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Asymptotic models of anisotropic heterogeneous elastic walls of blood vessels

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Abstract. Using the dimension reduction procedure in the three-dimensional elasticity system, we derive a two-dimensional model for elastic laminate walls of a blood vessel. The wall of arbitrary cross-section consists of several (actually three) elastic, anisotropic layers. Assuming that the wall's thickness is small compared with the vessel's diameter and length, we derive a system of the limit equations. In these equations, the wall's displacements are unknown given on the two-dimensional boundary of a cylinder, whereas the equations themselves constitute a second order hyperbolic system. This system is coupled with the Navier–Stokes equations through the stress and velocity, i.e. dynamic and kinematic conditions at the interior surface of the wall. Explicit formulas are deduced for the effective rigidity tensor of the wall in two natural cases. The first of them concerns the homogeneous anisotropic laminate layer of constant thickness like that in the wall of a peripheral vein, whereas the second case is related to enforcing of the media and adventitia layers of the artery wall by bundles of collagen fibers. It is also shown that if the blood flow stays laminar, then the describing cross-section of the orthotropic homogeneous blood vessel becomes circular.

1 Introduction

1.1 Formulation of the problem

Blood vessels form one of the most complicated and important systems (circulatory system) in human body, which is exposed to various risks and is poorly amenable to medical treatments. Mathematical modelling of blood transport in arteries, veins, capillaries and other blood vessels is a classical problem which is still very actual nowadays, see [6], [29], [9] and [8, section 8]. Although the existing models are usually based on the anisotropic and composite structure of blood vessel-walls (see Fig. 1 and cf. the monographs [8] and [10]), the analysis in this direction is far from being completed yet. In this way our paper makes a next step in derivation of adequate governing relations that carefully take into account both, the laminated structure of an elastic wall of a blood vessel and the complicated composite structure of each laminate layer of the wall. For this purpose we consider a flow of a viscous incompressible fluid (blood) in a cylindrical vessel having an arbitrary cross-section. The wall of vessel can consist of several layers of anisotropic materials. Our aim is twofold: first, to derive a model, in which a three-dimensional but thin anisotropic wall of the vessel is replaced by a boundary surface; and, second, to give an explicit relation between Hooke's tensors for the three- and two-dimensional models. We obtain such a model under the assumption that the wall's thickness is small compared with the diameter of the vessel, whereas the latter diameter is small compared with the length of the part of the vessel under consideration. In this part of the vessel, the blood flow is assumed to be laminar in view that the hydrostatic pressure prevails over hydrodynamic forces. This, in particular, allows us to conclude in Sect. 4 that the circular cross-section of a blood vessel is optimal in a certain sense. Moreover, the fact that a flow is laminar and the elastic wall material is strong and tough, results in a small deviation of wall from the cylindrical reference shape, and so a dimensional reduction can be applied to the elastic vessel's wall.

The dimension reduction of the three-dimensional Navier-Stokes equations in a blood vessel has been developed in the paper [3], where the two-dimensional wall model was taken for granted. Our results, especially explicit formulas in Section 4, give, in particular, concrete values for the elastic moduli used in [3] in the orthotropic rigidity tensor of the vessel's wall. Thus, in this paper, the main attention is paid to the formal asymptotic analy-

Figure 1: The wall of blood vessel consisting of three layers reenforced by collagen fibres.

sis resulting in these explicit formulas while a scheme of their justification, routine but cumbersome, is omitted for several reasons. First, the dimension reduction for a thin elastic cylindrical shell under fixed external loading, that is, with prescribed hydrodynamical forces, follows a standard scheme; see the papers [28], [5], [17], the monograph [4] and other publications. In particular, the paper [21] contains a detailed proof of the error estimate in a similar situation. Second, the evaluation of the effective elastic properties of blood vessels is considered as the most urgent problem in simulation of circulatory system, see [27].

The mathematical formulation of the problem is as follows. Let ω be a two-dimensional, simply connected bounded domain enveloped by a smooth contour γ . In a neighborhood of γ , say $\mathcal{V}$, we introduce the natural curvilinear orthogonal coordinate system (n, s) , where n is the oriented distance to γ ($n > 0$ outside ω and $n < 0$ inside ω) and s is the arc length along γ measured counterclockwise. Let H be a smooth, positive function on γ , and let h be a small positive parameter. Putting

$$\gamma_h = \{y \in \mathcal{V}, n = hH(s)\},$$

we denote by ς_h the domain between γ and γ_h , see Fig. 2. Then the lumen of the vessel is given by $\Omega = \omega \times \mathbb{R}$ and its wall is $\Sigma_h = \varsigma_h \times \mathbb{R}$. An appropriate rescaling makes parameters and the coordinates dimensionless.

Figure 2: The cross-section of the blood vessel

The flow in the vessel is described by the velocity vector $\mathbf{v} = (v_1, v_2, v_3)$ and by the pressure p which are subject to the Navier-Stokes equations:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} = -\nabla p \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega. \quad (1)$$

Here ρ_b is the density of the fluid and ν is the kinematic viscosity related to the dynamic viscosity μ by $\nu = \mu/\rho_b$. The stress state of the linear elastic wall is described by the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ and by the stress tensor $\sigma = \{\sigma_{jk}\}_{j,k=1}^3$ subject to the nonstationary elasticity equations

$$\frac{\partial \sigma_{j1}}{\partial x_1} + \frac{\partial \sigma_{j2}}{\partial x_2} + \frac{\partial \sigma_{j3}}{\partial x_3} = \rho \frac{\partial^2 u_j}{\partial t^2} \quad \text{in } \Sigma_h, \quad j=1,2,3, \quad (2)$$

and Hooke's law

$$\sigma_{jk} = \sum_{p,q=1}^3 A_{jk}^{pq} \varepsilon_{pq}, \quad j, k = 1, 2, 3, \quad \varepsilon_{pq} = \frac{1}{2} \left(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p} \right). \quad (3)$$

Here ρ is the mass density, $\varepsilon = \{\varepsilon_{jk}\}_{j,k=1}^3$ is the strain tensor, while the rigidity tensor $\mathbf{A} = \{A_{jk}^{pq}\}$ (also called Hooke's tensor), consists of the moduli of elasticity of the wall material and has the standard symmetry and positivity

properties:

$$A_{jk}^{pq} = A_{pq}^{jk} = A_{pq}^{kj},$$

$$\sum_{j,k,p,q=1}^3 A_{jk}^{pq} \xi_{jk} \xi_{pq} \geq C_A \sum_{j,k=1}^3 |\xi_{jk}|^2.$$

Here $C_A > 0$ and $\{\xi_{jk}\}$ is an arbitrary symmetric 3×3 -matrix. In the Eulerian framework (this is our simplifying assumption), the actual position of the interior surface $\Gamma = \gamma \times \mathbb{R}$ at the moment t is given by $\{x + u(x, t) : x \in \Gamma\}$; it corresponds to stretching of the elastic wall caused by the pulsatory flow of blood.

The exterior surface $\Gamma_h = \gamma_h \times \mathbb{R}$ is assumed to be traction free¹, i.e.

$$\sigma_{j1}n_1 + \sigma_{j2}n_2 = 0 \quad \text{on } \Gamma_h \text{ for } j=1,2,3, \quad (4)$$

where $\mathbf{n} = (n_1, n_2, 0)$ is the outward unit normal to Γ_h . On the interior surface Γ , it is natural to impose two conditions. The first one says that the velocity of fluid and the velocity of the elastic wall coincide with each other, i.e. the kinematic no-slip boundary condition holds

$$\mathbf{v} = \partial_t \mathbf{u} \quad \text{on } \Gamma, \quad (5)$$

whilst the second one, dynamic condition, says that the hydrodynamic force equals the normal stress vector (traction comes with minus because the normal $\mathbf{n}$ is interior for Σ_h):

$$\sigma_\Gamma := \boldsymbol{\sigma} \cdot \mathbf{n} = \rho_b \mathbf{F}. \quad (6)$$

Here $\mathbf{F} = (F_n, F_z, F_s)$ and

$$F_n = -p + \nu \frac{\partial v_n}{\partial n}, \quad F_s = \frac{\nu}{2} \left(\frac{\partial v_n}{\partial s} + \frac{\partial v_s}{\partial n} - \kappa u_s \right), \quad F_z = \frac{\nu}{2} \left(\frac{\partial v_n}{\partial z} + \frac{\partial v_z}{\partial n} \right), \quad (7)$$

where v_n and v_s are the velocity components in the direction of the normal $\mathbf{n}$ and the tangent $\mathbf{s}$, respectively, whereas v_z is the longitudinal velocity component ($z = x_3$); finally, $\kappa(s)$ is the curvature of the contour γ at the point s .

¹it can be used also the Robin boundary condition to describe the interaction of the surrounding tissue and the vessel of the blood, see, e.g., [24] and [23]

We assume that

$$\rho = \frac{1}{h}\rho\left(\frac{n}{h}, s, z\right) \quad \text{and} \quad \mathbf{A} = \frac{1}{h}\mathbf{A}\left(\frac{n}{h}, s, z\right)$$

satisfy one of the following conditions:

(I) (*A heterogeneous wall material*): the function $\rho(\zeta, s, z)$ and the matrix $\mathbf{A}(\zeta, s, z)$ are smooth on $\overline{\Sigma}_1$, where

$$\Sigma_1 = \{(\zeta, s, z) : s \in \gamma, \zeta \in (0, H(s)), z \in \mathbb{R}\};$$

(II) (*A laminate wall with layers of piecewise constant thickness*): we have $H(s) = 1$, whereas ρ and $\mathbf{A}$ are defined as follows. Let $h_1, \dots, h_N$ be the given numbers that satisfy the following relations:

$$h_1, \dots, h_N > 0, \quad h_1 + \dots + h_N = h, \quad a_0 = 0, a_j = a_{j-1} + h_j, \quad j = 1, \dots, N,$$

then

$$\rho(\zeta, s, z) = \rho^j(s, z), \quad \mathbf{A}(\zeta, s, z) = \mathbf{A}^j(s, z) \quad \text{for } \zeta \in (a_{j-1}/h, a_j/h),$$

where ρ^j and $\mathbf{A}^j$ do not depend on ζ .

