Methods to Predict Structural Response due to Random Sound Pressure Fields

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Methods to Predict Structural Response due to Random Sound Pressure Fields

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Abstract
To predict structural responses due to random sound pressure fields are of great interest within many fields of aircraft development, particularly within acoustic fatigue problems and definition of vibration requirements. Today there exist some methods to quantify sound pressure fields affecting the air-fighters. Some of them are considered to be expensive, time consuming or with high computational cost. Examples of this would be to measure a real flight, produce data from wind tunnels, use Computational Fluid Dynamics (CFD) or obtain data from an engineering database. Once the sound pressure levels are known they can be applied as loads to structural models and this is the area studied in this work. To study these problems a new working tool is made using MATLAB. The tool’s main purpose is to give an opportunity to study structural responses caused by random sound pressure fields with different correlation methods.

Because of the complexity of both the sound pressure and different structures of the aircraft a few limitations are considered. The plate is used since this makes it easy to produce different mode shape functions. The mode shape function is an important part in this work as it can be used to create all possible frequency response functions in a structure. Then, to determine a structure response, different methods to produce pressure fields are used. The methods are called correlation-models and five different models are considered: uncorrelated, fully correlated and moving correlated load (MCL) and two empirical models due to the similarity to real sound pressure fields called Turbulent Boundary Layer (TBL) and a diffuse excitation model.

To prove the accuracy of the created working tool, an independent FE-solver is used called Abaqus. Abaqus is used to validate the mode shape- and the frequency response-fucntions. Another advantage with Abaqus is that the solver already includes three of the correlation models which therefore simplify the verification of the new tool.

Finally, a simulation study is carried out in order to validate the MATLAB functions and test the sensitivity to different correlation models. In order to do this, the sound pressure field is to be reasonable approximated and therefore data from the database ESDU (acronym of Engineering Sciences Data Unit) is used that predicts sound pressure fields for different flight envelopes. In the simulation study all correlation models are compared to TBL due to its sound pressure and here it can be seen that fully correlated loads fails to predict response due to certain modes. On the other hand, the MCL model increases this accuracy for low Mach numbers and even more for high Mach numbers due to its velocity dependence. The diffuse model, which is supposed to imitate a real pressure chamber load, is often believed to be conservative but in this study it can be seen that this is not always the case.

Keywords: Random Vibration, Power Spectral Density, Cross Correlation, Turbulent Boundary Layer, Random Response
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  Peter Schmidt

My opponent,

  Joakim Hägglund

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<table>
<thead>
<tr>
<th>Abbreviation</th>
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<tr>
<td>ACF</td>
<td>Autocorrelation Function</td>
</tr>
<tr>
<td>ESDU</td>
<td>Engineering Sciences Data Unit</td>
</tr>
<tr>
<td>CSD</td>
<td>Cross Spectral Density</td>
</tr>
<tr>
<td>C-C-C-C</td>
<td>Clamped at all four edges (plate)</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree of Freedom</td>
</tr>
<tr>
<td>FC</td>
<td>Fully Correlated</td>
</tr>
<tr>
<td>FE</td>
<td>Finite Element</td>
</tr>
<tr>
<td>FRF</td>
<td>Frequency Response Function</td>
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<tr>
<td>MDOF</td>
<td>Multiple Degree of Freedom</td>
</tr>
<tr>
<td>MI/MO</td>
<td>Multiple Input/Multiple Output</td>
</tr>
<tr>
<td>MCL</td>
<td>Moving Correlated Load</td>
</tr>
<tr>
<td>PSD</td>
<td>Power Spectral Density</td>
</tr>
<tr>
<td>RMS</td>
<td>Root Mean Square</td>
</tr>
<tr>
<td>SDOF</td>
<td>Single Degree of Freedom</td>
</tr>
<tr>
<td>SI/SO</td>
<td>Single Input/ Single Output</td>
</tr>
<tr>
<td>SS-SS-SS-SS</td>
<td>Simply Supported at all four edges (plate)</td>
</tr>
<tr>
<td>TBL</td>
<td>Turbulent Boundary Layer</td>
</tr>
<tr>
<td>UC</td>
<td>Uncorrelated</td>
</tr>
<tr>
<td>Notation</td>
<td>Meaning</td>
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</tr>
<tr>
<td>$\alpha_x$</td>
<td>Coherence loss $x$-direction</td>
</tr>
<tr>
<td>$\alpha_y$</td>
<td>Coherence loss $y$-direction</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Angular excitation</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Frequency</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>Natural frequency</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Critical damping factor</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Time shift</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Mode shape coefficient</td>
</tr>
<tr>
<td>$\Lambda^{-1}$</td>
<td>Inverse pole matrix</td>
</tr>
<tr>
<td>$c$</td>
<td>Viscous damping</td>
</tr>
<tr>
<td>$c_s$</td>
<td>Speed of sound</td>
</tr>
<tr>
<td>$f$</td>
<td>Frequency</td>
</tr>
<tr>
<td>$f_n$</td>
<td>Natural frequency</td>
</tr>
<tr>
<td>$k$</td>
<td>Stiffness</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass</td>
</tr>
<tr>
<td>$r$</td>
<td>Absolute distance between two DOFs</td>
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<tr>
<td>$s_r$</td>
<td>pole</td>
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<tr>
<td>$t$</td>
<td>time</td>
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<tr>
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<td>$x_k$</td>
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<tr>
<td>$y$</td>
<td>displacement</td>
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<tr>
<td>$C$</td>
<td>Convection velocity</td>
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<tr>
<td>$F$</td>
<td>Input</td>
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<tr>
<td>$G_{xx}$</td>
<td>Excitation</td>
</tr>
<tr>
<td>$G_{xy}$</td>
<td>Random cross response</td>
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<tr>
<td>$H$</td>
<td>Frequency Response Function</td>
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<tr>
<td>$L_x$</td>
<td>Convection length $x$-direction</td>
</tr>
<tr>
<td>$L_y$</td>
<td>Convection length $y$-direction</td>
</tr>
<tr>
<td>$R_{xx}$</td>
<td>Autocorrelation Function</td>
</tr>
<tr>
<td>$U_{\infty}$</td>
<td>Free stream velocity</td>
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<tr>
<td>$X$</td>
<td>Input force in spectral domain</td>
</tr>
<tr>
<td>$Y$</td>
<td>Output vibration in spectral domain</td>
</tr>
<tr>
<td>Words</td>
<td>Meaning</td>
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<td>--------------------</td>
<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Dynamic system</td>
<td>Time dependent system</td>
</tr>
<tr>
<td>Ensamble averages</td>
<td>Group of averages</td>
</tr>
<tr>
<td>Ensamble data</td>
<td>Group of data</td>
</tr>
<tr>
<td>Excitation</td>
<td>Force, pressure field or applied energy</td>
</tr>
<tr>
<td>Free stream</td>
<td>Stream not affected by surrounding environment</td>
</tr>
<tr>
<td>Frequency</td>
<td>Occurrences per unit time</td>
</tr>
<tr>
<td>Laplace domain</td>
<td>See spectrum</td>
</tr>
<tr>
<td>Modal anti-node</td>
<td>Position where a maximum displacement is found in a mode</td>
</tr>
<tr>
<td>Modal node</td>
<td>Position where the displacement is zero</td>
</tr>
<tr>
<td>Mode</td>
<td>Characteristic shape at a certain frequency</td>
</tr>
<tr>
<td>Natural frequency</td>
<td>The rate at which a system tends to oscillate in absence of any driving force</td>
</tr>
<tr>
<td>Periodic vibration</td>
<td>Repeating vibration</td>
</tr>
<tr>
<td>Random vibration</td>
<td>Stochastic vibration</td>
</tr>
<tr>
<td>Turbulent flow</td>
<td>Chaotic flow, varies in pressure and velocity</td>
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1. Introduction

1.1 Background

The company Saab was founded 1937 with the primary aim to provide military aircraft for Sweden. Now Saab operates across the world with the vision that every human has the right to feel safe and the mission to make people safe by pushing intellectual and technological boundaries.

At the department of Environmental Engineering the main responsibility is to qualify and specify the requirement for equipment. To reach the tough requirement present at an aircraft certain areas are considered which involves thermal-, climatic and mechanical environments.

The mechanical environment involves impacts and vibrations. As a simplification, the vibrations can typically be divided into two categories; low frequency (<100 Hz) and high frequency (>100-2000 Hz). Low frequency vibrations are considered by global models and associated with performance and stability of an aircraft as well as structural loads on the aircraft. High frequency vibrations on the other hand are typically related to high cycle fatigue problems such as acoustic fatigue and are also of special interest when it comes to protecting equipment from severe environment.

The aircraft structure can be affected by intense sound pressure levels, which can lead to high frequency vibration problems. Today experiments in form of real flights, experiments in wind tunnels or simulations by Computational Fluid Dynamics (CFD) can be utilized to determine data describing the sound pressure levels. To provide data from real flight cases would be the most desirable but though very expensive. The tools using CFD are also expensive and require great knowledge of the theory, but also the computational power and the lack of accuracy in the high frequency range is a problem.

1.2 Aim and Scope

The aim of this work is to investigate a methodology of how to determine a structural response caused by random sound pressure fields. It is assumed that the sound pressure levels are known and the task is then to predict structural response. The challenge is then to properly define the sound pressure fields in a model accurately enough and this should be done with correlation models.

With the task to predict structural responses the applications could be used to:

- Predict environmental requirements
- Predict acoustic fatigue
- Predict the effect of structure design changes

This work is limited to prediction of random vibrations caused by sound pressure excitation. These vibrations are believed to be the largest source of vibrations in an aircraft. To be able to describe the sound pressure excitation, empirical models are used provided by ESDU (acronym of Engineering Sciences Data Unit). These excitations are to be combined and verified with a FE-solver to get deeper knowledge of how to predict the structural responses in an aircraft. The parameters describing the correlation of the excitation are also studied. How these parameters affect this excitation is to be investigated throughout the work.
1.3 Method
The methodology to produce a tool suitable to predict vibration due to sound pressure field excitation is shown in Figure 1-1 and to perform this task the toolbox is created in the numerical computing program MATLAB. The toolbox utilizes a modal description of the structural system, which is used to define frequency response functions. The geometries chosen in this work are two plates with two different boundary conditions, simply supported and a fixed plate due to its well-known physics.

To validate the toolbox, the FE-solver Abaqus is used. To perform a simulation study random pressure fields are applied to the structure. These pressure fields are defined by empirical models produced by ESDU which describes frequency spectrum and the correlation of the pressure fields. When the pressure fields are applied to the structure the random responses can be investigated.

