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Abstract
We derive a bound on the order of accuracy for interpolation schemes used in energy stable summation-by-parts discretizations on non-conforming multi-block grids. This result explains the suboptimal accuracy of such schemes reported in previous works. Numerical simulations confirm a corresponding reduced convergence rate in both maximum and $L_2$ norms.

Keywords: Summation-by-parts, High order finite difference methods, Multi-block discretizations, Interpolation operators.

1. Introduction

Summation-by-parts (SBP) operators used together with weak boundary conditions of simultaneous-approximation-term (SAT) type provide the most natural setting in which to construct energy stable high order finite difference discretizations on curvilinear and multi-block grids. See [1] and the references therein for a review of the development of SBP-SAT discretizations.

In [2], the concept of SBP preserving interpolation operators was introduced, allowing for stable couplings of computational domains with non-conforming grid interfaces. The technique was extended in [3] to more general couplings between several blocks involving so called T-junctions, and also in ([4]) to second order hyperbolic problems such as the wave equation. In [5], a modified formulation was presented in which dissipation could also be included at non-conforming interfaces using SBP preserving operators. Hybrid couplings between finite difference and discontinuous Galerkin methods posed on SBP form were also demonstrated in that paper.
The references cited above all employ SBP preserving operators with suboptimal accuracy, in the sense that they increase the truncation error of the scheme by one order at certain points in the domain. However, no theoretical proof has been presented that explains why this is necessary. Although it has been acknowledged that this suboptimal accuracy could impact the convergence rate negatively [3, 4], the numerical results presented in the literature are so far ambiguous as to whether such a drop actually occurs, and if so to what extent.

In this note we present a theoretical proof that explains the suboptimal accuracy of SBP preserving interpolation schemes previously reported in the literature. We also present new numerical results using three different classes of interpolation operators. These results clearly demonstrate a corresponding reduction in asymptotic convergence rate by one order in the maximum norm, and by a half order in the $L_2$ norm.

2. SBP operators in 1D

We begin by introducing the concept of SBP operators in one dimension. Let $x = (x_0, x_1, \ldots, x_{N_x}) \in \mathbb{R}^{N_x+1}$ discretize the segment $[\alpha, \beta]$ of the real line. A first derivative operator $D_x$ on SBP form acting on this grid is defined by the decomposition $D_x = P_x^{-1}Q_x$, where $P_x$ is a positive definite matrix defining a numerical quadrature $\|\phi\|^2_{P_x} = (\phi, \phi)_{P_x}$ approximating the continuous $L_2$ norm given by $\|\phi\|_{[\alpha,\beta]} = \int_{\alpha}^{\beta} |\phi|^2$. For this reason, we will also refer to $P_x$ as the norm associated with the SBP operator. Moreover, the matrix $Q_x$ satisfies the so-called SBP property:

$$Q_x + Q_x^T = B_x = e_{x,\beta}e_{x,\beta}^T - e_{x,\alpha}e_{x,\alpha}^T,$$

where $e_{x,\alpha}$ and $e_{x,\beta}$ restrict $x$ to the boundary points of the continuous domain:

$$e_{x,\alpha}^T x = \alpha, \quad e_{x,\beta}^T x = \beta.$$

In this note we will focus on SBP finite difference operators with diagonal norms $P_x$, since they are needed in order to guarantee stability on curvilinear grids and for problems involving non-constant coefficients [6, 7]. By using a stencil of order $2s$ in the interior, the order at the boundaries of such operators is limited to $s$, in the sense that they differentiate grid polynomials up to this order exactly [8]:

$$D_x x^j = j x^{j-1}, \quad j = 0, \ldots, s.$$
Note that the exponent $j$ in (3) above acts elementwise on the grid vector $x$. Note also that (3) leads to truncation errors of order $s$ when differentiating a smooth function restricted to the grid using the operator $D_x = P^{-1}_x Q_x$, since $P_x$ is scaled by the grid size.