Our aim is to derive a two-dimensional model of the blood vessel wall under Assumption (I) which simplifies the demonstration to some extent. However, the walls of veins and arteries involve composite laminate elastic structures, and so we give explicit formulas under Assumption (II) attributed mainly to peripheral veins (see Section 4.1). In arteries and voluminous veins, bundles of collagen fibres must be taken into account as well, see Section 4.2. Notice that the dimension reduction procedure intrinsically admit passing to various limits and a straightforward approach is to approximate composites with piecewise constant elastic moduli by those having smooth heterogeneous properties and then, in the final integral formula for effective moduli (see section 3.3), to return to the piecewise constant case.

1.2 Results

The dimension reduction plays an important role in mathematical modelling of engineering problems in which certain elements have small size in some directions. Theory of rods, plates, shells, elastic multi-structures etc. are examples worth mentioning. There are a lot of papers on this topic that

describe approximate models and justify them mathematically to a different extent of rigor using various methods and approaches. Note that there are many classical engineering theories for laminated plates and shells (see, e.g., [11]).

Here we apply the rigorous procedure of the dimension reduction; it was developed for problems in elasticity (see [28], [5], [4], [27], [15] and other papers), and for general elliptic problems in [20]. The main difficulty of the present problem stems from the fact that the anisotropic wall has laminated structure. Our approach is based on several important ideas, see Sect.2: 1) application of matrix notation for equations in the elasticity theory referred to as the Voigt-Mandel notation in mechanics; 2) rearrangement of components of stress and strain vectors, which reflects different order asymptotic behavior of "normal" and "tangent" components of corresponding tensors and closely related to the notion of surface enthalpy [22].

The crucial point of our asymptotic approach is to construct an operator $\mathbf{U} \rightarrow \sigma_\Gamma$ of the Dirichlet-to-Neumann type, where $\mathbf{U}$ is a given displacement on the boundary Γ and σ_Γ is the corresponding normal stress vector on Γ . These relation is obtained in Sect.3 and the leading term of σ_Γ on Γ is expressed through a hyperbolic operator on Γ applied to $\mathbf{U}$. Taking this into account the equilibrium equation (6) becomes a hyperbolic system with respect to $\mathbf{U}$ and with the right-hand side $-h^{-1}\rho_b\mathbf{F}$, see (54). Combining it with the Navier-Stokes system (1) and the kinematic condition (5), one obtains a system of constitutive relations that describes the interaction of the blood flow in the vessel with its elastic wall. Similar models have already appeared in mathematical research (see [8, Ch. 8] and [3] and references therein), but only for vessels with circular cross-section whose walls are isotropic and homogeneous.

In the last Sect.4 we present an analysis of our model. In particular, we discuss the connection between elastic coefficient in our model with elastic coefficient of the vessel's wall.

Various laminate composite structures of blood vessel walls are well-known (see [8, Chapter 8]) and, as outlined above, we apply the procedure of the dimension reduction to approximate a thin anisotropic elastic wall by an anisotropic shell in order to derive an explicit formula for the limit rigidity matrix (23). In contrast to usual mathematical models of vessels, we do not assume *a priori* that the cross-section is circular, see Sect.4.4 and 4.5. This allows us to consider wall strains caused by such damages of blood vessels as irregular calcification (hyalinosis, arterial calcinosis), oblong atherosclerotic

deposits (atherosclerotic plaque) and/or various surgical exposures.

2 Elastic walls

The immediate objective of our asymptotic analysis of the elasticity problem (2), (3) supplied with the boundary conditions (4) and

$$u_j = U_j \quad \text{on } \Gamma, \quad j=1,2,3, \quad (8)$$

is to compute the normal stress vector σ_Γ at the boundary Γ . Here, $\mathbf{U} = (U_1, U_2, U_3)$ is a given displacement vector on Γ .

We use the following notations for points inside Σ_h : $x = (x_1, x_2, x_3) = (y, z)$, where $y = (y_1, y_2) = (x_1, x_2)$ and $z = x_3$.

2.1 Elastic fields in the curvilinear coordinates.

We introduce the orthogonal system of curvilinear coordinates (n, s, z) in $\mathcal{V}$, where n and s are defined in the introduction. In particular, the contour γ is given by $(x_1, x_2) = \zeta(s)$, $0 \leq s \leq |\gamma|$, where $|\gamma|$ is the length of γ , which, by rescaling, is assumed to be equal to 1. Let also (n_1, n_2) be the unit outward normal vector to the boundary γ of ω . Then $n_1 = \zeta_2'(s)$, $n_2 = -\zeta_1'(s)$, and

$$(x_1, x_2) = (\zeta_1(s), \zeta_2(s)) + n(\zeta_2'(s), -\zeta_1'(s)), \quad x_3 = z \quad (9)$$

in the neighborhood $\mathcal{V}$. Since this system of coordinates is orthogonal we can use notations and general formulae from [14, Appendix C], for presenting all elasticity relations in this local coordinate system. In particular, the corresponding orthonormal basis is

$$\mathbf{n} = (-\zeta_2'(s), \zeta_1'(s), 0), \quad \mathbf{s} = (\zeta_1'(s), \zeta_2'(s), 0), \quad \mathbf{z} = (0, 0, 1)$$

and the scale factors are given by

$$H_n = H_z = 1, \quad H_s = 1 + n\kappa(s),$$

where $\kappa(s) = \zeta_2''(s)\zeta_1'(s) - \zeta_1''(s)\zeta_2'(s)$ is the curvature of the curve γ . The Jacobian of transformation (9) is denoted by J and $J = H_n H_s H_z = 1 + n\kappa(s)$. The components of the displacement vector in this coordinate system are

$$u_n = n_1 u_1 + n_2 u_2, \quad u_s = -n_2 u_1 + n_1 u_2, \quad u_z = u_3.$$

The components of the strain tensor are given by

$$\begin{aligned}\varepsilon_{nn} &= \frac{\partial u_n}{\partial n}, \quad \varepsilon_{ss} = \frac{1}{J} \left(\frac{\partial u_s}{\partial s} + \kappa u_n \right), \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \\ \varepsilon_{ns} &= \varepsilon_{sn} = \frac{1}{2} \left(\frac{\partial u_s}{\partial n} + \frac{1}{J} \left(\frac{\partial u_n}{\partial s} - \kappa u_s \right) \right), \\ \varepsilon_{sz} &= \varepsilon_{zs} = \frac{1}{2} \left(\frac{1}{J} \frac{\partial u_z}{\partial s} + \frac{\partial u_s}{\partial z} \right), \quad \varepsilon_{zn} = \varepsilon_{nz} = \frac{1}{2} \left(\frac{\partial u_z}{\partial n} + \frac{\partial u_n}{\partial z} \right).\end{aligned}\tag{10}$$

We need also derivatives of the basis vectors:

$$\begin{aligned}\frac{\partial \mathbf{n}}{\partial n} &= \frac{\partial \mathbf{n}}{\partial z} = \frac{\partial \mathbf{s}}{\partial n} = \frac{\partial \mathbf{s}}{\partial z} = \frac{\partial \mathbf{z}}{\partial n} = \frac{\partial \mathbf{z}}{\partial s} = \frac{\partial \mathbf{z}}{\partial z} = 0, \\ \frac{\partial \mathbf{n}}{\partial s} &= \kappa(s) \mathbf{s}, \quad \frac{\partial \mathbf{s}}{\partial s} = -\kappa(s) \mathbf{n}.\end{aligned}$$

Using these relations we obtain the elasticity equations in Σ_h

$$\begin{aligned}\frac{\partial \sigma_{nn}}{\partial n} &+ \frac{1}{J} \kappa (\sigma_{nn} - \sigma_{ss}) + \frac{1}{J} \frac{\partial \sigma_{sn}}{\partial s} + \frac{\partial \sigma_{zn}}{\partial z} = \rho \partial_t^2 u_n, \\ \frac{\partial \sigma_{sn}}{\partial n} &+ 2 \frac{1}{J} \kappa \sigma_{sn} + \frac{1}{J} \frac{\partial \sigma_{ss}}{\partial s} + \frac{\sigma_{sz}}{\partial z} = \rho \partial_t^2 u_s, \\ \frac{\partial \sigma_{zn}}{\partial n} &+ \frac{1}{J} \kappa \sigma_{zn} + \frac{1}{J} \frac{\partial \sigma_{sn}}{\partial s} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \partial_t^2 u_z,\end{aligned}\tag{11}$$

see again [14, Appendix C].