![Diagram](image)

Figure 1-1: Presents an overview of the methodology. At first an analytical model is needed which is chosen as a plate. Then the modal parameters along with the natural frequencies are determined of the plate. The modal parameters are determined to describe the frequency response function that serves as the equation describing the relationship between the random pressure field and the response.
2. Theory

This chapter includes theory in a wide range, from statistics to structural dynamics. Therefore this theory serves as an important part to understand the upcoming work structure. At first, random vibrations will be discussed along with different methods to analyze them. Modal analysis is then discussed, since this is an important part to describe the engineering system characteristic. Then the system response is shown along with different methods to describe a pressure excitation. The abbreviations, nomenclature and glossary used are given in Table A, B and C.

2.1 Vibration

Vibration is a mechanical phenomenon, which is defined as something that oscillates about an equilibrium point. Vibrations can be unpleasant or harmful, as an example, unbalance in machines with rotating parts such as fans and rotors, washing machines and even wind and earthquakes. For many engineering systems, operation at resonance would be undesirable and could be destructive. Resonance is a phenomenon that occurs when a system oscillates with greater amplitudes for certain frequencies. Suppression or elimination of these resonances is desired and a general goal for a vibration engineer [1].

Periodic motion is a simple form of vibration, a motion that repeats itself with a fundamental time period. For example, a misaligned motor coupling that is loose could have a bump once per revolution of the shaft. The more erratic motion that contains all different frequencies in a particular frequency band is called random motion and is not repeated. For example, a sound pressure excitation present on an aircraft [2]. Generally, for engineering applications the random motion is described with statistics, where it is assumed that the data is normally distributed [3].

![Figure 2-1: A random time sample of a displacement of an arbitrary system. To get an overview of how a random sample would look like this is just a small part of a longer sample. The motion in its present form is very complex and should therefore be considered to be described in another way.](image)

If a system is subjected to a random excitation as in Figure 2-1, the response will also be a random phenomenon. Because of the complexity involved, the description of a random phenomenon as a
function of time does not appear particularly meaningful and is seldom used. Instead statistical methods of analysis can be adopted [1]. One method often used to describe random vibration is called the **Power Spectral Density** (PSD), which instead describes the power of the vibration as a function of frequency. This statistical method will serve as an input excitation for all systems in this work.

### 2.2 Power Spectral Density

The **Power Spectral Density** (PSD) is used to understand random signals by abandoning the time domain description and instead describe how the power of the signal is distributed over the frequencies [4]. As an example, in the previous Figure 2-1, the time signal is a **bandpass** signal, which only produces a certain band of frequencies. In this case the band is between 50 Hz and 100 Hz. This knowledge can easily be obtained with the use of the PSD.

![Power Spectral Density](image)

**Figure 2-2**: Two Power Spectral Density functions, the blue line is an estimation of the time signal from Figure 2-1 and the green line is the theoretical estimation of the time signal in Figure 2-1. The PSD always has the unit of [unit²/Hz] so in this example, the units of the PSD will be [m²/Hz]. The time signal in Figure 2-1 is a bandpass signal which only has frequencies between 50 [Hz] and 100 [Hz], this can be seen in the figure at the x-axis. At the y-axis the power of the signal is shown, the PSD, and in this case it is of $1 \cdot 10^{-6}$ [m²/Hz]. The area under the graph will represent the square of the root mean square value, RMS, of the time signal.

Figure 2-2 shows a PSD of the random time signal in Figure 2-1. The PSD has the power of $1.5 \cdot 10^{-6}$ m²/Hz and is present in the frequency band between 50 Hz and 100 Hz. The area under the graph is the square of the root mean square value, RMS, of the time signal. Instead of the random time signal the PSD represents something that has the same property throughout the whole signal.

The mathematical way to describe a PSD is to use a Fourier transformation of an **autocorrelation function**, ACF. We can consider two ways of computing the ACF, $R_{xx}(\tau)$. The first way is to obtain $n$ random sample functions $x_k(t)$ ($k = 1, 2, \ldots, n$) and calculate the average over the entire collection of sample functions, where such quantities are called **ensemble averages** [5]. The other way is to have one long random sample of the same experiment $x(t)$, which is believed to have the same statistical properties as the ensemble averages. An example of this could be to sample vibrations at
the wing of an aircraft when its doing a tight turn many times, (ensemble data), or trying to do this turn for a longer period to get a longer time sample.

The autocorrelation function is then obtained by summarizing the products of two times \( t \) and \( t - \tau \), (where \( \tau \) is the time shift), and dividing the result by number of sample functions or the expected value of the product\(^1\). The description of the ACF with ensemble data \( x_k(t) \) is given in Eq. (2.1) and the description of an ACF for one long time signal \( x(t) \) is given in Eq. (2.2).

\[
R_{xx}(\tau) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k(t) \cdot x_k(t - \tau) \quad (2.1)
\]

\[
R_{xx}(\tau) = E[x(t) \cdot x(t - \tau)] \quad (2.2)
\]

\( R_{xx}(\tau) \) is defined for both negative and positive values of \( \tau \). We also know that \( R_{xx}(\tau) \) is mirrored and therefore it is possible to write the PSD as the standard Fourier transformation of the ACF as follows

\[
G_{xx}(f) = 2 \cdot \mathcal{F}\{R_{xx}(\tau)\}, \quad f > 0 \quad (2.3)
\]

because \( G_{xx}(f) \) only includes positive frequencies and half the power is at negative frequencies, thereby the factor 2. This relationship can also be written as:

\[
G_{xx}(f) = 2 \cdot \mathcal{F}[E[x(t) \cdot x(t - \tau)]], \quad f > 0 \quad (2.4)
\]

Another common definition of the PSD is instead based on spectral averages \([6]\) and written as follows

\[
G_{xx}(f) = 2 \cdot \lim_{T \to \infty} \frac{1}{T} E[X_k(f) \cdot X_k^*(f)], \quad f > 0 \quad (2.5)
\]

Equation (2.5) is typically used in practice when PSD is estimated from data.

### 2.3 Modal Analysis

Modal analysis is the process to determine the characteristics of a system in forms of natural frequencies, damping factors and characteristic displacement, namely mode shapes. The mode shape is of great use when calculating vibration responses in an engineering system. The definition of a mode shape can be described as a way of vibrating, or a pattern of vibrations, when applied to a system with several degrees of freedom (DOF) \([7]\). There are two different concepts of a mode where the characteristic displacement is zero or if it is a maximum. The zero displacement occurs at a node and the maximum displacement occurs at an anti-node. For each mode this node and anti-node positions is always present in the same positions. An example of a mode shape is given in Figure 2-3 where the first three modes are illustrated for a one-dimensional beam.

\(^1\) \( E[.] \) Stands for the mathematical expectation

\[
E[x] = \int_{-\infty}^{\infty} xp(x)dx
\]

where \( p(x) \) is the probability density function of the random variable \( x \).
The natural modes occur when the system is not exposed to any external excitation and are therefore completely determined by the properties of the system itself, and each mode corresponds to a natural frequency.

![Figure 2-3: Three first modes for a simply supported beam where each shape occurs at a specific natural frequency. The nodes are present where the displacement is zero and the anti-node are present at the maximum displacement.](image)

Later in this work a plate with different boundary conditions will be used as an engineering system exposed to different loads. How to determine the mode shape function along with its natural frequency is shown in Appendix A. The two boundary conditions used are a simply supported and fully fixed plate.

### 2.4 Mechanical Response Functions

For every mechanical system there is a function that will describe the characteristics for a system exposed to an arbitrary excitation. This function is called a *transfer function* and is in this work denoted by $H(s)$, where $s$ is the Laplace variable. If we suppose that a system has an excitation $X(s)$ of a sinusoid. The response $Y(s)$ can then be enhanced, weakened and/or in a different phase compared to the input excitation. This is described by the transfer function $H(s)$.

For simple systems like a *single-degree-of-freedom* (SDOF) system this is rather easily computed and derived in Appendix A. There it can be seen that the transfer function $H(s)$ is the ratio of the output response $Y(s)$ and the input excitation $X(s)$ and is given by:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/m}{s^2 + sc/m + k/m} \quad (2.6)$$

where $m$, $c$ and $k$ are mass, damping and stiffness respectively.

When we measure a dynamic system in order to identify it, we cannot measure the transfer function as in Eq. (2.6), because it is a nonphysical entity. Instead, we usually measure the *frequency response* which is defined as the spectrum of the output and the spectrum of the excitation [8].
By letting the Laplace variable \( s \) in Eq. (2.6) be equal to \( s = j\omega = j2\pi f \) we get

\[
H(f) = \frac{Y(f)}{X(f)} = \frac{1/m}{\omega^2 + j2\zeta \omega_n \omega + \omega_n^2}
\]  

(2.7)

where \( \omega_n \) is the natural frequency (rad/s) and \( \zeta \) is the critical damping factor.

Eq. (2.7) can be rewritten as follows which represents the Frequency Response Function (FRF)

\[
H(f) = \frac{Y(f)}{X(f)} = \frac{1/k}{1 - (f/f_n)^2 + j2\zeta (f/f_n)}
\]  

(2.8)

where \( f/f_n \) is the relative frequency. Now the response for the SDOF-system can be computed for each frequency.

When it comes to Multiple Degrees of Freedom (MDOF)-systems there are two ways to describe the frequency response, either directly from Newton’s equation, or by using modal parameters [8]. In this work the modal parameters will be of interest because many commercial Finite Element (FE) programs can compute the mode shape \( \psi \) in each DOF in a structure. The FRF’s can then be calculated from the modal parameters as shown next.

If we let \( \psi_{pr} \) be the mode shape function for the input DOF \( p \) and \( \psi_{qr} \) the output DOF \( q \) for the mode \( r \), then the FRF can be written as follows for a MDOF-system

\[
H_{pq}(s) = \sum_{r=1}^{N} \frac{\psi_{pr}\psi_{qr}}{m_r(s-s_r)(s-s_r^*)}
\]  

(2.9)

where \( m_r \) is the modal mass and \( s_r \) is the pole

\[
s_r = -\zeta_r \omega_r + j\omega_r \sqrt{1 - \zeta_r^2}
\]  

(2.10)

\( \zeta_r \) the modal damping and \( \omega_r \) is the natural frequency for each mode. In Appendix A Eq. (2.9) is derived in more detail by using Eqs. (A.37)-(A.45).

