Designing Mathematics Lessons Using Japanese Problem Solving Oriented Lesson Structure

A Swedish case study

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To my parents

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Content

Abstract	1
Chapter 1	3
Introduction.....	3
1.1 Background	3
1.1.1 Swedish students' knowledge of mathematics	3
1.1.2 "Structured problem solving" – a Japanese teaching method in mathematics.....	5
1.1.3 Souma's "problem solving oriented lesson structure"	6
1.2 Aim.....	8
Chapter 2	10
Literature review	10
2.1 The development of the structured problem solving.....	10
2.1.1 The "Green Book" — a new era for school mathematics	10
2.1.2 Raising students' "mathematical way of thinking"	13
2.1.3 Development of problem solving centred teaching	15
2.1.4 Whole-class discussions	15
2.1.5 Development of the open-ended approach	17
2.1.6 The structured problem solving in elementary school	19
2.1.7 The structured problem solving and lesson study	19
2.1.8 Problems for teachers using the structured problem solving.....	20
2.2 Research on classroom studies.....	21
2.2.1 Linkage between research and development.....	21
2.2.2 Design research	22
2.2.3 Problems, guessing and whole-class discussions	24
Chapter 3	27
Theoretical framework.....	27
3.1 An epistemological approach.....	27
3.2 The anthropological theory of didactics (ATD) and didactic transposition theory.....	28
3.2.1 The anthropological theory of didactics	29

3.2.2 The didactic transposition theory	31
3.2.3 Study and Research Course as a tool of didactic engineering	33
3.3 The theory of didactical situations	34
3.1.1 Didactical and adidactical situations	34
3.3.2 The didactical contract	35
Chapter 4	37
Research questions and methodological considerations	37
4.1 Research questions	37
4.2 Study one: a theoretical analysis of the PSO approach	38
4.2.1 Japanese and Swedish curricula	38
4.2.2 The lesson plan example “sums of exterior-angles”	39
4.3 Study two: an empirical study of Swedish mathematics classrooms	39
4.3.1 The process	39
4.3.2 The study with a grade seven class	40
4.3.3 The study with a grade eight class	42
4.4 Selection of lessons and data collection	43
4.4.1 The theoretical studies	43
4.4.2 The classroom observations	43
4.4.3 Interviews with the students	44
4.4.4 Questionnaire to the students	45
4.4.5 Interview with the teacher	45
4.5 Method of analysis	45
4.5.1 Analytical tools	45
4.5.2 Transcripts	46
4.6 Reflection on the quality of the study	46
4.7 Ethical considerations	47
Chapter 5	49
Results from the theoretical study	49
5.1 Japanese and Swedish curricula in Arithmetic, Algebra and Geometry	49
5.1.1 The Guidelines for the “Course of Study”	49
5.1.2 The description of the chapters “Numbers and Algebraic Expressions” and “Geometrical Figures” in the Guidelines	51
5.1.3 The Swedish curriculum	55

5.2 The structure of PSO approach based lesson plans.....	58
5.2.1 The basic structure of PSO	58
5.2.2 The structure of the sections in the chapter “Parallels and congruence”	60
5.2.3 The mathematical organisation of the sequence of lesson plans in Geometrical figures	61
5.2.4 The flow of the lesson “Sums of exterior-angles”	62
Chapter 6	66
Designing lessons in Swedish mathematics classrooms	66
6.1 The lessons “Operations on negative numbers”	66
6.1.1 Observations in the grade eight classroom	69
6.1.2 Observations in the grade seven classroom	74
6.2 The lessons “Introduction to finding the general solution”	81
6.2.1 Observations in the grade eight classroom	83
6.2.2 Observations in the grade seven classroom	88
6.3 The results of the questionnaire to the students	92
6.3.1 Grade seven	92
6.3.2 Grade eight	93
6.3.3 Summary and reflections on the result from the questionnaires ...	94
6.4 The results of the interviews with the students	95
6.4.1 Students’ responses prior to the problem solving lessons	95
6.4.2 Students’ reflections on mathematics after the project	96
6.4.3 Summary and reflections on the interviews	100
6.5 The results of the interview with the teacher	100
6.5.1 Reflections on the teaching approach	100
6.5.2 Reflections on the students	101
6.5.3 Other reflections	102
Chapter 7	103
Discussion and conclusions	103
7.1 The didactic transposition and praxeologies of the PSO based lessons	103
7.1.1 The analysis of Japanese and Swedish curricula	104
7.1.2 The complexity of mathematical organisation of Souma’s lesson plans	105

7.2 Didactical praxeologies used in the lessons and student's mathematical contributions	106
7.2.1 The lessons about operations on negative numbers	106
7.2.2 The case of introducing a general solution.....	108
7.2.3 Encouraging students' mathematical contribution	110
7.3 Conclusions	115
7.4 Further research.....	117
References.....	119
Appendices	

Abstract

This licentiate thesis is concerned with applying the Japanese problem solving oriented (PSO) teaching approach to Swedish mathematics classrooms. The overall aim of my research project is to describe and investigate the viability of PSO as design tool for teaching mathematics. The PSO approach is a variation of a more general Japanese teaching paradigm referred to as “structured problem solving”. These teaching methods aim to stimulate the process of students’ mathematical thinking and have their focus on enhancing the students’ attitudes towards engaging in mathematical activities. The empirical data are collected using interviews, observations and video recordings over a period of nine months, following two Swedish lower secondary school classes. Chevallard’s anthropological framework is used to analyse which mathematical knowledge is exposed in the original Japanese lesson plans and in the lessons observed in the classrooms. In addition, Brousseau’s framework of learning mathematics is applied to analyse the perception of individual students and particular situations in the classroom.

The results show that the PSO based lesson plans induce a complex body of mathematical knowledge, where different areas of mathematics are linked. It is found that the discrepancy between the Japanese and Swedish curriculum cause some limitations for the adaptation of the lesson plans, especially in the area of Geometry. Four distinct aspects of the PSO approach supporting the teaching of mathematics are presented.

Chapter 1

Introduction

1.1 Background

1.1.1 Swedish students' knowledge of mathematics

Since 1964, the first time Sweden participated in an international survey of IEA (International Association for the Evaluation of Educational Achievement), the academic standard of Swedish students' knowledge of mathematics has gradually weakened. In the second study of the IEA in 1980, Swedish 13 year olds students were at the lowest level together with Swaziland, Nigeria and Luxemburg (Skolinspektionen, 2009). In TIMSS 2007 (Trends in International Mathematics and Science Study), Swedish students in grade 4 ended below the average for the OECD/EU (Skolverket, 2008; Skolinspektionen, 2009). This situation was perceived as so serious that the teaching of mathematics in schools became a state focus area. For these reasons, in 2008 the Swedish Schools Inspectorate implemented quality assessment of the lessons in mathematics in the elementary and lower secondary schools (Skolinspektionen, 2009), and in 2009 to 2010 in the upper secondary schools (Skolinspektionen, 2010). The focus of the assessment has been to investigate how the learning environment encourages students to develop the competencies specified in the syllabus for mathematics.

The investigation reports that the students are not trained to develop crucial skills such as problem solving, the ability to see mathematical connections, to reason and express themselves both orally and in writing, or to deal with mathematical procedures. A root cause is thought to be that the teaching is largely characterized by students working individually with their text book and that the teachers do not have enough time to help all students during classes. The discourse that takes place is often a one-way communication from the teacher and many students are quiet and passive during the lessons. In these classes, students do not feel secure enough to dare express their opinions and ideas (Skolinspektionen, 2009; 2010).

In the report *Pisa 2012 Result: Ready to learn. Students' engagement, drive and self-beliefs* (OECD, 2013) the teachers' strategies to foster student learning are distinguished in four categories: *use of cognitive activation strategies, teacher-*

directed instruction, student orientation, and use of formative assessment (ibid., p. 124).

Teachers' use of *cognitive activation strategies* are characterized in 8 situations: the teacher asks questions that make students reflect on the problem; the teacher gives problems that require students to think for an extended time; the teacher asks students to decide, on their own, procedures for solving complex problems; the teacher presents problems in different contexts so that students know whether they have understood the concepts; the teacher helps students to learn from mistakes they have made; the teacher asks students to explain how they solved a problem; the teacher presents problems that require students to apply what they have learned in new contexts; and the teacher gives problems that can be solved in different ways (ibid., p. 124). *Teacher-directed instruction* is constructed so that the teacher sets clear goals for students' learning. For instance, the teacher asks students to present their reasoning, asks questions to check if the students understood the contents of the lessons, and tells the students what they have to learn. The index of *teachers' student orientation* entails that the teacher gives different tasks to different students depending on their varied stage of learning abilities. The teacher gives group work projects to the class and asks students to plan the classroom activities together. The index of *teachers' use of formative assessment* means that the teacher gives the students feedback on their work, telling their strengths and weakness, and what is needed to do to be better in mathematics.

The report shows, for those different strategies and on average across OECD countries, that the students whose teachers often use a large variety of *cognitive activation strategies* have particularly greater levels of perseverance and openness to problem solving compared to those students whose teachers use *teacher-directed instruction, teachers' student orientation, or teachers' use of formative assessment*. It is also reported that the students who consider mathematics as their favourite study field, to a higher extent than other students, reports that their teachers use *cognitive activation strategies* more frequently (ibid., pp. 139-140). Those results hold for Sweden as well as the other OECD countries (Skolverket, 2013a). In Sweden, the index of *teachers' use of formative assessment* and the index of *teacher-directed instruction* are used in about the same extent as other OECD countries. The index of *cognitive activation strategies* are used less and the index of *teachers' student orientation* are used in a much greater extent than in the other OECD countries (ibid.). These results indicate that a teaching method with an emphasis on *cognitive activation strategies* could be beneficial for mathematics education in Sweden. One such method is presented in the next section and will be developed further for the purpose of this thesis.

1.1.2 “Structured problem solving” – a Japanese teaching method in mathematics

There is a need for concrete teaching projects that can be used to influence the way we organise the didactical work, so that the new ideas can address problems in mathematics education such as those outlined above. I agree with Sfard (2000) who states that one of the basic problems in mathematics education is to find ways to organise the classroom discourse so as to motivate the students to enthusiastically participate in the lessons and to make them active learners of mathematics, without losing focus on the mathematical content.

Due to my Japanese background, I have in particular paid attention to the research concerning Japanese teaching methods in mathematics. During recent decades, the development in Japan of teaching methods with the focus on problem solving and whole-class discussions (henceforth called “structured problem solving”; cf. Stigler & Hiebert, 1999, and Shimizu, 1999) is motivated by issues as those discussed above (cf. Stigler & Hiebert, 1999). In recent years, there has been an increased interest in structured problem solving and it has been discussed as a possible model to develop within the Swedish school system (Dagens Nyheter, 2008, 2009, 2010; Svenska Dagbladet, 2010; Skolvärlden, 2010). The Swedish National Agency for Education (Skolverket, 2013b) has produced teaching modules for teachers of different grades in a national project called “The boost for mathematics” (in Swedish, “Matematiklyftet”) of which the overall purpose is didactic training for teachers in service. In one of the modules, “Teaching mathematics with problem solving” for teachers of upper secondary school, the Japanese structured problem solving method has been applied.

Structured problem solving aims to have a whole classroom discussion on the various solution methods proposed by the students (Hiebert, Stigler & Manaster, 1999; Stigler & Hiebert, 1999). Shimizu (2003) describes the common framework of Japanese mathematics lessons with structured problem solving (*ibid.*, p. 206):

- Posing of a problem
- Students’ work on the problem, individually or in groups
- Whole-class discussions of various solutions
- Summing up of the lesson
- Exercises or extension, which are optional, depending on time available and students’ facility in solving the original problem

According to Shimizu, Japanese teachers consider this kind of lesson structure, with challenging problems and time to reflect on the solutions, as the one that best serves to give students learning opportunities. He explains that the following didactical terms (in Japanese, with English translations), describing the teachers’ key roles, are used by Japanese teachers on a daily basis (Shimizu, 1999, pp. 109-111):

- *Hatsumon*: to ask a key question that provokes students' thinking at a particular point in the lesson.
- *Kikan-shido*: teachers' instruction at students' desk. Scanning by the teacher of students' individual problem solving process
- *Neriage*: whole-class discussions. A metaphor for the process of polishing students' ideas and of developing an integrated mathematical idea through the whole-class discussions.
- *Matome*: summing up. The teacher reviews what students have discussed in the whole-class discussion and summarizes what they have learned during the lesson.

The existence of a common didactical terminology indicates that Japanese teachers have an institutionalised perception about the teacher's role in the classroom

In Japan, this basic lesson structure comes in several versions, e.g. "Open-ended approach" by Becker and Shimada (1977), "Open approach method" by Nohda (1983, 1991, 1995), "Lessons with problem solving" by Tesima (1985)¹ and "Problem solving oriented lesson structure" by Souma (1997). In my empirical study in this thesis, I applied Souma's "Problem solving oriented lesson structure" to design the units of lessons in Swedish mathematics classroom. My reasons to employ Souma's approach among the various alternatives was that, firstly, it is a teaching method where teachers emphasise the process of mathematical thinking and focus on how to enhance the students' attitude towards engaging in mathematical activities. The method shares many common aspects to those listed in *teachers' use of cognitive activation strategies* in OECD's categorisation of teaching strategies. A second reason was that there were already a lot of fine grained materials ready to apply in lessons. In the next section, I will describe the structure of Souma's method.

1.1.3 Souma's "problem solving oriented lesson structure"

Kazuhiko Souma is a professor of mathematics education in Asahikawa in Japan. He established "The problem solving oriented" lesson structure (Souma, 1997) (the author's translation of "*Mondaikaiketsu no jugyou*", in Japanese; shortened to PSO). As his source of inspiration, Souma refers (1995; 1997) to John Dewey's theory of reflective thinking (Dewey, 1933) and Polya's work on problem solving (Polya, 1957); in particular, Polya's emphasis on the importance of guessing and making conjectures.

Souma has written and edited a number of practical books, including textbooks, where he proposes lesson plans using the PSO approach and which also contain collections of problems, suitable to the approach, to work with. His book,

¹ Tesima focuses on mathematics lessons for elementary school pupils.

“*The problem solving oriented approach*” (Souma, 1997) has reached its 13th edition since 1997 and his task collection (Souma, 2000), is now the 9th edition since 2000, which is a quite unusual phenomenon within publishing of books on mathematics education in Japan. This fact suggests that teachers in service in Japan actively use the PSO approach. It has however received little attention from the academic community, perhaps because of its practical aspect and the lack of a clear theoretical base.

The didactical technique, proposed by the PSO, is to start up the lesson by presenting challenging problems that are carefully chosen so as to lead to new mathematical understanding.

Some points, where the PSO differs from the other Japanese problem solving approaches are that Souma is specialised to lower secondary school lessons and that he describes the method, with examples, in quite some detail. He also supplies appropriate problems for every content area from grade seven to grade nine, and later, in 2011, he published a problem collection for grades one to six. Souma synchronised even more detailed lesson plans (Kunimune & Souma, 2009a; 2009b; Souma, Kunimune & Kumakura, 2011) to mathematics textbooks which he published as one of the editors (Seki, Hiraoka & Yoshida, 2006; Seki, Yoshida & Souma, 2012). These lesson plans are organised so that they follow the textbooks order of content. In that way, teachers can start to apply the PSO method relatively easily in their everyday lessons.

With regard to the didactic techniques, the main difference between the PSO and the other Japanese problem solving based approaches is that the PSO has an emphasis on an initial phase of *guessing* and that the teacher is advised to construct problems that are both *open* and *closed*. According to the PSO, one should always let the students guess or, otherwise, publicly make some observations about the task before they start to work with the task. The criterions of “open-closedness” means that the student response, the answers, in the later stage after they make the guesses, should be fairly predictable, but, hopefully, still give rise to a variation of the methods. The insistence on posing “open-closed” tasks distinguishes it from the more “open-ended” approaches. This didactical technique will be described in more detail in Chapter 5.

In Souma’s books, the example lessons are usually presented in a context of a sequence of lessons, but the PSO does not come with an elaborate epistemological model for the long term didactic design; the focus of the PSO is on problems and generated problems that cover individual chapters in a textbook. Souma (1987) proposed to name the design of a sequence of lessons and associated tasks over a longer period of time as “total mathematics” (*Sogoo suugaku*, in Japanese). His idea being that *after* the students have covered a subject in the textbook, they get to start to work with complex open-ended problems, including several sub-problems where one may connect, say, geometric problems with algebra and functions.

Souma's approach fits well to address those problems that, according to the Swedish school inspection, are present in Swedish mathematics education; it is a teaching method where both the teacher and the students are supposed to be active, for a sizeable part of classroom time, in an interactive mathematical discourse. It also contains elements for forming a norm in the classroom, a didactical contract (Brousseau, 1997), which is a necessary condition for establishing such a discourse. On the other hand, it is perhaps not a radical break with the established practice in Swedish classrooms, since Souma also stresses the need of individual work based on textbooks (Souma, 1997).

1.2 Aim

The aim of the thesis is to investigate the viability of the PSO approach as a design tool for mathematics lessons within the Swedish school system, with purpose that students acquire both mathematical knowledge and a positive attitude for participating in the lessons. My intention is not to discuss the meaning of "problem solving" in a wider context. Previous research in the teaching and learning of mathematics through problem solving focuses on how it helps to develop students' mathematical thinking (cf., Schoenfeld, 1985, 1992, 2007; Kilpatrick, 1985; Silver, 1985; Stanic & Kilpatrick, 1988; Lester, 1996). What I am trying to achieve in this study is to use theoretical frameworks to give a coherent analysis of Souma's version of structured problem solving approach and to explicate its mathematical and didactical structure. I also intend to describe how I applied Souma's method to a Swedish classroom as a study and to explain in which ways we could use the method directly and in which ways we could not.

Several Japanese research articles that analyse the structured problem solving approach are rather normative and are often stating opinions that are based on the researchers'/educators' own experiences. The reason might be that the researchers in mathematics education in Japan often are authors of textbooks or are taking part in the development of the curriculum. Hence their considerations are frequently on the practice, rather than on a theoretical analysis (Miyakawa, 2009).

This thesis includes one theoretical and one empirical study. Firstly, using Chevallard's theoretical framework, the anthropological theory of didactics (ATD), I try to analyse and clarify the praxeologies² (Chevallard, 1992) of the PSO based lessons in order to ask questions on whether the approach is suitable for using as a tool of teaching design. In this context I also make a comparison of the Japanese and Swedish mathematics curricula for lower secondary school. Secondly, I describe how I implemented a sequence of the PSO based lessons in a Swedish mathematics classroom and investigated if this approach can encourage

² I will explain the term "praxeology" in Chevallard's meaning, in Chapter 3.

students' mathematical contribution. The related more specific research questions are formulated in section 4.1.

Chapter 2

Literature review

This chapter is divided into two sections: The first part contains an historical overview on how the problem solving oriented learning and teaching in mathematics developed in Japan. The second part relates the present research context relevant for this thesis.

2.1 The development of the structured problem solving

The PSO has a similar basic structure as Japanese structured problem solving based lessons – start up the lesson by presenting problems that can be solved by various methods and later have the whole class discuss the settlement options. It is described here how the teaching/learning of mathematics through problem solving developed in Japan and how the basic form of today's structured problem solving evolved. It helps us to understand the background for the establishing of a specific method such as the PSO. For the overview of change in Japans educational policy in Japan, I mostly refer to Eizo Nagasaki's book "*The power of mathematics: Beyond the mathematical thinking*" (Nagasaki, 2007b).

2.1.1 The “Green Book” — a new era for school mathematics

In modern Japan (at the end of the nineteenth century), mathematics education was based on both acquisition of practical skills and formal building of knowledge (Nagasaki, 2007b). When the educational reform, which focused on children's spontaneous learning and initiative and was initiated by John Dewey and Edward Lee Thorndike was introduced to Japan during the 1910's to 1920's (Nagasaki, 2011), several educators began to criticise the character of mathematics tasks presented in the textbooks for elementary school (Matsumiya, 2007). They meant that the tasks lacked enough links to pupils' everyday-life. For example, the tasks in the textbook “The Black Book”, so called because of the colour of the cover, for grade four look like (ibid., p. 9);

- If 36 people eat 50 persimmons, how many persimmons can one person eat?

- There is a cube, the length of one side is 1,7 m. What is the volume of this cube?

And a task for grade five in 1933 (ibid., p. 9);

- You put a pole into the water. At first, you put $\frac{2}{3}$ of the pole, and then you put $\frac{5}{8}$ of the rest of the pole into the water. Then you see that 0,6m of the pole is above the water. How long is the whole pole?

The criticism was for example, “It is unnatural to calculate an average number of the fruits in decimals”, “Where will our pupils find a big cube such as this?” and about the pole problem, “Lack of reality” (ibid., p. 9).

Reflecting on such criticisms, the first government-approved textbook in mathematics, the “Green Book” for the first grade, was published in 1936 with Naomichi Shioya as a main editor. Distinct from the Black Book, the Green Book was all colours with lots of illustrations, and was constructed so that pupils learn different concepts of mathematics and solving methods through pupils’ everyday-life related tasks. Shioya describes the focus of the Green Book as that of developing pupils’ understanding of mathematical concepts and helping them to interpret their everyday-life mathematically (Shioya, 1936, in Ministry of Education, 2007). The year after, in 1937, “the Green Book” for the second grade was published and the year after for grade three, and so on up to sixth grade. Takagi (1980) has categorised different types of tasks in the Green Book. Here is a short summary of Takagi’s description.

1. Imaginational tasks: reading a story or looking at pictures and letting the pupils guess answers. For example, showing a picture of rabbits with rice cakes asking “How many rice cakes did the rabbits make?” (A task for grade one).
2. Statistical tasks: focusing on the change of phenomena. For example, showing the data of some children’s length and letting the pupils compare the data of their own and also the data between last year and this year (A task for grade four).
3. Open-ended problems: depending on the ability of the pupils; showing a figure of a combination of triangles (see Figure 2.1) and asking “Which kind of quadrangles can you see, and how many?” (A task for grade four).

Figure 2.1: Illustration for the task; “Which kind of quadrangles can you see and how many?” (Ministry of Education, 2007, p. 33)

4. Text problems with arithmetic operations, which may arouse curiosity from pupils even without strong link to pupils’ everyday-life.

5. Limits problems: Giving an idea of concept of infinity and limits, in both arithmetical and geometrical form. There are several tasks, which use infinite geometric series. For example, “In which order are the triangles arranged in figure a? If you continue adding the smaller triangles unlimited times, what is going to happen with the area of a? Reason this problem with figure b” (A task for grade six. See Figure 2.2a and b).

Figure 2.2a and b: Illustration for the task; “In which order are the triangles arranged in this figure?” (Ministry of Education, 2007, p. 77).

6. Interesting problems for pupils: problems, which are fun to solve such as a magic square. For example, looking at the Figure 2.3 and asking “Which way do you take to get out the labyrinth?” (A task for grade three).

Figure 2.3: Illustration of the task “Which way do you take to get out the labyrinth?” (Ministry of Education, 2007, p. 63).

According to Matsumiya (2007), it was the Green Book, which gave the first opportunity to Japanese teachers to consider learning/teaching mathematics from the pupils’ perspective and to try to develop pupils’ ability to use mathematics independently. The Green Book was possibly one of the first factors that influenced the development of today’s Japanese structured problem solving.

2.1.2 Raising students' "mathematical way of thinking"

After the Second World War, the Course of Study of mathematics for lower secondary school in Japan emphasised the "social need" (Nagasaki, 2007a; Isoda, 2010). For instance, the contents of a mathematics textbook for lower secondary school "*Mathematics for everyday (Nichijo no Suugaku, in Japanese)*" from 1950, consists of everyday-life related chapters such as "*our school*", "*our food*" and "*our dwelling*" (Souma, 1997, p. 12). A task from "*our food*", for example, encourages students to examine the ingredients of the food Japanese people eat and its nutrients; *how much rice does one ordinary person eat? How much of the energy absorption is from one portion of rice?* To solve this kind of problems, the textbook suggests using percentages and diagrams to display the factors and it shows how to calculate energy absorption using the four basic arithmetic operations. "Here, mathematics is located as a tool to solve students' everyday-life related problem" (Souma, 1997, p. 13).

Solving these everyday-life related problems as a goal of mathematics education transforms into solving "text problems" (Souma, 1997, Nagasaki, 2011). In "a tentative plan for the curriculum in mathematics for elementary school" (Ministry of Education, 1951 in Nagasaki, 2011), problem solving is described as an effective method to foster students' ability of logical thinking and explains a process of solving text problems as: (1) Leading the problem; (2) Understanding what is asked in the problem; (3) Clarifying what facts are given in the a problem; (4) Considering what is needed to be clarified; (5) Considering which kind of operations are needed; (6) Assuming a conclusion; (7) Calculating to obtain a conclusion; and (8) Evaluating the conclusion (ibid., p. 35). This description seems to be influenced by Georg Polya's four steps for solving mathematical problems in "How to solve it" (1957, first edition published in 1945); however, the first translation of the book came to Japan in 1954 and this tentative plan for the syllabus was published in 1951.

This goal of mathematics education changes during the 1950's to "understanding the mathematical concepts" (Nagasaki, 2007a). The goal of "raising students' *mathematical way of thinking*" appears as a goal of mathematics education for Japanese students for the first time in the Course of Study for upper secondary school in 1955 and for elementary school in 1958 (Nagasaki, 2007b). According to Nagasaki (2007a), opinions about the interpretation of the notion "mathematical way of thinking" differed between mathematicians and researchers in mathematics education. Mathematicians interpreted the notion as something that one acquires through mathematical "activities" – observing, discussing solving problems, etc. and therefore they regard that the concept is not possible to be defined clearly. Researchers in mathematics education on the other hand, interpreted the notion as logical, rational and abstract way of thinking, which is raised by learning mathematics.

Yasuo Akizuki – one of the prominent mathematicians in Japan at that time – discusses mathematical way of thinking as manner of problem solving (Akizuki,

1966). He explains that the first step of a mathematical way of thinking is that of categorising and arranging; counting things and putting them in a system. Then, the next step is quantification; comparing all possible phenomenons, which are expressed in quantities. The third step is representation with symbols, generalising and formalisation to make a rule. Akizuki's view of a mathematical way of thinking is that it comprises these mental activities. Akizuki uses a metaphor to explain his statement; "The person who invented the abacus certainly had a mathematical way of thinking. But masters of abacus may not always have a mathematical way of thinking" (ibid., p. 8).

Kenzo Nakajima, who introduced the notion of "mathematical way of thinking" to the Course of Study for elementary school in the 1950's, summarised his conception of it in his book, *"Mathematical education and mathematical way of thinking"* (Nakajima, 1981). There, he describes raising students "mathematical way of thinking" as something that *"trains students so that they independently can generate creative activities (without teacher's help), which are mathematical"* (p. 82). "Mathematical" activities he means here is to explore the problems, to acquire methods for making some hypotheses, finding ideas which lead to solving problems, generalisation/extension of the mathematical ideas and doing evaluation of the ideas/solutions, etc. The expression *"independently can generate creative activities"* reminds us of Brousseau's notion of "adidactical situation" (Brousseau, 1997), which I will describe in Chapter 3.

Nakajima's statement was criticised by the Association of Mathematical Instruction (AMI, *Suugaku kyouiku kyougikai* in Japanese) which was established 1951 by a group of teachers, researchers, parents and students. AMI was critical to the governments' education policy at that time (Nagatsuma, 1966, in Nagasaki, 2007b). AMI's criticism was towards Nakajima's stance of "speaking of students' attitudes and way of thinking" in the Course of Study, instead of "speaking of contents of mathematics". AMI considers mathematics as a science in itself and that it implies a risk to be imposed on the policy makers' convenience, if the Course of Study illuminates not only what students should learn, but also how students should think. AMI also took issue with Nakajima's claim that one should expand one mathematical concept to another (cf. addition to multiplication). Furthermore, one opposed to Nakajima's statement that what is important, about the "mathematical way of thinking", is "how one thinks" rather than "how one calculates". AMI held this statement as too naive since, according to them, there is no "mathematical way of thinking" without taking account of correct calculation (ibid.).

The argument regarding the "mathematical way of thinking" continues into the 1970's. The mathematics educator Heichi Kikuchi argues, in his book *"Coaching students to develop their mathematical way of thinking"* (1969, in Nagasaki, 2007b), that the "mathematical way of thinking" is, "processes in which one finds mathematical facts through understanding of mathematical concepts, constructing some mathematical problems and solving them. Secondly, it is

processes which one organises mathematical facts logically and becomes to be aware of what kinds of solving methods are mathematical” (ibid., p. 175).

The Japan Society of Mathematical Education (JSME), which was established in 1919 and consists of groups of teachers, teacher educators and researchers, published a guideline for “How to instruct in mathematical thinking (for students/pupils)” for teachers of upper secondary school in 1969, for elementary school in 1970 and for lower secondary school in 1971 (ibid.). According to Nagasaki, the guidelines express the “mathematical way of thinking” as distinct from “mathematical thinking” (ibid.). In “mathematical thinking”, the focus is on acquiring the “methods” to handle mathematical problems, but in “mathematical way of thinking” the focus is on “attitudes towards mathematics”.

2.1.3 Development of problem solving centred teaching

Shimamori (1962, 1965 in Iida, 2010) studied students’ reasoning processes in problem solving for text problems. He emphasizes the importance of using problem solving with a broad approach and states that the students’ ability in solving problems must be trained in every chapter in a mathematics textbook, not only in a chapter of problem solving in everyday-life based text-problem (1965 in Iida 2010).

Akizuki writes in his book “*Mathematical Thinking*” (1968, in Nagasaki, 2007b), about his interpretation of the process of problem solving. “Provide *good problems* for students to raise their ability of reasoning. Especially, if we want to advance their skills for modelling, the best way is to give them *well thought out problems*” (ibid., p. 175).

Kikuchi (1969, in Nagasaki, 2011) tries to describe the process of problem solving: (1) Aim for sub-problems (to solve the main problem); (2) Observe (the sub-problem); (3) Classify the factors; (4) Make the hypothesis; (5) Examine the methods; (6) Proof (the methods); and (7) Develop (even better methods through evaluation). This process Kikuchi describes might have been influenced by Polya. Aoyanagi introduced Polya’s four steps for mathematical problem solving in his book “A complete book of modern mathematics education” (1971, in Nagasaki, 2011).

2.1.4 Whole-class discussions

In his book, Nagasaki (2011) stresses the importance of students’ open and free discussions in the classroom. The importance of students’ discussions will become one essential component in the Japanese didactics based on structured mathematical problem solving. There is, however, a lack of sources illuminating how the method of whole-class discussions developed within the structured problem solving in Japan. For instance, I could not find any sources that describe how the term “neriage” (whole-class discussions) was formed and used over time.

Inagaki, Hatano and Morita (1998) discuss the role of whole-class discussions in Japanese educational practice in mathematics from the viewpoint of a

“community of learners” (Brown, 1994) and a “caring community” (Lewis, 1995). They state that Japanese students have a tendency to share their classmates’ reasoning and to actively engage in learning from one another and that they are trained to listen to others carefully.