2.2 The matrix notation.

In what follows we use matrix, rather than tensor, notation. By $\mathcal{U} = (u_1, u_2, u_3)^T$ we denote a column vector with the components u_1 , u_2 and u_3 . Using the Voigt-Mandel notation (see, e.g., [13], [15] and [2]) we introduce the strain and stress columns

$$\begin{aligned}\boldsymbol{\varepsilon}(\mathcal{U}) &= (\varepsilon_{11}, \sqrt{2}\varepsilon_{12}, \sqrt{2}\varepsilon_{13}, \varepsilon_{22}, \varepsilon_{33}, \sqrt{2}\varepsilon_{32})^T, \\ \boldsymbol{\sigma}(\mathcal{U}) &= (\sigma_{11}, \sqrt{2}\sigma_{12}, \sqrt{2}\sigma_{13}, \sigma_{22}, \sigma_{33}, \sqrt{2}\sigma_{32})^T.\end{aligned}\tag{12}$$

The factor $\sqrt{2}$ is inserted here for equalizing the euclidian norm of columns and the norm of the corresponding tensors and the upper index T denotes the transpose of the corresponding vector/matrix. Moreover, Hooke's law in (3) converts into

$$\boldsymbol{\sigma}(\mathcal{U}) = A \boldsymbol{\varepsilon}(\mathcal{U}),\tag{13}$$

where A is a symmetric, positive definite matrix of size 6×6 , which is called the rigidity (Hooke's) matrix and whose elements are related to entries of the rigidity tensor $\mathbf{A} = \{A_{ij}^{pq}\}$ by

$$\begin{aligned} A_{11} &= A_{11}^{11}, A_{12} = \sqrt{2}A_{11}^{12}, A_{13} = \sqrt{2}A_{11}^{13}, A_{14} = A_{11}^{22}, A_{15} = A_{11}^{33}, A_{16} = \sqrt{2}A_{11}^{23}; \\ A_{21} &= \sqrt{2}A_{12}^{11}, A_{22} = 2A_{12}^{12}, A_{23} = 2A_{12}^{13}, A_{24} = \sqrt{2}A_{12}^{22}, \dots \end{aligned}$$

Let $\varphi \in [0, 2\pi)$. Consider the orthogonal transformation

$$x \rightarrow \hat{x} = \theta x, \quad \theta = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (14)$$

which is a rotation by the angle φ about z -axis. Then the displacement, strain and stress column vectors transform according to

$$\hat{\mathcal{U}} = \theta \mathcal{U}, \quad \hat{\boldsymbol{\varepsilon}} = \Theta^T \boldsymbol{\varepsilon} \quad \text{and} \quad \hat{\boldsymbol{\sigma}} = \Theta^T \boldsymbol{\sigma}, \quad (15)$$

where the 6×6 -matrix Θ is also orthogonal and given by

$$\Theta = \begin{pmatrix} \cos^2 \varphi & \sqrt{2} \sin \varphi \cos \varphi & 0 & \sin^2 \varphi & 0 & 0 \\ -\sqrt{2} \sin \varphi \cos \varphi & \cos^2 \varphi - \sin^2 \varphi & 0 & \sqrt{2} \sin \varphi \cos \varphi & 0 & 0 \\ 0 & 0 & \cos \varphi & 0 & 0 & \sin \varphi \\ \sin^2 \varphi & -\sqrt{2} \sin \varphi \cos \varphi & 0 & \cos^2 \varphi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\sin \varphi & 0 & 0 & \cos \varphi \end{pmatrix}.$$

This is straightforward to verify and can be found also in [15, Chapter 2]. Note that the orthogonality property of the transformation matrix Θ in (15) is due to the above mentioned factor $\sqrt{2}$ in (12).

Comparing (15) and (13), we conclude that the change of variables (14) leads to the following transformation of the rigidity matrix

$$A \mapsto \mathcal{A} = \Theta^T A \Theta.$$

Using notation (12), we write the last formula in (3) in the matrix form

$$\boldsymbol{\varepsilon}(\mathcal{U}) = D(\nabla_x) \mathcal{U},$$

where $\nabla_x = \text{grad}$ and $D(\nabla_x)$ is a 6×3 -matrix of first order differential operators,

$$D(\xi) = \begin{pmatrix} \xi_1 & \frac{1}{\sqrt{2}}\xi_2 & \frac{1}{\sqrt{2}}\xi_3 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\xi_1 & 0 & \xi_2 & 0 & \frac{1}{\sqrt{2}}\xi_3 \\ 0 & 0 & \frac{1}{\sqrt{2}}\xi_1 & 0 & \xi_3 & \frac{1}{\sqrt{2}}\xi_2 \end{pmatrix}^T.$$

2.3 The surface rearrangement for stresses and strains.

As shown, for example, in the paper [22] and many others for asymptotic analysis of problems in elasticity and fracture mechanics involving thin layer surface structures it is convenient to rearrange components in the stress and strain vectors.

First, let us introduce the strain and stress columns in the orthogonal curvilinear coordinates (n, s, z) :

$$\begin{aligned}\boldsymbol{\varepsilon}(\mathbf{u}) &= (\varepsilon_{nn}, \sqrt{2}\varepsilon_{ns}, \sqrt{2}\varepsilon_{nz}, \varepsilon_{ss}, \varepsilon_{zz}, \sqrt{2}\varepsilon_{zs})^T, \\ \boldsymbol{\sigma}(\mathbf{u}) &= (\sigma_{nn}, \sqrt{2}\sigma_{ns}, \sqrt{2}\sigma_{nz}, \sigma_{ss}, \sigma_{zz}, \sqrt{2}\sigma_{zs})^T,\end{aligned}$$

where $\mathbf{u}$ is the column vector $(u_n, u_s, u_z)^T$. Hooke's law then takes the form

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}), \quad (16)$$

where $\mathcal{A} = \Theta(\varphi)^T A \Theta(\varphi)$. Here φ is the angle between y_1 -axis and the normal $\mathbf{n}$, which depends on s but, of course, is independent of n and z .

Let us introduce two more columns

$$\begin{aligned}\boldsymbol{\eta}(\mathbf{u}) &= (\sigma_{nn}, \sqrt{2}\sigma_{ns}, \sqrt{2}\sigma_{nz}, \varepsilon_{ss}, \varepsilon_{zz}, \sqrt{2}\varepsilon_{zs})^T, \\ \boldsymbol{\xi}(\mathbf{u}) &= (-\varepsilon_{nn}, -\sqrt{2}\varepsilon_{ns}, -\sqrt{2}\varepsilon_{nz}, \sigma_{ss}, \sigma_{zz}, \sqrt{2}\sigma_{zs})^T.\end{aligned} \quad (17)$$

The important property of this rearrangement is that all components in $\boldsymbol{\eta}(\mathbf{u})$ are "observable" at the surface Γ . This means that the stress column

$$\boldsymbol{\sigma}^\dagger(\mathbf{u}) = (\sigma_{nn}, \sqrt{2}\sigma_{ns}, \sqrt{2}\sigma_{nz})^T$$

implies traction at Γ given in the elasticity problem data, and the strain column

$$\boldsymbol{\varepsilon}^\#(\mathbf{u}) = (\varepsilon_{ss}, \varepsilon_{zz}, \sqrt{2}\varepsilon_{zs})^T \quad (18)$$

can be evaluated from components of the displacement vector on Γ and their derivatives with respect to s and z , that is along the surface Γ only, see (10). The columns

$$\boldsymbol{\varepsilon}^\dagger(\mathbf{u}) = (\varepsilon_{nn}, \sqrt{2}\varepsilon_{ns}, \sqrt{2}\varepsilon_{nz})^T \quad (19)$$

and

$$\boldsymbol{\sigma}^\#(\mathbf{u}) = (\sigma_{ss}, \sigma_{zz}, \sqrt{2}\sigma_{zs})^T,$$

gathered into the column in (17), do not possess the above properties and can be regarded as "unobservable". Indeed, to compute the components in

(19) one has to differentiate displacements in n , see (10), therefore one needs to know the displacements inside the body which are unobservable.

We represent the rigidity matrix $\mathcal{A}$ blockwise

$$\mathcal{A} = \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\#} \\ A^{\# \dagger} & A^{\#\#} \end{pmatrix}, \quad (20)$$

where all blocks are 3×3 -matrices, the matrices $A^{\dagger\dagger}$ and $A^{\#\#}$ are positive definite and $A^{\dagger\#} = (A^{\# \dagger})^T$. Writing (16) as

$$\begin{aligned} \sigma^\dagger(\mathbf{u}) &= A^{\dagger\dagger} \varepsilon^\dagger(\mathbf{u}) + A^{\dagger\#} \varepsilon^\#(\mathbf{u}), \\ \sigma^\#(\mathbf{u}) &= A^{\# \dagger} \varepsilon^\dagger(\mathbf{u}) + A^{\#\#} \varepsilon^\#(\mathbf{u}), \end{aligned}$$

we then derive

$$\begin{aligned} \sigma^\#(\mathbf{u}) &= (A^{\#\#} - A^{\# \dagger} (A^{\dagger\dagger})^{-1} A^{\dagger\#}) \varepsilon^\#(\mathbf{u}) + A^{\# \dagger} A_{\dagger\dagger}^{-1} \sigma^\dagger(\mathbf{u}), \\ -\varepsilon^\dagger(\mathbf{u}) &= (A^{\dagger\dagger})^{-1} A^{\dagger\#} \varepsilon^\#(\mathbf{u}) - (A^{\dagger\dagger})^{-1} \sigma^\dagger(\mathbf{u}). \end{aligned}$$

Thus we get the following relation connecting the $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ columns:

$$\boldsymbol{\xi}(\mathbf{u}) = \mathbf{Q} \boldsymbol{\eta}(\mathbf{u}),$$

where

$$\mathbf{Q} = \begin{pmatrix} Q^{\dagger\dagger} & Q^{\dagger\#} \\ Q^{\# \dagger} & Q^{\#\#} \end{pmatrix}$$

and

$$\begin{aligned} Q^{\#\#} &= A^{\#\#} - A^{\# \dagger} (A^{\dagger\dagger})^{-1} A^{\dagger\#} > 0, \quad Q^{\dagger\dagger} = -(A^{\dagger\dagger})^{-1} < 0, \\ Q^{\# \dagger} &= A^{\# \dagger} (A^{\dagger\dagger})^{-1}, \quad Q^{\dagger\#} = (A^{\dagger\dagger})^{-1} A^{\dagger\#}. \end{aligned}$$

The positivity of $Q^{\#\#}$ follows from the relations

$$0 < \mathbf{a}^T \mathcal{A} \mathbf{a} = (a^\#)^T Q^{\#\#} a^\#, \quad \mathbf{a} = (-(A^{\dagger\dagger})^{-1} A^{\dagger\#} a^\#, a^\#)^T,$$

which are valid for any $a^\# \in \mathbb{R}^3 \setminus \{0\}$. Clearly, Q is symmetric and invertible but certainly no longer positive definite.

