighborhoods of boundary points, which flatten the boundary. In particular, if the boundary is of the class $C^{1, \delta}$ with $\delta \in (0, 1]$, then for solutions to the equation (1)\footnote{Here the coefficients $a^{ij}$ may depend on $x$ (namely, we assume $a^{ij} \in C(\Omega \to L_\infty(0, T))$.} in $\Omega \times (0, T)$ with zero initial and boundary conditions the
following coercive estimate is proved in Theorem 4 (see Remark 1):
\[
\| (\tilde{d}(x))^\mu \partial_t u \|_{p,q} + \sum_{ij} \| (\tilde{d}(x))^\mu D_i D_j u \|_{p,q} \leq C \| (\tilde{d}(x))^\mu f \|_{p,q},
\]
\[
\| (\tilde{d}(x))^\mu \partial_t u \|_{p,q} + \sum_{ij} \| (\tilde{d}(x))^\mu D_i D_j u \|_{p,q} \leq C \| (\tilde{d}(x))^\mu f \|_{p,q},
\]
where \( \mu, p, q \) and \( \delta \) satisfy \( 1 < p, q < \infty, \ 1 - \delta - 1/p < \mu < 1 - 1/p \).

Let us recall some notation: \( x = (x_1, \ldots, x_n) = (x', x_n) \) is a point in \( \mathbb{R}^n \); \( Du = (D_1 u, \ldots, D_n u) \) is the gradient of \( u \).

We denote
\[
Q_R(x^0, t^0) = \{(x, t) : |x - x^0| < R, \ 0 < t^0 - t < R^2\};
\]
\[
Q^+_R(x^0, t^0) = \{(x, t) : |x - x^0| < R, \ x_n > 0, \ 0 < t^0 - t < R^2\}.
\]
The last notation will be used only for \( x^0 \in \mathbb{R}^n_+ \).

Set
\[
\mathcal{R}_x = \frac{x_n}{x_n + \sqrt{t - s}}, \quad \mathcal{R}_y = \frac{y_n}{y_n + \sqrt{t - s}}.
\]

In what follows we denote by the same letter the kernel and the corresponding integral operator, i.e.
\[
(Kh)(x, t) = \int_{-\infty}^{t} \int \mathcal{K}(x, y; t, s) h(y, s) \, dy \, ds.
\]
Here we expand functions \( \mathcal{K} \) and \( h \) by zero to whole space-time if necessary.

We adopt the convention regarding summation from 1 to \( n \) with respect to repeated indices. We use the letter \( C \) to denote various positive constants.

## 2 Preliminary results: estimates of strong and weak solutions

### 2.1 The case of whole space

Let us consider equation (1) in the whole space \( \mathbb{R}^n \). Using the Fourier transform with respect to \( x \) one can obtain the following representation of solution
through the right-hand side:

\[ u(x, t) = \int_{-\infty}^{t} \int \Gamma(x, y; t, s)f(y, s) \, dy \, ds, \quad (6) \]

where \( \Gamma \) is the Green function of the operator \( L_0 \) given by

\[ \Gamma(x, y; t, s) = \frac{\det \left( \int_s^t A(\tau) d\tau \right) - \frac{1}{2}}{(4\pi)^{\frac{n}{2}}} \exp \left( -\frac{\left( \int_s^t A(\tau) d\tau \right)^{-1} (x - y), (x - y)}{4} \right) \]

for \( t > s \) and 0 otherwise. Here \( A(t) \) is the matrix \( \{a^{ij}(t)\}_{i,j=1}^{n} \). The above representation implies, in particular, the following estimates.

**Proposition 1.** Let \( \alpha \) and \( \beta \) be two arbitrary multi-indices. Then

\[ |D^\alpha_x D^\beta_y \Gamma(x, y; t, s)| \leq C (t - s)^{-\frac{n+|\alpha|+|\beta|}{2}} \exp \left( -\frac{\sigma |x - y|^2}{t - s} \right) \]

and

\[ |\partial_s D^\alpha_x D^\beta_y \Gamma(x, y; t, s)| \leq C (t - s)^{-\frac{n+|\alpha|+|\beta|}{2}} \exp \left( -\frac{\sigma |x - y|^2}{t - s} \right) \]

for \( x, y \in \mathbb{R}^n \) and \( s < t \). Here \( \sigma \) depends only on the ellipticity constant \( \nu \) and \( C \) may depend on \( \nu, \alpha \) and \( \beta \).

In the next proposition we present solvability results for equation (1) in the whole space.

**Proposition 2.** Let \( p, q \in (1, \infty) \).

(i) If \( f \in L_{p,q}(\mathbb{R}^n \times \mathbb{R}) \), then the solution of equation (1) given by (6) satisfies

\[ \|\partial_t u\|_{p,q} + \sum_{ij} \|D_i D_j u\|_{p,q} \leq C \|f\|_{p,q}, \quad (7) \]

where \( C \) depends only on \( \nu, p, q \).

(ii) If \( f \in \tilde{L}_{p,q}(\mathbb{R}^n \times \mathbb{R}) \), then the solution of equation (1) given by (6) satisfies

\[ \|\partial_t u\|_{p,q} + \sum_{ij} \|D_i D_j u\|_{p,q} \leq C \|f\|_{p,q}, \quad (8) \]

where \( C \) depends only on \( \nu, p, q \).
The first assertion is proved in [2] and the second one in [4].

Now we consider the equation

\[ \mathcal{L}_0 u = \text{div } (f) \quad \text{in } \mathbb{R}^n \times \mathbb{R} \]  

(9)

(here \( f = (f_1, \ldots, f_n) \)).