When using experimental modal analysis, Eq. (2.9) is often rewritten in form of a pole matrix \([\Lambda^{-1}]\) which is similar to the inverse pole matrix \([S^{-1}]\) shown in Appendix A in Eq. (A.43), but formulated in the frequency domain instead of the Laplace domain. The full matrix can then be written as:

\[
[H(j\omega)] = [\psi][\Lambda^{-1}][\psi]^T
\]  

(2.11)

Where \([\Lambda^{-1}]\) is written as:

\[
[\Lambda^{-1}(j\omega)] = \begin{bmatrix}
1 & \cdots & 0 \\
(j\omega - s_1)(j\omega - s_1^*) & \ddots & \vdots \\
\vdots & \ddots & 1 \\
0 & \cdots & (j\omega - s_N)(j\omega - s_N^*)
\end{bmatrix}
\]  

(2.12)
An example is shown next to illustrate the physical meaning of the FRF matrix. If we now consider a MDOF-system with multiple inputs and multiple outputs, the beam in Figure 2-4 is a simple way to illustrate this. The beam has two DOF’s with one excitation $X$ and one response $Y$ in each.

![Figure 2-4: Simple MDOF-system of a beam. There are one excitation and one response in each DOF $X_{1-2}$ and $Y_{1-2}$.](image)

With the use of the FRF matrix $[H]$ we can now describe the relationship between the two responses and the two inputs. For example, if $Y_2$ is a displacement, the magnitude of this displacement will increase if both excitations are active as $Y_1 = H_{11}X_1 + H_{12}X_2$.

### 2.5 Calculation of Random Response

The mechanical response described in Equation (2.7) is shown as the ratio of the output response and the input excitation in the frequency domain. However, if the input is random, and defined with a PSD, then so will the output be random and defined with a PSD. We therefore need to derive the relation between output and input for the PSD. We begin by looking at a simple single-output-single-input case. If Eq. (2.7) is considered the input and the output are related to each with the FRF as:

$$H(f)X(f) = Y(f)$$

We then multiply the right hand side with the complex conjugate of the response as we know $G_{yy}(f) = E\{Y(f) \cdot Y^*(f)\}$ to achieve the PSD response

$$HX \cdot (Y^*) = YY^*$$

as $Y^* = H^*X^*$ from Eq. (2.14) we get:

$$HX \cdot (H^*X^*) = YY^*$$

The next step is to take the expected value on both sides

$$E[HX \cdot H^*X^*] = E[YY^*]$$

Because there does not exist any statistical information in the FRF-matrix we get that $[HH^*] = HH^*$, or

$$HH^* \cdot E[XX^*] = E[YY^*]$$

(2.18)
With the expectation of $YY^*$ and $XX^*$ we get both $G_{yy}(f)$ and $G_{xx}(f)$, which then show us the relation between the output and input PSD.

$$G_{yy}(f) = |H(f)|^2 G_{xx}(f)$$  \hspace{1cm} (2.19)

This method is used for the SDOF-system and is called SI/SO (Single Input/Single Output), but as we are interested in MDOF-systems this derivation need to be adopted for MI/MO (Multiple Input/Multiple Output).

Again we want to convert the random response $Y$ to a PSD response $G_{yy}$ as in the SI/SO example Eq. (2.19). If this is adapted to the MI/MO example it would be written as Eq. (2.20), in matrices form. The corresponding method of complex conjugate for a matrix is called Hermitian matrix and is denoted as $[.]^H$. A MI/MO example of how to determine a response can be written as

$$[H] \cdot [G_{xx}] \cdot [H]^H = [G_{yy}]$$  \hspace{1cm} (2.20)

The Hermitian of the output matrix $[Y]^H = [H]^H [X]^H$ is multiplied at the right hand side of Eq. (2.20)

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} X_1^H & X_2^H \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} Y_1^H & Y_2^H \end{bmatrix}$$  \hspace{1cm} (2.21)

further rewritten

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} X_1^H & X_2^H \end{bmatrix} \begin{bmatrix} X_1^T & X_2^T \end{bmatrix} = \begin{bmatrix} Y_1^H & Y_2^H \end{bmatrix}$$  \hspace{1cm} (2.22)

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} X_1^T & X_2^T \end{bmatrix} \begin{bmatrix} X_1^T & X_2^T \end{bmatrix} = \begin{bmatrix} Y_1^H & Y_2^H \end{bmatrix}$$  \hspace{1cm} (2.23)

With the expectation of (2.23) as in (2.18) where the FRF matrix does not consist any statistical information and $G_{xx} = E\{XX^*\}$ we get that

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} G_{x_1x_1} & G_{x_1x_2} \\ G_{x_2x_1} & G_{x_2x_2} \end{bmatrix} = \begin{bmatrix} G_{y_1y_1} & G_{y_1y_2} \\ G_{y_2y_1} & G_{y_2y_2} \end{bmatrix}$$  \hspace{1cm} (2.24)

Equation (2.24) is an example that can represent systems with multiple inputs and multiple outputs (MI/MO). The FRF matrix can be calculated from the modal parameters with Eq. (2.9). The diagonal terms in $G_{xx}$ define the excitation at one point (PSD) and the off-diagonal terms $G_{xy}$ describes the statistical relationship between the excitations at the neighboring points [9]. This relationship is called correlation.

The input pressure PSD will be determined by ESDU and is there given in a pressure $Pa^2/Hz$. To generalize the code this is recalculated to a force PSD $N^2/Hz$, this can be seen in Appendix B.
2.6 Correlation

The correlation is a measurement of the statistical dependency of two random variables. Typically, but not always, the correlation is a function of the distance between the two measuring DOF’s. Decreased distance generates increased correlation. Similarly, if the distance is increased the correlation will typically decrease, which means that there is less dependency between the two DOFs.

As for an example there are two PSD signals \( G_{xx}(f) \) and \( G_{yy}(f) \) in two different DOF’s. The correlation between these signals is called Cross Power Spectrum (CSD) and is denoted \( G_{xy}(f) \) which can show shared power and phase shift between the two signals.

An input pressure distribution that has a correlation depending on distance is a rather complex case, therefore two idealized cases are first considered: Uncorrelated and Fully Correlated. Then to approach the actual pressure distribution of an aircraft surface with a more realistic model, other input models can be used. The following input models are discussed in Section 2.6.2 to 2.6.4: Moving Correlated Load (MCL)-, Turbulent Boundary Layer (TBL)- and diffuse-excitation.

2.6.1 Uncorrelated and Fully Correlated

The first pressure distribution method used is uncorrelated, where all anti-diagonal (CSD) terms are zero

\[
G_{xy}^{uc} = 0
\]  
(2.25)

as an example for a two-DOF system

\[
[G_{xx}]^{uc} = \begin{bmatrix}
E\{X_1X_1^*\} & 0 \\
0 & E\{X_2X_2^*\}
\end{bmatrix}
\]  
(2.26)

Usually the PSD is assumed to be equal in all DOF’s as it would be an uniform pressure field but this does not have to be the case. The second method used is fully correlated. The CSD is then defined as:

\[
G_{xy}^{fc} = \sqrt{G_{xx}G_{yy}}
\]  
(2.27)

as an example

\[
[G_{xx}]^{fc} = \begin{bmatrix}
E\{X_1X_1^*\} & \sqrt{E\{X_1X_2\}} \\
\sqrt{E\{X_2X_1\}} & E\{X_2X_2^*\}
\end{bmatrix}
\]  
(2.28)
2.6.2 Moving Correlated Load
Both uncorrelated and fully correlated is used because the opportunity to validate the models with Abaqus. The same goes for Moving Correlated Load (MCL), this is the third and last complete method presented by Abaqus and is thus used for validation. MCL can be described as a more complex case, where the aim is to simulate an aircraft traveling with the speed $C_0$. The pressure distribution present in DOF 1 will later, because the correlation is fully correlated, also be present in DOF 2 a moment later. This can be seen in Figure 2-5. The time difference $\tau$ will be dependent of the speed and the distance between the two DOF’s $\varepsilon_x$ which can be written as $\tau = \frac{\varepsilon_x}{C_0}$. Hence, it is assumed that the load is fully correlated everywhere and propagates over the structure with a certain speed.

![Figure 2-5: Delayed pressure distribution. The pressure present in DOF 1 will later be present in DOF 2.](image)

From the property of the Fourier transformations we know that:

\[
\mathcal{F}(x(t)) = X(f) \quad (2.29)
\]

\[
\mathcal{F}(x(t - \tau)) = X(f) \cdot e^{-2\pi jtf} \quad (2.30)
\]

We know from Eq. (2.5) that

\[
G_{xy} = E[X(f)Y^*(f)] \quad (2.31)
\]

Combined with Eq. (2.29) and (2.30) can be written as

\[
G_{xy} = E[X(f)X^*(f) \cdot e^{2\pi jtf}] \quad (2.32)
\]

and

\[
G_{xy} = e^{2\pi jtf} \cdot E[X(f)X^*(f)] \quad (2.33)
\]

Along with $\tau = \frac{\varepsilon_x}{C_0}$, this gives the method called Moving Correlated Load, (MCL)

\[
G_{xy}^{MCL} = \sqrt{G_{xx}G_{yy}} \cdot e^{\frac{j2\pi f\varepsilon_x}{C_0}} \quad (2.34)
\]
Turbulent Boundary Layer Excitation

With the moving correlated load-excitation the pressure distribution only propagates in the free stream direction which is not the case in reality, close to the wall (plate) a turbulent flow will occur that also contributes to velocity perpendicular to the free stream direction. This gives a highly stochastic pressure field, which is described with a group of models called turbulent boundary layer (TBL)-excitations. The TBL model used in this paper is CORCOS. TBL is also partially correlated, which gives a better description of the correlation compared to uncorrelated, fully correlated and MCL.

Along with the exponential term seen in the MCL case in equation (2.34) two independent operators $A$ and $B$ depends on the longitudinal separation $\varepsilon$ and angular excitation frequency $\omega = 2\pi f$

$$G_{xy}^{TBL} = \sqrt{G_{xx}G_{yy}} \cdot A \cdot B \cdot e^{\frac{j2\pi f \varepsilon}{c_0}}$$  \hspace{1cm} (2.35)

where

$$A = \exp\left(\frac{-\varepsilon_x}{L_x}\right), B = \exp\left(\frac{-\varepsilon_y}{L_y}\right)$$  \hspace{1cm} (2.36)

The correlation length $L_x$ and $L_y$ that uses the coefficients $\alpha_x$ and $\alpha_y$ that indicates the coherence loss in x- resp. y-direction. Commonly used values are $\alpha_x = 0.7$ and $\alpha_y = 0.1$ according to [10].

$$L_x = \frac{C_0}{\alpha_x 2\pi f}, L_y = \frac{C_0}{\alpha_y 2\pi f}$$  \hspace{1cm} (2.37)

$C_0$ is the convection velocity which is expressed as $0.7U_\infty$.