Remark 1. According to [22], the quantity

$$\frac{1}{2} \boldsymbol{\xi}(\mathbf{u})^T \boldsymbol{\eta}(\mathbf{u})$$

is the density of the surface enthalpy. This particular Gibbs functional naturally appears in the asymptotic analysis of thin layers and surface structures such as elastic coatings, phase interfaces, propagating cracks etc.

3 The dimension reduction

The main goal of this section is to show that the leading term of σ_Γ obtained from solution to problem (2), (3) subject to the boundary conditions (4) and $\mathbf{u} = \mathbf{U}$ on Γ has the following form:

$$\begin{aligned} \sigma_\Gamma = & -hD^\sharp(\kappa(s), -\partial_s, -\partial_z)^T \overline{Q}^\sharp(s, z) D^\sharp(\kappa(s), \partial_s, \partial_z) \mathbf{U}(s, z) \\ & -h\overline{\rho}(s, z) \partial_t^2 \mathbf{U}(s, z), \end{aligned} \quad (21)$$

where κ is defined after formula (7),

$$D^\sharp(\kappa, \partial_s, \partial_z) = \begin{pmatrix} \kappa & \partial_s & 0 \\ 0 & 0 & \partial_z \\ 0 & \frac{1}{\sqrt{2}}\partial_z & \frac{1}{\sqrt{2}}\partial_s \end{pmatrix} \quad (22)$$

and

$$\overline{Q}^\sharp(s, z) = \begin{pmatrix} \overline{Q}_{11}^\sharp & \overline{Q}_{12}^\sharp & \overline{Q}_{13}^\sharp \\ \overline{Q}_{21}^\sharp & \overline{Q}_{22}^\sharp & \overline{Q}_{23}^\sharp \\ \overline{Q}_{31}^\sharp & \overline{Q}_{32}^\sharp & \overline{Q}_{33}^\sharp \end{pmatrix} (s, z) = \int_0^{H(s)} Q^\sharp(\zeta, s, z) d\zeta, \quad (23)$$

$$\overline{\rho}(s, z) = \int_0^{H(s)} \rho(\zeta, s, z) d\zeta. \quad (24)$$

The matrix $Q^\sharp$ is the Schur complement of the block $A^{\dagger\dagger}$ of the matrix $\mathcal{A}$, defined in (20), i.e.

$$Q^\sharp = A^\sharp - A^{\dagger\dagger}(A^{\dagger\dagger})^{-1}A^{\dagger\dagger}. \quad (25)$$

3.1 The asymptotic ansatz and the leading term.

We suppose that ρ and $\mathbf{A}$ satisfies one of the conditions (I) or (II) from the introduction. Therefore, equation (16) takes the form

$$\boldsymbol{\sigma}(\mathbf{u}; n, s, z) = \mathcal{A}(\zeta, s, z) \boldsymbol{\varepsilon}(\mathbf{u}; n, s, z), \quad (26)$$

where $\zeta = h^{-1}n$ is regarded as a fast variable or the stretched transversal coordinate.

We search for an asymptotic solution of problem (2),(3) supplied with the boundary conditions (4) and (8) in the form

$$\mathbf{u}^h(n, s, z) = \mathbf{u}^0(s, z) + h\mathbf{u}'(\zeta, s, z) + h^2\mathbf{u}''(\zeta, s, z) + \cdots \quad (27)$$

The subscript h on the left-hand side of (27) emphasizes the dependence of solution on the small parameter h ; $\mathbf{u}^0$ stands for the leading term and we explain later why it is independent of the fast variable. In next sections we find the correction terms $\mathbf{u}'$ and $\mathbf{u}''$ and derive a limit system of differential equations for $\mathbf{u}^0 = \mathbf{U}$. All functions may depend also on t as a parameter, but we do not indicate this dependence explicitly in what follows.

Using the relation $\partial_n = h^{-1}\partial_\zeta$ in formulae (10), we obtain

$$\varepsilon(\mathbf{u}^h) = h^{-1}D(\partial_\zeta, 0, 0)\mathbf{u}^0 + \dots \quad (28)$$

Here and in the sequel, dots stand for higher-order terms which are of no importance at the actual step of the asymptotic procedure. In the same way, the elasticity equations (11) read as follows:

$$h^{-1}D(\partial_\zeta, 0, 0)^T \boldsymbol{\sigma}(\mathbf{u}^h) + \dots = \dots \quad (29)$$

Moreover, since the gradient operator ∇_x in the curvilinear coordinates turns into $(\partial_n, J^{-1}\partial_s, \partial_z)$, the normal $\mathbf{n}^h$ at the exterior boundary $\Gamma_h = \gamma_h \times \mathbb{R}$ has the form

$$\mathbf{n}^h(n, s) = (1 + J(n, s)^{-2}h^2|\partial_s H(s)|^2)^{-1/2}(1, -hJ(n, s)^{-1}\partial_s H(s), 0)^T. \quad (30)$$

Therefore, $n_n^h = 1 + O(h^2)$, $n_s^h = O(h)$ and $n_z^h = 0$ which converts the boundary condition (4) into

$$D(1, 0, 0)^T \boldsymbol{\sigma}(\mathbf{u}^h) + \dots = 0. \quad (31)$$

As a result of (29), (31), (28), (26) and (8), we get the mixed boundary value problem for a system of ordinary equations in ζ with the parameters $(s, z) \in \Gamma$:

$$\begin{aligned} -D(\partial_\zeta, 0, 0)^T \mathcal{A}(\zeta, s, z)D(\partial_\zeta, 0, 0)\mathbf{u}^0(\zeta, s, z) &= 0, \quad \zeta \in \Upsilon(s), \\ D(1, 0, 0)^T \mathcal{A}(H(s), s, z)D(\partial_\zeta, 0, 0)\mathbf{u}^0(H(s), s, z) &= 0, \\ \mathbf{u}^0(0, s, z) &= \mathbf{U}(s, z). \end{aligned} \quad (32)$$

Since the matrix $\mathcal{A}$ is symmetric, positive definite and rank of $D(1, 0, 0)$ is equal to 3, the 3×3 -matrix

$$\mathbf{a} = D(1, 0, 0)^T \mathcal{A} D(1, 0, 0) \quad (33)$$

is also symmetric and positive definite. Using this notation the differential operator in the first line of (32) takes the form $-\partial_\zeta \mathbf{a}(\zeta, s, z) \partial_\zeta$ and in the second line $\mathbf{a}(\zeta, s, z) \partial_\zeta$. Hence, problem (32) has a unique solution, which does not depend on ζ :

$$\mathbf{u}^0(\zeta, s, z) = \mathbf{U}(s, z). \quad (34)$$

3.2 The first correction term.

Since $\mathbf{u}^0$ does not depend on ζ , we get

$$\boldsymbol{\varepsilon}(\mathbf{u}^h; n, s, z) = \boldsymbol{\varepsilon}^0(s, z) + D(\partial_\zeta, 0, 0)\mathbf{u}'(\zeta, s, z) + \dots$$

where

$$\boldsymbol{\varepsilon}^0 = \left(0, \frac{1}{\sqrt{2}}(\partial_s u_n^0 - \kappa u_s^0), \frac{1}{\sqrt{2}}\partial_z u_n^0, \partial_s u_s^0 + \kappa u_n^0, \partial_z u_z^0, \frac{1}{\sqrt{2}}(\partial_s u_z^0 + \partial_z u_s^0) \right)^T. \quad (35)$$

Collecting coefficients of order h^{-1} in the elasticity equations, we arrive at the system of ordinary differential equations

$$\begin{aligned} & -D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z) D(\partial_\zeta, 0, 0) \mathbf{u}'(\zeta, s, z) \\ & = D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z) \boldsymbol{\varepsilon}^0(s, z), \quad \zeta \in \Upsilon(s). \end{aligned} \quad (36)$$

The boundary conditions (4) at the exterior boundary Γ_h imply

$$\begin{aligned} & D(1, 0, 0)^T \mathbf{A}(H(s), s, z) D(\partial_\zeta, 0, 0) \mathbf{u}'(H(s), s, z) \\ & = -D(1, 0, 0)^T \mathbf{A}(H(s), s, z) \boldsymbol{\varepsilon}^0(s, z). \end{aligned} \quad (37)$$

Furthermore, we derive the second boundary condition

$$\mathbf{u}'(0, s, z) = 0 \quad (38)$$

because on the right of (8) there is no term of order h . Since the matrix differential operator on the left-hand side of (37) can be written as

$$D(1, 0, 0)^T \mathbf{A}(\zeta, s, z) D(1, 0, 0) \partial_\zeta = \mathbf{a}(\zeta, s, z) \partial_\zeta,$$

solving (36), (37) and using matrix (33), we have

$$\partial_\zeta \mathbf{u}'(\zeta, s, z) = -\mathbf{a}(\zeta, s, z)^{-1} D(1, 0, 0)^T \mathbf{A}(\zeta, s, z) \boldsymbol{\varepsilon}^0(s, z). \quad (39)$$

Taking into account (38), we obtain

$$\mathbf{u}'(\zeta, s, z) = - \int_0^\zeta \mathbf{a}(\tau, s, z)^{-1} D(1, 0, 0)^T \mathbf{A}(\tau, s, z) \boldsymbol{\varepsilon}^0(s, z) d\tau.$$