**Lemma 1.** Let \( 1 < p, q < \infty \) and \( \mu \in (-\frac{1}{p}, 1 - \frac{1}{p}) \).

(i) Suppose that \( f \in L_{p,q}(\mathbb{R}^n \times \mathbb{R}) \). Then the function

\[ u(x, t) = -\int_t^t \int_{-\infty}^\infty \nabla_y \Gamma(x, y; t, s) \cdot f(y, s) \, dy \, ds \]  

(10)

gives a weak solution of equation (9) and satisfies the estimate

\[ \| Du \|_{p,q} \leq C \| f \|_{p,q}. \]  

(11)

(ii) Suppose that \( f \in \tilde{L}_{p,q}(\mathbb{R}^n \times \mathbb{R}) \). Then the function (10) gives a weak solution of equation (9) and satisfies the estimate

\[ \| Du \|_{p,q} \leq C \| x_n f \|_{p,q}. \]  

(12)

**Proof.** The function (10) obviously solves (9) in the sense of distributions. Next, the symmetry of \( \Gamma \) with respect to \( x \) and \( y \) implies \( \nabla_x \nabla_y \Gamma(x, y; t, s) = -\nabla_x \nabla_x \Gamma(x, y; t, s) \), and estimates (11) and (12) follow from (7) and (8), respectively.

\[ \square \]

### 2.2 The case of the half-space under Dirichlet boundary condition

We formulate two auxiliary results on estimates of integral operators. The first statement is a particular case \( m = 1 \) of [4, Lemmas A.1 and A.3 and Remark A.2], see also [9, Lemmas 2.1 and 2.2].

**Proposition 3.** Let \( 1 \leq p \leq \infty, \sigma > 0, 0 < r \leq 2, \lambda_1 + \lambda_2 > -1 \), and let

\[ -\frac{1}{p} - \lambda_1 < \mu < 1 - \frac{1}{p} + \lambda_2. \]  

(13)
Suppose also that the kernel $T(x, y; t, s)$ satisfies the inequality

$$|T(x, y; t, s)| \leq C \frac{R_x^{\lambda_1 + r} R_y^{\lambda_2}}{(t-s)^{n+2r}} \frac{x_n^{\mu - r}}{y_n^\mu} \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right),$$

for $t > s$. Then the integral operator $T$ is bounded in $L_p(\mathbb{R}^n \times \mathbb{R})$ and in $\tilde{L}_p(\mathbb{R}^n \times \mathbb{R})$.

The next proposition is a particular case $m = 1$ of [4, Lemma A.4], see also [9, Lemma 3.2].

**Proposition 4.** Let $1 < p < \infty$, $\sigma > 0$, $\kappa > 0$, $0 \leq r \leq 2$, $\lambda_1 + \lambda_2 > -1$ and let $\mu$ be subject to (13). Also let the kernel $T(x, y; t, s)$ satisfy the inequality

$$\frac{R_x^{\lambda_1 + r} R_y^{\lambda_2}}{(t-s)^{n+2r}} \frac{x_n^\mu}{y_n^\mu} \left( \frac{\delta}{t-s} \right)^\kappa \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right),$$

for $t > s + \delta$. Then for any $s^0 > 0$ the norm of the operator

$$T : L_{p,1}(\mathbb{R}^n \times (s^0 - \delta, s^0 + \delta)) \to L_{p,1}(\mathbb{R}^n \times (s^0 + 2\delta, \infty))$$

do not exceed a constant $C$ independent of $\delta$ and $s^0$.

We denote by $\Gamma_D(x, y; t, s)$ the Green function of the operator $L_0$ in the half-space $\mathbb{R}^n_+$ subject to the homogeneous Dirichlet boundary condition on the boundary $x_n = 0$.

The next statement is proved in [4, Theorem 3.6].

**Proposition 5.** For $x, y \in \mathbb{R}^n_+$ and $t > s$ the following estimate is valid:

$$|D_x^\alpha D_y^\beta \Gamma_D(x, y; t, s)| \leq C \frac{R_x^{2-\alpha_n-\varepsilon} R_y^{2-\beta_n-\varepsilon}}{(t-s)^{n+|\alpha|+|\beta|}} \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right),$$

where $\sigma$ is a positive number depending on $\nu$ and $n$, $\varepsilon$ is an arbitrary small positive number and $C$ may depend on $\nu$, $\alpha$, $\beta$ and $\varepsilon$. If $\alpha_n \leq 1$ (or $\beta_n \leq 1$) then $2 - \alpha_n - \varepsilon$ (or $2 - \beta_n - \varepsilon$) must be replaced by $1 - \alpha_n$ ($1 - \beta_n$) respectively in the corresponding exponents.

Since $(\partial_s + a_{ij}(-s) D_y D_y) \Gamma_D(x, y; t, s) = 0$ for $s < t$, we obtain
Corollary 1. For \( x, y \in \mathbb{R}^n_+ \) and \( t > s \)
\[
|D_x^\alpha D_y^\beta \partial_s \Gamma^D(x, y; t, s)| \leq C \frac{R_x^{2-\alpha_n-\varepsilon} R_y^{-\beta_n-\varepsilon}}{(t-s)^{n+2+|\alpha|+|\beta|/2}} \exp \left( -\frac{\sigma|x-y|^2}{t-s} \right). \tag{15}
\]
If \( \alpha_n \leq 1 \) then \( 2 - \alpha_n - \varepsilon \) must be replaced by \( 1 - \alpha_n \).

Now we consider the problem
\[
L_0 u = f_0 + \text{div}(f) \quad \text{in} \quad \mathbb{R}^n_+ \times \mathbb{R}; \quad u\big|_{x_n=0} = 0. \tag{16}
\]