Diffuse excitation

The diffuse pressure distribution is often used for acoustic excitation. Just as TBL-excitation this load is partially correlated. It represents a sound pressure which is instead only dependent of the speed of sound $c$ as $\kappa_0 = \omega/c_s$, where $r$ is the absolute distance between two DOF’s.

This load model is often used to represent a sound pressure chamber. Its CSD can be written as

$$G_{xy}^{diffuse} = \sqrt{G_{xx}G_{yy}} \cdot \frac{\sin(\kappa_0 r)}{r \kappa_0}$$  \hspace{1cm} (2.38)
3. Verification

MATLAB tools have been developed (see Appendix B) to run simulations based in the theory presented in Chapter 2. The aim of this chapter is to verify the solutions produced by MATLAB. To do this Abaqus is used. Abaqus is a commercial *Finite Element* (FE)-program, which is able to produce mode shapes and natural frequencies on various engineering systems. A simple system is the plate with different boundary conditions which is used in this study. The plate is set up by two boundary conditions: simply supported (SS-SS-SS-SS) and fully fixed (C-C-C-C) at all edges. At first MATLAB is used to reproduce these results and later further the correlation models: uncorrelated, fully correlated, MCL, TBL and diffuse loads are also made in MATLAB. In Abaqus only three of the models are present: uncorrelated, fully correlated and MCL (called moving noise in Abaqus) which is therefore the only models verified between the programs.

The setup used in Abaqus is a simple plate which can be seen in Figure 3-1. The mesh is of 32x32 elements and element type S4R which is robust and known to be suitable for a wide range of applications. The dimensions and material properties in both MATLAB and Abaqus can be seen in Table 3-1.

![Abaqus mesh. 32x32 elements and element type S4R.](image)

Table 3-1: Material properties of Aluminum used for the plate in the verification study. The data are used for both MATLAB and Abaqus.

<table>
<thead>
<tr>
<th>Plate properties</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>( a = 0.2 , [m] )</td>
</tr>
<tr>
<td>Width</td>
<td>( b = 0.1 , [m] )</td>
</tr>
<tr>
<td>Thickness</td>
<td>( h = 0.002 , [m] )</td>
</tr>
<tr>
<td>Young’s Modulus</td>
<td>( E = 70 \cdot 10^9 , [Pa] )</td>
</tr>
<tr>
<td>Density</td>
<td>( \rho = 2800 , [kg/m^3] )</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>( \nu = 0.3 )</td>
</tr>
</tbody>
</table>
3.1 Natural Frequency

The first verification is for the simple supported plate where the natural frequency \( f_n \) of the first 10 modes for both MATLAB and Abaqus is calculated. This is shown in Table 3-2, also the error between the two cases is presented in the right column. The low error indicates that the results from MATLAB are acceptable.

Table 3-2: Validation of the natural frequency \( f_n \) between Matlab and Abaqus. The model used is a simply supported plate (SS-SS-SS-SS). The table presents the first 10 modes and the error to the right show that the error does not increase with an increased number of modes.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( f_n ) [Hz]</th>
<th>( f_{n,abq} ) [Hz]</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>594.2</td>
<td>591.52</td>
<td>0%</td>
</tr>
<tr>
<td>2</td>
<td>950.7</td>
<td>944.15</td>
<td>1%</td>
</tr>
<tr>
<td>3</td>
<td>1544.9</td>
<td>1535.6</td>
<td>1%</td>
</tr>
<tr>
<td>4</td>
<td>2020.2</td>
<td>2013.1</td>
<td>0%</td>
</tr>
<tr>
<td>5</td>
<td>2376.7</td>
<td>2360.9</td>
<td>1%</td>
</tr>
<tr>
<td>6</td>
<td>2376.7</td>
<td>2366.2</td>
<td>0%</td>
</tr>
<tr>
<td>7</td>
<td>2970.9</td>
<td>2944.1</td>
<td>1%</td>
</tr>
<tr>
<td>8</td>
<td>3446.2</td>
<td>3436.2</td>
<td>0%</td>
</tr>
<tr>
<td>9</td>
<td>3802.8</td>
<td>3765.1</td>
<td>1%</td>
</tr>
<tr>
<td>10</td>
<td>4396.9</td>
<td>4374.2</td>
<td>1%</td>
</tr>
</tbody>
</table>

The second verification for the fully clamped plate, and is presented in Table 3-3, the error is slightly larger than for the simply supported plate, but still acceptable. The analytical setup used in MATLAB is known to have received voluminous treatment [11] and is very hard to predict due to the boundary condition. This is why the mode shape also should be considered in this verification.

Table 3-3: Validation of the natural frequency \( f_n \) between Matlab and Abaqus. The model used is a fully clamped plate (C-C-C-C). The table presents the first 10 modes. To the right the error present and is slightly higher than for the simply supported case.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( f_n ) [Hz]</th>
<th>( f_{n,abq} ) [Hz]</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1149.5</td>
<td>1181.5</td>
<td>3%</td>
</tr>
<tr>
<td>2</td>
<td>1428.7</td>
<td>1529.5</td>
<td>7%</td>
</tr>
<tr>
<td>3</td>
<td>1996.5</td>
<td>2156.2</td>
<td>7%</td>
</tr>
<tr>
<td>4</td>
<td>2859.4</td>
<td>3063.3</td>
<td>7%</td>
</tr>
<tr>
<td>5</td>
<td>3037.1</td>
<td>3073.8</td>
<td>1%</td>
</tr>
<tr>
<td>6</td>
<td>3256.4</td>
<td>3408.9</td>
<td>4%</td>
</tr>
<tr>
<td>7</td>
<td>3685.5</td>
<td>3990</td>
<td>8%</td>
</tr>
<tr>
<td>8</td>
<td>3995.9</td>
<td>4245.5</td>
<td>6%</td>
</tr>
<tr>
<td>9</td>
<td>4381.3</td>
<td>4833.9</td>
<td>9%</td>
</tr>
<tr>
<td>10</td>
<td>5366.1</td>
<td>5702.5</td>
<td>6%</td>
</tr>
</tbody>
</table>
3.2 Mode Shapes

The simply supported- and the fixed plate modes are both computed in MATLAB and are then compared to Abaqus. This is shown in Figure 3-2 and Figure 3-3 where the comparison for the simply supported plate is Figure 3-2 and the fixed plate in Figure 3-3. Both figures present the natural frequency for MATLAB and for Abaqus ($f_M$ and $f_A$).

If now the natural frequencies are compared for simply supported and the fixed plate it can be seen that the frequencies for the simply supported plate is considerably smaller because the fixed plate is stiffer.
Even if the natural frequencies produced for the fixed plate differs, the mode shapes characteristics are identical. The error presented in Table 3-3 is quite large, but the fixed plate does not show any tendency of either divergence or convergence for an increased number of modes. This proves that this case is hard to replicate but is still usable for further verification.

Figure 3-3: First 12 modes of a fully fixed plate. $f_M$ and $f_A$ are the natural frequencies from MATLAB and Abaqus for their related mode shape. The modes shown are taken from MATLAB and they are identical to the ones produced by Abaqus.
3.3 Frequency Response Function

In this section the frequency response function (FRF) between two specific DOF’s of a MDOF-system is tested. The MDOF-system is represented as the plates presented previously, which consider 1681 DOF’s. As every two DOF could measure a FRF between each other and even with itself. The number of response functions of this plate will be $1681 \cdot 1681 = 2825761$ functions. Due to the large number of FRF’s in the plate, the choice of only verifying two DOF’s, is considered enough.

The functions considered in this verification has both input and output in the same point. Then two DOF’s are chosen, see Figure 3-4. The first DOF, marked black, is positioned in a modal-node, i.e. the characteristic displacement will be zero for certain modes (See the second mode for both the simply supported and fixed plate in Figure 3-2 and Figure 3-3 etc.). The second DOF, marked red, is positioned in a position where the lower order modes will be nonzero. The results of these two DOF’s are calculated in both MATLAB and Abaqus for both the simply supported and fully fixed plate.

![Figure 3-4](image)

Figure 3-4: The MATLAB plate illustrates the plate with 1681 DOF’s (nodes) and the red dot represents DOF 1035 and the black dot DOF 841. The FRF in the red resp. black dot will describe how this plate reacts in these particular points to an excitation in the same point for a chosen frequency span. The FRF is calculated in a way that makes it possible to determine the response function for any combination of inputs and outputs, and even multiple combinations at the same time.
The data used for this verification is shown in Table 3-4, where the number of modes that should be included are $N = 9$ to limit the frequency range. The position of the input/output (the red dot in Figure 3-4) is in the $1035^{th}$ DOF which is positioned in $(x, y) = (0.045, 0.0625)$ and the black dot is in the $841^{th}$ DOF which is positioned in $(x, y) = (0.1, 0.05)$. Now the damping in the structure is relevant, as it is directly tied to the response. For this work it is assumed that the modal damping is $5\%$ for all modes.

Table 3-4: The setup used to produce the FRF shown in Figure 3-5 and Figure 3-6.

<table>
<thead>
<tr>
<th>FRF setup</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 9$</td>
<td>Number of Modes</td>
</tr>
<tr>
<td>1681</td>
<td>Number of Nodes</td>
</tr>
<tr>
<td>1035, $(x, y) = (0.045, 0.0625)$</td>
<td>Red DOF</td>
</tr>
<tr>
<td>841, $(x, y) = (0.1, 0.05)$</td>
<td>Black DOF</td>
</tr>
<tr>
<td>$\zeta = 0.05$</td>
<td>Modal Damping</td>
</tr>
</tbody>
</table>

The frequency response functions from the two points in Figure 3-4 are presented in Figure 3-5 for the simply supported plate. There it can be seen that MATLAB agrees with the result from Abaqus, especially for DOF 1035 marked red. For higher frequencies DOF 841 in the middle of the plate deviate from the Abaqus result. This is probably because the Abaqus mesh is not accurate enough due to the discretization of the structure.

If the red curve is considered which is the FRF from DOF 1035 we know that all modes are active in this point, as it is not positioned in a modal-node, this information can be obtained from the 7 maximums the red curve produces. If Table 3-2 is considered again it tells us that the first nine modes occur in the frequency span of 594 $Hz$ to 3802 $Hz$, therefore all 9 modes should be distinguished in this curve. The first and second mode occur at 594 $Hz$ and 950 $Hz$, and they correspond to the two first maximums in the red line. The third and fourth maximum got corresponding modes at 1544 $Hz$ and 2020 $Hz$. Now the fifth and sixth mode occurs at the same frequency and then probably corresponds to the same maximum, and if we look closely at 3800 $Hz$ where the ninth mode occur a small maximum can be distinguished. This confirms that DOF 1035 is not present in a modal-node.
If we take a closer look at the first nine mode shapes for the simply supported plate in Figure 3-2, six out of nine modes pulsates around midpoint which then will not influence the FRF in this position. If the black line for DOF 841 (Figure 3-5) is considered again the three maximums occur at 500 Hz, 1500 Hz and 3700 Hz and according to Table 3-2 this is the first, third and ninth mode.