Inagaki et al. (1998) studied the “*Hypothesis-experiment-instruction method*”³ proposed by Itakura (1967 in Inagaki, et al., 1998) to illustrate the positive effect of having whole-class discussions in the classroom. The method is originally aimed at science teaching and was developed during the 1960’s in Japan. A lesson begins with the teacher showing a problem that has several alternative answers from which the students can choose. For instance, “If we close a certain electric circuit with a paper, coin, or magnet, will the circuit allow the miniature bulb to glow?” The problem is formulated in a so called “lesson text (*jigyo-sho* in Japanese)”, which is a combination of a lesson plan, a text book and a notebook. The teacher provides such lesson texts for the students at every lesson and students are not allowed to see the lesson text in advance. These alternatives like “it does”, “it does not”, “it does, but not strong enough to light the bulb” etc., are the “hypotheses” in Itakura’s meaning. When the students have decided upon the alternatives, the teacher writes on the blackboard how many have guessed each one of the alternatives. Then every group of different alternatives present the reasons for their choice. After that, the groups are supposed to debate in the class and to formulate questions to the other groups and corresponding answers to the questions. During the debate, students may change their choice of alternatives. Finally, the class implement an experiment; students observe the outcome and confirm the right answer. At the end of the lesson, they write a lab report concerning the result (Itakura, 1977).

To some extent, Souma’s problem solving oriented approach might have been inspired from the Hypothesis-experiment-instruction method. The different points are that Itakura’s method *always* gives students alternatives in lesson text, while Souma’s PSO varies the way of letting the students make a guess. Also, the PSO does not have “debate” as a fixed part of the process. The form that the whole-class discussion takes varies in the PSO and depends on the problems and classes. Itakura’s method is still used in Japan. The Hypothesis-Verification-Through-Experimentation Learning System Research-Group has around 1300 members (Hypothesis-Verification-Through Experimentation Learning System Research-Group, 2010). But for most of today’s younger generation of Japanese science teachers, this method is rather unknown (Ueshima & Hiroki, 2009). According to Ueshima and Hiroki, possible reasons might be that Japanese teachers consider that the lesson plans are not suitable for younger than secondary school students and that they do not match the current Japanese Course of Study.

³ Translated by Inagaki, et. al. “Kasetsu jikken jigyo” in Japanese; often “Hypothesis-Verification-Through Experimentation Learning System” is used as English translation.

2.1.5 Development of the open-ended approach

Hino (2007) gives an overview on how the approach with problem solving has influenced the Japanese mathematics education. The first recommendation of “An Agenda for Action” by the National Council of Teachers of Mathematics (NCTM) in 1980 stated that “Problem solving should be the focus of school mathematics in the 1980’s,” and it had, according to Hino, a strong impact on Japanese educators. Since then researchers in mathematics education have given a lot of attention to mathematical problem solving. One of the representative methods of problem solving is that of teaching with “*open-ended problems*”. An open-ended problem is a conditional or incomplete problem, which leads to multiple correct answers and the search for an answer develops different methods. The teaching method called the “*Open-ended approach*”, based on open-ended problems, is proposed by Shigeru Shimada in 1977 in his book, “*The open-ended approach for mathematics teaching*” and conducted in Japan with other researchers⁴.

For instance, Nohda developed the open ended – approach to “*Open-approach*” (Nohda, 1983, 1995), where the focus was on to attract students’ interest in participating in mathematical activities and at the same time to foster their mathematical thinking and to motivate their joy to learn new knowledge. Between 1971 and 1976, Japanese researchers approved many developmental research projects regarding methods for evaluating higher-order-thinking skills (Becker and Shimada, 1997) in mathematics education. Higher-order-thinking comprises complex learning processes, which need more than a mechanical response to the situation. For instance, when students face a problem based on a realistic situation, he or she can formulate it mathematically (mathematical modelling) and critically analyse and evaluate its results and situation. According to Sawada (1997, p. 23), the process of classroom activities, using the open-ended approach, are structured to help to develop students’ skills in;

- mathematizing situations appropriately;
- finding mathematical rules or relations by making use of their previous knowledge and skills;
- solving the problem;
- checking the results;

while;

- seeing other students’ discoveries or methods;
- comparing and examining the different ideas;
- modifying and further developing their own ideas accordingly.

⁴ In Japanese, Shimada’s book had the title “Sansuu, suugakuka no open-end approach”; this book was translated into English by Becker and Shimada and was published in 1997 by NCTM.

The pattern of such classroom activities with interaction with other students explaining their reasoning and comparing their different solutions are going to be carried on to structured problem solving based lessons.

In addition, Sawada lists advantages and disadvantages with the open-ended approach. He considers advantages to be: (1) students become active participants and express their ideas frequently; (2) students get opportunities for extensive use of their mathematical knowledge and skills; (3) the method can give even low-achieving students a chance to respond in their own ways; (4) students become motivated for constructing proofs; and (5) it brings pleasure to students when they discover on their own and when they receive approval from their classmates.

Disadvantages Sawada mentions are: (1) it is difficult to prepare meaningful mathematical problem situations, and (2) it is difficult to communicate to every student the kind of mathematical activities the class demands. For the disadvantage (3), Sawada expresses that “some students with higher ability may experience anxiety about their answers”, and (4), “students may feel that their learning is unsatisfactory because of their difficulty in clear summarising” (Sawada, 1997, p. 24). These disadvantages Sawada perceives might be caused by the character of the open-ended approach that the answer is not unique, so it leads that students to feel uncertain to grasp the goal of the lesson. As Shimizu (2003), the advantages and disadvantages that Sawada considers, are commonly mentioned with respect to structured problem solving,

Another disadvantage I could add is that the approach is difficult to apply to every lesson. Nohda notes that “We do the teaching with the open-approach once a month as a rule” (Nohda, 1991, p. 34). Bosch, Gascón and Rodríguez (2007) have discussed the risk with open-ended activities, which are often introduced at school without any connection to a specific content or discipline. They state that this type of didactic technology suffers the risk of ending up in the construction of mainly localised mathematical organisations, since this is what students are trained to study.

Sawada continues further to classify types of problems, which apply to open – end approaches (Sawada, 1997, p 27):

Type 1. Finding relations. Students are asked to find some mathematical rules or relations.

Type 2. Classifying. Students are asked to classify according to different characteristics, which may lead to formulate some mathematical concepts.

Type 3. Measuring. Students are asked to assign a numerical measure to a certain phenomenon. Problems of this kind involve several facets of mathematical thinking. Students are expected to apply mathematical knowledge and skills they have previously learned in order to solve the problems.

These types of problems in open–end approaches have a similarity to types of problems of the PSO approach. The differences between them are that in the PSO,

the answer is not open but unique and the didactical technique, letting the student guesses the answer first. I will describe this issue in more detail in Chapter 5.

2.1.6 The structured problem solving in elementary school

Katsuro Tesima, professor of mathematics education in Tokyo, is one of the frontiers who have developed structured problem solving in elementary school. In his book “*The problem solving approach – the subject of elementary school mathematics*” (1985), Tesima describes his teaching methods to motivate the pupils to be active participants. Through presenting examples of pupils’ reactions from his own lessons in arithmetic and geometry, he describes how to foster his pupils to develop mathematical way of thinking.

The basic style of Tesima’s lessons has also the standard four phases, *hatsumon*, *kikan-shido*, *neriage* and *matome*. Tesima uses both open-ended and open-closed problems to start lessons and his focus is on letting pupils start reasoning, making sense of a mathematical concept (for instance, “What is a circle?” “Why do you think the sum of inner angles of triangles is 180° ?”) and finding some patterns and algorithms by their own (“How can you calculate $73 \cdot 77$ without using a calculator or algorithms we have learned before?”). Souma indicates by personal interview (2013-0629) that he got influence and inspiration for developing his own teaching method through observing Tesima’s lessons during the 1970’s. Tesima’s pupils are very much encouraged to be active learners through getting opportunities to think and discuss and finding out different solving methods by themselves and get involved to participate in the lessons.

2.1.7 The structured problem solving and lesson study

Hino (2007) discusses how the institution of “*lesson study*” is a driving force for the improvement of lessons with problem solving in Japan. Lesson study (*jigyo-kenkyu* in Japanese) is a long term teaching improvement process, which is implemented usually by groups of teachers. Teams of teachers collaboratively plan and study their instructions of the lessons aiming to determine how students learn best (cf. Stigler & Hiebert, 1999; Fernandez & Yoshida, 2004; Isoda, Stephens, Ohara & Miyakawa, 2007). The basic form of lesson study within one school (*konai kenshu* in Japanese) is the following (Baba, 2007):

1. Preparation – designing the lesson–planning the task, selecting appropriate materials to the class, tying all information together into a lesson plan collaboratively with the team.
2. Study/research lesson (*kenkyu jigyo* in Japanese) – the actual lesson performed by one of the teachers in the team. Other members of the team (or all the colleagues in the school) observe the class together. Sometimes they invite university instructors and supervisors from the board of education.
3. Review session – discussion and evaluation of the lesson after the study lesson

The scale of lesson study varies. Every elementary school and most of lower secondary schools have own lesson study groups within the school. It can be varied year by year which subject will be studied, but mathematics is one subject that is more emphasized than most other subjects. Some schools have study/research lessons open for teachers of other schools in the region or even for teachers from the whole country (the class may be moved to a gym-hall and may have more than several hundreds of observers in some study lessons). In such cases, the review session always has a chairman, a secretary and mostly, a university instructor who gives comments to the performed teacher. Through this kind of open study/research lessons, teachers can get ideas for different teaching methods.

Almost all lesson-studies in mathematics are conducted with a base in structured problem solving (Hino, 2007) and are quite often realized using the “open approach method” (Miyakawa & Winsløw, 2009, p. 200). In their study, Miyakawa and Winsløw compare Japanese lesson studies and Brousseau’s Theory of Didactical Situations (TDS) as two different didactical designs (ibid.). They conclude that the lesson study is a systematic engineering framework for development of teaching practice: “The collaboration between researchers and teachers in such a format – which may be transposed to other cultural settings, albeit with significant adaptations – could lead to new and more practice-oriented forms of “didactical engineering”, if retaining at least parts of the theoretical basis (TDS)” (p. 217). They consider that analysing a design of lesson study, which is structured with problem solving, supported by a fundamental theoretical framework would be worth studying.

2.1.8 Problems for teachers using the structured problem solving

As the final remarks of this section, I describe Japanese teachers’ concerns about a difficulty in applying structured problem solving. Hino (2007) designates that many of the teachers still implement such lectures, where they explain and demonstrate directly every detail of the solution methods. There, it does not exist any space for students to think through and discuss the problems. The reason for that might be that applying structured problem solving demands more preparation time, or some teachers simply do not see a value using the method. Even when one applies structured problem solving, it does not give a guarantee that the lesson provides a substantial result for the students, because managing lessons with structured problem solving requires a skill by the teachers. Some Japanese educators warn that using structured problem solving “turns into a formality” (Tanaka, 2011). It means that lessons follow the pedagogical pattern of *hatsumon*, *kikan-shido*, *neriage* and *matome* but it does not employ active whole-class discussions during the lesson. For instance, the discussion is made by between the teacher and only one or two students so the rest of the class do not express anything; the discussion is focused on mostly different solution methods and do not develop into a more mathematical discussion. Some students have nothing to do during *kikan-shido* (teachers’ instruction at students’ desk/monitoring students’

different strategies) moment, because they do not have any idea for how to solve the problem, or some students have already solved the problem and just wait for the teacher to move to the next neriage (whole-class discussions) stage, and so on.

2.2 Research on classroom studies

This research project set out to investigate the viability of the PSO approach as a design for teaching in Swedish mathematics classrooms. There are some essential aspects of this intervention project to analyse: Firstly, does the proposed sequence of teaching acts during lessons achieve both the mathematical and didactical goals? Secondly, how does the practical didactical strategies and tactics proposed by PSO, that is guidelines on how the teacher should behave in the classroom, effect the outcome?

The PSO approach is not a learning theory; it is a tool for lesson design. I have not used any additional, more theory-oriented framework, for the design of the lesson sequences. The PSO does not include, as an integral part, any method for iterative evaluation and improvement of upcoming lessons. For those reasons, the project is not a design research in the proper meaning of the term (see below). However, in order to understand the context of this design study in relation to design research in general, I will present previous research and classroom studies that deal with intervention study. But first, I will describe theoretical studies concerning the relationship between this type of educational research and teaching practice, since it is a highly relevant issue.

2.2.1 Linkage between research and development

Wittmann (1995) discusses how to develop mathematics education as a “design science”, where research and development have specific connections to practice. He stresses the importance of finding ways to design teaching units and related empirical research projects. Artigue and Perrin-Glorian (1991) suggests the use of *didactical engineering* as a “theory related” controlled intervention that organises the didactical process. They propose the following definition of the term *didactical engineering* (ibid., p. 13):

The term was introduced to label a form of didactic work: work comparable to that of an engineer who, in order to carry out a particular project, uses scientific knowledge, agrees to submit to a scientific type of control, but who at the same time finds himself obligated to work on objects which are much more complex than the refined objects of science and which he therefore must attack in a practical way and with all the means at his disposal, problems which science does not want to or cannot take care of.

According to Artigue and Perrin-Glorian, this notion of didactical engineering brings up questions about “the organisation of a rational relationship between

research and action on the teaching system” and “the role which should be given to ‘didactic productions’ in classrooms within the methodology of didactic research” (ibid., p. 13). They address the need for a genuine body of didactic knowledge that can supply solutions for problems within the actual education system.

Lesh and Sriraman (2005) point to the lack of research studies that aim to develop tools for building an “infrastructure” (ibid., p. 494) which eventually solve complex problems in mathematics education. They state that many research articles and doctoral dissertations have focused on “quick fix” interventions which are very specific and therefore impossible to modify and adapt in a continually changing environment. Lesh and Sriraman explain the importance of having the different elements of innovation working together as a whole; “...when developing and assessing curriculum innovations, it is not enough to demonstrate THAT something works; it also is important to explain WHY and HOW it works, and to focus on interactions among participants and other parts of the systems” (ibid., p. 494).

Amettler, Leach & Scott (2007) discuss the difficulty of using educational research and scholarship as a guide for teaching practices. They mean that statements of general guidance in pedagogy are usually “*large grain*” which do not expose to educators the details of the proposed didactical organisation proposed. They state that there is a need for proposals of more “*fine grained*” didactical research – providing educators with instructive help on how to develop specific strategies that aim for concrete pedagogical goals.

Silver and Herbst (2007) investigated the way theory is used in mathematics education research from a practice-oriented perspective. They describe criticism from practicing teachers in schools in the U.S., who perceive mathematics education research as something which is “too theoretical”; what is true in theory does not necessarily have useful practical implications. On the other hand, as shown by Silver and Herbst, some research studies receive criticism from the academic communities of being “too applied”: too close to the teaching practice and lacking a sufficient theoretical basis. Silver and Herbst hold that there is a legitimate tension between theory and practice.

2.2.2 Design research

Van den Akker, Gravemeijer, McKenney and Nieveen (2006) see the term design research as “a common label for a ‘family’ of related research approaches with internal variations in aims and characteristics” (ibid., p. 4). They also describe different ways to label such studies as *Design* studies; *Design* experiments, *Development/Developmental* research, *Formative* research; *Formative* evaluation and *Engineering* research. The design research offers iterative and normative approaches in a practice-based context (Edelson, 2002).

Educational design research is an organised study of educational interventions which aim, at the same time, to develop the theoretical insights and the practical

situations (McKenney & Reeves, 2012). Plomp (2009, p. 13) defines the educational design research as “the systematic study of designing, developing and evaluating educational interventions (such as programs, teaching-learning strategies and materials, products and systems) as solutions for complex problems in educational practice, which also aims at advancing our knowledge about the characteristics of these interventions and the processes of designing and developing them”.

The *Design Experiment* was introduced by Brown (1992) and Collins (1992). It is developed to implement formative research with an iterative cycle that test and improve the theory-based designs settled by prior research cycles (Collins, Joseph & Bielaczyc, 2004). Cobb, Confrey, diSessa, Lehrer and Schauble (2003) identify five “crosscutting features” of the various Design Experiment as: (1) The purpose of design experimentation is to develop a class of theories on intended subject/object in the relevant system to support (e.g. a learning process in a community of classroom, a community of teachers); (2) Its interventionist methodology is based on prior research; (3) Its prospective and reflective faces; (4) Its iterative design; and (5) Its pragmatic roots. Cobb et al. (2003) summarise the primary goal of the Design Experiment as a way “to improve the initial design by testing and revising conjectures as informed by ongoing analysis of both the students’ reasoning and the learning environment.” (ibid., p. 11). They discuss how the scale of the study team (often cooperated with a teacher) varies depends on the type and purpose of the experiment.

Ruthven, Laborde, Leach and Tiberghien (2009) discuss the design of teaching sequences where the designer applies theoretical perceptions on the epistemological and cognitive dimensions. They state that a key aim of didactical design is “to devise teaching sequences that not only are suitable for widespread use in ordinary classroom circumstances but are sufficiently comprehensive and robust to achieve their intended effects in a reliable way” (ibid., p. 329). Ruthven et al. compare three particular didactical frameworks and discuss how one can develop “overarching” ideas of the relationship between the grand theories (Cobb et al. 2003), the intermediate frameworks and the design tools. They state that these frameworks serve to establish a connection between epistemology, teaching and learning through its design of “domain-specific teaching sequences” (Ruthven et al., 2009, p. 334). Thus it orientates explicitly how a specific content in mathematics/science can be taught and learned in the optimal way. Ruthven et al. note that there are some essential difficulties, caused by several environmental factors, to design teaching sequences in this way. Creating mathematical activities using the design tools (e.g. didactical situations) and to locate those activities into a coherent sequence are both important steps.

2.2.3 Problems, guessing and whole-class discussions

George Polya's contribution

Polya (1950; 1968; 1978) encourages students and teachers of mathematics to “Guess and test” and to “First guess, then prove”. He explains the art of guessing; “Our first guess may fall wide off the mark, but we try it and, according to the degree of success, we modify it more or less. Eventually, after several trials and several modifications, pushed by observations and led by analogy, we may arrive at a more satisfactory guess” (Polya, 1968, p. 158). In his well-known book “*How to Solve It*” (1957), Polya states that a “guess of a certain kind” (ibid., p. 99) should be taken seriously since such guesses, based on consideration on the features of the problems, usually contain “a fragment of the truth” (ibid., p. 99) and it may lead to the whole truth, if one examine it in the proper way. He holds what is really bad is “to have no idea at all” (ibid., p. 99).

In the filmed lecture from 1965 (Polya & MAA, 1965), Polya tries to show us how to teach the art of guessing. He declares his “slogans” regarding the attitudes on teaching mathematics: “Teaching is to give opportunity to the students to discover and to think by themselves”, “Let students guess first, then, prove” and “Finished mathematics consists of proofs. But mathematics in the *making* consists of guesses”. The lesson exemplifies with a problem: *into how many parts is space divided by five planes?* Polya revises the problem to an instance which is more manageable for the students and starts with *one* plane. He lets the students pose guesses, observe the patterns, generate the rules and test the rules. In doing so, Polya demonstrate how the guess leads to the solution of a mathematical problem. In this film, Polya describes those guesses which are worth making as “rational/educational guesses”. Guessing without any mathematical deliberations on the problem he refers to as “wild guessing”. Leinhardt and Schwarz’s study (1997) on Polya’s lecture characterises two important aspects of the lesson. First, the rich variation of representations and models, which stimulates the students’ intuitions to see what is problematic in the five planes problem. Secondly, the students’ high degree of commitment in the lesson, which Leinhardt and Schwarz explain by “the way Polya links each of the episodes and coordinates between symbolic and representational presentations” (ibid., p. 431).

Polya introduces “modern heuristic” (Polya, 1957, p. 129) as a tool for understanding the problem solving process as mental operations guided by both logic and psychology. He seeks to explain how activities in the process of problem solving improve the students’ learning of mathematics. He states that “experience in solving problems and experience in watching other people solve problems must be the basis on which a heuristic is built” (ibid., p. 130). Schoenfeld (1992) discusses Polya’s conception of mathematics as an activity and his tenet that, in mathematics, the epistemology and the pedagogy are deeply interlaced. Thus Polya holds that students’ experience of mathematics must be “consistent with the way mathematics is *done*” (ibid., p. 339).

Whole-class discussions

Conducting a whole-class discussion is considered a crucial part of teaching mathematics through problem solving. The students tackle rich problems, make conjectures, present their diverse ideas and deepen their understanding of mathematical ideas (e.g. Lampert, 1990; Stigler & Hiebert, 1999; Lampert, 2001; Stein, Engle, Smith & Hughes, 2008; Jackson & Cobb, 2010). Yackel and Cobb (1996) state that learning opportunities arise when children “attempt to make sense of explanations given by others, to compare other’s solutions with their own and to make judgments about similarities and differences” (p. 446). They also remark that whole-class discussions are demanding for teachers to manage. It is necessary to align and make sense of the pupils’ variety of disparate solution methods.

Franke, Kazemi, and Battey (2007) give an overview of different research contributions concerning the teacher’s role of managing classroom discourse, especially in mathematical conversations. They advocate that *revoicing* – reuttering of students’ remark by the teacher or by other students – can be a way of orchestrating the mathematical conversation. Frank et al. hold that several case studies of classroom practice provide support for the claim that revoicing “structures students participation as students work together and connects students to the mathematics” (ibid., p. 234).

In the study on collective learning by Cobb, Stephan, McClain and Gravemeijer (2001), it is mentioned that *circulating* of the teacher(s) (and research staffs) in the classroom during individual/group work in order to monitor students’ different interpretations before joining the whole-class discussion, is a good pedagogical technique. “In doing so, they routinely focus on the qualitative differences in students’ reasoning in order to develop conjectures about mathematically significant issues that may, with the teacher’s proactive guidance, emerge as topics of conversation” (ibid., p. 117).

Stein et al. (2008) propose five practice models for orchestrating productive whole-class discussions. Their intention is to support teachers who are new to this form of lesson, so that even those teachers can *plan in advance* the discussion facilitation. Stein et al. explains that they are “purposely de-emphasizing the improvisational aspects of discussion facilitation” (ibid., p. 321). Their five practice models are: (1) anticipating likely student responses; (2) monitoring students’ responses during the exploring phase; (3) selecting particular students to present their responses during the whole-class discussion; (4) sequencing the students’ responses; and (5) making connections between different students’ “key ideas”. The practices Stein et al. suggest are clearly connected to the structured problem solving approach.

How to establish “norms” in the classroom community in the context of whole-class discussions is a frequently discussed subject (e.g. Lampert, 1990; Wood, 1993; Yackel & Cobb, 1996; Lampert, 2001; Franke, et al., 2007). Social norms (Yackel, Cobb & Wood, 1991) are characteristics of the classroom

community and jointly constituted by the participants. Wood (1998) explains that “social norms underline the patterns and routines that become established in the classroom and that enable the students and teacher to interact harmoniously” (p. 170). Those *patterns* are tacit rules that navigate the participants’ acts in the classroom. Cobb et al. (2001) illustrate how classroom social norms relate to students’ individual reasoning. They further discuss the role of the sociomathematical norms (Yackel & Cobb, 1996) on individual students’ intellectual autonomy (Cobb et al., 2001). What distinguishes sociomathematical norms from general social norms in the classroom is that they are subject-specific and are concerned with the mathematical aspects of activities (Yackel & Cobb, 1996).

Chapter 3

Theoretical framework

In this chapter, I will describe two French theoretical frameworks of didactics that form the grounds for my analysis in this thesis. I begin this section with a description of basic theoretical assumptions about learning mathematics.

3.1 An epistemological approach

Sierpinska and Lerman (1996) point out that issues in epistemology are central to mathematics teaching: “What is knowledge and what characterizes the process of development of knowledge?” (p. 828). Mathematics education theories that generate didactic principles, whether they are constructivist, socio-cultural, interactional, linguistic or anthropological, are, to some extent, trying to answer basic epistemological questions; any discourse on didactics is based on a description of the learning object and the learning process. Bergsten (2008) states, that the epistemological approach focuses on “the structure and use of mathematical knowledge and its dissemination in educational institutions” (p. 191). The point of the epistemological approach is, therefore, that the focus is on “*what is taught*” in mathematics.

In France, since the mid-1970s, several researchers in mathematics education have developed theories with an epistemological approach (Sierpinska & Lerman, 1996). In the late 1960s, the research field called “*Didactique des mathématiques*” was developed with the intention to research mathematical education in a scientific manner (Miyakawa, 2009). The French word “*didactique*” has several translations into English. Chevallard (1999b) criticises the tendency of using the word *didactique* on the assumption that it is untranslatable into English. Instead, he and many others use “*didactics*” for *didactique* and defines it as “the science of the diffusion of knowledge in any institution, such as a class of pupils, society at large, etc.” (Chevallard, 2007, p. 133) and, more particularly, the “scientific study (and the knowledge resulting thereof) of the innumerable actions taken to provoke (or impede) the diffusion of such and such body of knowledge in such and such institution” (p. 133).

It comprehends that to study mathematics education one applies a theory, which holds systematic knowledge with universality (Miyakawa, 2009). French theories with an “epistemological approach” that have received most attention are: “*Theory of didactical situations*” (TDS) (Brousseau, 1997); “*Didactic transposition theory*” (Chevallard, 1985; 1991); “*The anthropological theory of didactics*” (ATD) (Chevallard, 1999a); “*Theory of conceptual fields*” (Vergnaud, 1991); and “*Theory of linguistic registers*” (Duval, 1995).

One of my intentions with this thesis is to describe “what is taught” and “what is learned” during lessons where the PSO approach is applied and to illustrate the complex of learning activities of mathematics that the teacher sets in motion. The anthropological theory of didactics (ATD) supplies a framework for the analysis of how the didactic process relates to and transforms the mathematics taught in educational institutions. It is based on the concept of a “*praxeology*” as the basic body of knowledge. My assumption is that ATD is useful for the purposes of didactic planning: It can work as a “fine-grained” theory that can be used for didactic design of individual lessons and can also be used for mapping the structure and transformation of knowledge at a larger scale. The epistemological analysis of sequences of Souma’s mathematics lesson plans is carried out with respect to the framework suggested by ATD.

Didactical techniques of the PSO are often motivated from the perspective of the individual student and the particular situation in the classroom. I will discuss certain aspects regarding the PSO using the *Theory of didactical situations* (TDS) (Brousseau, 1997), since this theory will more naturally allow a discussion of motivational points of view on individual episodes. The theories of Brousseau and Chevallard share basic norms with the problem solving approach; both have focus on inciting autonomous work on behalf of the student.

3.2 The anthropological theory of didactics (ATD) and didactic transposition theory

Chevallard studies mathematical knowledge in teaching systems at the macro-level within “*institutions*” (cf. Chevallard, 1989, 1992). The anthropological theory of didactics (ATD) provides a way to describe an epistemological object (a body of knowledge) as a praxeology; a set of tasks and corresponding techniques and a technology and theory that justify those techniques. A praxeology can, recursively, often be described in terms of more local praxeologies. The didactic transposition theory has a dynamical point of view; it aims to explain how the epistemological objects, the praxeologies, are formed when they are adopted by educational institutions. Bosch and Gascón (2006) summarise the contribution of these two theoretical frameworks:

1. The theories make it clear that, in order to study school mathematics properly, it is necessary to concern oneself with the phenomena of how a school *reconstructs* mathematical knowledge.

2. The concept of a *praxeology* makes it possible to describe the mathematical bodies of knowledge in relation to didactic activities in teaching/learning mathematics.
3. The concepts of the *ecology* of mathematical and didactical *praxeologies* and *scale of levels of co-determination* describe from where the conditions and restrictions that regulate the activities of teaching and learning in the classroom stem.

In the next two sections, I will further elaborate these theories.

3.2.1 The anthropological theory of didactics

Chevallard's anthropological attempt to study the mathematical knowledge in institutional contexts extended into "*the anthropological theory of didactics*" (ATD) (*Théorie anthropologique du didactique* in French, Chevallard, 1999a). It holds that mathematics learning, like any other learning, can be modelled as the construction within social institutions of *praxeologies*. Chevallard describes a praxeology as "the basic unit into which one can analyse human activity at large" (Chevallard, 2006, p. 23).

A praxeology supplies both methods for the solution of a domain of problems (*praxis*) and a framework (*the logos*) for the discourse regarding the methods and their relations to a more general setting. Chevallard refers to a French anthropologist Marcel Mauss (1872-1950), who states that a praxeology is a "*social idiosyncrasy*", that is, "an organised way of doing and thinking contrived within a given society – people don't walk, let alone blow their nose, the same way around the world" (ibid., p. 23).

A praxeology of mathematical knowledge is also called a *mathematical organisation* (MO) (Barbé, Bosch, Espinoza & Gascón, 2005). As I mentioned above, every praxeology consists of a block of "praxis", usually considered as "know-how" and a block of "logos", considered as knowledge, or "know-why". The praxis can be described by the set of *tasks* (T) and *techniques* (τ) and the logos part consists of a *technology* (ϑ) and a *theory* (Θ):

T: Set or type of tasks, i.e. problems to be solved.

τ : Techniques are solutions, that achieve the tasks.

ϑ : Technology. It constitutes the tools for discourse on and justification of the *techniques* for solving these tasks, like theorems that ensures the correctness of the techniques.

Θ : Theory. It provides further justification of the *technology*, like definitions and proofs, and connects the praxeology with other praxeologies.

A praxeology does not always contain all four parts; it may happen that the logos block is empty, or very implicit, and it may apply only the know-how block ("silent knowledge").

Praxeology is a notion that scales well; larger more complex praxeologies can be woven from simpler smaller praxeologies. To analyse the intricacy of different

praxeologies, Chevallard classified the mathematical praxeologies into increasing complexity as: *specific* (or, *point*), *local*, *regional* and *global* praxeologies (Garcia, Gascón, Higuera & Bosch, 2006). In line with Barbé et al. (2005) I will refer to the following three praxeology levels and characterisations:

- A *specific* praxeology is characterised by a single type of task, where a specific technique applies and where the technology usually is implicit.
- A *local* praxeology is the integration of several specific praxeologies that are connected by a common technological discourse.
- A *regional* praxeology is a collection and coordination of local praxeologies where a common mathematical theory is formulated.

Chevallard proposes to model the *didactical process* in which a mathematical praxeology is constructed within an educational institution (Barbé et al., 2005). The dynamic of the didactical process is described as occurring in six “moments” (ibid., p. 238), which are different modes of learners’ activity:

1. (FE) The moment of *first encounter* (or re-encounter) of a certain type of task associated to the praxeology;
2. (EX) The *exploratory* moment of finding and elaborating techniques suitable for the task;
3. (T) The *technical-work* moment of using and improving the techniques;
4. (TT) The *technological–theoretical* moment in which alternative techniques are assessed and a technological discourse is taking place;
5. (I) The *institutionalisation* moment, where one is trying to identify and discern the elaborated praxeology;
6. (EV) The *evaluation* moment which aims to examine the value of the constructed praxeology.

A complete realisation of these six moments ensures the quality of the resulting praxeology, so that it leads to at least a local MO (ibid.).

According to Chevallard (Chevallard, 2002 in Bosch & Gascón, 2006), this complexity of praxeologies depends on the *ecology* of the mathematical and didactical praxeologies: “Bringing in the notion of ecosystem makes it possible for the researcher in didactique of mathematics to consider in relation with mathematics several new objects outside mathematics” (Wozniak, Bosch, & Artaud, 2009, n.p.). The form of a praxeology takes depends upon a structuring schema in nine levels (I put together the first two levels in the description below) in a “*hierarchy of levels of co-determination*” (Chevallard, 2002 in Bosch & Gascón, 2006). The term “co-determination” suggests that those two aspects – mathematical and didactical organisations – cannot be separated because they are determined in their mutual interaction (Dorier & Garcia, 2013).