The accuracy relations of the diagonal quadrature $P_x$ are given by [9]:

$$1^T P_x x^{i+j-1} = \frac{\beta^{i+j} - \alpha^{i+j}}{i+j}, \quad i + j = 1, \ldots, 2s,$$

(4)

where $1 = x^0$ is a vector containing the entry 1 in each position. These relations follow from combining (3) and the SBP property (1). For ease of presentation, we will later refer to discretizations using this class of operators with diagonal norms as SBP$(2s,s)$.

3. SBP preserving interpolation operators

Consider the scalar constant coefficient advection equation in two space dimensions:

$$u_t + au_x + bu_y = 0, \quad (x, y) \in \Omega,$$

(5)

defined on the rectangular domain $\Omega = [-1, 1] \times [0, 1]$ with boundary $\partial \Omega$. We divide this domain into two computational blocks with a common interface along the line $x = 0$, and discretize using the four grid vectors $x_L, x_R, y_L$ and $y_R$. To each one of these grids we associate a corresponding one-dimensional SBP operator.

In order to couple the discrete solution between the two blocks, we additionally need interpolation operators acting between $y_L$ and $y_R$. We thus introduce the operators $\mathcal{I}_{L,R}$ and $\mathcal{I}_{R,L}$ for this purpose. In [2], the following condition was identified, guaranteeing stability for correctly imposed weak SAT interface couplings:

$$\mathcal{I}_{R,L} = P^{-1}_{y_L} \mathcal{I}^T_{L,R} P_{y_R}.$$

(6)

Operators satisfying (6) are referred to as SBP preserving in [2]. Moreover, we say that a pair of interpolation operators $\mathcal{I}_{L,R}$ and $\mathcal{I}_{R,L}$ are accurate to order $p_1$ and $p_2$ respectively, if they interpolate grid polynomials up to these orders exactly:

$$\mathcal{I}_{L,R} y_L^i = y_R^i, \quad i = 0, \ldots, p_1,$$

(7)

$$\mathcal{I}_{R,L} y_R^j = y_L^j, \quad j = 0, \ldots, p_2.$$

(8)
We discretize (5) on the two subdomains using the following SBP-SAT formulation, where we have ignored the boundary conditions to the problem for clarity of presentation:

\[
\begin{align*}
U_t + a(D_{xL} \otimes I_{yL})U + b(I_{xL} \otimes D_{yL})U &= \sigma_L P_{xL}^{-1} e_{xL,0} \otimes (u_I - I_{L,R}v_I), \\
V_t + a(D_{xR} \otimes I_{yR})V + b(I_{xR} \otimes D_{yR})V &= \sigma_R P_{xR}^{-1} e_{xR,0} \otimes (v_I - I_{L,R}u_I),
\end{align*}
\]

where \(u_I = (e_T^x \otimes I_{yL})U\) and \(v_I = (e_T^x \otimes I_{yR})V\) both represent the solution along the common grid interface.

Remark 1. Note that the accuracy conditions of the interpolation operators defined in (7) and (8) leads to truncation errors of order \(p_1\) and \(p_2\) in (9), respectively. This results from the fact that the penalty terms are multiplied with the inverses of \(P_{xL}\) and \(P_{xR}\), which in turn are scaled by the grid sizes. Similarly, the differential terms in (9) lead to truncation errors of order \(s\), due to the accuracy conditions (3) of the SBP operators.

The discrete energy method is employed by multiplying the first equation in (9) with \(U^* (P_{xL} \otimes P_{yL})\) and the second one with \(V^* (P_{xR} \otimes P_{yR})\), adding the results together and finally adding the conjugate transpose. This approach leads to the following estimate, ignoring any contributions from the outer domain boundary \(\partial \Omega\):

\[
\frac{d}{dt} \|U\|_{P_{xL} \otimes P_{yL}} + \frac{d}{dt} \|V\|_{P_{xR} \otimes P_{yR}} = (u_I) \tilde{A} (u_I)^T,
\]

where

\[
\tilde{A} = \begin{pmatrix}
(-a + 2\sigma_L) P_{yL} & -\sigma_L P_{yL} I_{R,L} - \sigma_R I_{L,R} P_{yR} \\
-\sigma_R P_{yR} I_{L,R} - \sigma_L I_{R,L} P_{yL} & (a + 2\sigma_R) P_{yR}
\end{pmatrix}.
\]

Assuming that the interpolation operators satisfy the SBP preserving condition (6), then \(\tilde{A}\) vanishes with the choice \(\sigma_L = a/2\) and \(\sigma_R = -a/2\) (6), leading to a stable imposition of the interface conditions.