Now we can calculate the trace of the leading term of the normal stresses on Γ :

$$\begin{aligned}
D(1, 0, 0)^T \boldsymbol{\sigma}(\mathbf{u}^h; 0, s, z) &= D(1, 0, 0)^T \mathbf{A}(0, s, z) (\boldsymbol{\varepsilon}^0(s, z) \\
&+ D(1, 0, 0) \partial_s \mathbf{u}'(0, s, z) + \dots) = D(1, 0, 0)^T \mathbf{A}(0, s, z) \boldsymbol{\varepsilon}^0(s, z) \\
&- D(1, 0, 0)^T \mathbf{A}(0, s, z) D(1, 0, 0) \mathbf{a}(0, s, z)^{-1} D(1, 0, 0)^T \mathbf{A}(0, s, z) \boldsymbol{\varepsilon}^0(s, z) \\
&+ \dots = 0 + \dots
\end{aligned}$$

Here, we have used equality (33) to show that the leading term vanishes. In other words, the first couple of asymptotic terms in ansatz (27) brings zero traction at the interior surface contacting blood. In the next section we show that traction generated by the third term $h^2 \mathbf{u}''$ becomes non-trivial and it is given by a matrix differential operator applied to the vector (33). In the asymptotic analysis, it is convenient to endow formally the rigidity matrix $\mathbf{A}$ with the order h^{-1} , which means that we consider the elastic wall to be thin but hard; certainly, this is true for both arteries and veins.

3.3 The second correction term.

In the same way as in Section 3.2 we conclude that the term $\mathbf{u}''$ in (27) satisfies the same mixed boundary value problem for the system of ordinary differential equations in ζ but with new right-hand sides $\mathbf{f}''$ and $\mathbf{g}''$:

$$\begin{aligned}
-D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z) D(\partial_\zeta, 0, 0) \mathbf{u}''(\zeta, s, z) &= \mathbf{f}''(\zeta, s, z), \quad \zeta \in \Upsilon(s), \\
D(1, 0, 0)^T \mathbf{A}(H(s), s, z) D(\partial_\zeta, 0, 0) \mathbf{u}''(H(s), s, z) &= \mathbf{g}''(s, z), \\
\mathbf{u}''(0, s, z) &= 0.
\end{aligned} \tag{40}$$

In order to find out $\mathbf{f}''$ and $\mathbf{g}''$, we have to take into account the lower order terms in the strain columns $\boldsymbol{\varepsilon}(\mathbf{u}^0)$ and $\boldsymbol{\varepsilon}(\mathbf{u}')$. First, we obtain

$$\boldsymbol{\varepsilon}(\mathbf{u}^0; n, s, z) = \boldsymbol{\varepsilon}^0(s, z) + h \boldsymbol{\varepsilon}^1(\zeta, s, z) + \dots \tag{41}$$

where $\boldsymbol{\varepsilon}^0$ is given by (35) and according to (10) we set

$$\boldsymbol{\varepsilon}^1 = -\zeta \kappa \left(0, \frac{1}{\sqrt{2}} (\partial_s u_n - \kappa u_s), \frac{1}{\sqrt{2}} \partial_s u_z, \partial_s u_s + \kappa u_n, 0, 0 \right)^T. \tag{42}$$

Here, the factor $-\zeta\kappa(s)$ comes from the decomposition

$$J(n, s)^{-1} = (1 + n\kappa(s))^{-1} = 1 - h\zeta\kappa(s) + O(h^2).$$

For $\boldsymbol{\varepsilon}(\mathbf{u}')$, we have

$$\boldsymbol{\varepsilon}(\mathbf{u}'; n, s, z) = h^{-1}D(\partial_\zeta, 0, 0)\mathbf{u}'(\zeta, s, z) + \boldsymbol{\varepsilon}'(\zeta, s, z) + \dots \quad (43)$$

where analogously to (35)

$$\boldsymbol{\varepsilon}' = \left(0, \frac{1}{\sqrt{2}}(\partial_s u'_n - \kappa u'_s), \frac{1}{\sqrt{2}}\partial_z u'_n, \partial_s u'_s + \kappa u'_n, \partial_z u'_z, \frac{1}{\sqrt{2}}(\partial_s u'_z + \partial_z u'_s)\right)^T. \quad (44)$$

Formulae (41) and (43) allow us to compute the traction on Γ

$$\begin{aligned} D(1, 0, 0)^T \boldsymbol{\sigma}(\mathbf{u}^h; 0, s, z) &= h(D(1, 0, 0)^T \mathbf{A}(0, s, z) D(\partial_\zeta, 0, 0) \mathbf{u}''(0, s, z) \\ &+ D(1, 0, 0)^T \mathbf{A}(0, s, z) (\boldsymbol{\varepsilon}^1(0, s, z) + \boldsymbol{\varepsilon}'(0, s, z))) + \dots \\ &= h(D(1, 0, 0)^T \mathbf{A}(0, s, z) D(\partial_\zeta, 0, 0) \mathbf{u}''(0, s, z) \\ &+ D(1, 0, 0)^T \mathbf{A}(0, s, z) \boldsymbol{\varepsilon}'(0, s, z)) + \dots \end{aligned} \quad (45)$$

Here, we used that $\boldsymbol{\varepsilon}^1(\zeta, s, z) = 0$ for $\zeta = 0$ due to the factor ζ in (42). By solving (40), we obtain

$$D(1, 0, 0)^T \mathbf{A}(0, s, z) D(\partial_\zeta, 0, 0) \mathbf{u}''(0, s, z) = \int_0^{H(s)} \mathbf{f}''(\zeta, s, z) d\zeta + \mathbf{g}''(s, z).$$

Therefore for calculating the next term for the traction on Γ it suffices to determine the right-hand sides $\mathbf{f}''$ and $\mathbf{g}''$.

In order to compute $\mathbf{f}''$ we need the terms (42) and (44) and asymptotic expansion of the matrix differential operator on the left-hand side of the equilibrium equations (11) which is

$$-h^{-1}D(\partial_\zeta, 0, 0)^T - h^0(D(0, \partial_s, \partial_z)^T + \kappa(s)\mathbb{K}) + \dots$$

where

$$\mathbb{K} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned}\mathbf{f}''(\zeta, s, z) &= D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z) (\boldsymbol{\varepsilon}^1(\zeta, s, z) + \boldsymbol{\varepsilon}'(\zeta, s, z)) \\ &+ (D(0, \partial_s, \partial_z)^T + \kappa(s)\mathbb{K}) \mathbf{A}(\zeta, s, z) (\boldsymbol{\varepsilon}^0(s, z) + D(\partial_\zeta, 0, 0) \mathbf{u}'(\zeta, s, z)) \\ &+ \rho(\zeta, s, z) \partial_t^2 \mathbf{u}^0(s, z),\end{aligned}\tag{46}$$

where the right-hand side of (11) has been taken into account.

In order to calculate $\mathbf{g}''$ we recall that according to (30)

$$D(\mathbf{n}^h(s, z)) = D(1, 0, 0) - hD(0, \partial_s H(s), 0) + \dots$$

Therefore,

$$\begin{aligned}\mathbf{g}''(s, z) &= -D(1, 0, 0)^T \mathbf{A}(H(s), s, z) (\boldsymbol{\varepsilon}^1(H(s), s, z) + \boldsymbol{\varepsilon}'(H(s), s, z)) \\ &+ D(0, \partial_s H(s), 0)^T \mathbf{A}(H(s), s, z) (\boldsymbol{\varepsilon}^0(s, z) + D(\partial_\zeta, 0, 0) \mathbf{u}'(H(s), s, z)).\end{aligned}\tag{47}$$

From (46) and (47) it follows that

$$\begin{aligned}& \int_0^{H(s)} \mathbf{f}''(\zeta, s, z) d\zeta + \mathbf{g}''(s, z) \\ &= \left(\int_0^{H(s)} D(\partial_\zeta, 0, 0)^T \mathbf{A}(\zeta, s, z) (\boldsymbol{\varepsilon}^1(\zeta, s, z) + \boldsymbol{\varepsilon}'(\zeta, s, z)) d\zeta \right. \\ &\quad \left. - D(1, 0, 0)^T \mathbf{A}(H(s), s, z) (\boldsymbol{\varepsilon}^1(H(s), s, z) + \boldsymbol{\varepsilon}'(H(s), s, z)) \right) \\ &\quad + \left(\int_0^{H(s)} (D(0, \partial_s, \partial_z)^T + \kappa(s)\mathbb{K}) \mathbf{A}(\zeta, s, z) (\boldsymbol{\varepsilon}^0(s, z) + D(\partial_\zeta, 0, 0) \mathbf{u}'(\zeta, s, z)) d\zeta \right. \\ &\quad \left. + D(0, \partial_s H(s), 0)^T \mathbf{A}(H(s), s, z) (\boldsymbol{\varepsilon}^0(s, z) + D(\partial_\zeta, 0, 0) \mathbf{u}'(H(s), s, z)) \right) \\ &\quad - \int_0^{H(s)} \rho(\zeta, s, z) d\zeta \partial_t^2 \mathbf{u}^0(s, z) =: I_1 + I_2 - I_3.\end{aligned}$$

Integrating by parts leads to

$$I_1 = -D(1, 0, 0)^T \mathbf{A}(0, s, z) \boldsymbol{\varepsilon}'(0, s, z),$$

which cancels the last term in (45). Furthermore, one can directly check that

$$I_2 = (D(0, \partial_s, \partial_z)^T + \kappa(s)\mathbb{K}) \int_0^{H(s)} \mathbf{A}(\zeta, s, z) (\boldsymbol{\varepsilon}^0(s, z) + D(\partial_\zeta, 0, 0) \mathbf{u}'(\zeta, s, z)) d\zeta.$$

Using now (39), we get

$$I_2 = (D(0, \partial_s, \partial_z)^T + \kappa(s)\mathbb{K})\mathbf{M}(s, z)\boldsymbol{\varepsilon}^0(s, z), \quad (48)$$

where

$$\mathbf{M}(s, z) = \int_0^{H(s)} (\mathbf{A}(\zeta, s, z) - \mathbf{A}(\zeta, s, z)D(1, 0, 0)\mathbf{a}(\zeta, s, z)^{-1}D(1, 0, 0)^T\mathbf{A}(\zeta, s, z))d\zeta \quad (49)$$

is a symmetric 6×6 matrix.