The levels of co-determination are defined as below:

- *civilisation/society* (e.g. political, or cultural orientation in education)
- *school* (e.g. curriculum)

- *pedagogy* (e.g. general teaching principles)
- *discipline* (e.g. mathematics, physics,...)
- *domain* (e.g. algebra, geometry,...)
- *sector* (e.g. equations, similarity,...)
- *theme* (e.g. triangles, root,...) and
- *subject* (e.g. one simple question)

These levels generate the conditions and restrictions that influence the complexity and form of praxeologies. For instance, Barbé et al. (2005) have shown how restrictions on teachers' practice in Spanish high schools are related to this hierarchy of levels of co-determination. In their international comparative study in 12 European countries, Dorier and Garcia (2013) use four levels of didactical co-determination (society, school, pedagogy and discipline) to investigate from which level the conditions and constraints are derived from in each country by an implementation of inquiry-based mathematics and science education within their project.

Since ATD is a general epistemological model, it can be applied to other areas besides mathematics. A *didactical organisation* (*DO*) is a *praxeology* developed by educators and students to organise the work of teaching and learning, i.e. with the purpose of supporting a *didactical process*. Winsløw and Møller Madsen (2008) describe that the didactic transposition is “mediated” (p. 2382) by the *DO* and that it “can be located in the transition from the regional mathematical organisation MO_m of the professional mathematician (...), to the local mathematical organization MO_s to be enacted by students” (p. 2382).

3.2.2 The didactic transposition theory

According to Sierpinska and Lerman (1996), Chevallard's “*anthropology of knowledge*” (*anthropologie des savoirs*) is an extended epistemology, which recognises that knowledge usually is an object that is to be *used* and *taught* (p. 856). The process of adapting the knowledge content for the purpose of being taught within a given institution, is called a “*didactic transposition*”. It means a transposition from the “*scholarly knowledge*” (Chevallard, 1989), which comprise of praxeologies produced and used in the community of mathematicians, scientist and engineers, into praxeologies that are adapted to different levels within the education system. Chevallard considers that the scholarly knowledge is nothing else but the *used knowledge* (Chevallard, 1989) and is transposed when it applies, for example, to a curriculum or in classrooms.

Chevallard developed the conceptualisation of the didactic transposition into “the didactic transposition theory” (“*Théorie de la transposition didactique*” in French, 1985; 1991), before the anthropological theory of didactics. It was stated first by the French sociologist Michel Verret (1975) as the notion of the “*transmission didactique*”, which emphasises that an object of knowledge cannot be taught in the same way as it is produced and used in the scientific community

(Bergsten, Jablonka & Klisinska, 2010). Chevallard clarifies the difference between *used knowledge* and *taught knowledge* as below:

As long as you only use knowledge in doing something, you need not justify nor even acknowledge the used knowledge in order to endow your activity with social meaning. Its meaningfulness derives from its outcome, judged by pragmatic standards. Knowing something in this case, is close to, and even inseparable from, knowing *how to do* something. Knowledge and know-how enjoy the status of means to an end, which is the standard by which their *relevance* as tools of the trade will be judged. In contrast, teaching requires the social acknowledgement and legitimation of the knowledge taught. In going from used knowledge to taught knowledge, relevance gives way to *legitimacy*. Teaching some body of knowledge cannot be justified only on the grounds that the knowledge taught could be useful in such and such social activities. (Chevallard, 1989, p.59)

These notions, knowing “how to do something” – know-how, contra “justifying some body of knowledge to be taught” is (later) carried over to the concept of praxeology in ATD.

Chevallard explains, that by a “transposition” he does not mean that two or more things simply change place. He uses a metaphor of music, where a piece is transposed from one key to another. Then “no two things change places” (1999b, p. 2). Further, he continues to discuss about how knowledge transposes according to the theory of didactic transposition:

Knowledge is *not a substance* which has to be transferred from one place to another; it is a world of experience which, through a creative process, has to be... transposed, to be adapted to a different ‘key’ – the child – and to a new ‘instrument’ – the classroom (1999b, p. 2).

According to the didactic transposition theory, knowledge is “a changing reality” (Chevallard, 2007) that will take distinct forms in different teaching systems. The theory makes it possible to point out and problematize the attitude, which French sociologists used to call the “*illusion of transparency*”; the attitude that one believes the reality of the world is already known and so that learning takes an interpretive form. In the mathematics educational reality, it means that a teacher, say, does not question what the knowledge to be taught is or should be, since he feels that he already knows it (Chevallard, 1992).

This didactic transposition process underlines “the institutional relativity of knowledge and situate didactic problems on an institutional level, beyond individual characteristics of the subjects of the considered institutions” (Bosch & Gascón, 2006, p. 56). Every process of transposition is caused by different kinds of restrictions. Bosch and Gascón argue that pointing out such restrictions “contributes to explain, in a more comprehensive way, what teachers and students do when they teach, study and learn mathematics” (ibid., p. 53).

Bosch and Gascón (2006) illustrate the steps of a didactic transposition process through different institutions:

Figure 3.1: The didactic transposition process (Bosch & Gascón, 2006, p. 56)

The first step, the “*scholarly knowledge*” has already been discussed. The “*knowledge to be taught*” is designed by institutions, which Chevallard calls *noosphere*, – sphere of those who think (*noos* is Greek for “mind”) – which is a non-structured set of different types of stake holders within the education system; like educators, curriculum developers, politicians, textbook authors and others (Chevallard, 1992). The “*taught knowledge*” is created by the institution of the teachers’ praxis. Finally, the “*learned, available knowledge*” is formed by the teacher and learners in the classroom.

3.2.3 Study and Research Course as a tool of didactic engineering

Chevallard further proposes (2006) a tool of didactical engineering; the *Study and Research Course (SRC)*⁵ to establish a new type of didactical organisation, which is based on a long term study process, where the students study by researching a sequence of generating questions. In Winsløw, Matheron & Mercer (2013), the distinction is made between a *task* and a *generating question*: A generating question usually presupposes the construction of a new technological environment and it will generate both new sub-questions and new *tasks*; for each task the formulation being the first step of an *answer*, which means the construction of a local praxeology. The study of the generating question and the following sub-questions leads students to construct complex praxeologies. Winsløw et al. (2013) propose to use decision trees of generating questions as a technique for the didactic design of Study and Research Courses.

Artigue (2009) describes two important constructions of SRC as “the *scale of levels of didactic codetermination* and the *dialectic of media and milieus*”. She states that the scale of levels of didactic codetermination can make didactical design “sensitive” to the diverse constraints from the civilization level to the subject level. Also, Artigue argues that the dialectic of *media* and *milieus* “a priori appears as a powerful tool for taking in charge the way technological evolution radically changes our access to information, and the characteristics of the *milieus* we can interact with for developing knowledge” (p. 14).

⁵ SRC has recently been termed *Study and Research Path* (Wozniak, Bosch, & Artaud, 2009).

3.3 The theory of didactical situations

Guy Brousseau is one of the developers of *Didactique des mathématiques* in France. According to Bosch and Gascón (2006), Brousseau was the first who “postulated the existence of *didactic phenomena* which appears in the forms of unintentional regularities in the processes of generation and diffusion of mathematics in social institutions and are irreducible to the corresponding cognitive, sociologist or linguistic ones” (ibid., p. 54). Bosch and Gascón state that Brousseau’s *theory of didactical situations* (Brousseau, 1997) caused radical change in our way of studying problems related to the teaching and learning of mathematics. The *theory of didactical situations* (TDS) has contributed to apply a methodology to study mathematical knowledge in educational institutions within the conception of “*situations*”.

Artigue (1994) considers the theory of didactical situations and the didactic transposition theory as tools for didactical engineering, and claims that both theories emphasise the need to study the didactical phenomena within a systematic approach. She states that TDS is situated at a more “local level” and aims to be a tool for “modelling” the teaching situations, so that such situations can be developed and managed “in a controlled way” (ibid., p. 28). In an e-mail to Anna Sierpinska, Brousseau describes how TDS relates to didactical engineering (Sierpinska, 1999). There he remarks that TDS is not a method of teaching. It aims to study and analyse all kinds of learning and teaching. TDS contains “models” which may support and inspire teachers in planning lessons, which aim at making the students (re)discover some mathematics, so in that way, “the theory can make suggestions for engineering” (ibid., no. 8, p. 1).

3.1.1 Didactical and adidactical situations

I will apply Brousseau’s theory of didactical situations as a complement of ATD to analyse how the PSO motivates the individual student to become an active participant at particular situations in the classroom, namely, didactical situations and a-didactical situations. Brousseau’s notions of milieu and didactical contract will also be shortly discussed.

From the individual learner’s perspective, the didactical process is described in TDS as a dialectic between a *didactical situation* – a situation which the teacher creates with the intention for her students’ learning – and an *adidactical situation* – where the students are released from the teacher’s expectations and autonomously solve tasks by interacting with the *milieu* (Brousseau, 1997). I quote Sierpinska on the meaning of a *milieu*:

‘Milieu’ should perhaps be understood in an ecological sense, as in ‘water is the natural milieu of fish’. Thus the ‘didactic milieu is the natural milieu of students’ (Sierpinska, 1999, no 1, p. 1).

The *milieu* consists of the physical surroundings of students, for instance, the teacher, reviews and explanations by the teacher, the classmates, the textbook, the

working materials, etc. The *media/medium* is “any social system pretending to inform some segment of the population or some group of people about the natural or social world” (Chevallard, 2006, p. 9). The student and the medium, in the form of, say, the teacher or the textbook, play the “didactic game”. The goal of this game is to eventually create an *adidactical* situation. It means, so to speak, forces, or induces a move in the adidactical situation. According to Brousseau, students’ learning actually happens in the adidactical situation and takes place when the students adapt themselves to the milieu:

The teacher’s work therefore consists of proposing a learning situation to the student in such a way that she produces her knowledge as a personal answer to a question and uses it or modifies it in order to satisfy the constraints of the *milieu*, and not just the teacher’s expectation. There is a great difference between adapting to a problem raised by the *milieu*, an unavoidable problem, and adapting to the teacher’s expectations (Brousseau, 1997, p. 228).

In order for the didactic game to take place, the milieu usually must include some prerequisite “body of knowledge” and a major part of the didactical game is to set up these components of the milieu. Brousseau initiates the book “Theory of didactical situations in mathematics” with an introduction chapter, “Setting the scene with an example: the race to twenty” (ibid., p. 3). The didactical situations the teacher has proposed are that the students play a game “the race to twenty”. Here, the concept of “*the situation of action*” presents itself clearly (Warfield, 2006): When the teacher presents the rules of the race, the students start to interact with the didactic milieu and start to develop their “implicit model of strategy” or “elaborating the winning strategies” (ibid., p. 40). Brousseau means that the learning takes place by considering the winning strategies during the play of the game (the situation of action) and that further learning takes place in the follow up discourse (e.g. *situation of formulation and validation*, see Warfield, 2006; Sierpiska, 1999, no 1). Warfield clarifies the situation of action further as:

The Situation of action lays the essential foundation for the explicit models and formulations which follow. ...A specific requirement in the case of *Didactique* is that the Situation of action provides feedback to the student on which to base, and against which to test, his models. In this case, the feedback consists simply of the winning or losing of the race. (Warfield, 2006, p. 22).

The situation of action plays the crucial role for promoting the student’s own efforts in the didactic process of developing their problem solving strategies and knowledge.

3.3.2 The didactical contract

The concept of “*didactical contract*” contributes to Brousseau’s intention to illuminate *didactic phenomena*. It points out how the paradox of learning in a

school situation arises for the teacher and the students. The didactical contract implies implicit and mutual rules between the teacher and the students. In Brousseau's description, it is "the set of (specific) behaviours of the teacher which are expected by the student and the set of behaviours of the student which are expected by the teacher" (Brousseau & Warfield, 1999, p. 47). As Sierpiska describes, the didactical contract differs in different cultures (1999, no. 3, p. 1):

...in every didactic situation there is a didactic contract, and that across different cultures, classrooms and time some rules remain constant, such as, for example, that the teacher is expected to perform teaching actions such as giving the students tasks that are specific to the knowledge he or she aims at, and the student is expected to attend to the tasks given by the teacher

The teacher tries to suggest to the students what *she* wants her students to do to solve the problems, or answer to her questions. For the students' acquisition of the knowledge, the teacher creates sufficient conditions for the students to recognise her intentions. Then the students have no choice to try any other methods and are not given any chance to acquire the knowledge by her own way. The students have only been given "an illusion" of learning (Brousseau, 1997, p. 41).

To avoid the negative effect of the didactical contract, Brousseau proposes the action of "*devolution*" – the teacher acts so that she hands responsibility for learning to the students themselves. Brousseau states that the value of acquired knowledge varies depending on the quality of the milieu similar to the "real" cultural functioning of knowledge and apparently free from teacher's didactical intentions so that it leads the learning situation to be adidactical (*ibid.*, p. 230). It is therefore necessary that the students themselves want to achieve higher and more complex structures of knowledge.

Chapter 4

Research questions and methodological considerations

This chapter begins with the presentation of the research questions. Then I report methodological aspects on the design and implementation of the two studies; the procedures for data collection, how the analytical tools were used in the investigation of the data and, finally, some ethical considerations.

4.1 Research questions

This thesis consists of two related studies. In the first study I present a theoretical analysis of elements in the PSO adapted lesson plans and how they relate to the didactical process. I will then report an empirical study of applying the PSO in Swedish lower secondary school. In the first study, I address the following two research questions:

1. To what extent does the discrepancy of Japanese and Swedish curricula influence the adaptation of Souma's lesson plans?

This question will be used to illuminate how the didactic transposition process (Chevallard, 1989) acts on a sequence of Souma's lesson plans. It will give us a picture of institutional differences between the Japanese and the Swedish teaching systems.

2. What is the structure of the didactical praxeologies in Souma's lesson plans?

Question two deals with the description and analysis of the construction of the PSO approach. My intention is to illustrate the epistemological and didactical structure of a sequence of lesson plans and the didactical structure of two particular lesson plans.

The empirical study has been guided by the following research question:

3. When applying the PSO approach in Swedish mathematics classrooms;
 - a. What can the mathematical and the didactical organisations look like?

- b. How does this approach encourage students' mathematical contributions?

This third question thus addresses the issue whether applying the PSO approach has any positive effect on Swedish mathematics classrooms and, if so, what the most significant outcome of employing this approach might be. My answer to this question will be based on an intervention study involving collaboration with one mathematics teacher in two lower secondary school classrooms.

4.2 Study one: a theoretical analysis of the PSO approach

In this part of the thesis, I refer to the Japanese and Swedish curricula in an attempt to study the didactic transposition that leads to the lesson plans of Souma and to the one we adopted in Sweden. I also refer to some of Souma's basic template lesson plans and study their implied mathematical organisations.

4.2.1 Japanese and Swedish curricula

The Japanese curriculum from 2008 and the Swedish curriculum in mathematics for lower secondary school from 1994 (revised in 2000) are reviewed in arithmetic, algebra, and geometry. My intention is not *to compare* the two countries curricula. I *relate* the contents of the Japanese curriculum to the Swedish counterpart in order to understand why some parts of Souma's lesson plans, namely, those in arithmetic and algebra, are easier to adapt to lessons in Sweden than other lesson plans (geometry). At the same time, I needed to analyse how *the knowledge to be taught*, steered by the Japanese *noosphere*, affects Souma's lesson plans (Kunimune & Souma, 2009b).

In Japan, the curriculum is called The Course of Study (Ministry of Education, Culture, Sports, Science and Technology, shortened to MEXT, 2008a). Teachers in lower secondary school (grade seven to nine, students' ages are between 13 to 15) are supposed to refer to and mind the guidelines ("Gakushu shido yoryo kaisetsu" in Japanese, MEXT, 2008b), which is a detailed commentary for The Course of Study for all subjects, not exclusively in mathematics. The main focus of my review of the guidelines is on the first chapter of the guidelines, "The goal and contents of mathematics", and on the second chapter, "Concept of content configuration". I describe how the guideline explains the relationship between the notion of "mathematical activities" and problem solving, also how the guidelines treat geometry, the number concept and algebra.

Regarding the Swedish curriculum, my main source is "Curriculum in mathematics for the compulsory school" (Skolverket, 2000), which is connected to "Curriculum for the compulsory school system, the pre-school class and the leisure-time centre Lpo94" (Skolverket, 1994).

4.2.2 The lesson plan example “sums of exterior-angles”

In order to investigate the structure of the didactical praxeologies of the PSO approach based lessons, I decided to examine to what extent the lesson plans realise the six different moments of the didactical process as ATD proposes (Barbé et al., 2005). For this study, I chose one of Souma’s lesson plans⁶ for grade eight: The “sums of exterior-angles”, from an organised collection in “Practical lesson plan collection for mathematical activities” (*Suugakuteki katsudou no jissen plan shu*, in Japanese) (Kunimune & Souma, 2009b). The lesson plan contains the following headings:

1. The concept of this lesson
2. The lesson disposition
3. The goal of this lesson
4. The mathematical activities of this lesson
5. The flow of the lesson

Souma notes (ibid., p. 55) that the “The flow of the lesson” part is based on empirical data from real lessons implemented in Hokkaido by a Japanese teacher. In my study, I focus on the flow of the lesson in order to illustrate the didactical process of the lesson. The form in which the flow of the lesson is reported is the noting of *teacher’s act* and *activities of the students*. Souma also report, as students’ activities, what kind of *mathematical activity* the students are engaged in.

4.3 Study two: an empirical study of Swedish mathematics classrooms

In this section, I will describe the empirical studies that were conducted in one grade seven class and one grade eight class.

4.3.1 The process

The general aim of this empirical research was to examine the viability of the PSO approach as a design tool for the Swedish mathematics classroom. In order to implement the study, a long-term cooperation with a mathematics teacher was needed. The intention was to apply teaching methods adapted from Souma’s problem collections and lesson plans, which are oriented for Japanese lower secondary school, to Swedish mathematics classrooms in grades seven to nine.

In April 2010, five months before the project began, I contacted a local mathematics developer (in Swedish, “matematikutvecklare”) who served to help

⁶ As I describe in section 4.4.1, the reason I chose this particular lesson plan is that this lesson shows some distinguishing traits of the PSO approach.

mathematics teachers in the region in further developing their professional skills. I asked the local mathematics developer to distribute my request, where I sought a mathematics teacher in lower secondary school, who would be willing to collaborate with me on a weekly basis for a period of one year. The local mathematics developer also offered the teacher involved in my project to finance a ten percent workload reduction.

The teacher who eventually engaged in the project worked at a school devoted to Montessori pedagogy. The school is located in the middle of Sweden in a socially stable area and covers grades one to nine. The school principal was positive to the collaboration of the teacher during the period. The teacher had worked as a qualified mathematics teacher for three years and had at the time been serving at that school for two years. In August 2010, the teacher and I started the implementation of the PSO approach in a grade seven class consisting of 15 students (one student moved to another school in the middle of September and one student in the end of November 2010). In January 2011, we started to implement the PSO also in a grade eight class (17 students), which the teacher took over.

4.3.2 The study with a grade seven class

The teacher and I had a first meeting in June 2010. I explained to her the structure of the PSO approach, as comprising of the following routines in the classroom:

1. State a problem in a way to incite a guess or immediate observations on part of the students.
2. Let all students state a guess or hypothesis.
3. Discuss the viability of guesses and let students motivate their guesses.
4. Let students, individually or in groups, work on the problem, or, perhaps, on a derived reformulation or a sub-problem.
5. Let students present their solutions in class and discuss the different solution techniques.
6. Turn to the textbook for an outline of related theory.

I translated parts of Souma's problem collection for her to see the character of the problem construction. In August 2010, the teacher and I started to plan the contents of the lessons for the whole year for the grade 7 class, which consisted of 10 girls and 5 boys. Six of the students had parents from other countries than Sweden.

The contents of the lessons were based on the results of a diagnostic test (McIntosh & NCM, 2009) implemented at the very beginning of the course in the end of August to diagnose the levels of the students' arithmetical knowledge and calculation skills. Since the diagnostic test showed that many of the students had a lack of the understanding of rational numbers, we decided to start with lessons that concerned problems on fractions. Soumas' task collections did not include fraction calculation related problems, since the fraction chapter is learned by grade 6 in Japan. Thus, we created our own problems having the open-closed character

prescribed by Souma. For example, one problem looks like: “Which of the expressions belong together among $\frac{3}{4}$, $\frac{6}{8}$, $\frac{6}{9}$, $\frac{4}{6}$ and $\frac{2}{3}$?”

A problematic issue was that the mathematics textbook the school was using did not contain enough of summaries of definitions and mathematical laws, which is prescribed by Souma. On several occasions, we had to write a compendium of theorems and definitions, for instance: “What is the distributive law?” “What is the absolute value?” Several lesson plans in Souma’s book were not possible to apply directly neither to grade seven or eight because they were not included in the Swedish curriculum. This was the case especially for geometry, as for example parallel shift of triangles, construction of perpendicular bisectors and all the solid geometry lesson plans, including the calculation of the volume of a sphere.

The contents we completed in grade seven from September 2010 to May 2011 were divided into the following topics:

1. Extension and abbreviation of rational numbers
2. Arithmetic operations on rational numbers
3. Fractions and decimal numbers
4. Arithmetic operations on percentages
5. Area of polygons
6. Angles of polygons
7. Property of circles
8. Area of circles
9. Number line and negative numbers
10. Arithmetic operations on negative numbers
11. Algebraic expressions
12. Equations
13. Prime numbers
14. Exponentiation
15. Properties of the square root and its arithmetic operations

After chapters 3 and 4, 8 and 9, 10 and 13, the teacher made tests to evaluate the students’ knowledge. I supervised the contents of the tests but the results of the test are not included in this study. Mathematics lessons (each 60 minutes) took place twice a week. In addition, they had once a week, according to Montessori pedagogy, a 60 minutes lesson without whiteboard demonstration. There, the students had an opportunity to ask the teacher for help when they trained on exercise problems. The teacher and I planned together almost every lesson: the goal of the lesson, the problems, the flow of the lesson and problems for homework. The teacher wrote a lesson plan for every lesson in line with the structure of the PSO. In the beginning, to train writing the PSO template lesson plan, the teacher wrote quite detailed lesson plans including: 1. *The goal*, 2. *The preparation*, and 3. *The flow of the lesson*. Part 3 includes three columns:

1. Teacher’s activity (what the teacher will do during the lesson)
2. Students’ activity (what the students will do during the lesson)
3. Something to consider doing

”Today’s problem”, which always was presented in the beginning of the lesson, was placed in the first column. In the second column, detailed anticipation of students’ solutions, argumentations, and likely errors are included. In column 3, the teacher entered mainly didactical/pedagogical considerations, such as “*make students write down sketches of the figure first*”, or “*if nobody comes up with the solution 1, then I will mention the subject of the previous lesson*”. After half a year, from the contents 9 (Number line and negative numbers), the teacher wrote less detailed lesson plans in order to save time. They still had three columns:

1. Lesson theme
2. Lesson process
3. Something to consider

4.3.3 The study with a grade eight class

After four months of lessons with the grade seven class, the teacher I worked with took over a class of grade eight from another teacher from January 2011. This class included six girls and eleven boys. Of these students, ten had parents from other countries than Sweden. At the moment the teacher took over the class, the class had already moved through the following contents: 1. Rational numbers and calculation of percent, 2. Plane and solid geometry with calculation of angles, area and volume, 3. Functions and graphs.

The teacher, who taught this class until January 2011, had chosen to apply some of the problems we had created for grade seven in his teaching, since he found that our problems should be appropriate. For that reason, the class obtained approximately the same content as the class of grade seven, in addition to functions and solid-geometry.

The teacher and I decided on the contents for the rest of the semester (five months) as follows:

1. Number line and negative numbers (1 lesson)
2. Arithmetic operations on negative numbers (3 lessons)
3. Algebraic expressions (3 lessons)
4. Equations (3 lessons)
5. Exponentiation (2 lessons)
6. Properties of the square root and its arithmetic operations (2 lessons)
7. Probability (2 lessons)

For our convenience, we planned the order of the contents in the same way as for grade seven. I did not take part in the construction of the lesson plans in the last subject probability since I could not attend those lessons. The students performed tests which the teacher had created for control and evaluation of the students’ learned knowledge. Those tests were implemented two times, after the lessons of “Arithmetic operations on negative numbers” and “Exponentiation” had been completed. As was the case for grade seven, I supervised the contents of the tests, but the results are not included in this study. The class had mathematics lessons

(each 60 minutes) twice a week and they also had a 60 minutes lesson to work on some exercise problems once a week. Before a lesson took place, the teacher wrote a simplified version of a lesson plan. During the lessons, the teacher usually used only a whiteboard and whiteboard markers to write down the problems, students' solutions and other notes.

4.4 Selection of lessons and data collection

4.4.1 The theoretical studies

I will present the result of the analysis of the mathematical and didactical organisations in the PSO based lesson plans in section 5.2. For this presentation, I selected the lesson “sums of exterior-angles” from the lesson plan collection “Practical lesson plan collection for mathematical activities” (Kunimune & Souma, 2009b) which is intended for Japanese students of grade 8. For this lesson plan collection, Souma is one of the two editors. There are several authors to this lesson plan collection, but for this particular lesson, Souma himself is the sole author. The reason I selected this particular lesson plan in geometry is that the lesson shows clearly some distinguishing traits of PSO based lessons.

I also adapted Souma's lesson plan “Introduction of variables” (Kunimune & Souma, 2009a) to grade seven and eight in Swedish classrooms, without changing any elements of the lesson plan. The analysis of this lesson plan is presented, together with the result of the classroom observation, in section 5.3.

The Japanese curriculum and curriculum guidelines I present here are from 2008, since the lesson plans I studied are based on these. I present in detail the sections “Numbers and algebraic expressions” for grade 7 and “Geometrical Figures” for grade 8. My intention is to show why Souma's lesson plans in arithmetic and algebra are easier to adapt to lessons in Sweden than lesson plans in geometry. For the same reason, I review the content of “Numbers and expressions” and “Geometry” for grade seven to grade nine in the Swedish curriculum from 1994 (Lpo94). Since the study finished in 2011 and in the same year the new curriculum Lgr11 was adopted, I give a short review on the contents of those three areas in Lgr11, which has its focus on students' mathematical competencies.

4.4.2 The classroom observations

I chose to analyse the lessons of operations on negative numbers and the introduction to finding general solutions for my observations of grade seven and eight classes. The main reason for this choice is that, in those lessons, the teacher and I applied a sequence of the first four of Souma's lesson plans of calculation with negative numbers and the first six lesson plans of algebraic expressions and equations taken directly from the lesson plan collection for Japanese grade seven students (Kunimune & Souma, 2009a). My interest is to learn how Souma's

sequence of lesson plans work out in our Swedish classroom. However, Souma's lesson plan "Multiplication of positive numbers and negative numbers", was modified before we applied it: We added a different technique and technology which were neither presented in Souma's lesson plan, nor in our Swedish text books. The reason for adding this technique/technology is described in section 6.1, together with the presentation of the observations.

In order to observe the classroom activities, I used video recordings. One camera with a stand was placed in the rear of the classroom behind the students. This camera followed the teacher and the students with whom she interacted. At the same time, I kept a written observation protocol, which included the teacher's actions and students' reaction. My notes about the lessons were intended to work as a backup to the video recordings. During the lessons I stayed behind the camera and virtually did not interact at all with the lessons. The lessons with the video recorded data collection for the study of the grade seven classroom lasted from September 2010 to March 2011, and for the grade eight classroom, from February to March 2011. Further classroom observation, without video, was finished by the end of May 2011 for both classes. The reason I did not continue with the video recording was that I could not attend every lesson during April and May.

4.4.3 Interviews with the students

In the beginning of September 2010, interviews with structured questions (Kvale, 1997) with 15 students in grade seven were conducted. The interview was based on four main questions with several sub-questions. The aim of these interviews were to learn how their mathematics lessons looked like in grade six in their previous schools, to probe their attitude toward mathematics and to compare to their reflections regarding the project. I asked questions and wrote down the answers without using audio recordings. The interview lasted approximately 20 minutes per student. All interview questions in Swedish are presented in appendix I.

During May 2011, at the end of the project, I conducted an additional set of interviews with ten grade seven students who had the opportunity to participate. At that time, the number of students in the class was 13. The aim of this interview was to record their reflections upon the PSO based lessons the teacher had implemented. I used the same method as in the first interview. The interview lasted approximately 20 minutes per student. All interview questions in Swedish are presented in appendix II.

In grade eight 11 students (of total 17) who had the opportunity to participate during the same period (May 2011) were interviewed. I asked them about their previous situations in grade seven. For this I applied the same pre-questionnaire as I used for grade seven in the beginning of the project. I also asked them to reflect upon their current lessons using the questionnaire presented above. The interview lasted approximately 20 to 25 minutes per student. The time line of the interview is summarised below:

During September 2010: 15 students from grade seven (interviews about their previous settings regarding mathematics lessons)

During May 2011: 10 students from grade seven (interviews about their current reflections on mathematics lessons)

During May 2011: 11 students from grade eight (interviews about their previous settings and about their current reflections on mathematics lessons).

4.4.4 Questionnaire to the students

11 students from grade seven and 14 students from grade eight answered a non-anonymous questionnaire at the end of May 2011. The teacher distributed the questionnaire during the final lessons of the semester and the students filled them out by themselves. The questionnaire consists of 10 questions for grade seven and 9 questions for grade eight. The students were supposed to mark the level of agreement (“I agree”, “agree” and “do not agree”) of each question on the blank line. Some questions had sub-questions that asked the students to motivate their answers. The aim for this questionnaire was to directly obtain the students’ impressions of the PSO-based lessons during the period. The main focus of the study is on the questions 1, 2, 5, 6, 7 and 9, concerning students’ attitudes toward the lessons conducted by the teacher, since those questions especially concerns my research question 3b, “How does this approach encourage students’ mathematical contribution?” The full questionnaires in Swedish are presented in appendix III and IV.

4.4.5 Interview with the teacher

A interviews with structured questions with the teacher was conducted in June 2011; one and half week after the last lesson. The interview included eight main questions with several sub-questions to get more elaborated answers. The interview lasted approximately one hour. The teacher was not given the interview sheet in advance. All interview questions in Swedish are presented in appendix V, along with an English translation. The rationale behind those questions was to probe the teacher’s reflection and evaluation of using the PSO approach and her perception of the students’ reaction and their mathematical contribution in the classroom.

4.5 Method of analysis

4.5.1 Analytical tools

The epistemological analysis of the lesson plan of “sums of exterior-angles” from Souma’s lesson plans collection was made according to ATD. The mathematical praxeologies are classified into increasing complexity (Garcia, et al., 2006) as *specific*, *local* and *regional*. The *dynamic of the didactical process* (Barbé, et al., 2005) is also used to describe different moments of learners’ activity. The *didactic*

transposition theory is applied to illuminate how the curricula influence the lesson plans.