Theorem 1. Let \( 1 < p, q < \infty \) and \( \mu \in \left( -\frac{1}{p}, 1 - \frac{1}{p} \right) \).
(i) Suppose that \( x_\mu^{\mu+1} f_0, x_\mu^\mu f \in \tilde{L}_{p,q}(\mathbb{R}^n_+ \times \mathbb{R}) \). Then the function
\[
u(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^n_+} \left( \Gamma^D(x, y; t, s) f_0(y, s) - D_y \Gamma^D(x, y; t, s) \cdot f(y, s) \right) dy ds \quad \tag{17} \]
gives a weak solution of problem (16) and satisfies the estimate
\[
\|x_\mu^\mu Du\|_{p,q} + \|x_\mu^{\mu-1} u\|_{p,q} \leq C(\|x_\mu^{\mu+1} f_0\|_{p,q} + \|x_\mu^\mu f\|_{p,q}). \tag{18}
\]
(ii) Suppose that \( x_\mu^{\mu+1} f_0, x_\mu^\mu f \in L_{p,q}(\mathbb{R}^n_+ \times \mathbb{R}) \). Then the function (17) gives a weak solution of problem (16) and satisfies the estimate
\[
\|x_\mu^\mu Du\|_{p,q} + \|x_\mu^{\mu-1} u\|_{p,q} \leq C(\|x_\mu^{\mu+1} f_0\|_{p,q} + \|x_\mu^\mu f\|_{p,q}). \tag{19} \]

Proof. First, function (17) obviously solves problem (16) in the sense of distributions. Thus, it is sufficient to prove estimates (18), (19).

Put
\[
\mathcal{K}_0(x, y; t, s) = \frac{x_\mu^{-1}}{y_{n+1}} \Gamma^D(x, y; t, s); \quad \mathcal{K}_1(x, y; t, s) = \frac{x_\mu^{-1}}{y_n} D_y \Gamma^D(x, y; t, s); \\
\mathcal{K}_2(x, y; t, s) = \frac{x_\mu}{y_{n+1}} D_x \Gamma^D(x, y; t, s); \quad \mathcal{K}_3(x, y; t, s) = \frac{x_\mu}{y_n} D_x D_y \Gamma^D(x, y; t, s).
\]

(i) By Proposition 5 the kernels \( \mathcal{K}_0 \) and \( \mathcal{K}_1 \) satisfy the conditions of Proposition 3 with \( r = 1 \) and
with \( \lambda_1 = -1, \lambda_2 = 1 \) and \( \mu \) replaced by \( \mu + 1 \) for the kernel \( \mathcal{K}_0 \);
with \( \lambda_1 = \lambda_2 = 0 \) for the kernel \( K_1 \), respectively.

This implies that for \( \mu \in (-\frac{1}{p}, 1 - \frac{1}{p}) \)

\[
\|x_n^{-\mu-1}u\|_p \leq C(\|x_n^{\mu+1}f_0\|_p + \|x_n^\mu f\|_p) \tag{20}
\]

and

\[
\|x_n^{-\mu-1}u\|_{p,\infty} \leq C(\|x_n^{\mu+1}f_0\|_{p,\infty} + \|x_n^\mu f\|_{p,\infty}). \tag{21}
\]

Interpolating (20) and (21) we arrive at

\[
\|x_n^{-\mu-1}u\|_{p,q} \leq C(\|x_n^{\mu+1}f_0\|_{p,q} + \|x_n^\mu f\|_{p,q}), \tag{22}
\]

for \( 1 < p \leq q < \infty \) and \( \mu \in (-\frac{1}{p}, 1 - \frac{1}{p}) \). Now duality argument gives (22) for all \( 1 < p, q < \infty \) and for the same interval of \( \mu \).

To estimate the first term in the left-hand side of (18) we use local estimates. We put

\[
B_{\rho,\vartheta}(\xi) = \{ x \in \mathbb{R}^n : |x' - \xi'| < \rho, \rho \vartheta < x_n < \rho \}.
\]

Localization of estimates (8) and (12) using an appropriate cut-off function, which is equal to 1 on \( B_{\rho,2} \) and 0 outside \( B_{2\rho,8} \), gives

\[
\int_{B_{\rho,2}(\xi)} \left( \int_{\mathbb{R}} |Du|^q dt \right)^{\frac{p}{q}} dx \leq C \int_{B_{2\rho,8}(\xi)} \left( \int_{\mathbb{R}} (|u|^q \rho^{-q} + \rho^q |f_0|^q + |f|^q dt) \right)^{\frac{p}{q}} dx.
\]

This estimate together with a proper partition of unity in \( \mathbb{R}^n_+ \) leads to

\[
\int_{\mathbb{R}^n_+} \left( \int_{\mathbb{R}} |Du|^q dt \right)^{p/q} x_n^{\mu p} dx \leq C \left( \int_{\mathbb{R}^n_+} \left( \int_{\mathbb{R}} |u|^q dt \right)^{p/q} x_n^{\mu p-p} dx \right.
\]

\[
+ \int_{\mathbb{R}^n_+} \left( \int_{\mathbb{R}} |f|^q dt \right)^{p/q} x_n^{\mu p} dx + \int_{\mathbb{R}^n_+} \left( \int_{\mathbb{R}} |f_0|^q dt \right)^{p/q} x_n^{\mu p+p} dx.
\]

This immediately implies (18) with regard of (22).

(ii) To deal with the scale \( L_{p,q} \), we need the following lemma.
Lemma 2. Let a function $h$ be supported in the layer $|s - s^0| \leq \delta$ and satisfy
\[ \int h(y; s) \, ds \equiv 0. \]
Also let $p \in (1, \infty)$ and $\mu \in (-\frac{1}{p}, 1 - \frac{1}{p})$. Then the operators $\mathcal{K}_j$, $j = 0, 1, 2, 3$, satisfy
\[ \int_{|t-s^0|>2\delta} \| (\mathcal{K}_j h)(\cdot; t) \|_p \, dt \leq C \| h \|_{p,1}, \]
where $C$ does not depend on $\delta$ and $s^0$.

Proof. By $\int h(y; s) \, ds \equiv 0$, we have
\[ (\mathcal{K}_j h)(x; t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} \left( \mathcal{K}_j(x, y; t, s) - \mathcal{K}_j(x, y; t, s^0) \right) h(y; s) \, dy \, ds \tag{23} \]
(we recall that all functions are assumed to be extended by zero).