![Graph](image)

Figure 3-5: The frequency response function for the simply supported case. The excitation- and response DOF’s are chosen in the same point. The red curve is DOF 1035 and the black curves are DOF 841, and the solid lines represent MATLAB and the dotted lines represent Abaqus. Both the solution from MATLAB and Abaqus coincide well. Each maximum occur slightly after the normal frequency for each mode, which then tells that in the chosen point the number of active modes is the same as the number of maximums seen in the FRF.

Now that we know that the maximums corresponds with the natural frequencies, in Figure 3-6 the natural frequencies of the fixed plate, which produced errors if MATLAB and Abaqus are compared. This was shown in Table 3-3. These errors explain offset seen in Figure 3-6 and because of these errors the simply supported plate will be the basis for the further verification and study.

![Graph](image)

Figure 3-6: The frequency response function for the fixed plate case in DOF 1035. Here the solutions are a bit more offset compared to Figure 3-5. This error can also be seen when comparing the natural frequency (Table 3-3) for MATLAB and Abaqus and these errors are probably related as the FRF is function of natural frequency.
3.4 Random Response

In total five pressure distributors are implemented in MATLAB but only three of these are present in Abaqus, which is uncorrelated, fully correlated, and moving correlated load. Therefore the responses of these three methods are compared to verify the results.

The PSD used for all distribution models is set to be 1 Pa through the whole frequency span and is multiplied with the element area, contributed from the mesh, to achieve a force PSD in N. This PSD is then converted to a CSD (Cross-Spectral Density) in an uncorrelated, fully correlated or MCL way. The CSD is present as input in all DOF's but the response shown in this verification is only shown from the mid-point of the plate.

In Figure 3-7 the plate response from an uncorrelated pressure distribution can be seen. There are three meshes with 16x16, 32x32 and 64x64 elements for both MATLAB and Abaqus, where each mesh represents one color, black, red and blue. If MATLAB and Abaqus are compared we can see that the FE-solution is in need of a denser mesh as Abaqus has problem to predict the third mode (last maximum). This is not the case for MATLAB as the modes are analytically determined, which then gives the same result for all meshes and should therefore be the more trustworthy solution. The most important thing about this plot is that the characteristics for the uncorrelated mesh show that it does not converge for denser meshes either for Abaqus or MATLAB. The doubled mesh gives an equally large error in power of 10 and this is not the case for the other methods. The divergence can be proved as an increased mesh decreases the distance between adjacent DOF’s and to say that two closely adjacent DOF’s are uncorrelated is unphysical. As a result, the assumption of uncorrelated loads and the way it is implemented in this study is unreliable and is therefore not used further in this work.

Figure 3-7: Uncorrelated plate response verification with three different meshes, 16x16, 32x32 and 64x64. The solid lines represent MATLAB and the dotted lines are Abaqus. If each mesh is considered separately MATLABs analytical solution is meshindependent which is not the case for the FEM solution. Abaqus needs atleast 32x32 elements or more in this matter. If only the meshes are considered the uncorrelated pressure distributor does not converge with a denser mesh.
From the results in Figure 3-7 for the uncorrelated plate response we can see that MATLAB produces a rather mesh independent solution for the natural frequency and therefore the 32x32 mesh is used in the coming verification studies.

In Figure 3-8 we can now see a plot of both MATLAB and Abaqus of the fully correlated solution, MATLAB aligns Abaqus well, which is good. Compared to uncorrelated this method converges which is shown in Appendix C. The error at the last maximum is in this case affordable as it is present at a high frequency range.

![Plot of MATLAB and Abaqus results](image)

**Figure 3-8**: Fully correlated with a 32x32 mesh for both MATLAB in black, and Abaqus in red. MATLAB here show that it produces a trustworthy result compared to Abaqus with only minor errors.
The moving correlated load method is verified and shown in Figure 3-9, where a comparison between MATLAB and Abaqus is made for two different free stream velocities which represents black and red lines. The black lines are of $100 \text{ m/s}$ and the red lines of $20 \text{ m/s}$. Compared to both the results from uncorrelated and fully correlated the MCL curves look rather chaotic. The answer to this is probably because MCL is also fully correlated along with the propagating pressure field. With a free stream velocity that is increased to infinity the MCL solution will converge to a fully correlated solution. Even if the solution is chaotic, the MATLAB result agrees well with the Abaqus solution.

Figure 3-9: Moving correlated load pressure distribution where the free stream velocity is changed between $20 \text{ m/s}$ and $100 \text{ m/s}$ for both MATLAB and Abaqus. The solutions are not identical but the characteristic of Abaqus is also shown by MATLAB.
To get a brief understanding, all models used in the simulation study is presented in the same plot, see Figure 3-10. The black lines are fully correlated and MCL that both could be compared and validated to Abaqus. The two red lines are two new models not validated but theoretically described and are called TBL and Diffuse. These two models are only compared in MATLAB. TBL is somewhat similar to the MCL model with a propagating flow of 100 m/s, but the correlation is distance dependent in the TBL model. The Diffuse model is in this master thesis considered as it represents sound pressure which should come in hand as experiments are often made in sound pressure chambers. As both TBL and Diffuse is found in the same range as the other methods these solutions is believed to be accurate. It can be seen that fully correlated and the TBL CSD agrees well for lower frequencies.

![Graph showing the response of fully correlated, Moving Correlated Load, TBL, and diffuse models](image)

Figure 3-10: The response of fully correlated (solid black), Moving Correlated Load (dotted black), TBL (solid red) and diffuse (dotted red) to get an overview of the simulation study. It is clearly seen that Moving Correlated Load under-predict, except for lower frequencies. Fully correlated and diffuse agrees well to the first mode, slightly after they separates to later align again.
4. Simulation Study

The purpose of this simulation study is to test different pressure correlation models. The goal is to see what effect the correlation model has on the final response of the structure. This is done by comparing them to each other in different environments and by different methods.

Before the simulation study a few assumptions are made. At first the flight envelope at different altitudes along with different Mach numbers is considered. Every combination exposes the structure with a new pressure field. The ESDU [12] standard is here used to give a PSD for each particular case. The main parameters used to estimate the PSD with the ESDU standard are the Mach number $M$, the flight altitude and distance from leading edge measuring the pressure. Leading edge distance is set to $5 \text{ m}$ and the altitude is set constant at $6000 \text{ m}$ as the altitude does not affect the result much. The structure tested in this simulation study is a simply-supported plate with different dimensions. The Mach number however is affecting the PSD and is therefore varied.

The TBL excitation uses properties of a propagating flow along with a damping correlation length. This method is believed to be the pressure distribution best suited for analyses of high speed. Therefore TBL is compared to the other methods fully correlated, MCL and diffuse load.

4.1 Mean Response PSD

To get a brief understanding of different excitation methods of different structures, the mean response PSD is studied. The first two figures will represent two flight cases where the first is of Mach $0.6$ (Figure 4-1) and the second is of Mach $1.1$ (Figure 4-2). The rows represent the method comparison and the two columns are two different plate sizes, left column $a = 0.4 \text{ m}$ and right column $a = 0.16 \text{ m}$. The number of modes used is for the large plate 44 and the smaller plate 8 modes, see Table 4-1 along with environmental parameters. This represents all modes up to $3000 \text{ Hz}$ for both plates.

<table>
<thead>
<tr>
<th>Simulation setup</th>
<th>Number of Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 44$ and $8$</td>
<td>Plate length streamwise</td>
</tr>
<tr>
<td>$a = 0.16 \text{ m} \text{ and } 0.4 \text{ m}$</td>
<td>Plate length cross-streamwise</td>
</tr>
<tr>
<td>$b = 2a/3 \text{ m}$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$M = 0.1 \text{ and } 1.1$</td>
<td>Convection velocity</td>
</tr>
<tr>
<td>$U_c = 0.7U_\infty$</td>
<td>Correlation length x-dir.</td>
</tr>
<tr>
<td>$\alpha_x = 0.7$</td>
<td>Correlation length y-dir.</td>
</tr>
<tr>
<td>$H = 6000 \text{ m}$</td>
<td>Flight altitude</td>
</tr>
</tbody>
</table>

In Figure 4-1 it can be seen that for certain frequencies the solutions agrees well. This information is important as some investigations have limitations or requirements in certain frequency ranges. If the fully correlated plots are considered, it can be seen that both TBL and fully correlated agrees well with at the first mode, the scale is though a log scale and the error could be quite large. But as the frequency increases fully correlated does not produce certain modes compared to TBL. In the right plot for the smaller plate this is clearly seen as fully correlated in the frequency span $0 - 3000 \text{ Hz}$ only predicts 2 active modes as TBL predicts 5 active modes.
If fully correlated then is compared to moving correlated load it can be seen that MCL handles the first modes well. This gives the solution that if a larger frequency span is needed MCL should be chosen before fully correlated as fully correlated fails to predict certain modes of the structure. For higher frequencies neither of the methods produces a good result where they both under-predict TBL.

The diffuse model on the other hand under-predicts for the large plate and over-predict for the smaller plate. This result show that one should be cautious when using a diffuse model as it could under some circumstances produce varying results.

Figure 4-1: Mean response comparison of TBL versus fully correlated, Moving Correlated Load and diffuse (in rows). The two columns represents two plate sizes, to the left \( a = 0.4 \) m and to the right \( a = 0.16 \) m. The PSD produced for these cases is for \( Mach \ 0.6 \).
In the second figure for Mach 1.1 Figure 4-2, almost the same result as in Figure 4-1 could be seen. Here fully correlated keeps to predict the first mode but fail to produce the second mode. The large plate for higher frequencies also under-predicts the TBL model as in Figure 4-1, and now for a higher Mach number the smaller plate also under-predict for higher frequencies.

If we now look towards the MCL solution it can be seen that the overall solution agrees fairly good. It though misses the third mode for the large plate but for higher frequencies it does not under-predict as shown in Figure 4-1. As for the smaller plate MCL now instead over-predicts which is good.

The diffuse model still presents the same trend where it under-predicts for the large plate and over-predicts for the smaller plate.