For the analysis of the empirical study, the focus is on mathematical organisations and the teacher's didactical organisation. I use ATD to describe the praxeologies of the mathematical content in the lessons and the teacher's didactical organisations when using the PSO approach.

In order to distinguish the notion of a *task* in the meaning of ATD from its "common" meaning, I use the term *problem* for those which were considered by teachers in the lesson plans and the classroom observations. There, a problem is viewed as a mathematical task, which is *not* a routine task (for the problem solver) and the methods of the solution is not known in advance (Schoenfeld, 1985).

Broussard's theory of didactical situations (TDS) is applied to analyse how the approach can motivate the individual student to become an active and independent participants in the lessons; TDS is used in both, the study of the lesson plan of "sums of exterior-angles" and in comments to the classroom observations.

4.5.2 Transcripts

According to Cohen, Manion and Morrisson (2007), it is inescapable that transcriptions "lose data from the original encounter" (p. 367). A transcription implicates the transformation from oral systems to written systems. In that meaning, data on transcripts are "*already interpreted*" (ibid., p. 367). That is why video recording may present a richer set of data, by showing non-verbal communication which the written text does not express. In my study, however, since the camera was placed behind the students, the video recording did not catch the details of students' facial expressions except in the case when they presented their solution by the whiteboard for the whole class. At the point of writing this thesis, the written observation protocols were quite useful in order to grasp what had been going on.

I did not use any transcription software but wrote down selected sequences myself. Transcribing was done in Swedish and then translated into English. All analysis was done before the transcripts were translated. The orthography of the transcription in this study includes the following symbols:

- ... A short pause
- (...) A longer pause
- (word) A description of an occurrence
- [word] A note
- T: Identity of the speaker. T is the teacher.

4.6 Reflection on the quality of the study

Kvale (1997) holds that the relation between the researcher and the participants in a research project is an important ethical issue. During the period of the study, my

existence became a natural part of the lessons, since I attended mathematics classes every week during the study of classes, grade seven and grade eight. My role converted to some kind of participant observer more than that of an observer from outside (Robson, 2002). In that way, the students seemed unaffected by my presence and the video-recordings. The relationship between the teacher and me as researcher was open and friendly with mutual respect and understanding; we cooperated and worked together several times per week to plan the lessons during ten months in the project. Basically, I never expressed my opinion about the teacher's skills of teaching except when the teacher asked me about my opinion, or when I needed to let the teacher understand the matter of following some prescription from the method of the PSO.

The work with the historical perspective for development of the structured problem solving, description and analysis of Souma's lesson plan, analysis of Japanese and Swedish curriculum is literature review, did not involve much ethical consideration, except when I referred to some specific content from a specific author to get agreement.

The research reported in this thesis is done within the *qualitative research paradigm* (Ernest, 1998). There, the word *paradigm* is interpolated as "set of basic beliefs" and it represents a "worldview" (Guba & Lincoln, 1994, p. 107). The focus of qualitative research is on the process of inquiry (Creswell, 1994) and using multiple methods for interpreting natural phenomena (ibid.; Guba & Lincoln, 1994). Robson (2002) points out that validity in a study with qualitative research might be a difficult issue, but it is possible to "recognize situations and circumstance which make validity more likely" (p. 170) through constructing the research with "good" flexible design. He describes "using multiple methods" (pp. 370-373) and states the importance of triangulation that is using multiple methods to gathering data to increase understanding of the complexity of the phenomenon; thereby enhancing the validity of the study. For that reason, both the theoretical and the empirical study on the PSO approach were made in this thesis. The intention was to estimate the outcome, so as to illustrate the whole picture of using this Japanese approach. The interpretation of the empirical study, the classroom observation, was theoretically driven in order to answer the research questions. A short summary of the results from the student questionnaire, the interviews with the students and the teacher were also implemented to support the analysis of the classroom observations.

4.7 Ethical considerations

Cohen et al. (2007) stress the researcher's responsibility of ethical behaviour towards participants of a study.

Whatever the specific nature of their work, social researchers must take into account the effects of the research on participants, and act in such a way as to preserve their dignity as human beings (p. 58).

Regarding the classroom research in junior secondary school, the teacher was asked to inform the students about the research study. In the classroom, the teacher explained the purpose of the study and the long-time presence of the researcher. All students accepted the researcher's presence in the classroom and the video recording. To parents, information was given in written form as a letter. I attended a parents meeting before the project began, where I informed about the intention and design of the study, the guiding ethical principles, and how the research material would be used. I also explained that the PSO approach will not spoil the concept of Montessori pedagogy in mathematics, which stresses the use of practical and laboratory material for a part of the lessons. All the parents were positive to the study and most of them signed written consent forms to allow their children to be filmed and interviewed. One pair of parents denied to their child being video recorded. The teacher then got permission that their child would be placed in a dead angle with respect to the video camera, so that the child would not be exposed in the video recording. In the text, the teacher is presented anonymously and students are given pseudonyms (preserving gender distinction). The questionnaires to the students were implemented non-anonymously, however, the students' names are not presented in the result.

The teacher elucidated the students several times during the first term that answering correctly was not an important part of mathematical lessons and encouraged them to try to explain how they reason in front of the class. They were explained that also their mistakes and difficulties are important for the learning process and also of interest for the research.

Concerning the security of storage and treatment of data, only the researchers involved in this licentiate study, that is myself and my two supervisors, have had access to the record of data which has been reported to Linköping University.

Chapter 5

Results from the theoretical study

In this chapter I describe the Japanese and Swedish curricula and I comment on the extent of which the discrepancies between these two curricula influence the adaptation of Souma's lesson plans. Secondly, I describe the structure of the didactical praxeologies in Souma's lesson plans.

5.1 Japanese and Swedish curricula in Arithmetic, Algebra and Geometry

In this section, I present an overview of the guidelines for the Japanese curriculum in mathematics. It is meant to explain how the chapters of Arithmetic, Algebra and Geometry are positioned in the curriculum. I also expose the guidelines explanation of the relationship between “*mathematical activities*” and “*problem solving*”. Finally, I give a detailed description of the chapters concerning Arithmetic, Algebra and Geometry. Then the corresponding sections within the Swedish curriculum are discussed.

5.1.1 The Guidelines for the “Course of Study”

In the chapter “The Goal and Contents of Mathematics” in the guidelines for The Course of Study (MEXT, 2008b), there is a discussion of the “significance of coaching the students *mathematical activities* (ibid., p. 32)”. What is called mathematical activity refers to, for example: to explore and grasp the nature of numbers and geometrical figures; to apply previous knowledge in new situations; and to communicate in the class using a mathematical language.

The guideline declares that the mathematical activities shall be carried out through the process of *problem solving* (ibid., p. 32). The process of problem solving is described in the same way as Polya described his four phases of problem solving (Polya, 1957). The guidelines also describe, in some length, the positive effects the experience of problem solving has on the students' self-esteem and their appreciation of mathematics.

In the section “Approaches to Content Organization”, the core concepts of the curriculum are listed as below (ibid., p. 26):

1. Concept of numbers and expanding the range of numbers
2. Euclidean spaces
3. Functions
4. Random outcomes
5. Algebraic expressions with letters
6. Mathematical reasoning
7. Explaining and communicating

According to the guidelines, the core concepts 1 to 3 belong to the “mathematical world” (MEXT, 2008b, p. 26). It means that the goal is for the students to use those concepts as mathematical objects in their modelling activities. The core concept 4, “random outcomes”, belongs to the “real world” (ibid., p. 27) and the goal is for students to model random phenomena mathematically. Concepts 5, 6 and 7 are said to be “frameworks” that support the learning of core concepts 1 to 4.

With regards to core concept one, the goal of “expanding of the concept of numbers” is emphasised. Students should learn to use the terms of natural numbers, integers, rational numbers and irrational numbers. The word *fraction* will be learned as *rational number* instead, so that it is considered as a *fractional representation* of a rational number.

Geometry in the Euclidean space is described as a model for working with forms existing in our world, such as lines and planes. The aim is also to figure out the logical relationship between the mathematical model and the real world. It is underlined that it is important to learn the appropriate terminology. Especially those terms that expresses figures and their position and orientation in space. The students should observe figures and express them in mathematical form as equations and be able to treat such figures as mathematical objects.

In “Algebraic expressions with letters”, *the use of letters* (variables) is designated as a tool to *describe* phenomena in our everyday life as mathematized relations. It is also treated as a method for “solving problems”, since the mathematical interpretation captured by mathematical expressions clearly exposes fundamental relations regulating the phenomena modelled.

In Core concept six, “Mathematical reasoning”, the inductive method, the deductive method and the analogical inference are explained. The skill of constructing a deductive proof is emphasised; for example, a proof for the sum of angles in a triangle. The cognitive process of mathematical communication is highlighted in the last Core concept “Explaining and communicating”.

The content description for grades 7, 8 and 9 is divided into four domains:

- A. Numbers and Algebraic Expressions
- B. Geometrical Figures (the term *geometry* is not used)
- C. Functions
- D. Making use of data

This division of the content is common for all grades from 7 to 9. For these domains, the learning goals and their significance are described in detail. Neither in The Course of Study nor in the guidelines is the term “*algebra*” found. Instead, the description of *using letters* is located under the domain “*Numbers and Algebraic Expressions*” together with arithmetic. The expression “*Algebraic Expressions*” is used in MEXT’s English translation of The Course of Study (MEXT, 2010). But the headline of this domain in the Japanese version, is actually “Numbers and Expressions”, without the term “algebraic” (MEXT, 2008a). The term “variables” is not used, neither in The Course of Study nor in the guidelines; instead, the term used is “letters”.

5.1.2 The description of the chapters “Numbers and Algebraic Expressions” and “Geometrical Figures” in the Guidelines

In the chapter “Numbers and Algebraic Expressions” in The Course of Study, it is emphasised that one should cultivate students’ ability to use variables to represent the relationships and rules of numbers and to represent quantities in algebraic equations. One also underlines the understanding of representations with expressions together with algebraic calculation skills.

Here, we will look at “A. Numbers and algebraic expressions” for grade 7 and “B. Geometrical Figures” for grade 8 in the guidelines (MEXT, 2008b).

A. Numbers and Algebraic Expressions (grade 7):

Key goals are stated as “expansion of the number concept” to integers with negative numbers as a larger set of numbers, “understanding the necessity and meaning of positive and negative numbers”, and “the meaning of the four fundamental arithmetic operations with positive and negative numbers” (ibid., p. 56). In the section “Terms and Symbols”, it is stated that students should become familiar with the terms “*natural numbers*”, “*signs*” and “*absolute values*”. Further, the guidelines continues, students should “understand the necessity and meaning of positive and negative numbers by relating these concepts to their own experiences in daily life; for example, by looking at temperature differences” (ibid., p. 57). Students should be aware of the convenience of using positive and negative numbers, for example (ibid., p. 57):

- To be able to represent opposite directions and opposite properties
- To be able to compare quantities
- To be able to express quantities on the number line
- To be able to use subtraction in all circumstances
- To be able to express the relationship between addition and subtraction coherently

It is also a goal to strengthen the confidence in using the operations of arithmetic with positive and negative numbers: “By using negative numbers, one can consider subtraction $3 - 2$, as an addition $(+3) + (-2)$. Considering addition

and subtraction coherently means that one considers arithmetic expressions as sums with positive and negative terms. Also it gives the possibility to express the distance between a and b on a number line, coherently as $a - b$ if $a > b$. For the students' further study, it is important to be able to do such considerations in calculations with variables and in the solutions of equations. It is hence necessary to train the skills properly" (ibid., pp. 57-58).

The description of the goals of learning negative numbers ends with a concern about applications of the concept in our daily-life: "For example, you place a goal for something. The value of this goal, you can consider as the starting point, or 'origin' and evaluate the current situation as the difference relative this origin; *positive* or *negative*, instead of *over* or *under* the goal" (ibid., p. 58).

The term *absolute value* is mentioned in "Terms and symbols", but is not present in the descriptions for grades 8 and 9. Thus, the concept of "absolute value" appears only once in relation to "positive and negative numbers".

Regarding algebraic expressions, it is underlined that in order to develop students' skills to express quantities and mathematical laws with letters, one needs skills to interpret and to calculate. The guidelines explain, in "Necessity and meaning of using letters" that:

Expressions with letters are necessary to generalise and express relationship of quantities and mathematical laws clearly. ...For instance, it is possible to show the commutative property of addition in an expression $2 + 3 = 3 + 2$ but it does not show how it can be generally applied. In this case, using letters makes it possible to show the commutative property of addition as $a + b = b + a$. Moreover, applying of letters makes it possible to consider relations of mathematical quantities in an abstract manner. For instance, an expression $s = ab$ can be considered as expressing *an area of rectangle = length times depth*, or, *a price of something = a unit cost times quantity*, or, *a distance = velocity times time*. Then we can transform $s = ab$ to $a = s/b$ and go on to consider derived relationships between the quantities. (ibid., p. 59)

The guidelines continue to explain the advantage of using letters when one wishes to communicate one's thought process with other people. It is illustrated by an example of showing that the number of matchsticks making squares will be of the form $4n - (n - 1)$, or $2n + (n + 1)$. It shows, not only the sum of number of matches, but also a visual process on how to reason to find out the solution.

Furthermore, the guidelines describe the use of variables as a tool for making models:

Using letters are an outstanding way of expression. It is important that students express quantities and mathematical laws with letters. By doing so, they become able to understand the meaning of said expressions and eventually they can keenly apply the tool of algebraic expressions. For example, if an entrance fee of a museum is a JPY/adult and b JPY/child,

then ‘the sum of the entrance fee for one adult and two children’ will be expressed as $a + 2b$. Also $a - b$ means ‘price difference between adult and child’ (ibid., p. 61).

Then the relationships between modelling, understanding equations and solving equations are described. In subsequent section, the focus of the guidelines moves to underline the necessity and significance of solving equations and especially to understand the meaning of an equation. The cancellation laws for equations are explained (ibid., p. 62):

1. $a = b$ is equivalent to $a + c = b + c$
2. $a = b$ is equivalent to $a - c = b - c$
3. if $c \neq 0$ then the equation $a = b$ is equivalent to $ac = bc$
4. if $c \neq 0$ then $\frac{a}{c} = \frac{b}{c}$ if and only if $a = b$

Here, it is noted that one can consider cancellation law one and two as the same by using negative numbers. It is also stated that one should consider 3 and 4 as inverse operations. The cancellation laws are to be used in the process of solving linear equations and the section ends with the explanation of a detailed process of solving linear equations.

In summary, the focus in the chapter “Numbers and Algebraic Expressions” is on the expansion of the number concept and on the use of algebra as a *tool* for *modelling* and *generalisation*.

B. Geometrical Figures (grade 8):

In the chapter “Geometrical Figures” (ibid., p. 39) one establishes initially that

1. It often happens that we consider/estimate phenomenon around us from the perspective of its *figure*, *dimension* and *position*. That is why it is important to deepen the students’ understanding of the basic properties and associated quantities of plain and solid figures.
2. The logical view and the logical way of thinking, which has a very important role for the learning of great many disciplines, should be trained through the process of making mathematical proofs. Students should study properties of figures using their intuitive perception by and knowledge of plain and solid figures. This will also raise their aptitude for reasoning and for making judgments. The topic of ‘Geometrical Figures’ should develop students’ abilities to inquire and to express their thoughts logically.

The focus is much on how to train students’ skills of observation and logical argument. It pervades all three grades. The content of “Geometrical Figures” for grade 8 consists of:

- Parallel lines and angles
- Property of polygons
- Congruence of polygons
- Mathematical assumptions

- Necessity and significance of using proofs
- Properties of triangles and parallelogram
- Reading a proof and finding out new properties

Here, I will describe the first three sections from the guidelines: “Parallel lines and angles”, “Property of polygons” and “Congruence of polygons”. These three sections are described in my analysis of Souma’s lesson plans.

Parallel lines and angles (ibid., p. 92):

“The students begin the learning of expressing mathematical proofs coherently and logically by studying the property of parallel lines and vertical angles” (ibid., 92). By learning the properties of vertical angles, parallel lines and transversal lines and corresponding angles, students will be able to draw, as a conclusion, the equality of alternate angles and vice versa: if the alternate angles are equal, then the two lines are parallel. After that, students should learn to prove that the angular sum of a triangle is 180° , by reasoning on a parallelogram.

Property of polygons (ibid., pp. 92 - 93):

It is stressed that one should use previous knowledge – the angular sum of a triangle – to deduce the corresponding result on the sum of the inner angles of polygons. The guidelines explain that it is a core mathematical strategy to use previous knowledge when reasoning and inquiring on new phenomena. The determination of the sum of polygons’ exterior angles is described in the same way.

Congruence of polygons (ibid., pp. 93 – 94):

The goal is to deepen the students’ perception of geometrical figures by understanding congruence of polygons. One starts with the “conditions of congruence for triangles”. In this way, one develops the students’ abilities of logical thinking and their ability to express their reasoning. Also, understandings of the following facts are stressed:

1. Significance of congruence of polygons and the conditions of congruence of triangles.
2. The necessity and the significance of using and understanding proofs.
3. Consider basic properties of triangles and parallelograms by studying conditions of congruence of triangles. Find new properties of figures by studying the method of proofs in the theorems.

The chapter “Geometrical Figures” in the Course of Study has a strong emphasis on logic. The skill of constructing deductive proofs by considering previously learned propositions and theorems is central.

5.1.3 The Swedish curriculum

Lpo94

The curriculum in mathematics for the compulsory school (Skolverket, 2000), which is part of the “Curriculum for the compulsory school system, the pre-school class and the leisure-time centre Lpo94” from 1994 (Skolverket, 1994), begins by stating “*Goals to strive towards*”, in which, regarding arithmetic, algebra and geometry, it states that a student should:

- develop the ability to use logical reasoning, drawing conclusions generalising, and orally and in writing explain and argue for their thinking
- develop ability to use simple mathematical models and, modelling, and critically examine the conditions of the models, limitations and its use
- develop understanding of use of numbers and spatial awareness, and use basic number conception and calculating with real numbers, approximations and proportionality
- develop ability to use measurement systems and measurement instruments to compare, estimate and determine major quantities
- understand basic geometrical concept, properties, relations and theorems
- understand basic algebraic concept, expressions, formulas, equations and inequalities

The observable goals listed in “*The goal for mathematical knowledge*” in Lpo94 stipulate that a student in grade six to nine should:

- have developed his basic number sense, which includes integers, rational numbers in fractions and decimal form
- have good skills in mental calculations with natural numbers, decimal numbers, percentages and proportionalities, with the help of written calculation algorithms and technical aids
- be able to use measurement system and measurement instruments in order to compare, estimate and determine lengths, areas, volumes, angles, masses, time and time differences
- be able to reproduce and describe major characteristics of common geometrical objects and to interpret and use figures and maps
- be able to interpret and use simple formulas, to solve simple equations, and interpret and use graphs of functions which describe real conditions and events

In comparison to the “*Goals to strive towards*”, the focus is on knowledge for practical use and calculating skills and the roles of *mathematical proofs* and *generalisations* are reduced. Understanding basic *geometric concepts, properties, relations* and *theorems* has changed to “reproduce and describe major

characteristics of common geometrical objects”. The use of the skills in modelling applications is emphasised.

The companion document “*The Commentary for Compulsory School’s Curriculum and Grading Criteria in Mathematics*” (in Swedish, ”Kommentar till grundskolans kursplaner och betygskriterier i matematik”) (Skolverket, 1997) is a much less detailed document in comparison to its Japanese counterpart: The Japanese Guidelines have 142 pages for the course in mathematics from grade 7 to 9, while the Swedish commentary consists of 38 pages for all grades in the compulsory school.

In the chapter “Aim” (in Swedish “Syfte”) in the Swedish commentary one declares that “developing the competency of problem solving” (ibid., p. 12) is a motive and aim for all mathematics education: “Problem solving is an important *medium* to develop concept and mathematical thinking – problem-based learning” (ibid., p. 12).

In the section “*Goals to strive towards*” in the Commentary, it is described how contents and aims in the curriculum have changed from previous curricula Lgr 80 and earlier, with respect to the following aspects (ibid., p. 13-21):

- students’ own confidence of abilities of learning mathematics
- historical development
- basic mathematical concepts and methods
- different representation forms”
- problem solving
- mathematical models
- use of electronic calculators and computers

In the same section, the Commentary underlines the importance of finding relations between different mathematical forms of representation, mathematical expressions and phenomena in everyday life and the importance of understanding the transition between reality and mathematical models and theories.

The contents of the mathematical knowledge are outlined in individual paragraphs titled “Number Sense”, “Space Sense”, “Measurement and Geometry” and “Algebra, Equations and Functions”. What characterises a “good number sense” is exemplified in some detail, but the proficiency of using different arithmetical representation form for calculating is the main characteristic. Regarding geometry, in the section “Measurement and Geometry” qualitative aspects are considered (ibid., p. 25):

- skills such as to recognise, describe, depict and construct, and patterns and aesthetic perspective
- logical reasoning and deductive study
- science of the space as a tool for describing and measuring in idealised models of the physical world and the phenomena around us
- as a way to illustrate the concepts and processes in other subjects.

One concludes by stating that geometry is a way of thinking, a way of understanding and also a field of science with its own comprehensive terminology and set of concepts.

These perspectives of geometry are quite similar to those discussed in the Japanese guidelines. However, unlike the Japanese guidelines, there is not given any set of examples illustrating the content.

When it comes to the short section named “Algebra, Equations and Functions” in the Commentary, it is mentioned that Swedish students have presented results that are worse than comparable countries in algebra. Algebra (ibid., p. 26) is important since it trains the skill of interpreting formulas and algebraic relations. The study of algebra should ease the transition from calculating with numbers and then letters. It is also important to learn to recognise the similarities and differences between these two ways of calculation.

The description of mathematics in the curriculum Lgr 11 (Skolverket, 2011a) begins with a section titled “The Purpose of the Subject”. There, it is stressed that the teaching should aim to develop the students’ ability to use mathematics in everyday life and to use it for studies in other subjects. The focus in Lgr 11 is on the student’s competencies in problem solving, communication, reasoning, conceptual analysis and the application of appropriate mathematical methods. The education should help students “to formulate and solve problems and also reflect over and evaluate selected strategies, methods, models and result” and also to help students “to argue logically and apply mathematical reasoning” (ibid., p 59).

The section titled “Central Contents” has sub-headings “Understanding and Use of Numbers”, “Algebra”, “Geometry” and “Problem Solving”. It is more detailed in comparison to the corresponding sections in Lpo94. In *Commentary Material* (Skolverket, 2011b), there is a section titled “Geometrical Theorems” (ibid., p. 21). It is declared that the students should have the opportunity to argue for the validity of equations and to show the relationship between different basic geometrical concepts: “this knowledge will give students a possibility to understand the significance of theorems and proofs, for their future study” (ibid., p. 21, author’s emphasis). Thus, to understand the significance of geometrical theorems and proofs is *not* required in lower secondary school. The outlined motivations for algebra in the *Commentary Material* for grade 7 to 9 (ibid., pp. 17-18) are:

- as a tool for general reasoning in problem solving
- as a tool for modelling
- as a tool for supporting other areas, especially geometry
- as a tool for mastering solutions of equations

The difference from Lpo 94, “general reasoning” is clear. As in Lpo94, the importance of modelling is also emphasised.

In summary, the Swedish curriculum, both Lpo94 and Lgr11 for grades 7 to 9, do not put any strong emphasis on mathematical proofs in geometry. The

Commentary for Lpo 94 mentions problem solving, logical reasoning and deductive thinking, and Lgr11 mentions general reasoning in problem solving and geometry. However, neither document mentions deductive *proofs*. In algebra, both Lpo94 and Lgr11 focus on it as a tool for modelling and arriving at general parameterised solutions.

5.2 The structure of PSO approach based lesson plans

5.2.1 The basic structure of PSO

The template lesson episodes, which are presented in Souma's lesson plans, have the following structure:

1. State a problem in a way to incite a guess or immediate observations on part of the students. (*Hatsumon*/stating of the problem)
2. Let all students state a guess or hypothesis. (*Yosou*/guessing)
3. Discuss the viability of guesses and let students motivate their guesses
4. Let students, individually or in groups, work on the problem, or, perhaps, on a derived reformulation or a sub-problem. The teacher monitors students' various solutions and plan in which order, students should present their solutions in the next whole-class discussions. (*Kikan-shido*/monitoring)
5. Let students present their solutions in the class and discuss the different solution techniques. (*Neriage*/whole-class discussions)
6. Turn to the textbook for an outline of the theory. (*Matome*/summing up)

Thus, the structure of the PSO lesson follows, largely, that of Japanese structured problem solving as it is described in Chapter 1. It differs in that it initially lets the students state a guess or hypothesis and that it induces them to reformulate the problem and clarify the main problem. Souma describes the four different types of this initial open-ended problem as follows (Souma, 1997):

- Demands an answer: "How many cm is ~?", "What kind of triangle is this?"
- Students choose a response from the few alternatives: "Which of these expressions are actually same?", "Which of these expressions are right /wrong?"
- Right or wrong: "Is it correct that ~?" "Is it the same as ~?"
- Discovering phenomenon/patterns: "What can you say about the following expressions?"

An example of the last type of problem would be: "What can you say about following expressions: $(5^2 - 4^2)$, $(8^2 - 7^2)$, $(4^2 - 3^2)$?" Souma means that, if the problem takes the form "Show that the difference of the squares of two integers that follow each other is equal to the sum of the two numbers", there will be some

students that have no idea where to begin. However, if the problem is stated as above, they may come up with several initial guesses, for example “the differences equal the sum of the integers”, “the differences equals the first integer times two minus one”, or “the last integer times two plus one”. When they have clarified if each guess is correct or not, the teacher can present the actual mathematical task to the class. “Do those statements *always* hold?” The formulated problems should have many possible solutions. Some students may use the formula for expanding the square of a sum; and some others, using x to the first integer and y for the second integer, the rule of the conjugate.

Souma explains different ways of driving the whole-class discussions (1997, pp. 66-71):

Aa. *Present different students’ different solution-methods one by one.* During Kikan-shido (monitoring of students’ solutions), a teacher decides which students’ solution will be presented in which order. This approach reflects the teacher’s intentions best. Usually, the most common solution method is selected first. Then “smarter” or more specific methods are presented later. Students will listen to different solutions and eventually understand the properties of the problem.

Ab. *Letting students raise the hands and let them present their solutions.*

A teacher gives some minutes to consider the problem and letting the class raise the hands. Unlike **Aa**, the teacher cannot assume exactly which kind of solutions students are going to present. It requires a teacher’s flexible correspondence depending on students’ response. On the other hand, it will produce a kind of excitation and curiosity in the class since the students know that the ideas were not picked and filtered by the teacher.

Ba. *Presenting different students’ different solution-methods at once on the blackboard.*

Same as the pattern **Aa**, a teacher chooses some individual students’ solutions or group-solutions during the Kikan-shido moment and then let all of them write their methods on the blackboard at once. Students/groups explain their methods one by one and then, a teacher lets the class compare the different methods. To write the whole solution might be too time consuming: Often a teacher instructs students to write a part of the solution or to just illustrate the core idea with some geometrical construction.

Bb. *Letting students raise the hands and present their solutions on the blackboard at once.*

(This approach is a combination of **Ab** and **Ba**.) A teacher selects students from those that raise hands and have all of them write their solutions on the blackboard. It might lead to an overlap of solutions and therefore the teacher often asks the individuals beforehand. A teacher could have eye on students’ solutions during the Kikan-shido moment and selects the students with different ideas. In that case, it actually is the same as the pattern **Ba**.

5.2.2 The structure of the sections in the chapter “Parallels and congruence”

I illustrate the PSO approach with an example lesson plan titled “*Sums of exterior-angles*”. The context of this lesson plan is the chapter “*Parallels and congruence*” in Book 2 (Kunimune & Souma, 2009b), which is intended for students in grade eight (Book 1 is for grade seven and Book 3 is aimed at grade nine). The content of Book 2 is:

Chapter Arithmetic and algebraic expression:

Section 1. Expressions and calculations

Section 2. Applying algebraic expressions

Chapter System of linear equations:

Section 3. First degree equations with two variables and their solutions

Section 4. Systems of linear equations and their solutions

Section 5. Applications of systems of linear equations

Chapter A linear function:

Section 6. The meaning of linear functions

Section 7. Graphs of linear functions

Section 8. A linear function and a first degree equation with two variables

Section 9. Applications of linear functions

Chapter Parallels and congruence:

Section 10. Parallel lines and angles

Section 11. Angles of polygons

Section 12. Congruency of polygons

Chapter Triangles and quadrilaterals:

Section 13. Proof for the construction of a bisector

Section 14. A condition for a parallelogram

Section 15. Variations of quadrilaterals

Chapter Probability:

Section 16. Meaning of probability

Section 17. Methods for solution in probability problems

Section 18. Uncertain phenomena and its probability

The lesson plan “*Sums of exterior-angles*” is thus located in section 11 “*Angles of polygons*” in the chapter “*Parallels and congruence*”. Sections 10 and 11 are strongly linked to each other and the lesson plans associated to section 10 and 11 are:

- Parallel lines and angles (3 lessons)
- Angles of triangles (1 lesson)

- Angles of polygons (2 lessons)
- Activity project (laboratory work) on polygons (2 lessons)

The first three lesson plans, “*Parallel lines and angles*”, have the intention to introduce the laws about parallel lines and angles. “*Angles of triangles*” is a problem solving lesson that leads to establish the sum of triangles’ inner and exterior angles. The lesson preceding “*Sums of exterior-angles*” is about calculating the sums of a general polygon’s *inner angles* using the algebraic method where the number n of vertices of the polygon is the parameter of the solution. This formula is supposedly a crucial technological component used to find out the sum of exterior angles in the lesson “*Sums of exterior-angles*”.

The content of sections 10 and 11 is used as a preparation for later proving theorems on congruency of polygons. The succeeding lesson nine plans in section 12 are:

- Congruency of polygons (1 lesson)
- Conditions for congruency of triangles (3 lessons)
- Proof of congruency (3 lessons)
- Construction/drawing (of triangles) and proofs (2 lessons)

After those lessons, the chapter finishes with lessons on standard exercises from the chapter (2 lessons).

There are subsequent lesson plans for geometry in section 13-15: Section 13 treats properties of isosceles triangles, the congruency of right triangles and the proof for the construction of a bisector. Section 14 treats properties of parallelograms, right triangles, isosceles triangles and circles. In section 15, one covers other kinds of quadrilaterals and their properties.

5.2.3 The mathematical organisation of the sequence of lesson plans in Geometrical figures

The technological components of the regional praxeology (Garcia et al., 2006) under construction are the terms and the theorems that describe parallel lines, polygons, alternate angles, interior/exterior angles, for example. In this lesson a key technological component is the previously established formula for interior angles. Also, one also depends on the technology of linear expressions in the integer variable n , the number of vertices of the polygon. The complexity of this regional praxeology is substantial.

In a typical lesson plan, it is assumed that students should apply previously established techniques and technological elements, in order to explore and solve new tasks and learn new techniques. Many lessons aim to establish some new technological components. In the chosen example lesson one aims for a formula for the sum of exterior angles. The solution methods of the proposed problems are readily generalised to arguments that establish more general statements.