Lemma 1. *The matrix $\mathbf{M}$ has the form*

$$\mathbf{M}(s, z) = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \overline{Q}^{\sharp\sharp}(s, z) \end{pmatrix}, \quad (50)$$

where $\mathbb{O}$ is the null matrix of size 3×3 and $\overline{Q}^{\sharp\sharp}(s, z)$ is given by (23) and (25).

Proof. Using formulae (20), (32) and (52), we see that the integrand in (49) is equal to

$$\begin{aligned} & \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} - \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} \begin{pmatrix} \mathbb{E} \\ \mathbb{O} \end{pmatrix} \left[\begin{pmatrix} \mathbb{E} & \mathbb{O} \end{pmatrix} \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} \begin{pmatrix} \mathbb{E} \\ \mathbb{O} \end{pmatrix} \right]^{-1} \\ & \times \begin{pmatrix} \mathbb{E} & \mathbb{O} \end{pmatrix} \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} = \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \\ A^{\sharp\dagger} & A^{\sharp\sharp} \end{pmatrix} - \begin{pmatrix} A^{\dagger\dagger} \\ A^{\sharp\dagger} \end{pmatrix} (A^{\dagger\dagger})^{-1} \begin{pmatrix} A^{\dagger\dagger} & A^{\dagger\sharp} \end{pmatrix} \\ & = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & A^{\sharp\sharp} - A^{\sharp\dagger}(A^{\dagger\dagger})^{-1}A^{\dagger\sharp} \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & Q^{\sharp\sharp} \end{pmatrix}, \end{aligned}$$

where $Q^{\sharp\sharp}$ is defined by (25) and

$$\mathbb{E} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (51)$$

The above relation together with (23) gives (50). □

Let $D^\sharp(\kappa(s), \partial_s, \partial_z)$ be defined by (22). Lemma 1 together with (34) shows that (48) can be written as

$$I_2 = -D^\sharp(\kappa(s), -\partial_s, -\partial_z)^T \overline{Q}^{\sharp\sharp}(s, z)\boldsymbol{\varepsilon}^\sharp(\mathbf{U}; s, z), \quad (52)$$

where $\boldsymbol{\varepsilon}^\sharp$ is given by (18). Using the evident equality $J(0, s) = 1$, we get

$$\boldsymbol{\varepsilon}^\sharp(\mathbf{U}) = \left(\partial_s u_s + \kappa u_n, \partial_z u_z, \frac{1}{\sqrt{2}}(\partial_s u_z + \partial_z u_s) \right)^T = D^\sharp(\kappa, \partial_s, \partial_z)\mathbf{U}.$$

Noting finally that

$$I_3 = \bar{\rho}(s, z) \partial_t^2 \mathbf{U}(s, z),$$

where $\bar{\rho}$ is defined by (24), we arrive at the following expression for the normal stress vector on Γ :

$$\begin{aligned} \sigma_\Gamma &= D(1, 0, 0)^T \boldsymbol{\sigma}(\mathbf{u}^h; 0, s, z) = -h \bar{\rho}(s, z) \partial_t^2 \mathbf{U}(s, z) \\ &\quad - h D^\sharp(\kappa(s), -\partial_s, -\partial_z)^T \bar{Q}^\sharp(s, z) D^\sharp(\kappa(s), \partial_s, \partial_z) \mathbf{U}(s, z) + \dots \end{aligned} \quad (53)$$

This together with (34) leads to (21).

Remark 2. *Writing Hooke's law (13) in the form*

$$\boldsymbol{\varepsilon}(u) = \mathbf{B} \boldsymbol{\sigma}(u),$$

where $\mathbf{B} = \mathbf{A}^{-1}$ is the compliance matrix, we observe that $Q^\sharp = B^\sharp$.

3.4 The model of the vessel wall

The flow in the cylindrical vessel Ω is described by the Navier-Stokes equations (1), where $\mathbf{v}$ is the velocity vector, p the pressure and ν the kinematic viscosity. On the walls of the vessel we assume that the velocity of wall coincides with the velocity of the fluid, which is described by relation (5), and hydrodynamic force is equal to the normal stress vector on the boundary. Due to (53) the latter means

$$\begin{aligned} &D^\sharp(\kappa(s), -\partial_s, -\partial_z)^T \bar{Q}^\sharp(s, z) D^\sharp(\kappa(s), \partial_s, \partial_z) \mathbf{U}(s, z) \\ &+ \bar{\rho}(s, z) \partial_t^2 \mathbf{U}(s, z) = -h^{-1} \rho_b \mathbf{F}(s, z) \end{aligned} \quad (54)$$

on $\partial\Omega$, where $\rho_b \mathbf{F}$ is the hydrodynamic force with components given by (7), ρ_b is the density of blood and $D^\sharp$ is defined by (22).

4 Analysis of the proposed model

4.1 An additive property of the rigidity matrix

Let $\Sigma_h = \varsigma_h \times \mathbb{R}$ be a laminated wall consisting of K layers having thickness h_k and in each of these layers the rigidity matrix $A_{(k)}$ is constant. Relations (24) and (23) take the form

$$\bar{\rho}(s, z) = \sum_{j=1}^N \frac{h_j}{h} \rho^j(s, z), \quad \bar{Q}^{\#\#} = \sum_{k=1}^K \frac{h_k}{h} Q_{(k)}^{\#\#}$$

where h_k/h is normalized thickness of k th layer and $Q_{(k)}^{\#\#}$ is a block of the matrix $Q_{(k)}$ constructed by using the matrix $A_{(k)}$ according to (25).

4.2 Rigidity matrix for the arteria wall

The laminate structure of the wall depends on a type of blood vessels. The most studied are arteries (see [6]-[10]) whose walls consists of three layers: intima, media and adventitia. The internal layer, which is just a very thin film, has no influence on elastic properties of the wall. However, the media and adventitia layers are composites formed by bundles of collagen fibers in a homogeneous material consisting of muscle cells. The bundles in each layer are usually modeled by two families of fibres wound around the cylinder under the angles $\pm\varphi_m$ and $\pm\varphi_a$ to the z axis respectively. Here $\varphi_m, \varphi_a \in (0, \pi/2)$. As a result we obtain composite materials reinforced by periodic families of rigid rods.

There are several approaches for determining elastic properties of laminated composite walls of arteries. For example, in [10] a non-linear rheological stress/strain relation is proposed for the entire arterial wall as well as in the case of the dissection of the media and adventitia. For the estimation of the elastic properties of the vessels by means of the solution of inverse problems see [25] and [1]. However, there is still no two-dimensional model for these rheological relations. Here, we use a technique based on the linear homogenization theory, which allows us to compute matrix (23) in the boundary condition (21) explicitly.

Application of the asymptotic homogenization procedure developed in [17] and [18] gives the following stiffness matrix

$$\bar{Q}^{\#\#} = Q_{(c)}^{\#\#} + Q_{(m)}^{\#\#} + Q_{(a)}^{\#\#}, \quad (55)$$

where

$$Q_{(m)}^{\#} = E_m \sum_{\pm} \Theta^{\#}(\pm\varphi_m) \mathbb{E} \Theta^{\#}(\pm\varphi_m)^T, \quad Q_{(a)}^{\#} = E_a \sum_{\pm} \Theta^{\#}(\pm\varphi_a) \mathbb{E} \Theta^{\#}(\pm\varphi_a)^T.$$

Here,

$$Q_{(c)} = \begin{pmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & 2\mu \end{pmatrix}$$

is the rigidity matrix of isotropic filler with the Lamé constants λ and μ , which are small with respect to Young's modulus E_m and E_a of the collagen fibers from the media and adventitia layers, i.e. $\mu, \lambda \ll E_m, E_c$. Furthermore, the matrix $\mathbb{E}$ is given by (51) and

$$\Theta^{\#}(\varphi) = \begin{pmatrix} \cos^2 \varphi & \sin^2 \varphi & -\sqrt{2} \sin \varphi \cos \varphi \\ \sin^2 \varphi & \cos^2 \varphi & \sqrt{2} \sin \varphi \cos \varphi \\ \sqrt{2} \sin \varphi \cos \varphi & -\sqrt{2} \sin \varphi \cos \varphi & \cos^2 \varphi - \sin^2 \varphi \end{pmatrix}$$

see [15, Ch.2] for details. In particular,

$$Q_{(a)}^{\#} = 2E_a \begin{pmatrix} \sin^4 \varphi_a & \sin^2 \varphi_a \cos^2 \varphi_a & 0 \\ \sin^2 \varphi_a \cos^2 \varphi_a & \cos^4 \varphi_a & 0 \\ 0 & 0 & 2 \sin^2 \varphi_a \cos^2 \varphi_a \end{pmatrix}. \quad (56)$$

Similar formula for $Q_{(m)}^{\#}$ obtained from (56) by changing φ_a for φ_m . Thus the composite material of the artery wall after averaging is orthotropic with the main orthotropy axes directed along the z - and s - axes.