We choose $\varepsilon > 0$ such that
\[ -\frac{1}{p} < \mu < 1 - \frac{1}{p} - \varepsilon. \tag{24} \]

For $|s - s^0| < \delta$ and $t - s^0 > 2\delta$, estimate (15) implies
\[ |\mathcal{K}_j(x, y; t, s) - \mathcal{K}_j(x, y; t, s^0)| \leq \int_{s^0}^{s} |\partial_s \mathcal{K}_j(x, y; t, \tau)| \, d\tau \]
\[ \leq C \frac{R_x^{\ell_1} R_y^{\ell_2-\varepsilon} x_n^{\ell_3} y_n^{\ell_4}}{(t-s)^{\frac{n+2-x}{2}}} \frac{\delta}{t-s} \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right), \]
with $r = 2$, $\ell_1 = 1$, $\ell_2 = 0$, $\ell_3 = \mu - 1$, $\ell_4 = \mu + 1$ for the kernel $\mathcal{K}_0$;
with $r = 1$, $\ell_1 = 1$, $\ell_2 = -1$, $\ell_3 = \mu - 1$, $\ell_4 = \mu$ for the kernel $\mathcal{K}_1$;
with $r = 1$, $\ell_1 = 0$, $\ell_2 = 0$, $\ell_3 = \mu$, $\ell_4 = \mu + 1$ for the kernel $\mathcal{K}_2$;
with $r = 0$, $\ell_1 = 0$, $\ell_2 = -1$, $\ell_3 = \mu$, $\ell_4 = \mu$ for the kernel $\mathcal{K}_3$.

On the other hand, estimate (14) implies
\[ |\mathcal{K}_j(x, y; t, s) - \mathcal{K}_j(x, y; t, s^0)| \leq C \frac{R_x^{\ell_1} R_y^{\ell_2+1} x_n^{\ell_3} y_n^{\ell_4}}{(t-s)^{\frac{n+2-x}{2}}} \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right). \]

Combination of these estimates gives
\[ |\mathcal{K}_j(x, y; t, s) - \mathcal{K}_j(x, y; t, s^0)| \leq C \delta^x \frac{R_x^{\ell_1} R_y^{\ell_2+1-x} x_n^{\ell_3} y_n^{\ell_4}}{(t-s)^{\frac{n+2-x}{2}+x}} \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right), \]
where \( \zeta = \frac{\varepsilon}{1 + \varepsilon} \). Thus, the kernels in (23) satisfy the assumptions of Proposition 4 with 
\( \lambda_1 = -1, \lambda_2 = 1 - \varepsilon \) and \( \mu \) replaced by \( \mu + 1 \) for kernels \( K_0 \) and \( K_2 \);
with \( \lambda_1 = 0, \lambda_2 = -\varepsilon \) for kernels \( K_1 \) and \( K_3 \), respectively.
Inequality (24) becomes (13), and the Lemma follows.

We continue the proof of the second statement of Theorem 1. Estimate (18) for \( q = p \) provides boundedness of the operators \( K_j, j = 0, 1, 2, 3 \), in \( L_p(\mathbb{R}^n \times \mathbb{R}) \), which gives the first condition in [1, Theorem 3.8]. Lemma 2 is equivalent to the second condition in this theorem. Therefore, Theorem 3.8 [1] ensures that these operators are bounded in \( L_{p,q}(\mathbb{R}^n \times \mathbb{R}) \) for any \( q \in (1,p) \). For \( q \in (p, \infty) \) this statement follows by duality arguments. This implies estimate (19).

\[ \square \]

3 Oblique derivative problem

3.1 The Green function

**Theorem 2.** There exists a Green function \( \Gamma^N = \Gamma^N(x,y;t,s) \) of problem (1), (5) and for arbitrary \( x, y \in \mathbb{R}^n_+ \) and \( t > s \) it satisfies the estimate

\[ |D_x^\alpha D_y^\beta \Gamma^N(x,y;t,s)| \leq C \frac{\mathcal{R}_x^{\hat{\alpha}_n} \mathcal{R}_y^{\hat{\beta}_n}}{(t-s)^{\frac{n+|a|+|b|}{2}}} \exp \left(-\frac{\sigma|x-y|^2}{t-s}\right), \tag{25} \]

\[ |D_x^\alpha D_y^\beta \partial_t \Gamma^N(x,y;t,s)| \leq C \frac{\mathcal{R}_x^{\hat{\alpha}_n} \mathcal{R}_y^{1-\beta_n-\varepsilon}}{(t-s)^{\frac{n-2+|a|+|b|}{2}}} \exp \left(-\frac{\sigma|x-y|^2}{t-s}\right), \tag{26} \]

where

\[ \hat{\alpha}_n = \begin{cases} 0, & \alpha_n = 0; \\ 2 - \alpha_n, & \alpha_n = 1, 2; \\ 3 - \alpha_n - \varepsilon, & \alpha_n \geq 3; \end{cases} \quad \hat{\beta}_n = \begin{cases} 0, & \beta_n = 0; \\ 1 - \beta_n - \varepsilon, & \beta_n \geq 1. \end{cases} \]

Here \( \sigma \) is a positive number depending on \( \nu \) and \( n, \varepsilon \) is an arbitrary small positive number and \( C \) may depend on \( \nu, \alpha, \beta \) and \( \varepsilon \).
Proof. Let \( u \) be a solution of problem (1), (5). Then the derivative \( D_n u \) obviously satisfies the Dirichlet problem (16) with \( f_0 = 0 \) and \( f = (0, \ldots, 0, f) \). Therefore,

\[
D_n u = - \int_{-\infty}^{t} \int_{\mathbb{R}^n_{+}} D_{y_n} \Gamma^D(x, y; t, s) f(y; s) \, dy \, ds,
\]

and we can write solution to problem (1), (5) as

\[
u(x; t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n_{+}} \Gamma^N(x, y, t, s) f(y, s) \, dy \, ds,
\]

where

\[
\Gamma^N(x, y; t, s) = \int_{x_n}^{\infty} D_{y_n} \Gamma^D(x', z_n, y; t, s) \, dz_n.
\]

Since \( D_{x_n} \Gamma^N(x, y; t, s) = - D_{y_n} \Gamma^D(x, y; t, s) \), we derive from (14) that

\[
|D_x^\alpha D_y^\beta D_{x_n} \Gamma^N(x, y; t, s)| \leq C \frac{\mathcal{R}^{2-\alpha_n-\varepsilon} \mathcal{R}^{1-\beta_n-\varepsilon}}{(t-s)^{\frac{n+1+|\alpha|+|\beta|}{2}}} \exp \left(-\frac{\sigma |x-y|^2}{t-s}\right), \quad (28)
\]

where \( 2 - \alpha_n - \varepsilon \) must be replaced by \( 1 - \alpha_n \) if \( \alpha_n \leq 1 \) and \( 1 - \beta_n - \varepsilon \) by 0 if \( \beta_n = 0 \). Estimate (25) with \( \alpha_n \geq 1 \) follows from (28).