The lesson plans serve appropriate questions to students as starting points for exploring solution methods. We show in the next section that the example lesson

realises all six modes FE, EX, T, TT, I, and EV of the didactical process (Barbé, et al., 2005). According to Garcia et al. (2006), such lessons establish local praxeologies. By its context, the local praxeology of the lesson is integrated in the larger regional praxeology that results from the contextual sequence of lessons. The formulation of the initial question is also motivated from a didactical perspective; it should stimulate discussions among students and allow each one to participate.

5.2.4 The flow of the lesson “Sums of exterior-angles”

The goals of this lesson are described as (Kunimune & Souma, 2009b, p. 55):

- To understand that the sum of exterior-angles of a polygon is 360° .
- To demonstrate how to express the sums of exterior angles by using previous knowledge.

The problem in this lesson plan is posed as follows:

Which of the sums of exterior-angles is the largest? The triangle's, or the pentagon's? (See Figure 5.1)

Figure 5.1: Illustration of the problem *Which of the sums of exterior-angles is the largest? In the triangle or the pentagon?*

The intention is to establish that the sum of any polygon's exterior angles is always 360° . Using the theorem of angular sums of a triangle, a previous lesson has established that the sum of a polygons inner angles is always $(n-2) \cdot 180^\circ$.

How to pose questions is an important aspect of the PSO. Souma proposes in his book (1995) several ways to formulate a question: “How many is that?”, “Is it correct that...?”, “Is it possible to calculate...?”, “Which of them are correct?”, and so on. He also points out the importance of encouraging students to *observe* the phenomenon. For example, by asking students; “What (kind of relationships, or patterns) can you see here?”

The PSO approach does not suggest to leave the textbook behind. The teacher can start from standard problems in the ordinary textbooks of mathematics and modify a part of the problem or change the way of stating them.

In this case, the students should come up with three alternatives: that the sum of the triangle's exterior angles is largest, that the pentagon's is largest or they are equal. The intention is to make the students curious to know which of the guesses are correct. Thus it is important to construct problems that lead to a variety of guesses (Kunimune & Souma, 2009b).

The teacher should let the students briefly present the reasoning for each alternative and ask them, “How can we decide which of the guesses is the correct one?” The sought reply from the students is something along the line: “We should determine the sum of exterior angles of each polygon”. The students should spontaneously focus on the sub-problem: “Demonstrate a technique that computes the sum of the exterior angles”.

Here, we will study the lesson plan using the *dynamic of the didactical process* (Barbé, et al., 2005). The guessing technique gives strength to the first encounter (FE) with the task. It also leads the students into the exploratory moment (EX), where students start to comprehend the problem and it motivates them to start to tackle the problem and to participate in the discourse. Souma refers to this part of the lesson plan as the “clarification of the main problem” (Kunimune & Souma, 2009b, p. 56). Then, the didactic process shifts to (T) – the *technical-work* moment. After a few minutes of individual considerations, the students discuss in groups to find out the methods for ascertaining the sum of angles.

The teacher walks around in a few minutes to check which groups come up with what kind of methods, and let some of the students present their reasoning. In preparing the lesson plan, the teacher is supposed to consider all anticipatable methods the students might come up with, so that she/he will be prepared to be able to evaluate the quality of the solutions and to give appropriate help and response.

Some students measured every angle using a protractor and certify that the sum of the triangle will be, say, 359° and 362° for the pentagon. Some students cut the paper as sectors along the exterior angles and show that it makes a circle, thus both sums must be 360° . Other students use their previous knowledge – that of the sum of the inner angles of a triangle. They calculate the sum of all inner and exterior angles as $n \times 360^\circ$ and subtract the sum of the inner angles of the triangle.

The students are then able to compare the different techniques and to evaluate the exposed methods. Hence, the didactic moment is driven to (TT) the *technological-theoretical* moment that looks at the solutions and decides if the techniques are viable and motivated by the existing technology.

The teacher lets the class find the sum of the exterior angles for the pentagon using the same method. The students observe the number of the angles and come up to $(180^\circ \cdot 5) - 180^\circ \cdot (5 - 2) = 360^\circ$. The class has finally certified that the answer to the first problem is “the same” and even the students who had a different guessing agree with the result. Then the teacher gives the class the core question: “Is the sum of exterior angles of *any kind of* polygons *always* 360° ?” Souma names this moment as “the generalisation of the phenomenon”. Some students suggest applying a variable n to count the number of vertices in a polygon as they learned at the previous lesson and find $180^\circ \cdot n - 180^\circ \cdot (n - 2)$. Those solutions are written on the blackboard by the teacher.

In Chevallard's description, the didactic process shifts to (TT), the *technological-theoretical* moment. It is also an instance of (I) the *institutionalisation* moment, since the question is here if one should/can extend the technology with the theorem stating that the exterior sum of angles equals 360° . Of course, the question posed also implies a new task of constructing a *proof* of the same theorem. It is a task which is somewhat external to the local praxeology concerning polygons constructed; it uses rules for handling algebraic expressions.

Then, the teacher asks the class to refer to the *textbook* and compares the explanations in the book relevant for the lessons. During this theoretical reflection, the didactic process steers towards (I), *institutionalisation*, to see what parts of the technology should be kept and canonised, and what should be reformulated. It is also a moment of evaluation (EV); a place to see for what purposes the obtained results can be used in other areas. Souma calls this moment "Control and application".

Souma (1997) states that studies in mathematics should be organised and based on a well-written textbook that gives a clear explanation of the mathematical definitions and theories. He claims that this process will also make students feel more confident about what they have actually learned in class. The textbook allows the students to recognize and get familiar with the theory, which the textbooks usually explain in more detail than during a classroom discourse.

Finally, the lesson finishes with exercise problems from the textbook and the students see if they can use the formula they have learned in order to solve similar tasks. The didactic process now oscillates between (FE), (EX), (T) and (TT).

The guessing technique

As an example, Souma states that if the teacher formulates the problem in this lesson as "In the previous lesson, we learned the method for calculating a sum of inner angles. Today, we will learn how to figure out the sum of exterior angles", or, "Prove that the sum of exterior angles is 360° ", then the task becomes "*make a proof for the sake of making a proof*" (Souma, 1997). The risk is that the students would not construct the result by themselves and would thus experience no "*necessity*" to explore the phenomena and to reason and they learn it just as a fact. Another motivational problem with the ordinary way to formulate the problems is that some students may not get any ideas on how to solve the problem and will then become alienated from the mathematical discourse. However, "guessing" is something that any students can participate in. Souma compares the phenomenon "guessing, which leads to motivate students to participate in the lesson" with a quiz show on TV programs, where both the participants and the viewer take a part in the game through "guessing" the right answers. One feels satisfaction when the guess was right. Even when it was not, it is fun to acquire new knowledge (Souma, 1995).

If the guessing technique is analysed using TDS, we understand the posing of the initial question as a play by the teacher in the didactical game. By force of the didactical contract, each student will give a guess. By using guessing, each student

can participate. The act of guessing will be a move to an adidactical “*situation of action*”, which is later followed up by a “*situation of formulation and validation*”. Souma’s motivation, that guessing leads to “necessity” to explore the phenomena, can be interpreted as a statement of the adidactical nature of the situation reached.

Of course, like any didactic technique, the PSO method suggests many other plays by the teacher during the discourse of the lesson, that *force* the students into adidactical situations, but it is the initial guessing technique that is significant for the PSO.

Chapter 6

Designing lessons in Swedish mathematics classrooms

In this chapter I present the result from the design study of applying the PSO approach in Swedish mathematics classroom. Detailed lesson processes of implemented lessons in the grade eight and grade seven classrooms are described here. The first case study concerns a lesson that introduces of subtraction and multiplication with negative numbers and the second case study concerns an introduction to finding general solutions. In both lessons, Souma's lesson plans are adapted. I will also present the results from the student questionnaire and interviews, as well as the results from the interview with the teacher.

6.1 The lessons “Operations on negative numbers”

It is a didactical challenge to convince students *why* the product of two negative numbers should be positive because of its lack of everyday-life character (Hefendehl-Hebeker, 1991). Sfard (2000) discusses why certain mathematical concepts and arguments are unconvincing to many students. She describes an example of this difficulty about teaching negative numbers and explains that the reason is based on the different recognition of the discourse of learning between a teacher and a learner.

The section “Multiplication of positive numbers and negative numbers” is located in the chapter “Arithmetic operations on positive and negative numbers” in Souma's lesson plans for grad 7, (book 1) (Kunimune & Souma, 2009a). It is observed in the book that the focus in this sector is usually on the calculating skills with negative numbers; developing students' ability to reason about it is commonly less emphasised. Thus the goal of this lesson is described as that of raising the students' ability of mathematical reasoning through observation on how multiplication of positive numbers generalises to the rule of multiplications with negative numbers (ibid., p.18).

The mathematical/didactical task in this lesson is to establish a local praxeology that gives the students opportunity to reason by themselves *why* the minus times minus operation works.

Some conceivable techniques (τ) that explain how the product of two negative numbers should be positive, are the following (quoted and arranged for our problem from Durand-Guerrier, Winsløw & Yoshida, 2010, p. 162-163):

- τ_1 : explanation by observing number patterns how the subtraction⁷ formula changes (e.g. $-6 - (+2) = -8$, $-6 - (+1) = -7$, $-6 - 0 = -6$, Then $-6 - (-1)$ it must be equal to -5 . Thus $-(-1) = +1$).
- τ_2 : explanation by drawing lines, e.g. $y = -6x$, then drawing first the half line for $x > 0$ forces $-6x$ to be positive for $x < 0$).
- τ_3 : explanation based on “parenthesis magic”, such as $(-6) \cdot (-2) = -(6 \cdot (-2)) = -(-12) = 12$.
- τ_4 : present a more or less complete proof based on the distributive law, e.g. as follows: $0 = 6 \cdot (2 - 2) = 6 \cdot 2 + 6 \cdot (-2) = 12 + 6 \cdot (-2)$; so $6 \cdot (-2) = -12$; and as $0 = (2 - 2) \cdot (-6) = 2 \cdot (-6) + (-2) \cdot (-6)$, we conclude that $12 = (-6) \cdot (-2)$.
- τ_5 : circular or otherwise unconvincing explanations, e.g. inappropriate real life explanations (such as “six times you remove a deficit of 2€ from my account”), travelling forth and back on the number line with negative speed, etc.
- τ_6 : explanations appealing explicitly to external authority – for example of the teacher (“trust me”), of convention (“this rule is used by everyone”), of technology (“try out what your calculator says”).

In Souma’s lesson plan, τ_1 is treated as a possible technique (Kunimune & Souma, 2009a, p. 21). Another technique the lesson plan takes up as a viable solution by students is to use a number line (ibid., p. 21):

If person A goes to “the right (plus)” from the origin 5 unit/step, after 3 steps, A comes to 15. In the same way, if person B goes to the “left (minus)” from the origin 5 unit/step, then after 3 steps, B comes to (–15) because it means:

(–5) + (–5) + (–5) = (–15). But if it was 3 steps before, B came to the origin, where were B? It indicates B was on (+15) on the number line.

Durand-Guerrier et al. (2010) regard the techniques τ_5 and τ_6 as “not appropriate” for students’ further learning. Our proposition is also to abandon strained everyday-life metaphors and instead, to promote students to work within the mathematical discipline. In this lesson, we introduce a technique τ_7 :

- τ_7 : explanation by viewing/representing positive numbers and negative numbers as vectors of opposite direction on the number line. Addition as

⁷ In Durand-Guerrier et al. (2010) it is described as “using number patterns e.g. look at $n \cdot (-3)$ for $n = 2, 1, 0, \dots$ ”.

juxtaposition and multiplication with scalars affect both length and direction.

It is a characteristic of the real number line that the points (the numbers) also can be interpreted as vectors that is translations or movements on the same line. The notion of “origin” and concept of “displacement” are also suggested in Souma’s lesson plan as mentioned above. Vectors are defined by their direction (right or left corresponding to plus and minus) and length (absolute value of the corresponding number). I considered τ_7 an appropriate technique as an introduction to operations on negative numbers due to its geometrical character and it does not require functional or algebraic pre-knowledge such as τ_2, τ_3, τ_4 . The vector visualisation can also be used to explain and illustrate other fundamental arithmetic laws, like the distributive law, associativity of addition, etc. This was done in later lessons during our study.

For this to proceed, it is necessary to establish a technology – this can be thought of the building up of a *milieu* for the students to interact with – based on the number line and the vector illustration of numbers. We accomplished this by following a sequence of lesson plans and examples provided by books on the PSO (Souma, 2000; Kunimune & Souma, 2009a). In the time before the presented lesson, the students had been introduced to the position and order of rational numbers on the number line and also to the interpretation of numbers as the corresponding vectors and addition of numbers as vector juxtaposition. However, we always used the term “distance”, “length” and “directions” instead of the term “vectors” in the lessons. They had also been introduced to the rules of the absolute value, the properties of negative numbers as those having opposite orientation (left, or negative direction) to that of positive numbers (right, or positive direction). It had also been established that multiplication by positive rational numbers can be interpreted as expansion/contraction of the lengths of the vectors, while multiplication by (-1) corresponds to reversing the direction. The illustration of multiplication as expansion/contraction is, from a mathematical standpoint, more problematic, due to dimensionality issues. However, Swedish students of this age have earlier been introduced to positive numbers as proportionalities between lengths and the group did not show any problems digesting this part.

The intention behind introducing this technology is to land in a praxeology largely decided by the current curriculum: The task domain consists of standard arithmetic tasks and, later, algebraic tasks like that of formulating and solving simple linear equations. *Techniques* in this praxeology are the arithmetic/algebraic operations informed by the milieu of the number line. The *technology* (justification and validation of the techniques) consists of the standard algebraic laws: associativity, distributivity, commutativity and, more specifically, the rule that “minus times minus is plus”. We *added* to this praxeology the interpretation of numbers as vectors and the basic interpretation of the absolute value. The vector illustration gives rise to a technological rule of “parsing” an arithmetic

expression conceptually; each *term* in the expression is represented by a vector giving displacement and the sum is the total displacement; one obtains the length of each term as the product of the length of the factors and one determines the sign by counting the number of negative factors.

The *theory* – the justification and expansion of the technology from a wider perspective – is constituted by basic arithmetic and algebra, the real or rational number system, and the theory of one-dimensional vectors as entities of direction and length. From the vector illustration we could establish the technological (theoretical) components such as associativity and commutativity of addition and the distributive law of multiplication over addition. In order to fully establish, say, the associative and commutative law for multiplication, an extended technology could be useful, for instance by interpreting multiplication as “area” or “volume” and deal with the issues of dimensions this give rise to.

However, as discussed later, the institutional constraints of the study would not allow us to further the institutionalisation, axiomatisation and expansion of the added technological components. Even on the technological level, the addition was not completely institutionalised; the term vector was for instance never used. The absolute value and the vector illustration are elements that are outside the current Swedish curriculum and not present in any of the textbooks that were available to us. Therefore, we had to supplement the literature with our own written material, constituted by a compendium of definition of absolute value, and definition of associativity and commutativity of addition and multiplication.

6.1.1 Observations in the grade eight classroom

In this section, I describe the lesson that introduces subtraction and multiplication with negative numbers in the grade eight class. This is the third lesson about the number line and on operations on negative number with the class in grade eight. Today’s problem is to compare the values of following expressions:

Problem A: In which of the following expressions is the value of the difference least?

(a) $+6 - (+2)$, (b) $+6 - (-2)$, (c) $-6 - (+2)$, (d) $-6 - (-2)$

Problem B: In which of the following expressions, is the value of the product largest?

(a) $(+2) \cdot (+3)$, (b) $(+2) \cdot (-3)$, (c) $(-2) \cdot (+3)$, (d) $(-2) \cdot (-3)$

The teacher mentions that she has noticed that many of the students worked on this kind of problems on their own during this week’s free choice lessons. Then she has the students to guess each value in problem A: They agree that the value of (a) is (+4) and that (c) should be (–8) and also that the value of (b) is (+8). Many of them already *know* the rule, still their guesses for (d) split into (–4) and (–8). The students notice that the solution has something to do with the direction of the negative number from the previous lessons. However, it is difficult for them

to explain it clearly. The teacher draws a number line on the whiteboard and asks the class if somebody can explain the argument they were using.

- T: Now we have two different proposals. Which one is correct? We should find out. Can someone explain?
- Frank: (comes up to the whiteboard and points at the two minus signs). Since minus times minus is plus, so, this part will be $(+2)$, and that's why it is (-4) .
- T: Ok, (-4) . Are there any other opinions?
- Ahmed: (points on the position of (-6) on the number line). Minus means going to the left, so, it should mean go to the left furthermore, so it is (-8) .
- T: How about this one? (points on the students' guess for question (c)) (-6) minus $(+2)$? Does someone want to explain?
- Ahmed: First, it was on (-6) , minus $(+2)$. Negative number minus positive number, isn't it?
- T: (...) once again, please?
- Ahmed: (-6) , then $(+2)$...
- T: It was on (-6) , (points on the number line), then which way it will go to?
- Ahmed: To the left.
- T: It reaches to (-8) then? (marks (-8) on the number line)
- T: Does everybody agree with Ahmed?
- Students: (several nod)
- T: In that case, I wonder... is it possible that the value of (d) reaches the same value as question (c)? Even (d) is $-(-2)$ and (c) is $-(+2)$? How many of you think that it is possible?
- T: (counts the number of the students who do not think it is possible). Many of you think that it is not possible. Then you mean that it is (-4) for (d)?
- Ahmed: But it feels...both are possible.
- T: You mean both (-4) and (-8) are correct?
- Ahmed: No, I mean, there are probably one answer but it feels that (-6) , minus (-2) ..., it may go to the left also. Because minus means go to the left...

Our previous lessons, which took up the treatment of positive and negative numbers as displacements on the number line with opposite directions, gave Ahmed a starting point to reason by himself. However, he and some other students still does not grasp the meaning of a subtraction with a negative number. The teacher then tries to have the class recall the interpretations of the positive and negative numbers that the class discussed in the previous lesson.

- T: What does the minus sign (in front of the number 2) actually express?
- Ahmed: It is less.

- T: It is less? And it means...? (...) It is called negative number. And the sign itself, we call it minus. But what does it mean negative, actually, in comparison to “positive”?
- Oliver: Mirrored.
- T: Mirrored. (writes on the whiteboard, “mirrored”) Overall, can we describe it in this way? Like positive person and negative person?
- Habib: A grouping? [Note: He said, “lagordning” in Swedish and probably, he meant “categorising in different groups”]
- T: A grouping? Yea, a kind of. Positive, contra negative.
- Some one: The opposite.
- T: Yes, in that way, we can disconnect what we have learned – negative – the less. If we look at a number line, to which direction is positive?
- Students: (several answer) to the right.
- T: And the negative?
- Students: (several answer) to the left.
- T: When there are negative (points on the operation sign $(-)$ in the question (d)) and negative (points on the minus sign on (-2)), and if negative means the mirroring, or opposite, to which direction it will go to?
- Ahmed: So, two minus means plus then?
- T: Yes. Oliver, you mentioned mirroring. Would you like to explain more?
- Oliver: It is supposed to go to the left side from (-6) by the subtraction, but because of the minus sign (-2) , conversely, it proceeds to the right. It means to move from the current position to the opposite side – like in a mirror.

When Oliver used the metaphor “mirroring”, the teacher wanted to utilise this term to let the class comprehend the subtraction with negative numbers by the vector illustration. She recalls for the class properties of negative numbers having opposite orientation to that of positive numbers using a metaphor of the poles of a battery.

Then the class solves the problem B, the multiplication problem. All students guess that the value of (a) $(+2) \cdot (+3)$ is $(+6)$ and that it is the largest product. Many guess the value of the question (b) and (c) as (-6) . The teacher then asks the class:

- T: You guessed it would be (-6) for question (b). Why did you think so?
- Oliver: We have (-2) . So actually, $(-2) \cdot (+3)$ means that we have (-2) three times. It is the same thing as $(-2) + (-2) + (-2)$. That is why it will be (-6) .
- T: (writes on the number line from (-2) and goes to the left) We start from (-2) and plus (-2) again, and (-2) , so it ends by here (-6) .
- Oliver: (-2) , three times.

T: (to the class) do you agree? Rachel?

Rachel: Yeah, it is correct.

T: The next one then? (c) $(+2) \cdot (-3)$? What does it mean, two times **minus** three?

Habib: We can reverse it to $(-3) \cdot (+2)$.

T: Can we do that?

Habib: I think so.

Ahmed: It will be same thing, (...) or ... not be?

T: How many of you think that it is possible to do so? (Looks at the class) not so many? (gives a compendium to Muhammad) Here is a mathematical law written. Could you read it?

Muhammad: (reads aloud the text) The commutative law says that $a + b = b + a$, at an addition.

T: And?

Muhammad: And $a \cdot b = b \cdot a$, at a multiplication.

T: (writes what Muhammad reads on the whiteboard) and in this case it is multiplication. Can we conclude as Habib says, that $(+2) \cdot (-3) = (-3) \cdot (+2)$?

Some one: I think so.

Oliver: We can do so, but I actually thought still $(+2) \cdot (-3)$. I mean, $(+2) \cdot (+3)$ is $(+6)$. But minus means mirroring, so it becomes (-6) . It is two times three, but in the negative scale.

T: So, you multiple the two and the three and...

Oliver: Turns to the other direction, because there's negative sign.

T: Ok... and you mean... (looks at Habib)

Habib: I revers $(+2) \cdot (-3)$ to $(-3) \cdot (+2)$...and...

T: Minus three, two times? (points on (-3) and moves to (-6))

Habib: Yes.

T: Well, then we have two different ways to think. Good.

In this episode, the students thought of $(-3) \cdot (+2)$ as repeated addition of minus 3. Habib did not interpret $(+2) \cdot (-3)$ as adding $(+2)$ minus 3 times, but used commutative law to write $(+2) \cdot (-3)$ as $(-3) \cdot (+2)$ and then to think of it as (-3) two times. However, Oliver applied absolute value and treated $(+2) \cdot (-3)$ as a product of numbers having opposite direction on the number line.

Now, the teacher moves to the last question, (d) $(-2) \cdot (-3)$ and asks the class:

T: Farid, what do you think?

Farid: No...

T: You can say something. It does not have to be a correct answer. In that case, someone can continue to think further.

Farid: Mm... (-2) and (-3) ... they are like the first one, $(+2) \cdot (+3)$.

T: They are same?

Farid: Yeah, it will be plus.

T: Ok, it will be (+6), since minus times minus is...

Farid: Plus.

The teacher knows from previous lessons that Farid is quite shy and Farid knows that the teacher does not accept him not trying to participate.

Ahmed, however, has no problem to formulate a reflection that goes counter to this, in spite that, in this situation, it looks like Farid has answered correctly.

Ahmed: Wait, that one, minus minus, is the same thing as the previous one but for times? [He means multiplication] But in this case, there are only minus signs.

T: So, you think it is wrong?

Ahmed: Yes, I will show it. (comes forwards to the whiteboard) (-2) times (-3), look, there are no plus signs. So it will be on the left side. As you say, (looks at Oliver) negative means mirroring. (points at (-6)) It cannot be plus.

Oliver: But I mean to mirror by origin. (comes forwards to the whiteboard)

T: Ahmed, say again how you thought.

Ahmed: When it is minus, as you [Oliver] said, it turns to the other side. [turns to the left] And here is **no** minus sign (points on the multiplication sign), so it cannot be plus.

T: OK, you mean it is [not subtraction, but] multiplication.

Oliver: I will show you here (points at the number line). We begin at (-2). Thus it is in this scale (points at below zero). Then, it times **minus** three. So it turns to the other side [of the origin] again. (points upper zero)

Ahmed: First, (-2), (points at the number line) then times (-3). (points at below zero)

Oliver: No, one must think mirror by origin.

Ahmed: I don't understand...

Ahmed argues that the clarification they got in the previous task was based on subtraction and it should not hold in the case of multiplication.

The teacher then tries to put the argument into shape, by repeating Oliver's way of reasoning on the solution for $(+2) \cdot (-3)$:

T: Ok, Oliver has explained about the question (c) that he considers the product of (+2) and (-3), without looking the signs, thus it is 6. Then he considers the signs. In this case (points at $(-2) \cdot (-3)$) the product of 2 and 3 is 6, then he said start from the origin. The first minus sign (points at (-2)), here, what it is say? In which position?

Students: (point to the left, below zero)

T: Opposite to plus? Then, what the second minus sign says?

Ibrahim: Opposite again. Turn to the right. So one should think the minus sign first, and...

Ahmed: The number [he means the absolute value of the product of $(-2) \cdot (-3)$] first.

- T: Yes, the number [absolute value of the product] first.
 Ibrahim: Then look at the signs.
 T: Indeed, the product of 2 and 3 is 6. Then if it is $(+2) \cdot (-3)$, so move to the negative direction, only once.

Ibrahim and Ahmed's dialogue shows that they begin to grasp the use of the absolute value and direction to determine the value of the product. When the teacher gives the class a training problem like $(-4) \cdot (-1) \cdot (-3)$, Ahmed gives the right answer by pointing at different directions on the number line:

- Ahmed: First of all, the number [the absolute value] will be 12. First, it starts from (-4) on the left [side of the origin] and then (turns his forefinger from the left to the right) changes to the plus direction because of [the multiplication with] (-1) , then (turns his forefinger from the right to the left) changes again to the negative direction because of [the multiplication with] (-3) , so the answer is (-12) .

The class gives Ahmed a round of applause. The students seem to have grasped the usefulness of the vector illustration, by separating the computation of length and direction to carry out the multiplications.

Thereafter, the teacher summarises the lesson by stating relevant passages in the textbook.

6.1.2 Observations in the grade seven classroom

In this section the lessons about the same operations on negative numbers in the grade seven class is presented. These lessons took place a week *before* the observations of grade eight presented above. We used the same tasks in grade eight and seven, however, in grade seven we allocated one lesson for subtraction and one lesson for multiplication with negative numbers. The second lesson was implemented 4 days after the first lesson.

The lesson on subtraction with negative numbers

The teacher wrote today's problem on the whiteboard:

In which of the following expressions is the value of the difference the least?

- (a) $+6 - (+2)$, (b) $+6 - (-2)$, (c) $-6 - (+2)$, (d) $-6 - (-2)$

She starts to ask the class what "the difference" (in Swedish, "differensen") meant. Some students answer that it is the "divergence" (in Swedish, "skillnaden") between the numbers. The teacher writes "term - term = the difference" and describes it as follows:

- T: If you think of a subtraction on the number line and the difference is a negative number which is large in magnitude (points the left direction) then the difference is small (call students' names who talk and asks them attention) ok? The difference is the divergence between two numbers.

She asks the students to guess in which of the expressions the difference is the least:

- T: Now guess! How many think that in a the difference is the least?
Linda? Do you think a is the least?
- Linda: Yeah, the a.
- T: You guess well. Anyone else? [looking at a student who was not looking at the whiteboard]
- T: X, [the student's name] in which one is the difference least?
- X: What?
- T: Can you guess? a, b, c or d?
- X: c.
- T: She says c. Samir, what do you think? a is $+6 - (+2)$, (b) is $+6 - (-2)$, [asks attention to students who talk] stop it [then continues to read out] c $-6 - (+2)$, d $-6 - (-2)$, which one is the least?
- Samir: c.
- T: [Samir answered while the teacher still was talking. That is why she could not hear what Samir said] What did you say?
- Samir: c.
- T: Now we have got two votes to the c, Kejo?
- Kejo: c.
- Linda: Take the a!!
- Some one: d!

The teacher asks all 11 students to guess and records the number of guesses for each alternative on the whiteboard: one student has guessed for (a), none for (b), five for (c) and four for (d). Then she asks the class to calculate the value of each expression. She mentions that everyone should know the value of expression (a). Here the teacher let the students start with "wild guessing" (Polya, 1965). In the lesson with grade eight, the students' guesses were more based on conjectures, (i.e. "rational guessing", *ibid.*) since some of them had been working with subtractions of negative numbers on their own. However, in this class, she uses the guessing to involve the students into participating; quite a few students were initially quite chatty and did not seem ready to join the lesson.

The teacher goes around between students and gives some hints and observes how the students solve the problem. After three minutes, she asks the class the value of the differences on every expression. The students agree that the value of (a) is +4. Then their answers for the value of (b) split into (-4) and (+4), for (c), (-8) and (-4). For (d) the answers presented are (-4) and (-8).

- T: Now we have so many alternatives, what shall we do?
- Ronja: We shall calculate them.
- Tina: Nooo! Can't you [the teacher] just say if it is right or wrong?
- T: Can I say that both are right? (Points the students' conjectures (-4) and (+4) for the expression (b) $(+6) - (-2)$). [looks at a student who

was talking with another student and asks her] Is it possible that the answer is both (-4) and $(+4)$?

A student: I don't know.

Ronja: We have $+6$, then we take minus 2 away (pointing to the left), so $+4$.

T: Y, [name of another student, who was chattering with his neighbor] now I ask you. Look at the a. You all agreed the value of this $(+6) - (+2)$ is $+4$, right? Is it possible then to have the same answer $(+4)$, for $(+6) - (-2)$? How many of you think it is possible?

Students: I do. [4 students raise their hands to agree that it is possible]

At this stage, some of the students' conjectures are still wild guesses and some like Ronja, treat $(+6) - (-2)$ as $(+6) - (+2)$. There are some students that do not concentrate on the lesson. The teacher tries to ask those students questions constantly. The teacher asks the students to consider the difference between $(+6) - (-2)$ and $(+6) - (+2)$:

T: What does negative sign mean on the number line?

Ronja: Under the origin.

T: (points to the left) Going to the left?

Ronja: (nodding)

T: If I must go to the left (pointing the minus sign after $(+6)$) and meet the minus sign (pointing the minus sign on the (-2)) which says I must go to the left again, to which direction, I must go to?

Tina: To upwards. I am just a kidding.

Ronja: Then my path is ending now...

Here, the teacher tries to steer the students to treat the integers as one-dimensional vectors with direction and length. She asks a student to draw a number line on the whiteboard. When the line is ready, she asks the class how one can show the operation of the expression (a). Students answer that to begin from $+6$ and go left for 2. She draws an arrow with the length 2 with the direction to the left from 6 to 4 (see Figure 6.1). Then she asks what the operation (b) $+6 - (-2)$ looks like on the number line.

Figure 6.1. Expression $(+6) - (+2)$ on a number line.

T: Marianne, from where shall we start? From which integer? (points on problem (b))

Marianne: 6 [in very quiet voice]

T: What did you say?

A student: Zero.
 Marianne: Zero.
 T: What did you say first?
 Marianne: 6.
 T: Yes, didn't you? You should say what *you* think, not what others thought.

Many times, during the whole semester, the teacher gave this kind of remark to students, with the intention to encourage them to express their actual thinking.

When the teacher asks if it is $+6 - (-2)$, where it should be landing from $+6$ on the number line, Ronja who was looking on the number line intensively, says:

Ronja: Then it is impossible to land on the other side of the origin, I think.
 T: What do you mean?
 Ronja: To obtain the negative alternative, [she means (-4)] we must have like minus 8.
 T: Did everyone listen to Ronja? She says that the alternative (-4) is impossible.
 Ronja: Then we must go back...10 steps.

The representation on the number line provides a clear explanation for arithmetical operations and gives Ronja an opportunity to think logically. The students agree that it cannot be (-4) . The question is, if the other alternative $(+4)$ is the correct answer. The teacher repeats the previous question: Is it possible that the values of $+6 - (+2)$ and $+6 - (-2)$ both coincide with $(+4)$? The students are not so sure.