The matrices $Q_{(m)}^{\#}$ and $Q_{(a)}^{\#}$ are not positive definite, for example the vector $(\cos^2 \varphi_a, -\sin^2 \varphi_a, 0)^T$ belongs to the kernel of the matrix (56). However, the sum $Q_{(m)}^{\#} + Q_{(a)}^{\#}$ and hence the matrix (55) are positive definite provided $\varphi_m \neq \varphi_a$. The last inequality is satisfied for arteries (see [6]-[10]). Thus the main elastic cyclic load is taken by collagen fibers and their location determines the orthotropic properties of the wall.

On the other hand each of the layers is a composite with contrasting properties which have a weak resistance to certain loads depending on φ_a and φ_m respectively. Under separation of the media and adventitia layers such loads caused by pulsation of blood lead to oscillations of a large amplitude which can be a reason of a dissonance in the media and adventitia layers. This can explain a well known fact in medical practice that an artery dissection (separation of layers) can lead to aneurysm and even may stimulate crushing of vessel walls.

4.3 Stability estimate and the Green formula for the limit problem

The aim of this section is to present the Green formula and a stability estimate for problem (1), (54), (5). To simplify exposition we assume here that the convective acceleration in the Navier-Stokes system is absent, i.e. instead of (1) we consider the Stokes system

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} = -\nabla p \text{ in } \Omega.$$

We choose a pair $(\mathbf{V}, \mathbf{W})$, where $\mathbf{V}$ is a solenoidal vector function on $\Omega \times [0, T]$ and $\mathbf{W}$ is a vector function on $\Gamma \times [0, T]$, where T is a positive number. We assume that $\mathbf{V}|_\Gamma = \partial_t \mathbf{W}$ and note that the pair $(\mathbf{v}, \mathbf{U})$ appearing in (1) and (54) also satisfies these properties. Multiplying this equation by a solenoidal vector field $\mathbf{V} = (V_1, V_2, V_3)$ and using the Green formula for the Stokes system (see [12], Ch.3, Sect.2) yield

$$\begin{aligned} \int_{\Omega} (\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p) \cdot \mathbf{V} dx &= \frac{\nu}{2} \int_{\Omega} \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) \left(\frac{\partial V_k}{\partial x_i} + \frac{\partial V_i}{\partial x_k} \right) dx \\ &- \int_{\Gamma} \mathcal{T}_{ik}(\mathbf{v}) V_i n_k dS_{\Gamma} + \int_{\Omega} \partial_t \mathbf{v} \cdot \mathbf{V} dx, \end{aligned} \quad (57)$$

where summation over repeating indexes is assumed, $\cdot$ denotes the inner product of two vectors and

$$\mathcal{T}_{ik}(\mathbf{v}) = -\delta_i^k p + \nu \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right).$$

Here δ_i^k is the Kronecker delta. Due to relations (6) and (54), we obtain

$$\int_{\Gamma} \mathcal{T}_{ik}(\mathbf{v}) V_i n_k dS_{\Gamma} = -\frac{h}{\rho_b} \left(a(\mathbf{U}, \mathbf{V}) + \int_{\Gamma} \bar{\rho}(s, z) \partial_t^2 \mathbf{U} \cdot \mathbf{V} dx \right)$$

with

$$a(\mathbf{U}, \mathbf{V}) = \int_{\Gamma} \bar{Q}^{\sharp\sharp}(s, z) D^{\sharp}(\kappa(s), \partial_s, \partial_z) \mathbf{U}(s, z) \cdot D^{\sharp}(\kappa(s), \partial_s, \partial_z) \mathbf{V} dS_{\Gamma}.$$

Applying this identity, we write (57) as

$$\begin{aligned} \int_{\Omega} (\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p) \cdot \mathbf{V} dx &= \frac{\nu}{2} \int_{\Omega} \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) \left(\frac{\partial V_k}{\partial x_i} + \frac{\partial V_i}{\partial x_k} \right) dx \\ &+ \frac{h}{\rho_b} \left(a(\mathbf{U}, \partial_t \mathbf{W}) + \int_{\Gamma} \bar{\rho}(s, z) \partial_t^2 \mathbf{U} \cdot \partial_t \mathbf{W} dS_{\Gamma} \right) + \int_{\Omega} \partial_t \mathbf{v} \cdot \mathbf{V} dx. \end{aligned} \quad (58)$$

If we put here $(\mathbf{V}, \mathbf{W}) = (\mathbf{v}, \mathbf{U})$ and integrate then over the interval $[0, T]$, we obtain

$$\begin{aligned} & \frac{\nu}{2} \int_0^T \int_{\Omega} \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) dx + \frac{1}{2} \int_{\Omega} |\mathbf{v}|^2 dx|_{t=T} \\ & + \frac{h}{2\rho_b} \left(a(\mathbf{U}, \mathbf{U}) + \int_{\Gamma} \bar{\rho}(s, z) |\partial_t \mathbf{U}|^2 dS_{\Gamma} \right) \Big|_{t=T} = \frac{1}{2} \int_{\Omega} |\mathbf{v}|^2 dx|_{t=0} \\ & + \frac{h}{2\rho_b} \left(a(\mathbf{U}, \mathbf{U}) + \int_{\Gamma} \bar{\rho}(s, z) |\partial_t \mathbf{U}|^2 dS_{\Gamma} \right) \Big|_{t=0}. \end{aligned} \quad (59)$$

where we recalled that the right-hand side in (58) vanishes due to (1). Since problem (1), (54), (5) must be supplied with initial conditions for $\mathbf{v}$ and $\mathbf{U}$ equality (59) implies stability estimate of a norm of the solution through norms of the initial conditions. Relation (58) can be used for a definition of a weak solution to problem (1), (54), (5).

4.4 On the shape of the vessel cross-section

Relations (21) and (23) allow us to find the geometry of the wall. This is based on two observations. First according to [7], [30], [31] and [19], [3] and others at low velocities, vessel walls are subject mainly to homogeneous hydrostatical pressure, i.e. the leading term for F in (6) and (54) is

$$F_n(s, z, t) = -p^0(z, t) + \dots, \quad F_s(s, z, t) = 0 + \dots, \quad F_z(s, z, t) = 0 + \dots.$$

Let us assume that both the derivatives of $p^0(z, t)$ in z and t are small, i.e. the hydrostatic pressure changes slowly in time and along the vessel. Under these conditions the system (54) reduces to

$$\begin{aligned} & \kappa(\bar{Q}_{11}^{\#}(\kappa u_n^0 + \partial_s u_s^0) + \bar{Q}_{13}^{\#} 2^{-1/2} \partial_s u_z^0) = h^{-1} p^0, \\ & -\partial_s(\bar{Q}_{11}^{\#}(\kappa u_n^0 + \partial_s u_s^0) + \bar{Q}_{13}^{\#} 2^{-1/2} \partial_s u_z^0) = 0, \\ & -2^{1/2} \partial_s(\bar{Q}_{31}^{\#}(\kappa u_n^0 + \partial_s u_s^0) + \bar{Q}_{33}^{\#} 2^{-1/2} \partial_s u_z^0) = 0. \end{aligned} \quad (60)$$

Comparing the first and the second equations, we conclude that κ does not depend on the variable s and hence $\kappa = \kappa_0 = \text{const}$, which implies that the cross-section is a disc. Now, from the first and third equations in (60) we get

$$\bar{Q}_{11}^{\#}(\kappa_0 u_n^0 + \partial_s u_s^0) + \bar{Q}_{13}^{\#} 2^{-1/2} \partial_s u_z^0 = h^{-1} \kappa_0^{-1} p^0$$

and

$$\overline{Q}_{31}^{\#\#}(\kappa_0 u_n^0 + \partial_s u_s^0) + \overline{Q}_{33}^{\#\#} 2^{-1/2} \partial_s u_z^0 = c_0.$$

Therefore,

$$u_n^0 = \frac{1}{q^\#} \left((\overline{Q}_{31}^{\#\#} \kappa_0 + \overline{Q}_{33}^{\#\#} 2^{-1/2}) \frac{p_0}{h \kappa_0} - (\overline{Q}_{11}^{\#\#} \kappa_0 + \overline{Q}_{13}^{\#\#} 2^{-1/2}) c_0 \right)$$

and

$$\partial_s u_z^0 = \frac{1}{q^\#} \left(\overline{Q}_{11}^{\#\#} \kappa_0 c_0 - \overline{Q}_{31}^{\#\#} p_0 \right),$$

where

$$q^\# = \overline{Q}_{11}^{\#\#} \kappa_0 (\overline{Q}_{31}^{\#\#} \kappa_0 + \overline{Q}_{33}^{\#\#} 2^{-1/2}) - \overline{Q}_{31}^{\#\#} \kappa_0 (\overline{Q}_{11}^{\#\#} \kappa_0 + \overline{Q}_{13}^{\#\#} 2^{-1/2}).$$

4.5 The vessel cross-section revisited

It is reasonable to assume that the vessel wall is subject to residual stresses. This can be explained in the following way. Blood circulation system is formed in embryonic state of organism and then develops, in particular, through growth of collagen fibres and muscle cells which do not occur consistently. This may lead to residual stresses in elastic walls, which can be described by the additional terms

$$D^\#(\kappa(s), -\partial_s, -\partial_z)(g_s(s), g_z(s), 0)^T = (\kappa(s)g_s(s) - \partial_s g_z(s), 0, 0)^T \quad (61)$$

on the right-hand side of (54) which do not destroy the structure of the system (60) and lead to the same conclusion about the circular shape of the cross-section of the vessel as in Section 4.2.