In a similar way we derive from (15) that

\[
|D_x^\alpha D_y^\beta D_{x_n} \partial_s \Gamma^N(x, y; t, s)| \leq C \frac{\mathcal{R}^{2-\alpha_n-\varepsilon} \mathcal{R}^{1-\beta_n-\varepsilon}}{(t-s)^{\frac{n+1+|\alpha|+|\beta|}{2}}} \exp \left(-\frac{\sigma |x-y|^2}{t-s}\right), \quad (29)
\]

where \( 2 - \alpha_n - \varepsilon \) must be replaced by \( 1 - \alpha_n \) if \( \alpha_n \leq 1 \). Estimate (26) with \( \alpha_n \geq 1 \) follows from (29).

To estimate derivatives with respect to \( x' \) we consider two cases.
Case 1: \(|x_n - y_n| \leq \sqrt{t - s}\). Then (28) implies

\[
|D^\alpha' x D^\beta' y \Gamma^N(x, y; t, s)| \leq \int_{x_n}^{\infty} |D^\alpha' x D^\beta' y D_{z_n} \Gamma^N(x', z_n, y; t, s)| \, dz_n
\]

\[
\leq \frac{C R^{\beta_n}}{(t - s)^{n + |\alpha'| + |\beta'| + 1}} \exp \left( -\frac{\sigma |x' - y'|^2}{t - s} \right) \int_{\mathbb{R}} \exp \left( -\frac{\sigma |z_n - y_n|^2}{t - s} \right) \, \frac{dz_n}{\sqrt{t - s}}
\]

\[
\leq \frac{C R^{\beta_n}}{(t - s)^{n + |\alpha'| + |\beta'| + 1}} \exp \left( -\frac{\sigma |x - y|^2}{t - s} + \sigma \right)
\]

(the last inequality is due to \(|x_n - y_n| \leq 1\), which gives (25) with \(\alpha_n = 0\) in the case 1. In a similar way we derive estimate (26) with \(\alpha_n = 0\) from (29) in the case 1.

Case 2: \(|x_n - y_n| > \sqrt{t - s}\). Then we rewrite equation \(L_0 \Gamma^N = 0\) as

\[
L'_0 \Gamma^N \equiv \partial_t \Gamma^N - \sum_{i,j=1}^{n-1} a_{ij}(t) D_{x_i} D_{x_j} \Gamma^N
\]

\[
= \mathcal{F} \equiv \left( 2 \sum_{j=1}^{n-1} a_{jn} D_{x_j} D_{x_n} \Gamma^N + a_{nn} D_{x_n}^2 \Gamma^N \right).
\]

From (28) and (30) it follows that

\[
|D^{\alpha'}_{x'} D^\beta' y \mathcal{F}(x, y; t, s)| \leq C \frac{R^{\beta_n}}{(t - s)^{n + 2 + |\alpha'| + |\beta'|}} \exp \left( -\frac{\sigma |x - y|^2}{t - s} \right).
\]
Using Proposition 1 for \( \Gamma' \) we get from (32) and (31)
\[
|D_x' D_y^\beta \Gamma^N(x, y; t, s)| \leq \int_s^t \int_{\mathbb{R}^{n-1}} \frac{C}{(t - \tau)^{n+1}} \exp \left( -\frac{\sigma |x' - z'|^2}{t - \tau} \right) 
\times \frac{R_{\gamma}^{\beta_n}}{(\tau - s)^{\frac{n+2+|\alpha'|+|\beta|}{2}}} \exp \left( -\frac{\sigma (z', x_n) - y|^2}{\tau - s} \right) \, dz' d\tau.
\]
We observe that \( R_y \) here has non-standard time argument: \( \tau - s \) instead of \( t - s \). However, since \( \beta_n \leq 0 \), we can estimate “non-standard” \( R_{\gamma}^{\beta_n} \) by standard one.

Integrating with respect to \( z' \) and using Fourier transform, we get
\[
|D_x' D_y^\beta \Gamma^N(x, y; t, s)| \leq \frac{C R_{\gamma}^{\beta_n}}{(t - s)^{\frac{n+1}{2}}} \exp \left( -\frac{\sigma |x' - y'|^2}{t - s} \right)
\times \int_s^t \frac{1}{(\tau - s)^{\frac{n+|\alpha'|+|\beta|}{2}}} \exp \left( -\frac{\sigma (x_n - y_n)^2}{\tau - s} \right) \, d\tau.
\]
Substituting \( \theta = \frac{t - \tau}{\tau - s} \), we arrive at
\[
|D_x' D_y^\beta \Gamma^N(x, y; t, s)| \leq \frac{C R_{\gamma}^{\beta_n}}{(t - s)^{\frac{n+|\alpha'|+|\beta|}{2}}} \exp \left( -\frac{\sigma |x' - y'|^2}{t - s} \right)
\times \int_0^\infty (\theta + 1)^{\frac{|\alpha'|+|\beta|}{2} - 1} \exp \left( -\frac{\sigma (x_n - y_n)^2}{s} (\theta + 1) \right) \, d\theta.
\]
Since \( \frac{|x_n - y_n|^2}{t - s} > 1 \), this implies
\[
|D_x' D_y^\beta \Gamma^N(x, y; t, \tau)| \leq \frac{C R_{\gamma}^{\beta_n}}{(t - s)^{\frac{n+|\alpha'|+|\beta|}{2}}} \exp \left( -\frac{\sigma |x - y|^2}{t - s} \right)
\times \int_0^\infty (\theta + 1)^{\frac{|\alpha'|+|\beta|}{2} - 1} \exp (-\sigma \theta) \, d\theta,
\]
which gives (25) with \( \alpha_n = 0 \) in the case 2.