T: Is it possible to land on the same value?
 Marianne: No.
 T: Who said no? Helena or Marianne?
 Samir: It was Marianne.
 T: Marianne says no.
 Marianne: I don't know...
 T: Samir, do you think it can be the same value?
 Samir: (...)
 T: How many steps should it move? [on the number line] Any proposals? Linda?
 Linda: I don't know...
 T: Guess?
 Linda: 2.
 T: Linda says 2.
 Linda: I don't know! I just said.
 T: You guessed. Believe your instinct. Linda says 2. Is it reasonable?
 Someone: Yes.
 Ronja: Yes, it is obvious.

- T: [gives warning to some non-attentive students] Linda says it must be two. How many of you agree? [many students raise the hands] Ok, and Marianne says it cannot land on (+4) as the first expression. In which direction should it then go?
- Tina: In the other direction.
- T: What did you say Tina?
- Tina: In the other direction.

The teacher has finally obtained the expression she needed.

After the students got the value (+8) to the expression $+6 - (-2)$, the teacher continues to go through the rest of the two expressions on the number line. The students could reason the expression (c) $-6 - (+2)$, as the expression (a) and concluded the value as (-8). Then the last expression (d) $-6 - (-2)$.

- T: Now we know that it starts from (-6), [on the number line] with how much does it move?
- Kejo: 2.
- T: Yes, and now the matter is to which direction it goes.
- Several: To the right!
- Ronja: It is a bit confusing.

The teacher draws an arrow of length two to the right of (-6) and asks the class.

- T: You say it goes to the right, why?
- Someone: Minus and minus is plus.
- T: Why is minus and minus is plus? (points to the right)
- Ronja: When...one takes something away...then it comes closer to the origin (pointing to the left)...so in the same way, when one takes something negative away from a negative,...minus and minus... on the negative side, then it also goes closer to the origin (points to the right).
- T: Helena, can you explain what Ronja means?
- Helena: Umm...if ... it is minus, after the minus ...then it goes to the plus (points to the right). And if it is a plus after the minus then it goes to the minus? (points to the left)
- T: Helena says that when it is minus, then one goes to the plus and when it is plus, then one goes to the minus. The question is then: In which case is this rule applied?
- Someone: Negative?
- Ronja: Subtractions. [many students talk and it is hard to hear]
- T: In which cases does this rule, which Helena formulated, hold?
- Kejo: Subtractions.
- T: Subtractions.

Ronja's remarks show that she started to associate the operation (subtraction of a number) as an operation having direction and length. The students' gestures (pointing to the right and the left) also indicate that they consider the subtraction as a directional operation. When $-6 - (-2) = (-4)$ has been settled, the teacher let

the students consider the expressions $7 + (-5)$ and $7 - (-5)$ on the number line. Then the teacher finishes the lesson by writing “*Ronja & Helena’s rule*”:

$- (+) \leftarrow \text{left } (-)$
 $- (-) \rightarrow \text{right } (+)$

The lesson on multiplication with negative numbers

The teacher applied this “rule” in the next lesson about multiplication with negative numbers. To begin with Robert explains the reason for why the value of $(+2) \cdot (+3)$ should be equal to $(+6)$:

Robert: I simply multiplied 2 and 3. It was 6. Then I looked at the sign. There were only plus signs. So, plus 6.

To handle the expression (b) $(+2) \cdot (-3)$ the students use repeated addition and treat it as $(-3) + (-3) = (-6)$. Then they use the commutative law for expression (c) $(-2) \cdot (+3)$ and treat this as $(-2) + (-2) + (-2) = (-6)$. At that point, the teacher asks Robert to repeat his idea concerning the operation for the first expression. Robert tells again his reasoning and the teacher asks some other students repeat again what Robert meant. The she asks the class:

T: Can we explain this $(-2) \cdot (+3)$, according to Robert’s model?
Samir: The same thing.
T: Can you explain with the word?
Samir: 2 times 3, then...We have those two basic numbers, right? Then we take the sign of the basic numbers...and it is minus...
Ronja: Are both the basic number?
Samir: It begins with (-2) ...
Ronja: Then the basic number is (-2) ?
Robert: If we take the signs away, anyway 2 times 3 is 6. Then we shall think about the sign.
T: Ok, you say it will be 6 anyway. What can we say about this 6? Do you remember this symbol? (writes a symbol of the absolute value)
Robert: Absolute value.
T: What did Robert say?
Someone: Absolute value!
T: Yes. Absolute value. That, Robert says, is calculating the absolute value of the product.
Ronja: So, all of them [the value of the expressions] are 6 something. Because they are 2 times 3.

Here, the teacher reminds the class about the absolute value from the previous lesson and many students make the connection directly. She then asks how one thinks about which sign the value of expression (d) $(+2) \cdot (-3)$ has. Several students think it is $+6$.

- T: Now Helena asked if “*Ronja & Helena’s rule*” can be applied to the multiplication.
- Kejo: Yeah. (raises his hand) [He wants to speak]
- T: What was “*Ronja & Helena’s rule*”?
- Kejo: Here! (waiving his notebook)
- T: Can you read aloud the rule from your notes for everybody?
- Kejo: (without looking at his notebook) if it is plus and minus, then it’s negative number. (points to the left by his finger) If it is minus and minus, then it is positive number (points to the right) Right? So this is +6.

Then the teacher writes some exercise problems $(-3) \cdot (-2) \cdot (+3)$, $(-3) \cdot (-2) \cdot (-3)$, $(+3) \cdot (-2) \cdot (+3)$ and $(+3) \cdot (-2) \cdot (-3) \cdot (+2)$. Several students look at the whiteboard and point with their fingers back and forth to the left and to the right.

- Ronja: The value is 18. 3 times 2 times 3. 18.
- T: Ok, the sign then? Is it positive or negative?
- Several: Positive!
- T: Why? What about the first two terms? (points $(-3) \cdot (-2)$)
- Kejo: Positive!
- Linda: Positive.
- T: Then (+6) times (+3)? (points (+3))
- Kejo: Positive!

The students use the term *positive* and *negative*, instead of *plus* and *minus*, and pointing back and forth with their fingers indicates that they have got a mental number line and consider those products with directions.

The last problem has 4 factors. Ronja says the absolute value is 36. Then Kejo explains that the product will be positive.

- Kejo: Can you cover the last two numbers? [the teacher covers the $(-3) \cdot (+2)$ by her hand] The first pair, $(+3) \cdot (-2)$ is -6. Negative. Then the next pair, [the teacher covers now the first pair $(+3) \cdot (-2)$] $(-3) \cdot (+2)$ is also negative, minus 6. Then $(-6) \cdot (-6)$ is plus 36.

The teacher then lets Kejo explain his idea:

- T: Kejo, can you describe what your rule is? It looks like very effective to ascertain to which directions the product lands on.
- Kejo: If there are two factors then one just multiplies those two. If there are three factors, one multiplies the first two factors then the last factor. If there are four factors, one multiplies two by two.

Several students seem to begin to grasp how to use the absolute value to perform the multiplications by separating the calculation of absolute value (length) and signs (direction).

6.2 The lessons “Introduction to finding the general solution”

In Souma’s lesson plans for grade seven (Book1), a section called “Introduction to using variables” is found in the chapter “Algebraic Expression”, after the chapter “Arithmetic Operations on Positive and Negative Numbers” (Kunimune & Souma, 2009a). Japanese school children have learned solving equations using variables in grade six. The section is an introduction to finding the general solution of geometrical problems and modelling using variables. The goal of the chapter is declared as “to express quantity and the relations of different quantities simply, clearly and generally, by using algebraic expressions” (ibid., p. 26). The lessons described in this section are, in sequence:

1. Finding a generalised formula to express regularity of figures
2. Training on calculation skills with variables
3. Training the modelling skills with variables that express different quantities (e.g. number of articles, price, weight, length, velocity etc.)

After these lessons, the next section, “Linear equations”, has the stated goals of training the students’ skills of operating on linear equations and the skills of modelling with equations. The focus of the section is to illuminate the use of variables and equations in modelling before starting to train on solving linear equations. The weight of this first lesson is on making the students realise the convenience of using formulas, for instance, the formula for calculating the area of a triangle. The goal of the lesson is stated as “to let the students understand the significance and usefulness of using variables instead of numbers” and “to develop students’ ability to explain to each other with the use of images/figures and formulas” (ibid., p. 27).

The initial problem of the lesson is formulated so that the students will generate a formula to determine the total number of stones of a square. First, the students are supposed to find out different ways to determine the total number of stones, which are arranged as a square with five stones on each side. Later, they will use an analogical argument to determine the total number of the stones of 20 on each side. These activities will lead to finding methods for stating a formula for the total number of stones in a square with n stones on one side.

The initial question (we call it the “Square Problem”) in the lesson plan is posed as follows:

We will make a square by putting stones as in the picture below. If one of the sides consists of 5 stones, how many stones are used in total? (ibid., p. 27) (See Figure 6.2)

Figure 6.2: Illustration of the “Square Problem”; “How many stones are used in total?”

The lesson plan assumes that students might come up with the following methods (see Figure 6.3) for the determination of the number (ibid., p. 28):

Figure 6.3: Illustration of students' possible solutions (Kunimune & Souma, 2009a, p. 28).

To anticipate techniques and technologies proposed by students is crucial when the teacher plans the flow of the lesson (Souma, 1997). When students have suggested the techniques, the teacher should state follow up problems:

“If one of the sides consists of 20 stones, how many stones are used in total?”

“If one of the sides consists of n stones, how many stones are used in total?”

The goal here is to make the students notice that it is possible to apply the same method as was used for five stones. The techniques (τ), the lesson plan suggests in the figures above are the following:

$$\tau_1: (n \cdot 4) - (1 \cdot 4)$$

$$\tau_2: (n - 1) \cdot 4$$

$$\tau_3: (n \cdot n) - (n - 2)^2$$

$$\tau_4: (n - 2) \cdot 4 + 4$$

Noticeable technologies are the following:

θ_1 : partitioning of figures

θ_2 : inclusion-exclusion in counting

θ_3 : the distributive law

θ_4 : the rule for expanding a square

θ_5 : informal induction from a table (not found in Souma's lesson plan)

The theories (Θ) that justify these technologies are the basic algebraic laws that govern arithmetic and algebraic operations and enumeration.

When we planned the lesson for the grade seven and eight classes, our goal was to illuminate the use of variables and to develop the students' skills to explain

the process of reasoning with the use of figures and formulas. We adopted this lesson plan virtually as it was, without adding any additional techniques or technology. In each grade we devoted three lessons for the implementation of the training and introduction of algebraic expressions: One lesson used the above mentioned lesson plan. One lesson (grade seven) was concerned with finding a formula for the sum of inner angles of polygons and one lesson with algebraic expression of a square involving the binomial $(a + b)^2$ with figures (grade eight). Another lesson compared four different algebraic equations to find out which were actually equivalent. Thereafter we started to implement modelling with linear equations.

As was said in section 5.1.3, the training of students' skill of interpreting formulas and algebraic relations is stressed in the Commentary for the Swedish Lpo94. One of the publications from the Swedish National Centre for Mathematics Education (NCM)⁸, "*Algebra for everyone*" (in Swedish, *Algebra för alla*) (Bergsten, Häggström & Lindberg, 1997), has a chapter named "Patterns and Generalisation". There, it is declared that "The competency of seeing the generality in the special case is perhaps the most fundamental mathematical form of thought" (ibid., p. 79). In this chapter there are 25 varied problems about finding patterns and analysing forms together with detailed suggestions to help readers to reason on the problems. One of the problems "Construct frames" is almost the same problem as the "*Square Problem*" we adopted from Souma's lesson plans. There, the authors suggest three different techniques; $(N + 2)^2 - N^2$, $4 \cdot (N + 1)$ and $4 \cdot N + 4$, which can be justified by the same technologies as I described above.

In Swedish textbooks for grade seven to nine, this kind of task, that is to find a pattern and a formula associated to certain figures, is often taken up in conjunction with the arithmetical sequence. The problems are placed in the section "Patterns and Formulas" or "Patterns and Expressions" in the chapters on algebra (e.g. "*Formula 9*", Mårtensson, Sjöström & Svensson, 2009, "*Matte Direkt 8*", Carlsson, Hake & Öberg, 2010). I will discuss the adequacy of such problems for interpreting formulas and algebraic relations in section 7.2.2.

6.2.1 Observations in the grade eight classroom

After four lessons of training the arithmetic operations with negative numbers, we initiated the introduction of a general solution with algebraic expressions to the class of grade 8. They have learned solving basic linear equations like $2x + x = 15$

⁸ NCM has published several monograph series *Tema Nämnaren* on their journal in mathematics education *Nämnaren*. "*Algebra for everyone*" is based on Lpo94. The text and the problems in the book are still highly referred and used by the Swedish teachers in service. Also, the book was used as a reference book for many mathematics courses within teacher education programs in different universities in Sweden.

in grade 7. The teacher begins the lesson to write the word “general solution” on the whiteboard. She asks the class:

- T: Has anyone heard the word “general solution” before? Does anyone associate the word “general” to anything?
- Oliver: Something common?
- T: Yea, something common, we can say that. (...) In this case, in mathematical terms, it means a solution, which is applicable for all values. You are going to see it during this lesson.

The teacher mentioned in the beginning, what students are going to learn today. She draws the first figure of the square with stones and asks the class what kind of figure she has drawn. A student answers “a square” and another student answers “four equal sides”. Then the teacher asks how many stones there are on the picture in total.

- T: We construct a square and if there are 5 stones on each side, how many stones in total?
- Nimra: 20.
- T: 20. Does everyone agree?
- Ahmed: Mm..., no.
- Alex: 16. (writes 16 beside the square)
- T: You say 16?
- Nimra: Well, maybe it is.
- T: How did you think, Nimra?
- Nimra: Well, you said, 5 stones on each side. And there are 4 sides, then 20. I was a little too quick...
- T: Ok, I understand exactly how you thought. (turns to the class) Do you understand how Nimra thought? ...Alex? Why did you think it was 16?
- Alex: (points at the squarer) 4 stones on the bottom ...then disappears [He seems to mean “shifts”] one stone on each of the sides.
- Nimra: Yea...
- T: You mean... (circles first, the four stones on the bottom, then all the other sides) here? [see Figure 6.3, no. 2]

In this episode, Nimra notices that something must be done to calculate the total number of the stones more than simply using 4 times 5. She understands this as soon as Alex starts his explanation.

The teacher then states the main task to the class:

- T: In which different ways can you determine the total numbers of stones without counting them?
- Nimra: One can take away one stone from each corner.
- T: You mean these? (circles the stones on each corner) [see Figure 6.3, no. 4]

Nimra: Yes, then three stones from the two sides ...six stones ...and another six stones from the another two sides. It makes $4 + 6 + 6$. 16 stones.

Here, the technique Nimra suggests is τ_4 , that is $(5 - 2) \cdot 4 + 4$. She probably focuses on the number 4 and the total number 16 and automatically counts the sides in pairs. The teacher later modified Nimra's expression to $4 + 3 + 3 + 3 + 3$ and furthermore to $4 + 3 \cdot 4$ by asking Nimra and the class.

Many hands have risen. Many students want to show their ideas.

Vincent: We can begin to take 5 stones on both (vertical) sides. Then we count the rest of 3s from the horizontal sides.

Habib: I thought in the same way. 5s and 3s.

This technique, that Vincent has mentioned, is not shown in Souma's lesson plan; $\tau_5: 2n + 2(n - 2)$. It is a variation of the $\tau_1: (n \cdot 4) - (1 \cdot 4)$. The teacher writes on the whiteboard $5 + 5 + 3 + 3 = 16$. Then later she modifies to $2 \cdot 5 + 2 \cdot 3$ through asking Vincent and the class.

Ahmed tries to express his idea, which he got a hint of the total number 16:

Ahmed: I count an area of those 16 stones (points to the 4 stones on the bottom and the 4 stones on the right).

T: Well, you mean this area with 16 stones? But how can we determine the area?

Ahmed: Count the squares then. 4 times 4.

T: Ok, (writes down $4 \cdot 4 = 16$ on the whiteboard). Any other ideas?

Oliver: This is a bit like Ahmed's. We skip one stone and count 4 stones times 4 rows.

T: Oh, who had a same idea in the beginning? Alex, it was you? (turns to Oliver) You mean like this? (circles the 4 stones on the each side) [see Figure 6.3, no. 2]

Oliver's idea is $\tau_2: (n - 1) \cdot 4$. He claims his idea is similar to Ahmed's idea, but actually this technique is totally different. Ahmed's idea seems to be based on the answer 16 rather than being driven by observation of the figures.

Now the teacher steers the lesson to its next step. The students are supposed to work in pairs and consider different ways to determine the total number of the stones in a square when we have 20 stones on each side. The teacher gives the students several minutes and walks around among the groups. She listens in on the discussions and asks four pairs of the students to write down their solutions on the whiteboard.

Vincent and Jal used $\tau_2: (n - 1) \cdot 4$ and wrote directly $19 \cdot 4$. (See Figure 6.4). Muhammad and Bint chose $\tau_5: 2n + 2(n - 2)$. They wrote a square with "20" by the two horizontal sides and 18 by the vertical sides (see Figure 6.5). Nimra and Rachel used $\tau_1: (n \cdot 4) - (1 \cdot 4)$. They wrote a square with "20" on all sides, then "- 4 corners" (see Figure 6.6). Oliver and Magda used $\tau_2: (n - 1) \cdot 4$. Oliver wrote

on the whiteboard $(20 - 1) \cdot 4$ which shows his process of mathematical reasoning (see Figure 6.7).

$$19 \cdot 4$$

$$9 \cdot 4 = 36$$

$$10 \cdot 4 = 40$$

(svar: 76 stema) 20 st

18 st 18 st

20 st

Figure 6.4: Vincent and Jal's solution. Figure 6.5: Muhammad and Bint's solution.

20 st

20 st 20 st

20 st

20 st - 4 höcn = 76 stema

$$(20-1) \cdot 4 = 76$$

Figure 6.6: Nimra and Rachel's solution. Figure 6.7: Oliver and Magda's solution.

Now, the teacher wants to talk about the “general solution” she mentioned at the beginning of the lesson.

T: Regarding “general solution”, the first case here, when it was 5 on the rows, I shall say, “ n is equal to 5” (writes $n = 5$). In this case, (points students' solutions for 20 stones on each side), it is “ n is equal to 20” (writes $n = 20$). What do you think n stands for?

Muhammad: Those stones. (points to the dots)

T: Those? (points to all dots on a side of the figure)

Muhammad: Yes, the numbers of the stones on one side.

Then the teacher asks the class in which way one can express the total number of dots using the variable n .

T: For instance, if the n is one million, how we can express the total number of stones?

Ahmed: One million times 999,998.

T: Ok, you mean this one? (points Muhammad and Bint's solution $2 \cdot 20 + 2 \cdot 18$) Anyone else?

Oliver: If we use our example, it will be n minus 1, times 4.

T: (writes down $(n - 1) \cdot 4$ below Oliver's solution $(20 - 1) \cdot 4$). Is that correct (asks the class)? Put in 5 in n and see if it will be 16 (points the formula). Must $(n - 1) \cdot 4$ be as it is now? Can we change the position of the 4?

Students: We can change the place.

T: Really? Ok. (writes $(n - 1) \cdot 4 = 4(n - 1)$)

Ahmed's answer shows that he has a sense for how the technique τ_5 works. But he does not express his reasoning correctly. On the other hand, Oliver has already seen how the technique τ_2 works. Thereafter, the teacher takes up Vincent and Jal's solution $19 \cdot 4$. She asks the class to consider how it could be expressed using n . Vincent explains that 19 came from $20 - 1$. The class establishes that $20 - 1$ can be expressed also $n - 1$, then this solution is also $4 \cdot (n - 1)$. Then the teacher asks the class to express Muhammad and Bint's solution, $20 + 20 + 18 + 18$ (See Figure 6.5).

- Ahmed: n is 20, so 18 is $20 - 2$.
 T: Yes, but we want to now to use n as a substitute for any number. We cannot use 20.
 Someone: $n - 2$.
 Ahmed: Ah, $n - 2$!
 T: $n - 2$, ok, then this one is (points on 20) n and here is one more n . How can we express those?
 Ahmed: 20 times 2.
 T: No 20, please.
 Ahmed: Ah, n ...and ... n ... times 2!
 T: Ok, n times 2. (writes $2n$) Then? This one? (points on $n - 2$)
 Ahmed: $n - 2$, and one more $n - 2$.
 T: Ok, one more $n - 2$. (writes $(n - 2) + (n - 2)$, after the $2n$) Ok, how can we simplify this expression?
 Someone: 2 and $(n - 2)$.
 T: Ok. (writes $2n + 2(n - 2)$) We can write in this way?

Finally, the teacher takes up Nimra and Rachel's expression. (see Figure 6.6)

- T: How can we express Nimra and Rachel's?
 Muhammad: $4n - 4$.
 T: (writes $4n - 4$) Ok.

This episode indicates that Ahmed begins to understand what is meant by a general solution. Some other students seem to understand it already.

Now, the teacher wants to show that all the expressions are identical.

- T: If we simplify Oliver's solution here ...do you remember how we can multiply in the 4? (points on $4(n - 1)$) 4 times n ... and?
 Muhammad: 4 times minus one.
 T: (writes $4n - 4$) Shall we take this one also? (points at $2n + 2(n - 2)$) If we multiply in this, (points on $2(n - 2)$) how do we do it?
 Someone: $2n$... (...)
 Muhammad: and $- 4$.
 T: (writes $2n + 2n - 4$) Can we simplify this? (points $2n + 2n$)
 Oliver: $4n - 4$.
 T: (writes $4n - 4$) What happens now? Now we have simplified those?

- Ahmed: They are...(…)
- T: Is there any similarity here?
- Someone: They are the same.
- T: Yes. They were different solutions, but when we put them in general form, all the solutions can be simplified to this. (points $4n - 4$)

Most of the techniques; $\tau_1: (n \cdot 4) - (1 \cdot 4)$, $\tau_2: (n - 1) \cdot 4$, $\tau_4: (n - 2) \cdot 4 + 4$ and additionally, $\tau_5: 2n + 2(n - 2)$ were suggested by the students. These techniques are technically justified by the technologies θ_1 : partitioning of figures, θ_2 : inclusion-exclusion in counting and θ_3 : the distributive law. Overall, it was observed that almost all students had commented and stated their solutions during this lesson.

6.2.2 Observations in the grade seven classroom

The students have learned to solve basic linear equations with one variable isolated on one side (like $4x + 6 = 26$) at grade six.

This lesson was observed one week before the lesson with grade eight presented above and the teacher's strategy of managing the lesson is quite different. The class is presented with the same problem: *We construct a square and if there are 5 stones on each side, how many stones are there in total?* Differently from the previous observation, the teacher does not talk about the "general solution", neither does she let the students guess the total number of the stones to begin with. Instead, she lets students consider the techniques directly. The proposed techniques from the students were:

- Robert's idea: $2 \cdot 5 + 2 \cdot 3$
Thus, it is $\tau_5: 2n + 2(n - 2)$
- Ronja's idea: $4 \cdot 4$
Thus, it is $\tau_2: (n - 1) \cdot 4$

The students, who came up with those expressions, described their ideas with the help of figures drawn by the teacher on the board. Thereafter the teacher asks the class about the case of where there are 100 stones on each side. The students fail to explain and come up with proposals. The teacher now draws a figure of 3 stones on one side and asks instead the case of 3 stones:

- T: How can we write according to this expression (points to Robert's idea $2 \cdot 5 + 2 \cdot 3$) if there are 3 stones on each side?
- Samir: 2 times 3 plus...2 times 2, it should be.
- T: Ok, (writes $2 \cdot 3 + 2 \cdot 1$) And it is equal to?
- Samir: 8.
- T: How about according to this expression? (points $4 \cdot 4$)
- Kejo: 2 times 4. Is equal to 8
- T: Ummh? (writes $2 \cdot 4 = 8$) Then I would ask you what does this 3 (points on 3 of $2 \cdot 3 + 2 \cdot 1$) mean?
- Samir: The 3 stones on the left side and on the right side. [see Figure 6.8]

- T: And this? (points on 1 on $2 \cdot 3 + 2 \cdot 1$)
- Samir: The rest of the stones in the middle. (points one on the top and one on the bottom) [see Figure 6.8]

Figure 6.8: Samir's explanation for the expression $2 \cdot 3 + 2 \cdot 1$.

When Kejo answered 2 times 4, the teacher was uncertain if Kejo had a figure in mind and associated τ_2 : $(n - 1) \cdot 4$, or simply answered 2 times 4 because he knew that the total number was 8. That is why she went back to the first formula $2 \cdot 3 + 2 \cdot 1$ and decided to ask the class about the cases of 4, 6, 7, 8 and 9 stones for letting the students to recognise the pattern.

She lists the answers from the students for several cases on the whiteboard:

3 stones	$2 \cdot 3 + 2 \cdot 1$
4 stones	$2 \cdot 4 + 2 \cdot 2$
5 stones	$2 \cdot 5 + 2 \cdot 3$
6 stones	$2 \cdot 6 + 2 \cdot 4$
7 stones	$2 \cdot 7 + 2 \cdot 5$
8 stones	$2 \cdot 8 + 2 \cdot 6$
9 stones	$2 \cdot 9 + 2 \cdot 7$

The students begin to see the pattern and grasp the relationship between the figure and the expression. Now again, she asks about the case of 100 stones.

- Tina: 2 times 100 anyway to begin with.
- T: Ok. (writes $2 \cdot 100 +$) How can we express this part (points on the rear term)?
- Kejo: 2 times 70. No, 2 times 80!
- Someone: 2 times 98!
- T: 2 times is agreed. (writes $2 \cdot 100 + 2 \cdot$) What is happening here? (points on 9 and 7 of $2 \cdot 9 + 2 \cdot 7$) 3 and 1, 4 and 2, 5 and 3, 6 and 4, 7 and 5, 8 and 6. Aisha?
- Aisha: 2 in interval. [Someone talked at the same time and could not be heard in the class]
- T: What Aisha said, Kejo?
- Kejo: 2 in interval.
- T: Then in that case, 2 times what?
- Kejo: 80!
- T: Aisha?
- Aisha: 80.

T: Is 2 in interval between 80 and 100?
 Kejo: 98!

Here, the teacher gives aimed questions to press out the correct answers. She asks the case of 1000 and one million. Kejo, who now can see the pattern clearly answers, $(2 \cdot 1,000,000) + 2 \cdot 999,998$. The teacher steers the direction of the lesson to come to the general solution.

T: In that way, we can calculate whatever, any number. That is called a general solution in mathematics. A general solution is something which we can apply to any of the cases. We look at this chart...as number 1 and 2..., we can first complete the chart. What is the case of 2?
 Ronja: 2 times 2 plus 2 times...0?
 T: Yes, then the case of 1 stone?
 Robert: 2 times 1 plus 2 times (-1).
 T: Can it be right? (writes $2 \cdot 1 + 2 \cdot (-1)$)
 Robert: There is no other way to express.

The teacher and the students check if the value of the case of one stone can be 0. The teacher explains that the reason for the phenomenon is that one stone would not construct a square of the considered type. Now the teacher describes using a variable n as the number of stones on one side. She ask the class how one can express the whole expression.

T: (writes n on the whiteboard). How can we express this one (pointing on 4 of the expression $2 \cdot 6 + 2 \cdot 4$) using n ? There are 2 in between to the n . How can we express it?
 Someone: n^2 ?
 T: n^2 ?
 Ronja: No, $n - 2$.
 T: (writes n [a space] $(n - 2)$) How can we write the whole expression? Compare to those expressions. n is for those (points the number of the stones on the chart) and $(n - 2)$ for those (points the rear part of the terms). What are missing?
 Someone: Plus.
 T: Yes, (writes $n +$ [a space] $(n - 2)$) and? What else?
 Someone: 2 times and 2 times.
 T: (writes $2 \cdot n + 2 \cdot (n - 2)$) Is that correct? Do you [all] understand?

Here, the teacher's act is again, directive. She lets the students complete the algebraic expression through comparing to the numerical patterns. The teacher gives the class a training task to calculate the value of $2 \cdot n + 2(n - 2)$ when $n = 20$ and $n = 40$. Her intention is to let the students understand that the form is applicable to calculate the total number of stones for any number n .

Thereafter, the teacher picks up on Ronja's idea $4 \cdot 4$ as an instance of $\tau_2: (n - 1) \cdot 4$. She draws the figure of 4 stones on each side and begins to let the students present the numerical chart again.

- T: How can we circle the stones according to Ronja's idea on this figure?
- Robert: We circle the 3 there, (points the stones on the bottom) 3 there 3 there and 3 there. 4 times 3.
- T: (circles 4 sides) [see Figure 6.9]

Figure 6.9. Robert's explanation for the expression $4 \cdot 3$.

The teacher finishes the numeric chart by asking the students and gives the class a question about the case of n stones on one side.

2 stones $4 \cdot 1$
 3 stones $4 \cdot 2$
 4 stones $4 \cdot 3$
 5 stones $4 \cdot 4$
 6 stones $4 \cdot 5$
 7 stones $4 \cdot 6$
 8 stones $4 \cdot 7$
 9 stones $4 \cdot 8$

- T: We must find the pattern of this as we did in the previous expression.
- Someone: 4 times something.
- Samir: When we look at it from the rear, the pattern is [4 times] 1 for 2 stones, 2 [4 times] for 3 stones, [4 times] 3 for 4 stones, [4 times] 4 for 5 stones and so on.
- T: Yes, and if those (points the number of the stones) are n , how do you express that (points the factors which are multiplied with 4) using n ?
- Samir: What?
- Ronja: $n - 1$!
- T: This is $n - 1$ (writes $(n - 1)$) and what is more? Because it is not only $n - 1$.
- Robert: There are $n - 2$, $n - 3$, $n - 4$?
- T: What is here (points all 4s)?
- Someone: 4.

The teacher completes the general form $4(n - 1)$ on the whiteboard. However there was no time left to compare this general form with the previous expression, $2n + 2(n - 2)$.

In this lesson, the techniques $\tau_2: (n - 1) \cdot 4$ and $\tau_5: 2n + 2(n - 2)$ were suggested by the students. These techniques are justified by the technologies θ_1 : partitioning of figures, θ_2 : inclusion-exclusion in counting and θ_5 : informal induction from a table. The distributive law (θ_3) was not applied in this lesson. The teacher used mainly the “*teacher-directed instruction*” strategy (OECD, 2013). It was observed that about the half of the class had commented and stated their ideas during this lesson.

6.3 The results of the questionnaire to the students

Here, I will present results of the questionnaires made in the end of the project including both classes. The questionnaires were non-anonymous and therefore all respondents’ names were known. The intention with making a non-anonymous questionnaire was to be able to compare the outcome with my classroom observations and the interviews with students.

6.3.1 Grade seven

11 of 13 students in grade seven answered to the questionnaire at the last lesson. Regarding the question “working with mathematics and participating in the lessons was fun”, the students’ answers are as follows: “agree strongly” (5 of 11), “agree” (5 of 11) and “do not agree” (1 of 11). Here are some student answers to the items “Give some example what you thought was fun” and “Give some examples the things you thought it was especially good/bad” (the texts within the parentheses are added by me):

It was much fun (than before) with math, since I understand (mathematics now) more.