Unfortunately, the energy estimate made possible by using SBP preserving interpolation operators comes at a price. In the following proposition we show that (6) leads to a bound on accuracy that reduces the order of the largest truncation errors in the scheme by one. For simplicity, the result is formulated for diagonal norm SBP(2s,s) operators satisfying the quadrature
conditions given in (4), but the same proof idea can be applied to non-diagonal norm operators satisfying more general compatibility conditions.

Before stating the proposition itself, we make

**Assumption 1.** Consider the semi-discrete discretization (9), using SBP(2s, s) operators satisfying (3) and (4). We will assume that the two diagonal norms \( P_{yL} \) and \( P_{yR} \) acting along the vertical coordinate does not satisfy

\[
1^T_R P_{yR} y_R^{2s} = 1^T_L P_{yL} y_L^{2s}.
\]

This assumption is reasonable given the fact that the norms are determined a priori by the SBP operators used in the discretization, and can thus in general not be tuned to satisfy additional constraints such as the one given above.

We can now prove

**Proposition 1.** Let \( \mathcal{I}_{L,R} \) and \( \mathcal{I}_{R,L} \) be a pair of SBP preserving interpolation operators (6) of order \( p_1 \) and \( p_2 \) respectively (see (7) and (8)), and assume that the corresponding norms \( P_{yL} \) and \( P_{yR} \) satisfy the accuracy condition (4). Then \( p_1 + p_2 \leq 2s - 1 \) given that Assumption 1 holds.

**Proof.** Let \( i \) and \( j \) be two integers such that \( i \leq p_1 \) and \( j \leq p_2 \). We begin by multiplying (8) with \((y^i_L)^T P_{yL}\) from the left:

\[
(y^i_L)^T P_{yL} \mathcal{I}_{R,L} y^j_R = (y^i_L)^T P_{yL} y^j_L.
\]

Next, we use the SBP preserving condition (6) to rewrite this into

\[
(\mathcal{I}_{L,R} y^i_L)^T P_{yR} y^j_R = (y^i_L)^T P_{yL} y^j_L.
\]

Finally, (7) leads to \((y^i_R)^T P_{yR} y^j_R = (y^j_L)^T P_{yL} y^i_L\), which is equivalent to

\[
1^T_R P_{yR} y^{i+j}_R = 1^T_L P_{yL} y^{i+j}_L. \tag{10}
\]

From (4) it follows that this identity is satisfied for \( i + j \leq 2s - 1 \), while if Assumption 1 holds it is not satisfied for \( i + j = 2s \). It follows directly from this that \( p_1 + p_2 \leq 2s - 1 \). \( \square \)

**Remark 2.** Proposition 1 implies that \( \min(p_1, p_2) = s - 1 \), since \( p_1 \) and \( p_2 \) are positive integers, leading to truncation errors of order \( s - 1 \) in (9) (see Remark 1). The SBP(2s, s) operators by themselves lead to truncation errors of at least order \( s \). However, the truncation error of order \( s - 1 \) can be restricted to a finite number of rows, corresponding to the number of linearly
independent ways of combining the two sets of equations (7) and (8) into (10) for $i + j = 2s$. The order of the truncation error thus exceeds $s - 1$ except at a finite number of points along the interface. Hence, we conclude that Proposition 1 implies a reduction by one order in the maximum norm of the truncation error for SBP($2s,s$) discretizations.