In a similar way we derive the estimate (26) with \( \alpha_n = 0 \) in the case 2, and the proof is complete. \( \square \)
3.2 Coercive estimates in $\tilde{L}_{p,q}$ and in $L_{p,q}$

**Theorem 3.** Let $1 < p, q < \infty$ and $\mu \in (-\frac{1}{p}, 1 - \frac{1}{p})$.

(i) If $f \in \tilde{L}_{p,q}(\mathbb{R}^n_+ \times \mathbb{R})$ then solution (27) to problem (1), (5) satisfies

$$\|x_\mu^n \partial_t u\|_{p,q} + \|x_\mu^n D(Du)\|_{p,q} \leq C \|x_\mu^n f\|_{p,q}. \quad (33)$$

(ii) If $f \in L_{p,q}(\mathbb{R}^n_+ \times \mathbb{R})$ then solution (27) to problem (1), (5) satisfies

$$\|x_\mu^n \partial_t u\|_{p,q} + \|x_\mu^n D(Du)\|_{p,q} \leq C \|x_\mu^n f\|_{p,q}. \quad (34)$$

The constant $C$ depends only on $\nu, \mu, p$ and $q$.

**Proof.** First, we recall that the function $D_n u$ satisfies the Dirichlet problem (16) with $f_0 = 0$ and $f = (0, \ldots, 0, f)$. Thus, Theorem 1 gives

$$\|x_\mu^n D(D_n u)\|_{p,q} \leq C \|x_\mu^n f\|_{p,q} \quad (35)$$

and

$$\|x_\mu^n D(D_n u)\|_{p,q} \leq C \|x_\mu^n f\|_{p,q}. \quad (36)$$

To estimate the derivatives $D'D'u$ in $\tilde{L}_{p,q}$-norm, we proceed similarly to Theorem 2. We rewrite equation (1) as in (30):

$$\mathcal{L}_0' u = \tilde{f} \equiv f + 2 \sum_{j=1}^{n-1} a_{jn} D_j D_n u + a_{nn} D_n D_n u. \quad (37)$$

Using Proposition 2 (ii) in $\mathbb{R}^{n-1}$ we obtain

$$\|D'D'u(\cdot, x_n)\|_{p,q} \leq C \|\tilde{f}(\cdot, x_n)\|_{p,q} \quad (38)$$

almost for all $x_n > 0$. Multiplying both sides of (37) by $x_\mu^n$ and taking $L_p$ norm with respect to $x_n$, we arrive at

$$\|x_\mu^n D'D'u\|_{p,q} \leq C \|x_\mu^n \tilde{f}\|_{p,q} \leq C \|x_\mu^n f\|_{p,q}. \quad (39)$$

where we have used estimate (35). The first term in (33) is estimated by using (35), (38) and equation (1), and the statement (i) follows.

For $L_{p,q}$-norm of $D'D'u$ this approach fails, so we proceed as in the part (ii) of Theorem 1. Let us introduce the kernels

$$K_4(x, y; t, s) = \frac{x_\mu^n}{y_\mu^n} D'_x D'_x \Gamma^N(x, y; t, s); \quad K^*_4(x, y; t, s) = K_4(y, x; t, s). \quad (40)$$
Estimate (38) with \( q = p \) means that the operator \( K_4 \) is bounded in \( L_p(\mathbb{R}^n \times \mathbb{R}) \). Choose \( \varepsilon > 0 \) such that relation (24) holds. Using estimates (25) and (26), it is easy to check that \( K_4 \) satisfies the same estimates as the kernel \( K_3 \) in Theorem 1. Verbatim repetition of arguments shows that this operator is bounded in \( L_{p,q}(\mathbb{R}^n \times \mathbb{R}) \) for any \( q \in (1, p) \).

Further, by duality the operator \( K_4^* \) is bounded in \( L_{p'}(\mathbb{R}^n \times \mathbb{R}) \). Using (25) and relation (24), it is easy to check that \( K_4 \) satisfies the same estimates as the kernel \( K_3 \) in Theorem 1. Verbatim repetition of arguments shows that this operator is bounded in \( L_{p,q}(\mathbb{R}^n \times \mathbb{R}) \) for any \( q \in (1, p) \).

Further, by duality the operator \( K_4^* \) is bounded in \( L_{p'}(\mathbb{R}^n \times \mathbb{R}) \). Using (25) and relation (24), we obtain

\[
|\partial_s K_4^*(x, y; t, s)| \leq \frac{C}{(s-t)^{n+2}} \frac{x_n^{-\mu}}{y_n^{\mu}} \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right).
\]

For \( |s-s^0| < \delta \) and \( s^0 - t > 2\delta \) this implies

\[
|K_4^*(x, y; t, s) - K_4^*(x, y; t, s^0)| \leq \frac{C \delta}{(t-s)^{n+2+1}} \frac{x_n^{-\mu}}{y_n^{\mu}} \exp \left( -\frac{\sigma |x-y|^2}{t-s} \right).
\]

The last estimate allows us to apply Proposition 4 with \( \alpha = 1, r = 0, \lambda_1 = \lambda_2 = 0 \) and \( p \) replaced by \( p' \). Therefore, Theorem 3.8 [1] ensures that for any \( q \in (p, \infty) \) the operator \( K_4^* \) is bounded in \( L_{p',q'}(\mathbb{R}^n \times \mathbb{R}) \). By duality the operator \( K_4 \) is bounded in \( L_{p,q}(\mathbb{R}^n \times \mathbb{R}) \).