I think “Today’s Problem” was fun.

To work in groups was good.

The problems posed on the whiteboard have been good. (it is not clear if the student means about the problems, or presenting their solutions using the whiteboard, or the both)

Two students gave some critical comments on the attitudes of their classmates:

Not so many (students) listen (to the teacher or the other students’ arguments during the lessons).

The lessons themselves have been good. But all the chatter and screaming (was not good).

Concerning the questions “I have received enough opportunities to explain my actual thoughts” and “I felt sufficiently secure to explain my actual thoughts during lessons”, about a half of the class marked “I agree strongly” (5 of 11), some marked “I agree” (4 of 11) and some marked “I do not agree” (2 of 11). On “When I was unsure about what to do or how to do it I have asked for help during lesson time”, more than half of the students marked “agree strongly” (7 of 11) and the rest of the students marked “agree” (4 of 11). To the question, “I want to continue with the same type of lessons at grade eight”, the majority of the students (7 of 11) marked “agree strongly” and the rest (4 of 11) “not agree”. Some proposals for the next semester are the following:

They (the lessons) have been very good. I think we should continue like now. But one should increase the level of the difficulty (of the problems). (I prefer to have) more difficult problems and to start with new areas in mathematics.

Here is a comment from one student who did not “agree” to continue the same type of the lessons at grade eight:

I think we shall have the textbook (to train the tasks) and I want to work with A and B (names of two students, whom she used to work with solving the tasks in the book at grade six).

Generally, those students that marked most of the statements as “I do not agree” did not give many comments to their answers.

6.3.2 Grade eight

Of the students 14 of 17 answered the questionnaire during the last lesson. To the statement “working with mathematics and participating in the lessons during the spring semester was fun”, the majority of the students marked “agree strongly” (10 of 14) and some students marked “agree” (4 of 14). Here I quote their comments on the question “Give some example what you thought was fun”:

Discussions.

It has been fun to be with the groups and to solve problems together.

Learning new things. Solving problems with others.

The teacher. She is an amazingly good teacher. One learns things quickly.

When I understand problems.

Here are their comments on the question “Give some examples the things you thought it was especially good/bad”:

It was good that T (the teacher’s name) has explained in a calm and reasonable way.

(It was good) that one was allowed to join the lesson and to answer. Good examples and the help for everybody.

(It was good) that one is disturbed by anyone anymore. Everybody joined the lesson and listened to what she (the teacher) explained. (It was good) that one could discuss.

I think that chapter Algebra and Equations was tough.

What has been bad was that people (the classmates) commented (on other's solution) all the time.

The last reflection was opposite to the rest of the class. The student might have felt that the lessons did not proceed quickly enough, since they were interrupted by comments from students. I did not have an opportunity to ask her.

Regarding the statements that they had received enough opportunities to explain their actual arguments and that they felt secure enough to do that, the majority of the class marked that they “agree strongly” (11 of 14), and only a few that they “agree” (2 of 14) and “not agree” (1 of 14). Some students wrote comments as those below:

Yes, if I had something I wanted to say in the class, then I definitely got opportunity to do so. I feel quite secure to say what I want to say. But not fully secure.

Now, when I understand (math) better, I dare to explain my thoughts too.

I ask if I do not understand.

6.3.3 Summary and reflections on the result from the questionnaires

Some tendencies stand out from the students' answers:

- Generally, the students appreciated solving problems by discussing with their classmates.
- The students from grade eight placed higher value on whole-class discussions than those of grade seven.
- Students in both classes found that they were given enough opportunities to formulate their solutions and ideas during the lessons.
- Half of the students in grade seven did not feel secure enough to explain their actual reasoning, while most of the students (except one) in grade eight did.

The reflections from the grade eight students were comparatively more positive in every category and they took more time to write comments than the grade seven students. Some students from grade seven complained about the “chatter” from their classmates.

6.4 The results of the interviews with the students

In this section, I will present the results of the interviews I made with the students.

6.4.1 Students' responses prior to the problem solving lessons

Grade seven

The students of grade seven came from four different elementary schools in the region. Seven of 15 students (at the time of September 2010) came from a grade six class in the current school (elementally school section).

In the interview, all students said that most of the time of the lessons in grade six they spent on solving exercises from their textbooks. Most of them had teachers giving short reviews (max 15 minutes) at the introduction of the lessons. About the half of the students used to solve the exercises together with their classmates. Two students from the same school told that they sometimes presented their solutions in front of the class. Only one student said that mathematics was a favorite subject in grade six and most of the students held that mathematics was "boring". However, of those students, only one student thought that mathematics was difficult. The remaining 13 students considered that mathematics was an easy subject for them or that they were at about the same level as their classmates in grade six:

Math lessons were boring, since almost everything we went through at the lessons, I already could. (Linda)

The class was quite messy. We did not have any kind of discussions in math at all. (Robert)

All the students answered that it is important to study mathematics so as to manage their future everyday-life.

Grade eight

Most students in grade eight, except some students who had transferred to the current school the previous year, had mathematics classes together with the previous teacher in grade seven until December 2010. All of the 11 students were used to some short reviews by the teacher at the start of the lessons and sometimes they also were asked to present their solutions in front of the class. However, all students said that most time was usually spent solving exercises from textbooks and they never had had any "discussions" about their solutions.

Compared with grade seven, their attitudes on mathematics were more positive. Four students mentioned mathematics as their favorite subject, four students considered that math was "ok" and three students considered it as "boring". Two of the students that did not like mathematics were Jal and Ahmed:

I usually did not understand teacher's explanation. (Jal)

I did not have much chance to say or discuss anything during the lessons. (Ahmed)

Similar to grade seven students, all students considered mathematics an important subject.

6.4.2 Students' reflections on mathematics after the project

Grade seven

Students' overall reflections about the mathematics lessons during the two semesters correlate quite strongly with their answers on the questionnaire. It means that those students, who had marked that they "agree strongly" with most statements in the questionnaire, consistently gave answers to questions of the interviews that showed a positive attitude to the project. However, the students, with a majority of "do not agree" or "agree" on the statements of the questionnaire, were not uniformly negative to the lessons in the interviews. This might be a quite natural reaction, since the interviewer in front of them had initiated the project and had also been present at lessons for nine months.

Here, I present some remarks from the interviews ordered according to aspects.

Thinking on problems

For Ronja, mathematics was not her favorite subject at grade six. But now it has been changed. To my first question "How do you think that the mathematics lessons have worked under the period?" she answered:

It has been great! Math is my most favorite subject in school now. I am much more interested during math lessons, since the lessons are not about learning answers but about thinking more. (Ronja)

Several other students gave similar remarks as Ronja:

The problems were interesting, because I had to think. (Robert)

It was fun to think on problems. (Linda)

One got to think more. (Tina)

I think that my development has been really good (during this period); both in mathematics and thinking. (Samir)

Whole-class discussions

Another common remark by the students concerned whole-class discussions. Of 10 students, eight expressed their positive experiences of it:

Actually, it is fun to express myself (in the class). We never had such a chance in grade six. (Robert)

Other's ideas gave me an opportunity to think in a new way. (Kejo)

I am listening to my classmates' explanations more (carefully) now, so that I understand how they think more. It is good since then, perhaps, I learn some methods I will never come up with by myself. (Ronja)

When I am in good mood, I like describing my solution to the class. It is good to listen to different ideas, since then I can reconsider my ideas (Linda).

When different solutions are presented, then it usually leads to a discussion. Then it is fun. When I listen to some good solutions, then I learn more. (Tina)

I learn how one shall think, through learning how they (other students) have thought. (Marianne)

I listen to (other students) more (now) and understand more. I think it is a short cut to learning things one did not think about previously (he means learning new solution methods). (Samir)

Sometimes it was much easier to understand from listening to other students' explanations than the teacher's adult-mannered one (she said "vuxet sätt" in Swedish). I liked it when different solutions came up in the class. (Helena)

Joining the whole-class discussions

Several students tried to describe how their attitudes towards formulating their solutions and arguments in front of the class had developed during the time:

Now I dare to explain more in class, since I knew that no one will laugh at me even though I say something wrong. (Helena)

I am not scared to say what I think any more. (Marianne)

I came to have more respect for my classmates and other things. The more I listened to others, the more I learned. I want everybody in the class to think forward. If everyone would do that, it would be perfect (to have good lessons). (Samir)

When I explain (my idea/solution) in the class, I remember better afterwards how I solved the problem. (Kejo)

However, some students never felt comfortable talking in front of the class. Aisha is the one who remarked that she was not so delighted to participate in the whole-class discussions:

I think the lessons were good. But I did not dare to say something in the front of my classmates, because W (a students' name) always give some derogatory remarks to me. That was very annoying. I want the teacher to give us a whiteboard demonstration first and then that we solve similar tasks in the book. (Aisha)

Bea is another student who did not appreciate the whole-class discussions. She remarked that the lessons were boring and that the problems were too difficult. According to Bea, she "did not understand at all what others (the classmates) meant when they presented their solutions". She said also that she was afraid of being commented by other classmates in case she said something wrong. Linda valued parts of the whole-class discussions, but stated that she was sometimes irritated by her "excited" classmates who "without raising hands" and "screamed their ideas" during the lessons. Six of the students stated that other students in the class usually listen to the teacher and classmates and four students stated that they

“sometimes” or “occasionally” listen. Kejo remarked on the teacher’s asking all students questions during the class:

I think it was good that T (the teacher’s name) often asked questions to us. But, sometimes it has been a bit tardy to ask everybody and it took too much time.
(Kejo)

Grade eight

The overall reflections of grade eight students on the mathematics lessons also coincide with their answers on the questionnaire. Unlike the interview in grade seven, all 11 students I interviewed showed some kind of appreciation of the lessons, also the one student who gave the lessons a quite “low” score on the questionnaire. No particularly critical reflection stands out among the answers in the interviews.

Students’ reflections

Neither in grade six nor in grade seven, did Nimra experience mathematics lessons as fun. She felt she always had difficulties on math lessons compared with her classmates. She says: “Usually, for me, it was difficult to understand what teachers meant when they explained some methods. That’s why I could not solve so many exercises during the lesson”. Nimra claims her class was pretty messy and that she was not used to listen to the short reviews very well. She reflects on the current situation after five months of the project: “It is interesting that the class concentrate much more during the lessons now. Also I myself think (on the problems) more now”. She finds that the mathematics lessons are now much more fun. To my question, “in what way is it fun?” she answers:

... those different reflections from different persons. Sometimes I understand better by listening to other students’ explanations rather than the teacher’s. Also, by comparing right answers and wrong answers, we can understand why it is right and wrong. (Nimra)

She now frequently goes to the mathematics room when it is the free-choice working pass (60 minutes lessons without reviews). “I want to have more working passes directly after mathematics lessons”. In the following quote Nimra reflects about herself:

I think I am going forward in the right way. I trust myself more now that I can think correctly. (Nimra)

Oliver says that mathematics was always easy for him and that it has always been his favourite subject. In grade six, he mainly solved exercise tasks in the book. He describes the lessons at grade seven:

At grade seven, lessons became better. We could tell little of our ideas during the lessons but not to the whole class. When we could discuss freely, I felt that was fun. But it (lessons) could be quite long-winded (“långrandig” in Swedish) sometimes”. (Oliver)

Regarding current lessons, Oliver remarked:

T (The teacher's name)'s lessons suit me very well. I think the problems are interesting. Some problems are little tangled, so I must think. Otherwise, they are quite easy. I like that the discussions are more open (in comparison to the previous teacher) for everybody. It is good that all students are allowed to say what they think and it is interesting to compare different ways of solving the same problem. We can see one problem from different views. It is also fun to explain my ideas to the class. (Oliver)

Vincent did not hate mathematics at grade seven but it had never been his favourite subject. His class in grade six usually had a short review by the teacher and then the students worked with the book. Vincent thinks that the current method is "pedagogical". His reflections about the problems and the whole-class discussions:

Compared to before, I explain in the group much more. I also learn from my friends. "Today's problems" have never been "easy" for me. So I learn often from other's ideas and their way of thinking. (Vincent)

One day, during the "prao" (work experience program) week, in the beginning of May, Vincent and one of his classmates joined the grade seven math class, since they did not have any practice work-place assigned. My impression of Vincent this lesson was that he was "unusually quiet", compared to how he used to present his ideas in the grade eight classroom. When I mentioned this to him, he agreed that he did not talk much that lesson:

I know that I did not say much there. I felt that I was not so...safe there, since I did not know many of those grade seven students in the class. (Vincent)

Jal remarked that not all students in the class listen to other students' presentations. He says, "I think most of them listen, but not everybody". According to Jal, he does not feel secure in expressing his ideas in front of the class:

Math still is not my favourite. I think I find math more difficult compared to my friends. But the lessons became a little bit more fun. I think it is good that we are allowed to say what we think. It (listening to the discussions) gives me a little. (Jal)

Several students praised the teacher:

T (the teacher's name) is the best teacher I ever had hitherto. The lessons are fun because I think a lot by myself. Listening to the discussions was good too; I think it was the best way to learn new things. Sometimes I got a real eye-opening moment ("aha-upplevelse" in Swedish) from the others' way of thinking. (Ibrahim)

I liked her way to let us always guess. Then I had to think more. (Alice)

It was good to have T's (the teacher's name) whiteboard demonstration; much better than just working with the book. (Fatima)

6.4.3 Summary and reflections on the interviews

Concerning the students' previous settings in mathematics lessons, they commonly spent most of the time solving the exercise tasks in the book. Several students from both classes compared those previous settings to the current settings – solving problems together with their classmate in the class – and stated that the current settings gives them more learning opportunities.

The result from the interviews shows that higher number of the students of grade eight appreciated the PSO based lessons in comparison to grade seven students. None of the students in grade eight that I interviewed had any critical remarks on the lessons or the attitudes of their classmates. In different ways, all students expressed their appreciations of the opportunity to “think” on the problems, to present their ideas during the whole-class discussions and to learn new ideas from other's explanations. Several students remarked that the whole-class discussions made them participate in the lessons. This tendency was also manifest in the interviews with the students of grade seven except for Aisha and Bea. I will discuss this further in Chapter 7.

6.5 The results of the interview with the teacher

The interview with the teacher was made after all lessons were completed in June 2011. The teacher has learned the PSO approach and implemented it, principally, on every lesson during a period of nine months. Overall, the teacher showed a positive attitude of applying the problems I proposed. She never found any problems “too difficult for my students”; she had high expectations of her students. She often came up with well-constructed and creative problems during the planning of the lessons.

6.5.1 Reflections on the teaching approach

About her teaching approach before this project started, the teacher described that she used to give a whiteboard demonstration first, showing the solution method for the class and she asked students questions during the demonstration. Afterwards she let them work with exercise problems in groups. She stated that she was not particularly aware of making students discuss problems during lesson-time. She commented that a lesson's length of 45 minutes was not enough to carry out such whole-class discussions. Her planning of the lessons usually followed the textbooks order.

The main reason, why the teacher had decided to participate in this project, was that she was concerned about the textbook her school used. According to her, the book was not good enough and something needed to be done about the situation. The spontaneous reaction, when the teacher read about the PSO approach, was that it seemed interesting to implement.

Concerning positive aspects using the PSO approach, the teacher mentioned that planning lessons became her “habit”. She began to plan her lessons much more carefully and in more detail so that the lesson-structure became more solid. As the reason for that she stated:

It depended on setting clear goals for the lessons and anticipating the students’ responses on the problems and also their arguments. It led to a deeper consideration of what kind of questions I would use.

As a negative aspect on the PSO approach, she found that there were not enough lessons with manipulatives (practical hand-on materials). That might be a natural reaction, considering the fact that her school worked with Montessori pedagogy, which stresses a constructivist learning model where different concepts are discovered through hands-on experiences with practical materials. Another negative factor was that working with the PSO was time consuming; she found that it demanded much more of her working hours.

6.5.2 Reflections on the students

At the beginning of the project, the teacher recognised that several students were not listening carefully to her expositions nor to their classmates’ explanations. She found them rather passive with respect to participating in the discussions and solving problems. However, after a while, more and more students became engaged in solving challenging problems and they started to take a more active part in the lessons. For her, it was delightful to see those students. She stated:

It was like they had got the wrong medicine before. When they got the right medicine: challenging problems and the dialogues with other students – and when they decided to join the discussions – their thoughts began to spark!

She believed that even their abilities to have an inner dialog developed. The teacher quoted Nimra:

Nimra said to me that, it was like thinking aloud together in the group and that it had helped her.

More students started to dare to come forward to the whiteboard and present their solutions. They learned that it was not so embarrassing to say something wrong, since they knew that they would always be able to correct the answer together during the discussion. She assumed that the guessing moment worked as a “warming up” to open their mouth:

It became easier for the students to think then.

In the beginning, the teacher enforced that the students took notes every lesson. But she said that, by now, they were used to doing it:

To write down what was written on the whiteboard was a good training for better communication. Not only to think in their head, but also to write down the reasoning process on a paper was good.

As it was described in the method chapter, there was a 60 minutes students' free-choice lesson without reviews once a week in her school, when students were allowed to choose any subject to, for instance, get help with homework by teachers. She noticed that more students came to her during these sessions, sitting and wanting to solve problems together on the whiteboard. There were often spontaneous discussions between the students and also between the students and the teacher.

6.5.3 Other reflections

Reflecting overall on the PSO, the teacher told me that the approach was mostly beneficial for those students who had a "will to learn". However, she stated that even students that initially lacked a strong confidence in their own mathematical skill became more assured of themselves and developed their interest and communicative ability. The teacher referred to a conversation she had had with a father of a student:

Marianne's father has shown gratitude for the fact that her attitude towards mathematics has changed to much more positive, since the class started to have my lessons.

Her colleagues have shown interest to learn the approach as well. The teacher told me that she, without doubt, will continue to use the problem solving oriented approach in her teaching.

Chapter 7

Discussion and conclusions

This study aimed to explore possibilities of applying a certain teaching method from Japan to the Swedish classroom, in order to improve the teaching and learning of mathematics. There are many factors to be considered if one adopts a teaching method from another culture. As Stigler and Hiebert (1999) state, teaching in itself is a cultural activity. The review of the historical background of the structured problem solving approach in Japan, gave an insight into how the Japanese curriculum, the knowledge to be taught, has been constructed during the twentieth century. This insight has helped to recognise how the praxeologies constructed within Souma's lesson plans are strongly connected to the praxeologies outlined in the Japanese curriculum. In that way, one can distinguish consequences of adapting the PSO approach to Swedish classrooms for the subject mathematics.

In this final chapter, the results are discussed in relation to the research questions posed in the beginning of chapter four. Proposals for further research are also suggested.

7.1 The didactic transposition and praxeologies of the PSO based lessons

Firstly, I discuss the act of didactic transposition on the Japanese curriculum that results in Souma's lesson plans. I also compare the content in the Japanese curriculum with the Swedish curriculum. This is an attempt to answer research question 1: *To what extent does the discrepancy of Japanese and Swedish curricula influence the adaptation of Souma's lesson plans?*

Secondly, I try to describe the complexity of the mathematical praxeology that results from the sequence of the lesson plans "Parallels and congruence". Thereafter, mathematical and didactical organisations of the lesson plan "Sums of exterior-angles" are put in relation to research question 2: *What is the structure of the didactical praxeologies in Souma's lesson plans?*

7.1.1 The analysis of Japanese and Swedish curricula

The review of the Japanese curriculum shows that giving students opportunities to experience mathematical activities through problem solving is emphasised inside the curriculum itself. The body of knowledge – the actual activities for learning and teaching mathematics, are described with detailed examples. Both the Japanese and the Swedish curricula emphasise the importance of applying problem solving as a motor that enables the students' mathematical development. However, regarding how it should be applied in practice, there are no examples of methods described in the commentary for the Swedish curriculum, while the guidelines for the Japanese Course of Study offer many such suggestions for every content area.

For grade eight, the descriptions of the area “Geometrical Figures” in the guidelines show that there is a focus on deductive proofs, where students are assumed to learn how to apply previously established theorems.

Miyakawa (2013) has made a comparative study between Japan and France, on the use of mathematical proofs in school geometry for lower secondary schools. He discussed the Japanese *knowledge to be taught* in the domain of mathematical proofs and discussed why some part of it is present in the curriculum and others parts are not. Miyakawa used the notion of *ecology of mathematical praxeologies* (Chevallard, 2002 in Bosch & Gascón, 2006). He points out that there are two elements in the *geometry to be taught* in Japanese lower secondary school that are characteristic: Firstly, it emphasizes generality and, secondly, it works with a “quasi axiomatic” geometry (Miyakawa, 2013).

By “emphasising generality” Miyakawa indicates that the treatment is usually general and that many tasks set out to prove various universal propositions. In the guidelines for the Course of Study, the definition of a proof is stated as “a method to clarify that it is always possible to apply a proposition without any exception”. The Course of Study's emphasis on generality is clearly notable within the domain of proofs (ibid., p. 350).

The second characteristic, “quasi axiomatic geometry”, is about the praxeology of geometry. According to Miyakawa (2013), similarly to the axiomatic system of Euclid's Elements, the system of Japanese school geometry is roughly axiomatic (that is why the term *quasi* is used). The quasi axiomatic geometry is a specific theory used as the regional praxeology of geometry, which is constructed in lower secondary school. It is a simplified version of Euclid's geometry for use in school mathematics. The theory contains axioms and definitions and allows the students to extend the technology and theory with deductively approved theorems. These theorems are proved one by one, and are derived from the definitions and a few axioms (ibid., p. 350).

The *geometry to be taught* in the Swedish lower secondary school, presented in the curriculum Lpo 94 (Skolverket, 1994), contains no connection to the learning of axiomatic proof. To learn and state geometrical theorems and to implement deductive proofs of such theorems is not required. Instead, the focus is

to link different aspects of geometry to students' everyday-life. This discrepancy constituted the main limitation for us, in our attempt to adapt Souma's lesson plans in geometry to Swedish classrooms. It was impossible to pick up some sequences from those lessons and apply them to Swedish lessons without breaking the continuity of the mathematical organisation. Souma's lesson plans in the domain of geometry are constructed to fit the Japanese program of finding deductive arguments that establish the "quasi-axiomatic" geometry.

In Japanese secondary school, the epistemological connections between different sectors in arithmetic and algebra are not as strong as in those in geometry. For example, absolute value of a real number is only described as a distance between numbers and is mentioned only shortly in connection the sector "negative numbers". Later it is never applied in other sectors or domains.

Deductive proofs and axioms are not obligatory in arithmetic and algebra in Japan. The basic statements are already accepted and the theory, like the existence of the real number line, is well established although implicit. Hence, in the "ecology of mathematical praxeologies" in Japan (Chevallard, 2002 in Bosch & Gascón, 2006), a theory based on axioms and a technology extended by deductive proofs is not indispensable for the legitimacy of the praxeologies in arithmetic and algebra. This is also recognised by the fact described in the section 5.1.1, that the area "Algebraic Expressions" is treated as "frameworks" in the "Approaches to Content Organization" in the Course of Study. It means that algebra is considered a medium to support the learning of other core concepts, for example geometry, which is considered to belong to the Platonic "mathematical world". That an axiomatic system is used in geometry is a distinct element of the didactic transposition from the scholarly knowledge to the knowledge to be taught in Japan. The domain Geometry in the Course of Study has received a special position compared to other domains in the Japanese curriculum.

That the sectors arithmetic and algebra are less explicitly connected is reflected in the sequence of Souma's lesson plans. Hence, we could adapt those lesson plans from arithmetic and algebra more easily than those from geometry. There are also many more commonalities between the Japanese and Swedish curricula in arithmetic and algebra. Both include an extension of the number concept to the negative numbers and rational numbers and also, as a goal, to acquire the basic skills of arithmetic operations on symbolic expressions. Both curricula treat the use of algebraic equations as a powerful tool for modelling. Both use identities between symbolic expressions for stating the arithmetic laws and formulas for general solutions and, similarly, to state equivalences between equations, as in the cancellation laws.

7.1.2 The complexity of mathematical organisation of Souma's lesson plans

The sequences of Souma's lesson plans in the chapter "Parallels and Congruence" is part of a process to establish a regional praxeology within the domain of

Euclidean geometry, regarding lines, angles, polygons, angular sums and, thereafter, congruencies. The set of *tasks* consists of making *proofs* and *arguments* within this domain, in addition to making computations on concrete geometrical figures. *Technological* and *theoretical* components are definitions, axioms and theorems used to describe lines, points, polygons and angles.

The sequence of the sections and the sequence of each lesson plans are epistemologically carefully planned, so that each section is connected to the next section and subsequent lesson plans are connected in order to successively extend the praxeology. As described in the previous section, with the analysis using the didactic transposition theory, the complexity of the praxeology is largely dependent on the epistemological strong connections between every sector in the domain of geometry outlined in the Japanese Course of Study.

The analysis of the lesson plan “Sums of exterior angles” shows that the complexity of the praxeology covered in the described lesson is local, but it is integrated into the larger praxeology under construction. The six moments of the dynamic of the didactical process (Barbé et al., 2005) are clearly realised. Technological discourse that covers and certifies varying techniques takes place and one generalises a geometrical theorem by using a partly algebraic proof.

7.2 Didactical praxeologies used in the lessons and student’s mathematical contributions

This section discusses the issue of the influence of the PSO approach which are found during the empirical study in relation to research question number three: *When applying the PSO approach in Swedish mathematics classrooms;*

- a. *What can the mathematical and the didactical organisations look like?*
- b. *How does this approach encourage students’ mathematical contributions?*

7.2.1 The lessons about operations on negative numbers

In the lessons on negative numbers, the mathematical and didactical tasks are to make the class starting to absorb and by themselves construct and establish the rule that “minus times minus is plus”. The guessing moment, which is realised at the first *encounter* (FE) and the *exploratory* moment (EX), therefore becomes rather different from the lesson plan in “sums of exterior-angles”. The latter examples aim to have the students eventually *find out* a geometrical proposition, while the lesson on negative numbers aims to *convince* the students of the rule “minus times minus is plus”.

The task and the techniques in the mathematical organisation are already described in the section 5.3.1. The technologies are the number line and the vector illustration of numbers. The theory is the real or rational number system (the number line).

The *didactical technologies* are rules, terminology and know-how that inform the structuring and the use of the *didactical techniques*. The didactical techniques (supplied by the PSO) are:

1. Consideration of suitable problems. In this lesson, the mathematical techniques the students suggest are τ_6 : to use the claim from an external authority that minus and minus is plus and τ_7 : treating the positive and negative numbers as vectors of opposite direction on the number line; our preparation aimed for τ_7 . We could let the student reach other techniques which may have followed if we had given an additional problem like, “Compare the value of the expressions: a. $(+3) \cdot (-3)$, b. $(+2) \cdot (-3)$, c. $(+1) \cdot (-3)$, d. $(0) \cdot (-3)$, e. $(-1) \cdot (-3)$, f. $(-2) \cdot (-3)$ and g. $(-3) \cdot (-3)$ ”.
2. Encouraging initial guesses (“in spite of the simple expressions, different guesses come out” and arouse a curiosity among the students).
3. Techniques to steer and invigorate the whole-class discussions.
4. Use of a textbook. Confirmation and institutionalisation of the mathematical praxeology by reflecting on the theory as it is outlined in a textbook.

The PSO approach uses the guessing technique and other motivational techniques to stimulate the whole class discussion. Without the discussion, students might just accept the statement of Frank that it follows “since minus times minus is plus”, without reasoning for themselves and actively participating in the construction. Also, when Farid and the teacher conclude that $(-2) \cdot (-3)$ should be +6, since “minus times minus is plus”, Ahmed protests and express his honest reflection that the case of subtraction might not hold in the case of multiplication. Even if many of the students might hold the same objections, it can happen that they think it is awkward to show that they do not understand and would therefore choose to simply accept the claim. The adidactical component of the didactical game might be lost for these students.

Ahmed’s reaction brought on a new discussion and through those discussions, students obtain a richer picture of the negative numbers. This phenomenon can be understood using the concept of a *didactical contract* (Brousseau, 1997) or *sociomathematical norms* (Yackel & Cobb, 1996); Ahmed understands that the teacher’s intention is not just to learn them to use the rule “minus time minus is plus”, but also to discuss the derivation and justification. He also knows that the class will allow him to do so.

These *contracts* and *norms* were not so developed in the grade seven classroom. For that reason the didactical game never leads to an adidactical situation in the same manner and the students conclude with the statement “minus and minus is plus”. The teacher does not try to formulate the answer directly and tries to pick up on the students’ remarks and repeat them. However, Tina’s remark “In the other direction”, is squeezed out by highly directed questions from the teacher. The session illustrates a phenomenon which Brousseau calls *Topaze-effect* (Brousseau, 1997). It refers to situations where the teacher simplifies the

activity and poses obvious questions in order to have the students reach a correct answer. During the first lesson in grade seven, Ronja gave some constructive remarks which show that she treats the subtraction of integer as an operation having direction and length. However, her statements are not elaborated in a whole-class discussion.

One didactical technique the teacher used in this lesson was to “label students’ method by their names” (Shimizu, 2003). The teacher labeled $- (+) \leftarrow$ left (then it is minus) and $- (-) \rightarrow$ right (then it is plus) as “*Ronja & Helena’s rule*” and referred it in the discussions in the following lessons. Shimizu states that this practical technique “may seem trivial, but it is very important to ensure the student’s ownership of the presented method and makes the whole-class discussion more captivating and interesting for students” (ibid., p. 213).

The mathematical goal of having the students make sense of positive and negative numbers as entities of direction and length was realised in both grade seven and eight. When Ahmed solved the training problem $(-4) \cdot (-1) \cdot (-3)$, his explanation showed clearly that the vector illustration of multiplication with absolute value as the length made sense. The same can be claimed to be valid for the grade seven students. When Kejo shows in which direction the product $(+3) \cdot (-2) \cdot (-3) \cdot (+2)$ should be oriented on the number line, he calculates products in pairs and finally correctly located $(-6) \cdot (-6)$ on the number line.

Regarding the initial subtraction problem, we should have treated the sum $-6 - (-2)$ as an addition and a multiplication, that is $-6 + (-1) \cdot (-2)$ and used the absolute value and the vector illustration. We could have demonstrated this after having completed the task on multiplication. It would have given a clear technological discourse on the relation between subtraction and multiplication by minus one. In the grade eight class, Oliver has already noted this technology using his own term “mirrored”. If we had demonstrated that technology in more detail, Oliver’s term could have been more profoundly interpreted in the class.

7.2.2 The case of introducing a general solution

The didactical process

The *mathematical and didactical tasks* in this lesson were to encourage modelling using a variable with help of figures and to develop the students’ skills in manipulating algebraic expressions. The “square problem” we used worked well as an introductory task: The six moments of the dynamic of the didactical process (Barbé et al., 2005) are evidently realised in this lesson with the grade eight students. The moment of first encounter (FE) began with guessing the total number of stones. When Nimra answered that the total number was 20, the didactical process already moved to the exploratory moment (EX); the class noticed that something must be considered to determine the total number correctly. Then the process shifted to the technical-work (T) moment, when the students started to consider different method to express the total number. They found out three of the four techniques which were presented in Souma’s lesson

plan and an additional technique τ_5 : $2n + 2(n - 2)$, which is a variation of the τ_1 : $(n \cdot 4) - (1 \cdot 4)$. The (T) moment continued in the next problem when students determined the number of stones with 20 on the each side.