Figure 4-2: Mean response comparison of TBL versus fully correlated, Moving Correlated Load and diffuse (in rows). The two columns represents two plate sizes, to the left \( a = 0.4 \) m and to the right \( a = 0.16 \) m. The PSD produced for these cases is for Mach 1.1.
4.2 Discussion: Mean Response PSD

By the results from Figure 4-1 and Figure 4-2 it can be concluded that fully correlated only produces a good result for the first mode. Moreover, it over-predicts the first mode, which is favored if this method is to be used for lower frequencies. The reason why fully correlated fails to predict certain modes is a known matter and occurs when the mode shape is in symmetry. When the mode shape is in symmetry the fully correlated excitation suppresses the response which is why we cannot see it in the figures.

MCL gives an acceptable result for the first modes up to almost 1250 Hz for Mach 0.6. If this is compared to fully correlated MCL produces a more trustworthy result for the lower frequencies and should then be used before fully correlated. When then the Mach number is increased MCL instead starts to follow the TBL solution and even for the smaller plate over-predicts. This information is important as this could show that the moving correlated load solution could be a good approximation for higher velocities. With the already implemented MCL method in the FE-solver Abaqus this could probably replace few experiments executed with fully correlated.

Sometimes when the sound pressure chamber is used it is believed that the pressure distribution created in this chamber always produces a higher pressure than in a real case. If this is the case and if Figure 4-1 for and Figure 4-2 is considered, it can be seen that the diffuse model does not always produce an over-predicting result when the plate size increases. If this result is true the experimental results should therefore be considered and maybe reworked if they are used to set qualifications for an aircraft.

4.3 Mean RMS Response Comparison

Instead of the mean response PSD shown previously, a new criterion is tested. The mean RMS of all DOF’s at the plate will be shown. As an example a TBL PSD for DOF n can be denoted as \( G_{n \text{TBL}}^{TBL} \). Remember that the RMS\(^2\) of a PSD is determined by the area under the graph. Then the mean RMS of the plate DOF’s is denoted as \( q \) and can be calculated as

\[
q_{\text{tbl}}(L, M) = \frac{1}{N} \sum_{n=1}^{N} \left( \int_{f_1}^{f_2} G_{n \text{TBL}}^{TBL}(f, L, M) \, df \right)
\]  

The mean RMS value is also calculated for fully correlated, MCL and diffuse load:

\[
q_{\text{fc}}(L, M) = \frac{1}{N} \sum_{n=1}^{N} \left( \int_{f_1}^{f_2} G_{n \text{fc}}^{fc}(f, L, M) \, df \right)
\]

\[
q_{\text{MCL}}(L, M) = \frac{1}{N} \sum_{n=1}^{N} \left( \int_{f_1}^{f_2} G_{n \text{MCL}}^{MCL}(f, L, M) \, df \right)
\]

\[
q_{\text{df}}(L, M) = \frac{1}{N} \sum_{n=1}^{N} \left( \int_{f_1}^{f_2} G_{n \text{df}}^{df}(f, L, M) \, df \right)
\]
The last models are to be compared towards TBL and a new factor is introduced called $\varepsilon(L, M)$. By changing both plate length $L$ and Mach number $M$ a new RMS value will be achieved and these will be stored and compared in $\varepsilon(L, M)$ as

$$
\varepsilon_1(L, M) = \frac{q_{tbi}}{q_{fc}} 
$$

(4.5)

$$
\varepsilon_2(L, M) = \frac{q_{tbi}}{q_{mcl}} 
$$

(4.6)

$$
\varepsilon_3(L, M) = \frac{q_{tbi}}{q_{df}} 
$$

(4.7)

These three factors will produce three color maps used to predict which method to use for certain environments. Red regions will represent an under-prediction towards TBL and blue regions will then be over-prediction, and over-prediction is in this case favored. TBL is again compared to fully correlated, MCL and diffuse and will be presented in Figure 4-3, Figure 4-4 and Figure 4-5. The simulation setup can be seen in Table 4-2.

Table 4-2: Simulation setup for the mean RMS response comparison.

<table>
<thead>
<tr>
<th>Simulation setup</th>
<th>Number of Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 20$</td>
<td></td>
</tr>
<tr>
<td>$a = 0.1 \text{ m} - 0.4 \text{ m}$</td>
<td>Plate length range, streamwise</td>
</tr>
<tr>
<td>$b = 2a/3 \text{ m}$</td>
<td>Plate length, cross-streamwise</td>
</tr>
<tr>
<td>$M = 0.1 - 1.1$</td>
<td>Mach number range</td>
</tr>
<tr>
<td>$U_c = 0.7U_\infty$</td>
<td>Convection velocity</td>
</tr>
<tr>
<td>$\alpha_x = 0.7$</td>
<td>Correlation length x-dir.</td>
</tr>
<tr>
<td>$\alpha_y = 0.1$</td>
<td>Correlation length y-dir.</td>
</tr>
<tr>
<td>$H = 6000 \text{ m}$</td>
<td>Flight altitude</td>
</tr>
</tbody>
</table>
In Figure 4-3 the first color map can be seen which presents a factor of TBL over fully correlated. In the lower left corner where the plate and Mach number are small, fully correlated over-predicts TBL. If then both variables are increased it can be seen that TBL over-predicts fully correlated. Here fully correlated instead under-predicts. Under-prediction should be taken with caution and then fully correlated should not be used for increased Mach number or large plates.

Figure 4-3: The color map represents mean RMS response-differences between TBL and fully correlated (TBL/FC) for different cases, see Eq. (4.5). The Mach number is varied between 0.1 and 1.1, and the plate length in streamwise direction varied between 0.1 and 0.4 m. The yellow region represents the flight cases where the two methods give the same result.
Figure 4-4 show a different trend compared to Figure 4-3 and Figure 4-5 where the magnitude is instead larger (red) in the lower left corner. This phenomenon was seen in the previous study where MCL produced under-predicting results for low Mach numbers and well agreeing result for higher Mach numbers (Figure 4-1 and Figure 4-2). There is a second region where MCL under-predicts TBL in the upper left region, the region is though small, but the reason for this behavior is unknown.

Figure 4-4: The color map represents mean RMS response-differences between TBL and moving correlated load (TBL/MCL) for different cases, see Eq. (4.6). The Mach number is varied between 0.1 and 1.1, and the plate length in streamwise direction varied between 0.1 and 0.4 m. In this figure the color is log scaled.
Again the diffuse model produces an interesting result. Compared to the previous study of the mean response PSD the diffuse model over-predicts at a plate size of 0.16 m and under-predicts at a plate size of 0.4 m for both Mach numbers, 0.6 and 1.1. This figure produces the same result. It can be seen that for smaller plates the Mach number does not affect the solution drastically as for the fully correlated color map (Figure 4-3). For larger plates the diffuse model instead over-predicts the response already at Mach 0.3-0.4.

Figure 4-5: The color map represents mean RMS response-differences between TBL and diffuse (TBL/Diffuse) for different cases, see Eq. (4.7). The Mach number is varied between 0.1 and 1.1, and the plate length in streamwise direction varied between 0.1 and 0.4 m.

4.4 Discussion: Mean Response PSD

In structural dynamics an over-prediction of the load is generally favored since this gives a safety margin. Though, if the dimensions of the structures are customized after a large over-prediction, it could be expensive. Assuming that TBL is the actual excitation, the figures presented in this chapter then make it possible to estimate what error one will obtain with other correlation models. Hence one could get the opportunity to choose the model depending on the environment. The most important knowledge given from the figures is the combination of Mach number and plate length that leads to an under-prediction of the response in comparison with a TBL excitation. In these cases, it is recommended to carry out the calculations with a TBL correlation model.

The plate length in the streamwise direction is an unsure parameter, if not the plate with these chosen dimensions is considered. With new dimensions the mode shapes will change and occur at new natural frequencies. This means that for each plate dimension maybe a new solution should be studied. Therefore another more general method should be considered and a possible parameter could be the plate wavelength. The wavelength and the length of the plate in streamwise direction could share a resonance at the first mode, which could produce important information. It is a possibility that a solution regarding this problem could be found, but more testing is necessary to achieve such a result.
5. Conclusion

Abaqus, the commercial FE-solver, gives the opportunity to work with three different pressure models: uncorrelated, fully correlated and moving noise. Moving noise is called MCL in this work. These models, depending on the situation are best suited for different sort of loads, simple pressure field, propagating flow pressure, acoustic pressure etc. Therefore these models are further investigated in a new toolbox that is able to simulate these pressure models at a plate surface. This work is limited to modal solutions in both MATLAB and Abaqus.

Due to the simplicity of applying fully correlated loads in FE-models, this method is often used but has later shown not to predict all needed data for certain cases. Therefore the other methods are considered and also a fifth model is introduced called turbulent boundary layer (TBL). TBL is a partially correlated load along with the diffuse load, but TBL represents a propagating flow where diffuse represents a sound pressure. That is why TBL later used as a basis for the validation in this work.

After the load models are validated towards Abaqus a simulation study is made. It begins with comparing the mean PSD of fully correlated, MCL and diffuse load with TBL load. The overall conclusion in this comparison is that all models produce good results for the lower frequencies. On the other hand, when the frequency is increased fully correlated quickly starts to miss a few modes compared to TBL. Here MCL and diffuse load actually produces a better overall result, but if the plate size is increased, it is important to know that the diffuse load starts to under-predict the TBL load. The diffuse model, which is supposed to imitate a real pressure chamber load, is often believed to be conservative but in this study it can be seen that it only applies to smaller plates. MCL is known to, along with TBL, to both be velocity dependent. This is also seen as for increased Mach-numbers the models starts to agree even better through the whole frequency span.

The second validation tool is using the mean RMS value through two dimensions, Mach-number and plate length. This creates a color map where the factor of the RMS value of TBL over each of the other models is studied. This produces a result that shows the models dependency of the air velocity. For low Mach-numbers fully correlated could be used as is then known to be conservative because it is over-predicting TBL. But as the Mach-number is increased fully correlated turn to a less good method. The same characteristics can be seen for the diffuse load. Once again the MCL model show promise with increased Mach-numbers.