Thus, we have

\[
\|x_n^\mu D'D'u\|_{p,q} \leq C \|x_n^\mu f\|_{p,q}
\]

for all \( 1 < q < \infty \). The first term in (34) is estimated by (36), (39) and equation (1), and the statement (ii) also follows.

4 Solvability of the oblique derivative problem in a bounded domain

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with boundary \( \partial \Omega \). For a cylinder \( Q = \Omega \times (0, T) \), we denote by \( \partial^n Q = \partial \Omega \times (0, T) \) its lateral boundary.

We introduce two scales of functional spaces: \( L_{p,q,(\mu)}(Q) \) and \( \tilde{L}_{p,q,(\mu)}(Q) \), with norms

\[
\|f\|_{p,q,(\mu),Q} = \|(\tilde{d}(x))^\mu f\|_{p,q,Q} = \left( \int_0^T \left( \int_\Omega (\tilde{d}(x))^\mu |f(x,t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}
\]

\( 16 \)
and
\[ \|f\|_{p,q,(\mu),Q} = \|\hat{d}(x)^\mu f\|_{p,q,Q} = \left( \int_0^T \left( \int_\Omega (\hat{d}(x)^\mu |f(x,t)|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \]
respectively, where \( \hat{d}(x) \) stands for the distance from \( x \in \Omega \) to \( \partial \Omega \). For \( p = q \) these spaces coincide, and we use the notation \( \|\cdot\|_{p,(\mu),Q} \).

We denote by \( W^{2,1}_{p,q,(\mu)}(Q) \) and \( \tilde{W}^{2,1}_{p,q,(\mu)}(Q) \) the set of functions with the finite seminorms
\[ \|\partial_t u\|_{p,q,(\mu),Q} + \sum_{ij} \|D_i D_j u\|_{p,q,(\mu),Q} \]
and
\[ \|\partial_t u\|_{p,q,(\mu),Q} + \sum_{ij} \|D_i D_j u\|_{p,q,(\mu),Q} \]
respectively. These seminorms become norms on the subspaces defined by \( u|_{t=0} = 0 \). For \( p = q \) we write \( W^{2,1}_{p,(\mu)}(Q) \).

We say \( \partial \Omega \in W^{2}_{p,(\mu)} \) if for any point \( x^0 \in \partial \Omega \) there exists a neighborhood \( \mathcal{U} \) and a diffeomorphism \( \Psi \) mapping \( \mathcal{U} \cap \Omega \) onto the half-ball \( B^+_1 \) and satisfying
\[ (\hat{d}(x))^\mu D^2 \Psi \in L_p(\mathcal{U} \cap \Omega); \quad x^0_n D^2 \Psi^{-1} \in L_p(B^+_1), \]
where corresponding norms are uniformly bounded with respect to \( x^0 \).

We set \( \hat{\mu}(p,q) = 1 - \frac{n}{p} - \frac{2}{q} \).

We consider the initial-boundary value problem
\[
Lu \equiv \partial_t u - a^{ij}(x,t)D_i D_j u + b^i(x,t)D_i u = f(x,t) \quad \text{in} \quad Q; \quad \gamma^i(x,t)D_i u|_{\partial^\prime Q} = 0, \quad u|_{t=0} = 0.
\]
The matrix of leading coefficients \( a^{ij} \in C(\overline{\Omega} \rightarrow L_\infty(0,T)) \) is symmetric and satisfies the ellipticity condition (2). We assume that the vector field \( \gamma \) is non-tangent, i.e.
\[ \gamma^i(x,t)\mathbf{n}_i(x) \geq \gamma_0, \quad (x,t) \in \partial^\prime Q, \quad \gamma_0 = \text{const} > 0 \]
(here \( \mathbf{n}(x) \) stands for the unit exterior normal vector to \( \partial \Omega \) at the point \( x \)).
Theorem 4. Let $1 < p, q < \infty$ and $\mu \in (-\frac{1}{p}, 1 - \frac{1}{p})$. Assume that the components $\gamma^i$ belong to the anisotropic Hölder space $C^{0,1;\frac{1}{2}}(\partial^p Q)$.

1. Let $b^i \in L_{\overline{p},\overline{q},(\overline{p})}(Q) + L_{\infty,\overline{q}}(Q)$, where $\overline{p}$ and $\overline{q}$ are subject to

$$\overline{p} \geq p; \quad \begin{cases} \overline{q} = q; & \hat{\mu}(p, q) > 0 \\ q < \overline{q} < \infty; & \hat{\mu}(\overline{p}, \overline{q}) = 0 \end{cases},$$

while $\overline{p}$ and $\overline{\mu}$ satisfy

$$\overline{\mu} = \min\{\mu, \max\{\hat{\mu}(p, q), 0\}\}; \quad \overline{\mu} < \mu + \frac{1}{p}. \tag{42}$$

Suppose also that either $\partial \Omega \in W^2_\infty(\overline{p})$ or $\partial \Omega \in W^2_p(\overline{p})$. Then, for any $f \in L_{p,q,(\mu)}(Q)$, the initial-boundary value problem (40) has a unique solution $u \in W^{2,1}_{p,q,(\mu)}(Q)$. Moreover, this solution satisfies

$$\|\partial_t u\|_{p,q,(\mu)} + \sum_{ij} \|D_i D_j u\|_{p,q,(\mu)} \leq C f\|f\|_{p,q,(\mu)},$$

where the positive constant $C$ does not depend on $f$.

2. Let $b^i \in L_{\overline{p},\overline{q},(\overline{p})}(Q) + L_{\infty,\overline{q}}(Q)$, where $\overline{p}$ and $\overline{q}$ are subject to

$$\overline{q} \geq q; \quad \begin{cases} \overline{p} = p; & \hat{\mu}(p, q) > 0 \\ p < \overline{p} < \infty; & \hat{\mu}(\overline{p}, \overline{q}) = 0 \end{cases},$$

while $\overline{p}$ and $\overline{\mu}$ satisfy (42). Suppose also that $\partial \Omega$ satisfies the same conditions as in the part 1. Then, for any $f \in L_{p,q,(\mu)}(Q)$, the problem (40) has a unique solution $u \in W^{2,1}_{p,q,(\mu)}(Q)$. Moreover, this solution satisfies

$$\|\partial_t u\|_{p,q,(\mu)} + \sum_{ij} \|D_i D_j u\|_{p,q,(\mu)} \leq C f\|f\|_{p,q,(\mu)},$$

where the positive constant $C$ does not depend on $f$.