Thereafter the teacher steered the process to the institutionalisation moment (I) by comparing and categorising the students' different techniques. The class moved to the technological–theoretical (TT) moment when the teacher led students to realise all expressions in general form. Finally, the class moved towards the (EV) *evaluation* moment and (I), institutionalisation moment. They examine the general formula by testing different values for n and institutionalised the technology during the theoretical reflection of the term “general solution”. The realisation of the six moments of the didactical process indicates that the complexity of the mathematical organisation of this lesson is at least local, which is obvious since the aim of this lesson was to connect different techniques τ_1 , τ_2 , τ_3 and τ_4 by having a technological discourse that justifies them.

In the lesson with the grade seven students, the (EX) moment was not realised by students' initiative. Making sense of the transition from the numerical to the algebraic interpretation of the mathematical model was not easy for them. The teacher therefore demonstrated an informal induction from a table to show the pattern explicitly to the class. Like in the lesson on negative numbers, the teacher steered the (T) moment by asking questions. Hence the realisation of the (TT) moment was not driven by the whole-class discussions. Instead it was driven by classical IRE (teacher-initiated question, student's *reply* and teacher's *evaluation*) pattern (Mehan, 1979) between the teacher and the students. The (I) moment was not fully realised since the class did not conclude that the two formulas presented by Robert and Ronja are actually the same.

Adequacy of the problem

As I noted in section 6.2, Swedish textbooks for grade seven to nine often connect this kind of mathematical task to an arithmetical sequence (see Figure 7.1). (e.g. “*Formula 9*”, Mårtensson, Sjöström & Svensson, 2009, “*Matte Direkt 8*”, Carlsson, Hake & Öberg, 2010).

Figure 7.1: Figures for “the match-stick problem”. “How many matches are in n th figure?” An example of a problem, in which students should determine the total number of the objects.

Problems such as the one in Figure 7.1 (a “match-stick problem”) have some different characteristics. It is slightly difficult to come up with different ways to determine a general solution, unless one has covered arithmetic sequences. The anticipated techniques (τ), which students should find out, are:

- τ_1 : $2n + (n + 1)$, where $2n$ is the number of horizontal match-sticks and the term $n + 1$ is the number of vertical match-sticks;
- τ_2 : $4 + (n - 1) \cdot 3$, where 4 is the number of match-sticks of the first square and $(n - 1) \cdot 3$ is the number of match-sticks placed afterwards;
- τ_3 : $3n + 1$, where 1 is the first placed match-stick and $3n$ is the number of match-sticks placed afterwards.

In my opinion, these techniques are harder to find out directly from observation, since the students are not acquainted with the arithmetic sequences nor the inductive construction. In order to introduce arithmetic sequences, students should probably already know how to model using algebraic expressions. Hence, we deemed this kind of problem as not appropriate to use, since the aim was to give an introduction to general parameterised solutions. The match-stick problems are more suitable for introducing arithmetic sequences.

Our focus was mainly on how to find the pattern by observation and how to translate it into an algebraic expression. *The square problem* has more focus on the modelling. Compared with the detailed suggestions in “*Algebra for Everyone*”, many textbooks show only one or, at maximum, two techniques. *Matte direct 8* suggests τ_3 and *Formula 9* suggests τ_1 and τ_2 without proper explanations. The main technology the textbooks suggest is to use a numeric chart rather than a partitioning of the figures. If teachers use those approaches, the lessons may become strongly *teacher-directed* (OECD, 2013) and it is difficult to realise an adidactical situation.

7.2.3 Encouraging students’ mathematical contribution

As stated in the beginning of this thesis, a basic problem in mathematics education is how to motivate the students to participate in the lesson and make them active learners. Naturally, there must also be a strong focus on the mathematical subject matter. As described in “*The Teaching Gap*” (Stigler & Hiebert, 1999), there can be a lively discourse during the lesson without the acquirement of mathematical knowledge. For this reason, I found it important to describe the mathematical and didactical organisations of the adapted lessons.

In my opinion, the adaptation of the sequence of the PSO-based lessons in arithmetic and algebra worked out well with respect to the epistemological goal of building a legitimate mathematical organisation, as well as the didactic goals of making students keener to express themselves in class and to discuss their reasoning in mathematical terms.

I list below some points I infer from the observations and the interviews about the way in which the PSO approach encourages students’ mathematical contributions.

The power of the problems

Both problems, in negative number and the square problem in the introduction of general solution, look very simple, but not so apparent that all students guess the

correct answer immediately. The problems fulfil the role of evoking curiosity and leading the students to the exploratory moment.

The Square Problem is somewhat open-ended and it generates several techniques, which are justified by technological arguments like the partitioning of figures. The techniques are then further justified and shown to give the same answer by algebraic arguments. According to Brousseau, *learning is adidactical*; learning takes place when a student tackles the problem as it is raised by the *milieu* and not by the teacher.

The grade eight students elaborate their “winning strategy” in that *situation of action* (Brousseau, 1997) where the students test their ideas and strategies on the figures. Later in the discussions the students will extend the milieu with the concept of a “general parameterised solution”. The process provides a feedback to students’ attempts and arguments. The explanations and discussions that follow in the class will also help them to develop their problem solving strategies.

The situation of action was not fully realised in the grade seven lesson. The students interacted mostly with the teacher and only a few students made, explicitly, autonomous experiments on the figures. However, they figured out two techniques and could investigate the viability of these techniques in the discussions with the teacher.

The problem presented in the lessons on negative numbers is not open-ended in a same way as the Square Problem. As explained in the previous section, the task of the lesson was to convince the students of the rule minus times minus is plus and the technique we prepared was τ_7 . Hence it was not constructed so as to generate different techniques. The realisation of the situation of action is a more difficult didactical task: The way students find out the “strategies” or “explanations” by themselves, depends largely on how the teacher manages the whole-class discussion.

The power of the guessing

The motivational perspective behind the guessing technique might be discussed using Polya’s notion of a “guessing game” that gives students “an opportunity to discover and think by themselves” (Polya & MAA, 1965). One may put it in relation to the notion “sense of game” used in TDS (Brousseau, 1997). The didactical technique of introducing a “sense of game” in the lesson is described as leading towards those “*adidactical situations*” (ibid.), where students have an independent motivation to solve the task. In Souma’s example lesson flow from section 5.2.4, the teacher lets the students guess which of the two sums of exterior-angles is the largest. By stating a guess, the students join the game. Even if some students initially start with wild guessing they might follow up with rational guessing later. The guessing moment leads to “*devolution*” (Brousseau, 1997), a situation in which the teacher hands over the responsibility for searching the solution to the students. Devolution is, according to Brousseau, required in order for the situations to become adidactical.

In the two example lessons, the guessing moment plays an important role to initiate the process of deliberations. Thereby, it gives an opportunity for all students to participate in the lesson. In the observations, the students made both “reasonable” and “wild” guesses during the lesson on negative numbers. Most of the students in grade eight made reasonable conjectures, while many students in the grade seven class made wild guesses without careful consideration of the problem. In the observation of grade seven, the teacher used the guessing moment to invite the students to start to think on the problem. In my opinion, even those wild guesses have the important function of making the students “pay attention” to the problems. Thereafter, when the students start to hear different guesses, they reflect on the guesses and further, considering which property of the problem led his classmate to make a different guess (Souma, 1997). Moreover, having *all students* state a guess, gives each of them a legitimacy to *be included* in the interactions during the lesson.

Beginning each lesson with some form of the guessing moment, will establish a routine in the mathematics lessons for students. The routine will build, a pattern of social norms (Yackel, Cobb & Wood 1991). One such norm is the consensus that “*the guessing does not have to be correct*”. Eventually, the class institutionalises a certain pattern of more refined sociomathematical norms (Yackel & Cobb, 1996), such as “*the guessing should lead us to start to think on the problem*”. Thereby, these sociomathematical norms might in the long term evolve to a further consent; *the guess should be rational, it should originate in passably mathematical deliberations*. I base this on the result from the interviews and the lesson observations. “I am much more interested in math lessons, since the lessons were not about learning the answers but about thinking more”; Ronja’s (grade seven) remark at the interviews reveals a segment of this phenomenon.

The power of the whole-class discussions

In the interview, the teacher quoted Nimra’s remark on the whole-class discussions as that of “thinking aloud together in the group” and Nimra and many other students placed high value on the whole-class discussions. Several students reflected on the benefit of listening to other students’ various solutions: “it is good that all students are allowed to say what they think and it is interesting to compare different ways of solving a same problem” (Oliver), “Sometimes I understand better by listening to other students’ explanations than the teacher’s. Also, by comparing the right answers and the wrong answers, we can understand why it is right and wrong” (Nimra).

From the grade eight lessons, especially, it could be observed that the whole-class discussion brought to the students opportunities to reconsider their interpretation of, say, negative numbers and general solutions. In the grade seven class, one student proposed to the teacher “Can’t you just say if it is right or wrong?” If the teacher had accepted this proposal and explained the arithmetical operations using a technique proposed by herself, “*Ronja & Helena’s rule*” would never have seen the light. The lesson would become “*teacher-directed*” (OECD,

2013) and the students' attitudes towards the lesson would probably remained more passive.

As described in sections 2.1.8 and 2.2.3, managing fruitful whole-class discussions is a great challenge for the teachers. It may easily fall into a type of discussion where only a few students are interacting with the teacher and where the main focus will be on presenting different solution methods without developing a deeper mathematical discussion (Tanaka, 2011). The whole-class discussions in PSO based lessons are strongly connected to the power of the problems and guessing moment. Without a well composed problem and an active guessing moment the chance to carry out a fruitful whole-class discussion risks becoming more limited.

The results from the observations and the interviews indicate that the regulations of the social and sociomathematical norms strongly affect the whole-class discussions (Wood, 1993; Yackel & Cobb, 1996; Franke et al., 2007). In the interview, both Aisha and Bea from grade seven state that they would prefer to have "normal" mathematics lessons starting with a short review and then moving to individual work with the textbook. They did not consider it to be fun to discuss the problems. They both remark that they did not feel secure to express their actual way of thinking in the class. Many other students in both grade seven and eight reflected on their attitude-development: "Now I dare to explain more in class, since I knew that no one will laugh at me even though I say something wrong" (Helena).

Also, Linda and Samir of grade seven remarked that the class lacked an attitude to listen carefully to each other. When Vincent from grade eight joined a grade seven lesson, he felt "not so safe" and did not want to present his ideas during this lesson. In grade eight, only Jal thought that some classmates were not listening to other students' presentation carefully. He also remarked that he was not confident to present his ideas in the class.

All those observations indicate that the possibility to evolve students' autonomy for participating in constructive discussions will be blocked if a student does not recognise the regularities for the discussions constituted by the social norms (Lampert, 1990; Wood, 1993). The norms are "the taken-for-granted ways of interacting that constitute the culture of the classroom" (Wood, 1998, p. 170). As presented in the result of the questionnaire in section 6.3, the social norms regulating the discussions in the grade seven class are not fully established. The remark from a student "Can't you just say if it is right or wrong?" also indicates that the norms with the purpose to enact intellectual autonomy (Cobb et al., 2001) in mathematical discussions are not fully established.

The instability of the social norms affects not only students' motivation to participate in the mathematical discussions. As is illustrated by Kejo's remark in the interview that he sometimes thinks the teacher asks too many questions to every student during the lessons, this instability influences the teacher's acts: The teacher faces the problem that not all students are taking a part in the lesson and

she tries to involve them by asking each a question. The risk is that this leads to the *teacher-directed instruction* (OECD, 2013) rather than *cognitive activation strategies* (ibid.).

The power of planning teaching sequences

In “The Teaching Gap” (Stigler & Hiebert, 1999), it is described that most Japanese teachers will explicitly state the connection between the different parts of the lesson. The authors explain that Japanese lessons are planned as “complete experiences – as stories with a beginning, a middle and an end” (ibid., p. 95). In such lessons, all activities are related to one another.

Ruthven et al. (2009) give illustrations of how to use a context in a teaching sequence. Their study on TDS based lessons shows the milieu is often designed to provide a context with which students are already familiar; “This familiarity guides the opening exchanges between situation and students” (ibid., p. 332). One of the crucial factors for the realisation of such context-emphasised teaching sequences is the teacher’s daily work of planning lessons.

In the interview, the teacher explained how “planning lessons” became her *habit*. She began to pay much more attention to the details of the lessons and she remarked that she found that the lesson-structure became more solid. As presented in chapter five, the teacher considers the following *components* as key to planning lessons according to the PSO approach; Firstly, the mathematical tasks and secondly appropriate initial problems. At that point, the teacher must consider how the problems shall be presented (the four basic patterns of initial problems in section 5.2.1). Which type of initial problems and presentation are most suitable, depends on classes’ different settings and circumstances. Thirdly, the teacher must then anticipate the students’ likely solutions and arguments. This is also a crucial part of the planning in order to realise meaningful whole-class discussions (Cobb et al., 2001; Stein et al., 2008). It makes possible to plan which types of students’ solutions should be picked up during the “kikan-shido” (moment of monitoring) for preparing the whole-class discussions (Shimizu, 1999). This anticipation also helps to plan which kind of questions she can state to follow up. Fourthly, the teacher plans the phases of the whole-class discussions (in section 5.2.1 I mention Souma’s suggestion of different types of whole-class discussions). Finally, the teacher must plan how she best institutionalises the knowledge they have learned: Through reading the book aloud? Showing a compendium?

The realisation of all those aspects in the lesson plans is a great challenge for teachers. However, without this planning, carrying out the lessons as “complete experiences” might be impossible.

The didactical situation created by the teacher when she posed the problem of computing the total number of stones with 100 on each side, did not lead to an adidactical situation in the grade seven class. The teacher then prepared another didactical situation for the lesson with the grade eight class, where she let the students first guess the total number of stones with five stones on each side and then had them work in pairs on the problem with 20 stones. This brought up more

varied ideas and more active discussions. Of course, the improvement might depend on other factors, such as students' better established mathematical content knowledge, or well established social and sociomathematical norms. But the daily work of planning lessons using the above mentioned components definitely will train the teacher's skills of judgment for using and adapting different strategies.

One can interpret Stigler and Hiebert's expression "complete experiences" as a lesson where the six moments of the didactical process (Barbé et al., 2005) are realised. As previously shown, all six moments of the didactical process were realised in the grade eight lesson. To aim for the realisation of these moments can be a design strategy for reaching a lesson with a "complete experience" while at the same time considering the complexity of the praxeology constructed in the lesson and previous lessons.

The long term planning of lessons, so that the lesson sequence is epistemologically well connected, is another challenge for teachers. Even if a teacher follows the order from a well composed textbook, individual lessons run the risk of being fragmented. It demands genuine mathematical knowledge and careful planning by teachers.

7.3 Conclusions

The background for today's structured problem solving in Japan goes back to when the Green Book was published in 1936. The Green Book gave an opportunity for Japanese educators to consider how to raise pupils' mathematical way of thinking in the lessons. Problem solving centred teaching was introduced and it was pointed out clearly in the curriculum. Eventually, the approach of structured problem solving reached the form it has today.

The PSO is a variant of the structured problem solving, with the technique of the task construction, guessing and whole-class discussions. The analysis of the lesson observations and interviews in section 7.2 gives an answer to my third research question. The PSO approach based lessons provide a complex mathematical praxeology and the teaching method has a focus on how to arouse students' curiosity about and motivation for participating in the lessons. The teaching methods proposed by the PSO are mainly motivated from a cognitive perspective and the emphasis is on the students' motivation to participate in the discourse.

There is, however, no contradiction with epistemological goals: That the proposed problems lead to multiple techniques means that these need to be discussed and justified, which leads to the use and establishment of a richer technology and theory. For epistemological reasons, Souma also proposes, at least for "everyday" lessons, that the problems used for problem solving should be sufficiently closed, so that one does not stray too far away from the path suggested by the curriculum. He also advocates the use of a textbook for the purpose of institutionalising and evaluating the knowledge produced during the lessons.

It might be stated that the act of didactic transposition from the scholarly knowledge to the knowledge to be taught is carefully done by the Japanese noosphere. The Japanese curriculum provides a relatively detailed fundament for the large mathematical (and also didactical) organisations. As a consequence, the transposition from the knowledge to be taught to the taught knowledge becomes quite explicit. The ATD based analysis in section 7.1 answers my first and second research questions. The taught knowledge in a sequence of Souma's lesson plans in geometry results in a complex mathematical praxeology. Partly, this stems from the extent of the mathematical organisation in the Japanese curriculum. It also causes a limitation for adapting such lesson plans in geometry to Swedish classrooms.

There are certain epistemological problems with this adaptation of the PSO. Some of these problems are probably generic for any intervention study of the same limited scale. The basic problem to deal with is the discrepancy between the Japanese curriculum, which the PSO lessons largely follow, and the Swedish counterpart. As stated, the Japanese curriculum and its guidelines are more ambitious both with respect to extent and detail and to what age basic concepts are introduced.

In our case, the main concern is the additional technology in the operations on negative numbers: the absolute value and the vector illustration, suffers the risk of hanging loose in the long term picture of the further studies of the class; the technology will not be fully institutionalised and it remains a way to explain and illustrate basic arithmetic and algebraic rules. In this way, the praxeology remains to be quite local. This is a pity, since both concepts have great potential significance in other mathematical topics (for instance, when dealing with linear functions and analytic geometry).

The basic problem that the PSO aims to address – that of individual and social motivation – is implicit in Garcia et al.'s (2006) description of what they call the “monumentalisation” of mathematical organisations; that, due to the *illusion of transparency*, students are commonly invited to *visit* the mathematical organisations, but not to *construct* them. The problem solving lesson structure addresses this motivational issue on the micro-level – on the scale of the didactic process constructing a local praxeology during an individual lesson. But it does not do so with respect to the overall picture.

An institutional explanation for the lack of overall epistemological analysis in the PSO is perhaps the high level of detail in the Japanese Guidelines for the Course of Study, which can make it hard for educators to leave an established epistemological order. It does also, perhaps, lessen the demand for such an epistemological analysis.

In contrast to the cognitive theoretical base of the PSO, ATD is a macro theory that views learning from an institutional perspective. I argue that the didactic organisation provided by the PSO could be complemented with ATD as a tool for epistemological analysis. ATD can give an epistemological motivation for each

lesson by describing the intended local praxeologies and their integration in overall regional praxeologies.

In particular, it could serve the purpose for the long term planning of lessons. If the tasks presented during a sequence of lessons are carefully constructed, it can lead to conjectures, new problems and methods that productively connect and coordinate the local praxeologies covered into more global ones. Although Souma extensively discusses the construction of questions and how it should both motivate (by openness) and direct (by closedness) students' response, the focus is on how the teacher should structure each lesson. The PSO approach proposes no intrinsic mechanism that guarantees that the constructed praxeologies are more than local. Of course, "following a textbook" could be such a safeguard, but it is contingent upon the quality of the textbook.

Some of the results presented in this study are not surprising. It is quite predictable that a discrepancy between the curricula of the two countries may cause certain limitations for an adaptation of a Japanese teaching method in a Swedish classroom. My theoretical study explicated some of these limitations. Also, it is not unexpected that an interventional attempt including some novelty brings about some "positive" result (the *Hawthorne effect*; see e.g. Adair, 1984). The outcome has been based, however, on a rather long term intervention period and a range of qualitative data analyses. It can also be argued that, while the comparison of the Japanese and Swedish curricula was done within a rather limited scale, using the combination of a theoretical and an empirical study as in this thesis has provided a broader perspective for the analysis of a body of mathematical knowledge which the PSO approach may make accessible to learners. The experiences from the empirical study reported in this thesis would need though, to be corroborated in similar intervention studies in other classroom contexts and topic areas.

7.4 Further research

One suggestion for future research concerns the design of lesson sequences supporting an elaborated epistemological model that provides a principled way to structure generative questions. The proposal by Souma of "Total Mathematics" (1987) puts the formulation and exploration of complex connecting problems *after* the construction of suitable praxeologies, not as in the proposed *Study and Research Courses* (Chevallard, 2006), as a motivation and driving force for the long term study. Study and Research Courses take a global view, where the whole body of knowledge, is motivated by students' initial research and guesses.

The use of ATD could therefore strengthen the proposed didactical organisation of the PSO as a tool for epistemological analysis. The demands for such is especially strong if the PSO is adapted to, say, the Swedish environment, which is not in the same way guided by a detailed curriculum. As an epistemological model, it would ensure that the local praxeologies constructed

during the discourse of individual lessons integrates to a regional praxeology; a sound body of knowledge.

The suggestions in Chevallard, (2006) and Winsløw et al. (2013) of a Study and Research Courses/Path offer a principled way to structure the generative questions posed according to the epistemology supplied by ATD and the establishment of knowledge other than that of “following a textbook”. The Study and Research Courses are intended to make the students’ autonomy of study less of an illusion and, during the initial phases, students are assumed to do autonomous work as groups or individuals by researching the root generating questions and generate and formulate sub-questions. To implement such courses one would probably need – during the research phase – to leave the basic lesson structure of the PSO and, in the Swedish context, revert to the standard lesson form of “guided individual/group work”. In the later phases of Study and Research Courses, the lesson form of the PSO could reappear in order to establish and present the *answers* in the form of local/regional praxeologies and also cover relevant parts of textbooks.

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Appendix I: Questions used in the interview about students' previous settings (September 2010)

Yukiko Asami-Johansson
Högskolan i Gävle

Datum;

Intervju om matematik/matematiklektion i mellanstadiet

Namn _____

Skola (mellanstadiet) _____

1. Hur gick det mattelektionen i årskurs 6?

- Hur började läraren lektionen? Genomgångar?
- Vad brukade du göra? Tysträkning?
- Diskussioner? Kommunikationer?
- Böcker? Stenciler?
- Regelbundna läxor?

2. Hur var miljön i klassen?

- Arbetsro? Lugn?
- Lyssnade klasskamraterna noga på läraren/klasskamraterna?
- Arbetstid?

3. Hur tyckte du om matematik/matematiklektionerna?

- **Roligt/tråkigt?** **Lätt/svårt** att förstå? Intressant att lära sig nya saker?
- Tror du att du hade lättare/svårare med matte jämfört med dina kamrater?

4. Är det viktigt att lära sig matematik?

- Varför? –Vad har du nytta av ?

English translation of interview questions September 2010 for grade 7 students

1. How did your mathematics class look like in grade 6?
 - How did your teacher use to begin the lessons?
Whiteboard demonstration?
 - What did you usually do during the lessons? Doing exercise tasks?
 - Did you usually have discussions during the lessons?
 - Did you use textbooks or compendiums?
 - Did you have homework regularly?
2. How was the working atmosphere in the class?
 - Did your class use to be peaceful/quiet?
 - Did your classmates usually listen to the teacher/other classmates?
 - How many minutes did a lesson usually go on?
3. How did you like the mathematics/ the mathematics lessons?
 - Fun/boring? Easy/difficult to understand? Interesting to learn new things?
 - Do you think you had it more easy/more difficult with mathematics compared to your classmates?
4. Is it important to learn mathematics?
 - Why? What benefit do you think you will get from it?

Appendix II: Questions used in the interview about students' reflections after the project (May 2011)

Yukiko Asami-Johansson
Högskolan i Gävle

Datum:

Intervju om matematiklektionen efter terminen

Namn _____

Skola _____

1. Hur tycker du att mattelektionen har gått under åren?

- Genomgångar:
- **Roligt/tråkigt?** **Lätt/svårt** att förstå? Intressant att lära sig nya saker?
- Uppgifternas svårighetsgrad?
- Diskussioner? Kommunikationer?
- Hade du tillräckligt med tid att räkna?
- Lyssnade klasskamraterna noga på läraren/klasskamraterna?
- Har du lär dig någonting genom att lyssna på andra elevers resonemang?
- Tror du att du hade lättare/svårare med matte jämfört med dina kamrater?

English translation of interview questions May 2011 for grade 7 students

1. How do you think the mathematics lessons have worked during the period?
 - The whiteboard demonstration?
 - Fun/boring?
 - Easy/difficult to understand the lessons?
 - Did you have interest of learning new things during the lessons?
 - How did you find the difficulties of the problems?
 - How did you find the discussions/communications during the lessons?
 - Did your classmates usually listen to the teacher/other classmates?
 - Do you think you have learned something through listening to other classmates' reasoning on the problems?
 - Did you have enough time to do your training tasks?
2. Do you think you had easier or more difficult math compared to your classmates?

Appendix III: Questionnaire to the students in grade seven

Utvärdering av matematikundervisningen i år 7 Namn: _____

Utvärderingen är till för att jag ska kunna hjälpa dig så bra som möjligt med matematiken. Det är viktigt att du svarar ärligt på alla frågor.

Jag tycker att det har varit roligt att arbeta med matematik i 7:an.

Håller inte med _____ Håller med helt

Ge exempel på vad du tycker har varit roligt:

Jag tycker att presentationerna/lektionerna har varit bra

Håller inte med _____ Håller med helt

Ge exempel på saker du tyckt varit speciellt bra/dåliga:

Jag tycker att svårighetsgraden på "dagens problem" har varit bra.

Håller inte med _____ Håller med helt

Motivera:

Jag har fått den hjälp jag behöver för att förstå

Håller inte med _____ Håller med helt

Jag tycker att jag har fått tillfälle att berätta mina tankar på lektioner.

Håller inte med _____ Håller med helt

Jag känner mig trygg nog att berätta mina tankar på lektioner.

Håller inte med _____ Håller med helt

Jag har frågat om jag är osäker på vad jag ska göra eller hur jag ska göra.

Håller inte med _____ Håller med helt

Jag har arbetat med matematik hemma

Håller inte med _____ Håller med helt

Önskemål och tankar inför nästa termin och annat jag vill säga om matematikundervisningen:

Appendix IV: Questionnaire to the students in grade eight

Utvärdering av matematikundervisningen på vårterminen år 8 Namn:

Utvärderingen är till för att jag ska kunna hjälpa dig så bra som möjligt med matematiken. Det är viktigt att du svarar ärligt på alla frågor.

Jag tycker att det har varit roligt att arbeta med matematik under vårterminen i år 8 (med Erika).

Håller inte med _____ Håller med helt

Ge exempel på vad du tycker har varit roligt:

Jag tycker att presentationerna/lektionerna har varit bra

Håller inte med _____ Håller med helt

Ge exempel på saker du tyckt varit speciellt bra/dåliga:

Jag tycker att svårighetsgraden på "dagens problem" har varit bra.

Håller inte med _____ Håller med helt

Motivera:

Jag har fått den hjälp jag behöver för att förstå

Håller inte med _____ Håller med helt

Jag tycker att jag har fått tillfälle att berätta mina tankar på lektioner.

Håller inte med _____ Håller med helt

Jag känner mig trygg nog att berätta mina tankar på lektioner.

Håller inte med _____ Håller med helt

Jag har frågat om jag är osäker på vad jag ska göra eller hur jag ska göra.

Håller inte med _____ Håller med helt

Jag har arbetat med matematik hemma

Håller inte med _____ Håller med helt

Önskemål och tankar inför nästa termin och annat jag vill säga om matematikundervisningen:

English translation of the questionnaires in Appendix III and IV

1. I think it has been fun working with mathematics and participating in the lessons during grade seven/during the spring semester in grade eight. Give some examples on what you thought was fun. Give some examples the things you thought it was especially good/bad.
2. I think the presentation/lessons have been good. Give some examples the things you thought it was especially good/bad.
3. I think the difficulty of “Today’s Problem” has been appropriate. Please motivate your answer!
4. I have got enough support (e.g. by the teacher) I need to understand (e.g. solution methods) during the lessons.
5. I have received enough opportunities to explain my actual thoughts
6. I felt sufficiently secure to explain my actual thoughts during lessons.
7. When I was unsure about to do or how to do it I have asked for help during lesson time.
8. I have worked with mathematics at home.
9. (only for grade seven students) I want to continue with the same type of lessons at grade eight.
10. Any proposal or thoughts for the next semester. Anything else I want to tell regarding lessons in mathematics.

Appendix V: Questions used for the teacher interview

Lärarintervju om japanska metoden i matematik/matematiklektion

1. Bakgrund
 - Utbildning, inriktning? Huvudämne?
 - Hur många år som arbetat som lärare?
 - I nuvarande skolan?
2. Hur brukade du genomföra mattelektionen tidigare?
 - Hur började du lektionen? Genomgångar?
 - Vad brukade du göra? Tysträkning?
 - Diskussioner? Kommunikationer?
 - Böcker? Stenciler? Miniräknare? Datorer?
 - Regelbundna läxor?
3. Vad tyckte du om metoden spontant i början?
 - Varför ställde du upp?
 - Vad var dina förväntningar inför denna metod?
4. Hur tyckte du om metoden?
 - Positiva aspekter
 - Negativa aspekter
 - Lyssnade klassen noga på läraren/klasskamraterna?
 - Delaktighet?
 - (Utökade) Matematiska diskussionen?
 - Skriftlig dokumentation av problemformulering?
 - Gissningsmoment?
 - Dagens problem?
5. Hur uppfattar du hur eleverna har mottagit metoden?
 - Positiva aspekter
 - Negativa aspekter
 - Lyssnade klassen noga på läraren/klasskamraterna?
 - Delaktighet?
 - (Utökade) Matematiska diskussionen?
 - Skriftlig dokumentation av problemformulering?
 - Gissningsmoment?
 - Dagens problem?
6. Prov? Studieresultat?
 - Tycker du att det har blivit något outcome?
 - Utvecklingssamtal?
7. När tycker du att du har haft en lyckad lektion?
 - Vad är faktorer att lektionen blev;
 - Lyckad?
 - Misslyckad?
8. Vill du fortsätta med metoden?
 - Möjligheter?
 - Hinder?
 - Om ämnesgruppen har blivit påverkad?

English translation of the questions used for the teacher interview

1. Your professional background.
 - Education and subjects
 - How many years have you worked as a teacher of mathematics?
 - How many years have you worked at the current school as a teacher of mathematics?
2. How did you use to implement your lessons before?
 - How did you usually begin your lessons?
 - What did you usually do during your lessons?
 - Did you use to let the student work with training problems in their textbooks?
 - Did you use to let the students discuss during the lessons?
 - Use of books, compendium, calculators, computers during your lessons?
 - Did you use to give the students home works?
3. What was your impression on PSO approach in the beginning?
 - Why did you decide to attend this project?
 - What was your expectation for applying the PSO approach?
4. What was your actual opinion about PSO approach?
 - Any positive aspects?
 - Any negative aspects?
5. How did you perceive the students' reactions on the PSO approach?
 - Any positive aspects?
 - Any negative aspects?
 - Did the students listen to you and the classmates well during the lessons?
 - The students' participation during the class?
 - Do you think the mathematical discussion of the students has been developed under the period compare to the beginning of the project?
 - Did the student make written documentation during the lessons?
 - About the "guessing" part?
 - About "today's problem"?
6. About the results
 - Do you think if the implementing of the approach gave any result?
 - Any reactions during the "parents/teacher meeting" on the approach?

7. When did you think you had a “successful” lesson?
 - Which factors do you think influenced to be “successful” and “unsuccessful” lessons?
8. Will you continue to apply the PSO approach in the future?
 - Any developing possibilities?
 - Any restrictions to apply?
 - Have your colleagues been influenced by you using the PSO approach?