4. Numerical results

From the convergence theory developed in [10, 11] and extended in [12], it is known that energy stable difference approximations of hyperbolic problems in general converges with one order higher than that of the truncation error at the boundaries. This means that for a single block implementation of the model problem (5) using SBP($2s,s$) without interpolation, the order of convergence should be $s + 1$. Since the introduction of SBP preserving interpolation operators reduces the order of the truncation error to $s - 1$ in maximum norm (see Remark 2), we should not be surprised to see a corresponding reduction of in convergence rate of the solution. However, as was mentioned in the introduction, the numerical results previously reported in the literature are ambiguous as to whether such a reduction is present, and if so to what level.

In this section we investigate the convergence rates of numerical solutions to (9), using SBP preserving interpolation schemes. We employ the exact solution $u = \sin(x + y - 2t)$ and measure the error at $t = 1$. In [2], the construction of SBP preserving operators based on a repeated interior stencil was discussed for SBP($2s,s$) discretizations, using a constant ratio between $\Delta y_L$ and $\Delta y_R$. Especially, a set of such operators for the 2 : 1 ratio was constructed and tested for convergence, of order $p_1 = p_2 = s - 1$ near both ends of the interface, and of order $2s - 1$ along the interior of the interfaces. These operators satisfy an even stricter bound (i.e. $p_1 + p_2 = 2s - 2$) than what is possible according to Proposition 1. However, this fact is of limited practical importance since the order of the largest truncation errors depends on $\min(p_1, p_2)$, which is still $s - 1$. In addition, a number of free parameters were introduced and tuned in order to minimize the leading order error terms. These optimized operators are denoted $OPT$ below.

For comparison, we have constructed a corresponding new set of minimal bandwidth operators of the same order as $OPT$, but not optimized for accuracy. These operators, denoted with $MBW$ below, can be found in Appendix A for the cases SBP($2,1$) and SBP($4,2$). We also compare with another new
set of interpolation operators for grid size ratios of 13 : 7. These are not constructed based on a repeated interior stencil as the previous ones, but are instead generated automatically by solving for all interpolation accuracy conditions directly on each grid. We restrict the order in the interior of these operators to \(s\), which we expect to be sufficient, given that the truncation errors resulting from the SBP operators in the same region of the domain are of the same order. We denote these operators with \(AUTO\).

The numerical results are shown in Figure 1 for the cases SBP(2,1), SBP(4,2) and SBP(6,3). As can be seen, the order is reduced to \(s\) for all three sets of operators if the error is measured in the maximum norm. In the \(L_2\) norm the error instead approaches \(s + 1/2\), and the drop from \(s + 1\) is only clearly visible at small error levels. This probably explains why no drop in convergence were reported in some of the previous works. We conjecture that the observed convergence rates in both maximum and \(L_2\) norms can be explained by the reduction in truncation error by one order in the maximum norm as explained in Remark 2, even though we have no direct proof.

A graphical plot of the error distribution is shown in Figure 2 for two discretizations of SBP(4,2), using the new \(MBW\) interpolation schemes. As
the figure suggests, the shape of the region that contains the increased error levels approaches a one-dimensional set. This observation is consistent with a drop by only a half order in the $L_2$ norm, even though it is reduced by one full order in the maximum norm.

Finally, we note that the three different types of interpolation operators show approximately the same performance. Especially, the automatically generated operators with order $s$ in the interior show comparable results to those with a repeated stencil of order $2s - 1$ in the interior. We thus expect that a continued development of automatically generated operators will enable both efficient and stable interface couplings of completely general multi-block and hybrid meshes.
Appendix A. New minimum bandwidth interpolation operators

The diagonal norm in the SBP(2,1) case is given by $P_x = \Delta x \text{Diag}(\frac{1}{2}, 1, \ldots)$. The corresponding new minimum bandwidth operator for the grid ratio of 2 : 1 between $y_L$ and $y_R$ is given by

$$I_{L,R} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \end{pmatrix}.$$

We recall that the operator $I_{R,L}$ is given by the SBP preserving formula (6).

In the SBP(4,2) case, the norm is given by $P_x = \Delta x \text{Diag}(\frac{17}{32}, \frac{219}{48}, \frac{43}{48}, \frac{49}{48}, 1, \ldots)$, and the interpolation operator $I_{L,R}$ is given by

$$I_{L,R} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \cdots \end{pmatrix}.$$

References