The situation is different when external forces are taken into account; they can be caused, for example, by an asymmetric position of a surgical suture. In this case the right-hand side of (54) has an additional stress term $\tau(s) = (\tau_n(s), \tau_s(s), \tau_z(s))$ and system (60) takes the form

$$\begin{aligned} \kappa(\overline{Q}_{11}^{\#\#}(\kappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\#\#} 2^{-1/2} \partial_s u_z^0) &= p^0 + \tau_n, \\ -\partial_s(\overline{Q}_{11}^{\#\#}(\kappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\#\#} 2^{-1/2} \partial_s u_z^0) &= \tau_s, \\ -2^{1/2} \partial_s(\overline{Q}_{31}^{\#\#}(\kappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{33}^{\#\#} 2^{-1/2} \partial_s u_z^0) &= \tau_z. \end{aligned} \quad (62)$$

Integrating the second equation, we get

$$\overline{Q}_{11}^{\#\#}(\kappa u_n^0 + \partial_s u_s^0) + \overline{Q}_{13}^{\#\#} 2^{-1/2} \partial_s u_z^0 = - \int_0^s \tau_s(\rho) d\rho + c_1 \quad (63)$$

and we will require that the right-hand side in (63) is a continuous, 1-periodic function, i.e.,

$$\int_0^1 \tau_s(\rho) d\rho = 0.$$

From (62) and (63) it follows that

$$\kappa \left(- \int_0^s \tau_s(\rho) d\rho + C_1 \right) = p^0 + \tau_n.$$

Since κ is a curvature of a closed curve with the unit length, we have

$$\int_0^1 \kappa(s) ds = 2\pi, \quad (64)$$

provided the origin is located inside the curve. Therefore

$$2\pi = \int_0^1 \left(p^0 + \tau_n(s) \right) \left(- \int_0^s \tau_s(\rho) d\rho + C_1 \right)^{-1} ds,$$

which is an equation to determine C_1 . After finding C_1 we proceed with the determination of the curve from the curvature.

Let the function $\kappa = \kappa(s)$ be given. It is convenient to assume that κ is given for all s and is periodic with period 1. We accept relation (64) to be valid. Let us reconstruct the curve $\zeta(s)$. We have the following explicit formulae for ζ (see, for example, [26, Sect.5, Chapter III]):

$$\zeta_1(s) = \int_0^s \sin \alpha(\rho) d\rho + x_0, \quad \zeta_2(s) = \int_0^s \cos \alpha(\rho) d\rho + y_0, \quad (65)$$

where the function α is defined by

$$\alpha(s) = \int_0^s \kappa(\rho) d\rho$$

and x_0, y_0 are some constants. Using periodicity of κ and formula (64), we get $\alpha(s+1) = 2\pi + \alpha(s)$. Therefore sufficient conditions for the curve (65) to be closed are

$$\int_0^1 \sin \alpha(\rho) d\rho = 0 \quad \text{and} \quad \int_0^1 \cos \alpha(\rho) d\rho = 0. \quad (66)$$

If we assume that the function κ is positive then we can make the change of variable $y = \alpha(\rho)$, $dy = \kappa(\rho)d\rho$, in integrals (66) which leads to the relations

$$\int_0^{2\pi} \frac{\sin y \, dy}{\kappa(s(y))} = 0 \quad \text{and} \quad \int_0^{2\pi} \frac{\cos y \, dy}{\kappa(s(y))} = 0.$$

The violation of conditions (66), which results in absence of the closed curves subject to requirement (61), means instability of the shape of the walls provoking the same artery pathologies as in the case of the dissection described in section 4.4.

The deviation of the cross-section shape from circular causes deterioration of the blood permeability: it is known that, among all cross-sections of the set perimeter, it is the circular cross section that provides the largest stream of fluid for the Poiseuille flow. However, the local distortion of the artery shape certainly represents a risk of secondary significance for the vascular system because the basic threat follows from a decrease in the cross-section area by means of the formation and accumulation of atherosclerotic damages.

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References

- [1] Bertoglio C, Moireau P, Gerbeau J.-F. Sequential parameter estimation for fluidstructure problems: Application to hemodynamics. *International Journal for Numerical Methods in Biomedical Engineering* 2012; **28**:434-455.
- [2] Bertram A. *Elasticity and plasticity of large deformations. An introduction*. Third edition. Springer: Heidelberg, 2012.
- [3] Canic S, Mikelić A. Effective equations modeling the flow of a viscous incompressible fluid through a long elastic tube arising in the study of blood flow through small arteries. *SIAM J. Appl. Dyn. Syst.* 2003; **2**:431-463 (electronic).

- [4] Ciarlet P.G. *Mathematical elasticity, II: Theory of Plates*. North-Holland: Amsterdam, 1997.
- [5] Ciarlet P.G, Destuynder P. A justification of the two-dimensional plate model. *J. Mécanique* 1979; **18**:315-344.
- [6] Dali J.P, Donald L.F. The Elastic Symmetry of Arterial Segments in Dogs. *Circulation Research* 1969, **24**:1-8.
- [7] David N. Ku. Blood flow in arteries. *Annu. Rev. Fluid Mech.* 1997; **29**:399-434.
- [8] Fung, Y.C. *Biomechanics: mechanical properties of living tissues*. Springer-Verlag: New York, 1993.
- [9] Grotberg J. B, Jensen O. E. Biofluid mechanics in flexible tubes. *Annual Review of Fluid Mechanics* 2004; **36**:121-147.
- [10] Holzapfel G.A.. Collagen in Arterial Walls: Biomechanical Aspects. In: P. Fratzl (ed.), "*Collagen. Structure and Mechanics*", Chapter 11, Springer-Verlag, Heidelberg 2008:285-324.
- [11] Jianqiao Ye. *Laminated Composite Plates and Shells: 3D Modelling*. Springer-Verlag: London Berlin Heidelberg, 2003.
- [12] Ladyzhenskaya O. A. *The mathematical theory of viscous incompressible flow. Second English edition, revised and enlarged*. Mathematics and its Applications, Vol. 2 Gordon and Breach, Science Publishers: New York-London-Paris, 1969.
- [13] Lekhnitskij S. G. *Theory of elasticity of an anisotropic elastic body*. Vol. 8 of Series in Mathematical Physics: San Francisco., Holden-Day, Inc., 1963.
- [14] Lurie, A.I. *Theory of Elasticity (Foundations of Engineering Mechanics)*. Springer-Verlag: Berlin Heidelberg, 2005.
- [15] Nazarov S.A. *Asymptotic Theory of Thin Plates and Rods. Vol.1. Dimension Reduction and Integral Estimates*. Nauchnaya Kniga: Novosibirsk, 2002. (Russian)

- [16] Nazarov S.A. Thin elastic coatings and the surface enthalpy. *Mekhanika tverd. tela* 2007; **5**: 6074 (English transl.: *Mechanics of Solids* 2007; **42**: 710–722).
- [17] Nazarov S.A, Slutskiy A, Sweers G. Homogenization of a thin plate reinforced with periodic families of hard rods. *Mat. sbornik* 2011; **202**: 4180 (English transl.: *Sb. Math.* 2011; **202**:11271168).
- [18] Nazarov S., Slutskiy A., Sweers G. Korn inequalities for a reinforced plate. *Journal of Elasticity* 2012; **106**:4369.
- [19] Nazarov S.A, Pileckas K.I. Reynolds flow of a fluid in a thin three-dimensional channel. *Litovsk. mat. sbornik* 1990; **30**: 772-783. (English transl.: *Lithuanian Math. J.* 1990; **30**:366-375).
- [20] Nazarov S.A. A general scheme for averaging self-adjoint elliptic systems in multidimensional domains, including thin domains. *Algebra Analiz* 1995; **7**:1-92. (English transl.: *St. Petersburg Math. J.* 1996; **7**:681-748).
- [21] Nazarov S.A. Asymptotics of solutions and modeling of the elasticity problems in a domain with the rapidly oscillating boundary. *Izv. Ross. Akad. Nauk. Ser. Mat.* 2008; **72**:103-158 (English transl.: *Math. Izvestiya* 2008; **72**:509564).
- [22] Nazarov S.A. Surface enthalpy. *Dokl. Ross. Akad. Nauk* 2008; **421**:4953. (English transl.: *Doklady Physics* 2008; **53**:383387.)
- [23] Nobile F, Pozzoli M, Vergara C. Time accurate partitioned algorithms for the solution of fluid-structure interaction problems in haemodynamics. *Computer & Fluids* 2013; **86**:470482.
- [24] Moireau P, Xiao N, Astorino M, Figueroa C.A, Chapelle D, Taylor C.A, Gerbeau J.F. External tissue support and fluid-structure simulation in blood flows. *Biomechanics and Modeling in Mechanobiology* 2012; **11**:1-18.
- [25] Perego M, Veneziani A, Vergara C. A variational approach for estimating the compliance of the cardiovascular tissue: An Inverse fluid-structure interaction problem. *SIAM J. Sc. Comp.* 2011; **33**:1181-1211.

- [26] Pogorelov A. V. *Differential geometry*. P. Noordhoff N. V.: Groningen, 1959.
- [27] Sanchez-Hubert J, Sanchez-Palencia E. *Coques élastiques minces: propriétés asymptotiques*. Masson: Paris, 1997.
- [28] Shoikhet, B. A., On asymptotically exact equations of thin plates of complex structure. *J. Appl. Math. Mech.* 1973; **37**:867877.
- [29] Stålhand J, Klarbring A, Karlsson M. Towards in vivo aorta material identification and stress estimation. *Biomechan. Model. Mechanobiol* 2004; **2**:169-186.
- [30] Nichols W, O'Rourke M, Vlachopoulos C. *McDonald's Blood Flow in Arteries 6th Edition: Theoretical, Experimental and Clinical Principles*. Oxford University Press, 2011.
- [31] Zwiebel W, Pellerito J. *Introduction to Vascular Ultrasonography*. Saunders, 2005.