Remark 1. It is well known (see, e.g., [7]) that if $\partial \Omega \in C^{1,\delta}$ for some $\delta \in (0,1]$, then $\partial \Omega \in W^2_{\infty,(1-\delta)}$. In this case the second inequality in (42) implies solvability of the problem (40) for $1 - \delta - \frac{1}{p} < \mu < 1 - \frac{1}{p}$. 

18
Proof. The standard scheme, see [6, Ch.IV, §9], including partition of unity, local flattening of $\partial \Omega$ and coefficients freezing, reduces the proof to the coercive estimates for the model problems to equation (1) in the whole space and in the half-space. These estimates are obtained in [3, Theorem 1.1] and our Theorem 3. By the Hölder inequality and the embedding theorems (see, e.g., [1, Theorems 10.1 and 10.4]), the assumptions on $b^i$ guarantee that the lower-order terms in (40) belong to desired weighted spaces, $L_{p,q,(\mu)}(Q)$ and $\tilde{L}_{p,q,(\mu)}(Q)$, respectively. By the same reasons, the requirements on $\partial \Omega$ imply $\partial \Omega \in C^1$ and ensure the invariance of assumptions on $b^i$ after flattening the boundary.

Next, after flattening of $\partial \Omega$ we can assume without loss of generality that $\gamma^i(0) = \delta^m_i$ and rewrite the boundary condition as follows:

$$D_n u |_{x_n=0} = \varphi \equiv (\delta^m_i - \gamma^i(x,t)) D_i u. \quad (43)$$

The inhomogeneity in boundary condition (43) will be removed if we subtract from $u$ some function satisfying the same boundary condition. By assumption $\gamma^i \in C^{0,1/2}(\partial''Q)$, the function $\varphi$ has the same differential properties as $Du$. Therefore, such a subtraction does not leave the space $L_{p,q,(\mu)}(Q)$ (respectively, $\tilde{L}_{p,q,(\mu)}(Q)$) of the right-hand side in (40). This completes the proof.

The assumption $\gamma^i \in C^{0,1/2}(\partial''Q)$ is not optimal. The sharp assumption here is that multiplication by the vector field $\gamma$ should keep the space of traces of gradients of functions from $\mathcal{W}_{p,q,(\mu)}^{2,1}(Q)$ (respectively, from $\tilde{\mathcal{W}}_{p,q,(\mu)}^{2,1}(Q)$). In other words, $\gamma$ should belong to space $\text{MTD}\mathcal{W}_{p,q,(\mu)}^{2,1}(Q)$ (respectively, $\text{MTD}\tilde{\mathcal{W}}_{p,q,(\mu)}^{2,1}(Q)$) of multipliers of traces of gradients of weighted Sobolev functions.

Unfortunately, to the best of our knowledge, these spaces are not described yet. In the isotropic case $p = q$ we can give rather sharp sufficient conditions in terms of the Besov spaces (the notation of the Besov spaces corresponds to [1, Ch.IV]). The following result can be extracted from the proofs of [1, Theorems 18.13 and 18.14], [10] and [8, 4.4.3].

**Theorem 5.** Let $1 < p < \infty$ and $\mu \in (-\frac{1}{p}, 1 - \frac{1}{p})$.  

Suppose that \( b^i \in L^p(\partial' Q) + L^\infty(\partial' Q) \), where \( \bar{p}, \bar{\mu} \) and \( \bar{\bar{\mu}} \) are subject to

\[
\bar{p} = \max\{p, n + 2\}, \quad \text{if} \quad p \neq n + 2; \quad \bar{p} > n + 2, \quad \text{if} \quad p = n + 2;
\]

\[
\bar{\mu} = \min\{\mu, \max\{1 - \frac{n + 2}{p}, 0\}\}; \quad \bar{\bar{\mu}} < \mu + \frac{1}{p}.
\]

Suppose also that either \( \partial \Omega \in W^2_{\infty, \bar{p}} \) or \( \partial \Omega \in W^2_{p, \bar{\mu}} \).

Finally, we assume that the components \( \gamma^i \) belong to the Besov space \( B^\lambda_{\bar{p}, \theta}(\partial' Q) \) with parameters

\[
\lambda \equiv (\lambda_1^1, \ldots, \lambda_n^{n-1}, \lambda_t) = \left(1 - \frac{1}{p}, \ldots, 1 - \frac{1}{p}, \frac{1}{2} - \frac{1}{2p}\right); \quad \theta = p;
\]

\[
\bar{\bar{p}} = \max\left\{p, \frac{n + 1}{1 - \mu - \frac{1}{p}}\right\}, \quad \text{if} \quad p \neq \frac{n + 2}{1 - \mu}; \quad \bar{\bar{p}} > \frac{n + 2}{1 - \mu}, \quad \text{if} \quad p = \frac{n + 2}{1 - \mu}.
\]

Then, for any \( f \in L^p(\mu)(Q) \), the initial-boundary value problem (40) has a unique solution \( u \in W^2_{p, \mu}(Q) \). Moreover, this solution satisfies

\[
\|\partial_t u\|_{p, \mu} + \sum_{ij} \|D_i D_j u\|_{p, \mu} \leq C\|f\|_{p, \mu},
\]

where the positive constant \( C \) does not depend on \( f \).

V. K. was supported by the Swedish Research Council (VR). A. N. was supported by RFBR grant 12-01-00439 and by St. Petersburg University grant 6.38.670.2013. He also acknowledges the Linköping University for the financial support of his visit in February 2012.

References