In summary, the fully correlated load could be used in low frequency ranges and is an easy model to apply. However, if critical modes are to be found this method is not to recommend in an environment exposed to propagating air flow. The moving correlated load, on the other hand, is best used at higher speeds. At lower speeds the frequency range should be considered and validated as the models could under-predict the solution at high frequency. The diffuse load is believed to always produce a conservative result (over-predict), however, this study show that this is not always the case.
6. Future Work

There are four interesting problems that should be considered in the future. As this thesis only considers methodologies for different excitation methods, there are still a few questions to be made. First, can the turbulent boundary layer model be implemented in a commercial FE solver? It is known that Abaqus can handle customized excitation models, where a TBL model could be made and this should be studied for future works. Second, the responses simulated with the new toolbox could be compared to real experimental data. Third, and maybe as doable as the first suggestion, is to compare the diffuse model to experimental data in a sound pressure chamber. A sound pressure chamber already exists at Saab and is rather easy to access. If this is possible, it could answer if the pressure chamber produces conservative results as believed. Fourth, is to study if the uncorrelated model would be applicable once again in MATLAB and Abaqus.
Bibliography


Appendix A

Appendix A first presents modal plate theory for two different boundary conditions, simply supported and fully clamped. Second and third the derivation of a FRF is made for both a SDOF-system and a MDOF-system.

A.1 Modal Properties for a plate

The derivation of two plates and how to determine the mode shape function along with its frequency responses.

A.1.1 Simply Supported

Assume a Harmonic response of a plate

\[ w(x, y, t) = W(x, y)e^{j\omega t} \]  (A.1)

and governing equation of motion

\[ D \left( \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + \rho h \frac{\partial^2 W}{\partial t^2} = 0 \]  (A.2)

If (A.1) and (A.2) are combined, we obtain:

\[ D \left( \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right)e^{j\omega t} - \rho \omega^2 Z e^{j\omega t} = 0 \]  (A.3)

or

\[ D \left( \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) - \rho \omega^2 W = 0 \]  (A.4)

Boundary conditions

\[ W(0, y) = Z(a, y) = 0 \]  (A.5)
\[ W(x, 0) = Z(x, b) = 0 \]  (A.6)
\[ M(0, y) = M(a, y) = 0 \]  (A.7)
\[ M(x, 0) = M(x, b) = 0 \]  (A.8)

Next step is to guess a mode shape function that satisfies Eq. (A.4) and the boundary conditions, Eqs (A.5-8). The function is used from Tom Irvine in [13], can be written as

\[ W = A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \]  (A.9)

Then if the derivatives \( \frac{\partial^4 W}{\partial x^4}, \frac{\partial^4 W}{\partial y^4} \) and \( \frac{\partial^4 W}{\partial x^2 \partial y^2} \) are solved from (A.9) and then put into (A.4) the frequency \( \omega \) can be obtained.

\[ \omega = \sqrt{\frac{D}{\rho h} \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right)} \]  (A.10)

The mode shape are normalized such that:
\[
\rho h \int_{0}^{b} \int_{0}^{a} [W(x, y)]^2 dx dy = 1 \quad \text{(A.11)}
\]

It then follows that \( A_{mn} \) in Eq. (A.9) is given by:

\[
A_{mn} = \frac{2}{\sqrt{\rho h ab}} \quad \text{(A.12)}
\]

### A.1.2 Fully Clamped

According to Leissa [11], Warburton was the first to present a complete collection of solutions for a rectangular plate with different boundary conditions. He used the Rayleigh method with deflection functions as products of beam functions

\[
W(x, y) = \alpha(x)\beta(y) \quad \text{(A.13)}
\]

The products \( \alpha(x) \) and \( \beta(y) \) are both fundamental mode shapes of a simple beam that having the same boundary condition as a plate. For the fully clamped case (C-C-C-C), the boundary conditions will be formulated as:

The displacement is zero at the edges

\[
W(0, y) = W(a, y) = 0 \quad \text{(A.14)}
\]
\[
W(x, 0) = W(x, b) = 0 \quad \text{(A.15)}
\]

And the angle is zero at the edges

\[
\frac{\partial W}{\partial x}(0, y) = \frac{\partial W}{\partial x}(a, y) = 0 \quad \text{(A.16)}
\]
\[
\frac{\partial W}{\partial y}(x, 0) = \frac{\partial W}{\partial y}(x, b) = 0 \quad \text{(A.17)}
\]

Now, the mode shape \( W(x, y) \) needs to satisfy the geometric boundary condition, which would be done by simply guessing \( \alpha(x) \) and \( \beta(y) \), but this is already done and the equations that satisfies the boundary conditions were found in [14].

The clamped beam function in \( x \) direction:

\[
\alpha(x) = C(\lambda_m x) - \frac{C(\lambda_m a)}{C(\lambda_m a)} C(\lambda_m x) \quad \text{(A.18)}
\]

where

\[
C(\lambda_m x) = \cosh(\lambda_m x) - \cos(\lambda_m x) \quad \text{(A.19)}
\]
\[
D(\lambda_m x) = \sinh(\lambda_m x) - \sin(\lambda_m x) \quad \text{(A.20)}
\]

where \( \lambda_1 a = \ldots = \lambda_2 a \leftrightarrow 4.73 = \ldots = 17.28 \) and then \( \lambda_m x = \frac{\lambda_m a}{a} \), these roots will be explained deeper in the next chapter. Also the clamped beam function in \( y \) direction:

\[
\beta(y) = C(\lambda_n y) - \frac{C(\lambda_n b)}{C(\lambda_n b)} C(\lambda_n y) \quad \text{(A.21)}
\]
where
\[ C(\lambda_n y) = \cosh(\lambda_n y) - \cos(\lambda_n y) \quad (A.22) \]
\[ D(\lambda_n y) = \sinh(\lambda_n y) - \sin(\lambda_n y) \quad (A.23) \]
where \( \lambda_1 b = \ldots = \lambda_5 b \Leftrightarrow 4.73 = \ldots = 17.28 \) and then \( \lambda_n y = \frac{\lambda_n b}{b} y. \)

As the clamped case has received voluminous treatment, various simplified methods for calculating the frequency can be considered. These cases often concern square plates at various modes or rectangular plates at a single mode. Therefore an equation that treats all cases seems more relevant which was explained by W. Soedel [14].

\[ \omega^2 = \frac{D}{\rho h} \left[ \lambda_m^4 + \lambda_n^4 + 2 \int_0^a \alpha(\partial^2 \alpha/\partial x^2) \, dx \int_0^b \beta(\partial^2 \beta/\partial y^2) \, dy \right] \quad (A.24) \]

This equation is to be calculated numerically in MATLAB where the integral is calculated as \( \sum_0^a (\alpha(\partial^2 \alpha/\partial x^2)) \cdot dx \) where \( dx \) is a chosen step length. The beam function (A.18) can be rewritten and along with its derivatives

\[ \alpha = [\cosh(\lambda_m x) - \cos(\lambda_m x) - \sigma_m [\sinh(\lambda_m x) - \sin(\lambda_m x)]] \quad (A.25) \]

\[ \frac{\partial \alpha}{\partial x} = \lambda_m [\sinh(\lambda_m x) + \sin(\lambda_m x) - \sigma_m [\cosh(\lambda_m x) - \cos(\lambda_m x)]] \quad (A.26) \]

\[ \frac{\partial^2 \alpha}{\partial x^2} = \lambda_m^2 [\cosh(\lambda_m x) + \cos(\lambda_m x) - \sigma_m [\sinh(\lambda_m x) + \sin(\lambda_m x)]] \quad (A.27) \]

where

\[ \sigma_m = \frac{\cosh(\lambda_m a) - \cos(\lambda_m a)}{\sinh(\lambda_m a) - \sin(\lambda_m a)} \quad (A.28) \]

which is done in the same way for \( \partial^2 \beta/\partial y^2. \)

\[ \frac{\partial^2 \beta}{\partial x^2} = \lambda_n^2 [\cosh(\lambda_n y) + \cos(\lambda_n y) - \sigma_n [\sinh(\lambda_n y) + \sin(\lambda_n y)]] \quad (A.29) \]

and

\[ \sigma_n = \frac{\cosh(\lambda_n b) - \cos(\lambda_n b)}{\sinh(\lambda_n b) - \sin(\lambda_n b)} \quad (A.30) \]

The mode shape is normalized as

\[ \rho h \int_0^b \int_0^a [W(x, y)]^2 \, dx \, dy = 1 \quad (A.31) \]

Which gives

\[ W(x, y) = \frac{1}{\sqrt{\rho ab h}} \alpha(x) \beta(y) \quad (A.32) \]
A.2 Transfer Function Single Degree-of-Freedom system

The derivation of a FRF for a SDOF-system will be shown. Consider a mass exposed by a force $x(t)$ which is also attached to a spring $k$ damper $c$. The derivation describes the transfer function $H(s)$ in form of force and displacement $y(t)$.

![Figure A.0-1 A Single-Degree-of-Freedom system](image)

\[
m\ddot{y} + c\dot{y} + ky = x(t) \tag{A.33}
\]

First step is to use the Laplace transform on both sides of Eq. (A.33)

\[
(ms^2 + cs + k)Y(s) = X(s) \tag{A.34}
\]

If $X(s)$ is the input excitation and $Y(s)$ the output response, then $1/(ms^2 + cs + k)$ is the transfer function $H(s)$.

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{1/m}{s^2 + sc/m + k/m} \tag{A.35}
\]

For second-order systems it is convenient to write the denominator the “standard form”. [8]

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{1/m}{s^2 + s2\zeta\omega_n + \omega_n^2} \tag{A.36}
\]

The transfer function describes the ratio between the input excitation and the output response of the system.

A.3 Transfer Function Multiple Degree-of-Freedom system

With little help from the SDOF-system derivation of a FRF it is possible to determine the MDOF-system FRF instead.

If we consider equation (A.23) from Appendix A as a MDOF-system we get

\[
[s^2[M] + s[C] + [K]]\{Y(s)\} = \{X(s)\} \tag{A.37}
\]

and if $H(s) = Y(s)/X(s)$, Eq. (A.26) can be rewritten as

\[
[s^2[M] + s[C] + [K]] = [H]^{-1} \tag{A.38}
\]

then multiply the left hand side with the transposed mode shape matrix $[\Psi]^T$ and the right hand side with $[\Psi]$
\[ [\Psi]^T [s^2[M] + s[C] + [K]] [\Psi] = [\Psi]^T [H]^{-1} [\Psi] \quad (A.39) \]

Next we can make use of that we have a proportional damping and the orthogonality criterion therefore makes the matrix on the left-hand side diagonal

\[ [s^2[M] + s[C] + [K]] = [\Psi]^T [H]^{-1} [\Psi] \quad (A.40) \]

If the inverse of a product is \([ABC]^{-1} = C^{-1}B^{-1}A^{-1}\), we inverse both sides of \((A.29)\) and obtain

\[ [s^2[M] + s[C] + [K]]^{-1} = [\Psi]^{-1}[H]([\Psi]^T)^{-1} \quad (A.41) \]

\[ [H] = [\Psi] [s^2[M] + s[C] + [K]]^{-1} [\Psi]^T \quad (A.42) \]

\([s^2[M] + s[C] + [K]]^{-1}\) is called the inverse pole matrix \([S^{-1}]\).

\[ [H] = [\Psi] [S^{-1}][\Psi]^T \quad (A.43) \]

Where each element in the inverse pole matrix is defined as for mode \(r\).

\[ s_{rr} = \frac{1}{s^2m_r + sc_r + k_r} = \frac{1/m_r}{(s - s_r)(s - s_r^*)} \quad (A.44) \]

The frequency response can then be written as

\[ H_{pq}(s) = \sum_{r=1}^{N} \frac{\psi_{pr}\psi_{qr}}{m_r(s - s_r)(s - s_r^*)} \quad (A.45) \]

\[ s_r = -\zeta_r \omega_r + j\omega_r\sqrt{1 - \zeta_r^2} \quad (A.46) \]

\(\zeta_r\) is the damping and \(\omega_r\) is the natural frequency for each mode.
Appendix B

Appendix B presents all MATLAB-scripts created which serves as the new toolbox to compute different pressure fields.

B.1 Matlab Toolbox

B.1.1 modalplate_ssss.m

```matlab
%modalplate_ssss   Get Mode Shape for a simply supported plate (SS-SS-SS-SS) plate with different configurations
%                  along with natural frequency for N modes.
%                  
% [W,fn,xy] = modalplate_ssss(N,a,b,Nx,Ny,h,E,rho,my)
%                  
% W = Mode Shape in columns for each mode number
% fn = Natural frequencies [Hz]
% xy = coordinates (x y)
% N = Number of modes
% a = Plate length x-direction [m]
% b = Plate length y-direction [m]
% Nx = Number of elements in x-direction
% Ny = Number of elements in y-direction
% h = Thickness [m]
% E = Young's Modulus [Pa]
% rho = Density [kg/m^3]
% my = Poisson's ratio [-]
%                  
% Each column in W represents a single mode form,
% then W consists N number of columns.
%                  
% To plot a specific mode form, use surfplate_ssss.
% see also:
% surfplate_ssss,modalplate_cccc,surfplate_cccc
```

B.1.2 modalplate_cccc.m

```matlab
%modalplate_cccc   Get Modal shape for a fully Clamped (C-C-C-C) plate with different configurations
%                  along with natural frequencies for N modes.
%                  
% [W,fn,xy] = modalplate_cccc(N,a,b,Nx,Ny,h,E,rho,my)
%                  
% W = Mode Shape in columns for each mode number
% fn = Natural frequencies [Hz]
% xy = coordinates (x y)
% N = Number of modes
% a = Plate length x-direction [m]
% b = Plate length y-direction [m]
% Nx = Number of elements in x-direction
% Ny = Number of elements in y-direction
% h = Thickness [m]
% E = Young's Modulus [Pa]
% rho = Density [kg/m^3]
% my = Poisson's ratio [-]
%                  
% Each column in W represents a single mode form,
% then W consists N number of columns.
```
To plot a specific mode form, use `surfplate_cccc`,
or `plot3(x(:,1),x(:,2),W(:,4),'.')`

see also: `surfplate_cccc`, `modalplate_ssss`, `surfplate_ssss`

**B.1.3 mode2frf.m**

Transfers a mode shape function \( W \) to a Frequency Response Function (FRF). The shape function is of unity modal mass scaling.

\[
[H] = \text{mode2frf}(W, fn, f, zeta, u)
\]

- \( H \): Frequency Response Function (FRF)
- \( W \): Mode Shape for each mode per column
- \( fn \): Natural frequencies [Hz]
- \( f \): Frequency vector
- \( zeta \): Damping for all modes, usually assumed to be \( zeta = 0.05 \)
- \( u \): FRF unit, choose between 1, 2 or 3 for displacement, velocity or acceleration

see also: `modalplate_ssss`, `modalplate_cccc`

**B.1.4 random_response.m**

Generates the PSD response of the given input \( G_{xx} \).

\[
[G_{yy}] = \text{random_response}(H, G_{xx}, f, dof)
\]

- \( G_{yy} \): Response
- \( H \): Frequency Response Function (FRF)
- \( f \): Frequency interval
- \( dof \): response for a specific degree of freedom

see also: `mode2frf`, `modalplate_cccc`, `modalplate_ssss`

**B.1.5 csd_uncorrelated.m**

Creates an uncorrelated input matrix \( G_{xx} \) from a given PSD vector. The Force PSD is set as the diagonal in the \( G_{xx} \) matrix and the antidiagonal is set to zero. The zero antidiagonal represents that none of the dofs have any correlation to each other and therefore the coherence is also zero in this case.

If the PSD is a Pressure PSD the it should be converted to a force PSD by simply multiplying with the square area, \( (PSD*A^2) \).

\[
[G_{xx}] = \text{csd_uncorrelated}(PSD, f, xy)
\]

- \( PSD \): Force PSD, Power Spectral Density [N^2/Hz]
- \( f \): Frequency vector
% xy = coordinates (x y)
%
% see also: csd_fullycorrelated, csd_movingnoise
% csd_movingnoise, random_response,
% csd_tbl

B.1.6 csd_fcorrelated.m
%csd_fcorrelated Creates a fully correlated input matrix Gxx from a
% given PSD vector. The PSD is set as the diagonal in the
% Gxx matrix and the antidiagonal is set sqrt(Gii*Gjj).
% This is the term that describes that all dofs are fully
% correlated to each other and therefore have a coherence
% of 1.
%
% The rewritten form of the PSD is called CSD, cross
% spectral density.
%
% If the PSD is a Pressure PSD the it should be converted
to a force PSD by simply multiplying with the square
% Area, (PSD*A^2).
%
% [Gxx] = csd_fcorrelated(PSD,f,xy)
%
% PSD = Force PSD, Power Spectral Density [N^2/Hz]
% f = Frequency vector
% xy = coordinates (x y)
%
% see also: csd_uncorrelated, csd_movingnoise,
% random_response, csd_diffuse,
% csd_tbl

B.1.8 csd_movingnoise.m
%csd_movingnoise Creates a Moving Correlated Load input matrix Gxx from a
% given PSD vector. The correlation in the Moving Correlated Load is
% also a fully correlated case but the difference is big
% because the correlation is time dependent and delayed
% between the dofs in the free stream direction, in this
case the x-direction.
%
% If the PSD is a Pressure PSD the it should be converted
to a force PSD by simply multiplying with the square
% Area, (PSD*A^2).
%
% [Gxx] = csd_movingnoise(PSD,f,xy,c_0)
%
% PSD = Force PSD, Power Spectral Density [N^2/Hz]
% f = Frequency interval
% xy = coordinates (x y)
% c_0 = freestream velocity
%
% see also: csd_uncorrelated, csd_fcorrelated,
% random_response, csd_diffuse, csd_tbl
B.1.8 csd_tbl.m

%csd_tbl     Creates a "Moving Correlated Load like" input matrix Gxx from a
given PSD vector called Turbulent boundary layer.
% The correlation between the dofs are
% now damped with factor that depends on the length
% between the dofs and the freestream velocity. Compared
% to the Moving Correlated Load CSD, the TBL (Turbulent boundary
% layer) depends also on the difference in the length in
% y-direction. The model used to describe the TBL case is
% a simplified law for the cross spectral density called
% CORCOS.
%
% If the PSD is a Pressure PSD the it should be converted
to a force PSD by simply multiplying with the square
Area, (PSD*A^2).
%
% [Gxx] = csd_tbl(PSD,f,xy,c_0,alpha_x,alpha_y)
%
%   PSD = Force PSD, Power Spectral Density [N^2/Hz]
%   f = Frequency interval, linspace(10,10000,100)
%   etc
%   xy = coordinates (x y)
%   c_0 = freestream velocity
%   alpha_x = coherence loss in x-dir coefficient
%   alpha_y = coherence loss in y-dir coefficient
%
% Tip: Commonly used values for alpha_x and alpha_y is
% ~0.7 and ~0.1
%
% see also: csd_uncorrelated,csd_fullycorrelated,
%          csd_movingnoise,
%          csd_diffuse, random_response

B.1.9 csd_diffuse.m

%csd_diffuse     Creates a Diffuse input matrix Gxx from a
given PSD vector. The method for creating the CSD
(cross-PSD), is said to be questionable but is still
considered to be the best representation of the acoustic
excitation field, according to Hekmati, Ricot and
Druault.
%
% If the PSD is a Pressure PSD it should be converted
to a force PSD by simply multiplying with the square
Area, (PSD*A^2).
%
% [Gxx] = csd_diffuse(PSD,f,xy,C_0)
%
%   PSD = Force PSD, Power Spectral Density [N^2/Hz]
%   f = Frequency vector
%   xy = coordinates (x y)
%   C_0 = Speed of sound adjacent through the fluid
%   adjacent to the material
see also: csd_uncorrelated, csd_fcorrelated,
csd_movingnoise, random_response,
csd_tbl
Appendix C
Appendix C presents two plots from the mesh independency study made in Abaqus for both fully correlated and moving correlated load.

C.1 Mesh Independency

Figure 0-1: Mesh independency for fully correlated pressure distribution response. This figure show that the mesh does not diverge like uncorrelated in Figure 3-7.

Figure 0-2: Mesh independency for Moving Correlated Load pressure distribution response.
Appendix D
Appendix D presents the Abaqus input files used to create mode shapes, FRF’s and random responses.

D.1 Abaqus Input File

```plaintext
**-----------------------------
** Modal Extraction
**
*INCLUDE, input=plate_16x16.inp
**
**
*STEP, name=modal_extraction, perturbation
*FREQUENCY, eigensolver=ams
,4000
**
** BOUNDARY CONDITIONS
**
** Name: Simply_Supported Type: Displacement/Rotation
*Boundary
 BC-set, 1, 3
**
*RESTART, WRITE
*OUTPUT, file, variable=PRESELECT
*END STEP
**-----------------------------
```

Figure 0-1: How to generate modal parameters in Abaqus.

```plaintext
**-----------------------------
*STEP, name=FRF calculation
*STeady State Dynamics
 1,4000,50,3,1
*Modal Damping, modal-direct
 1,5,0.05
*AMplitude, definition=tabular, name=amp1
 1,1
 400,1
*CLoad, amplitude=amp1
 9,2,1
*OUTPUT, field
*NODE OUTPUT
  TA
*END STEP
**-----------------------------
```

Figure 0-2: How to create a FRF in Abaqus.
Figure 0-3: How to do a random response analysis in Abaqus with fully correlated loads.