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Generalised Ramsey numbers and Bruhat order on involutions

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Abstract

This thesis consists of two papers within two different areas of combinatorics.

Ramsey theory is a classic topic in graph theory, and Paper A deals with two of its most fundamental problems: to compute Ramsey numbers and to characterise critical graphs. More precisely, we study generalised Ramsey numbers for two sets Γ_1 and Γ_2 of cycles. We determine, in particular, all generalised Ramsey numbers $R(\Gamma_1, \Gamma_2)$ such that Γ_1 or Γ_2 contains a cycle of length at most 6, or the shortest cycle in each set is even. This generalises previous results of Erdős, Faudree, Rosta, Rousseau, and Schelp. Furthermore, we give a conjecture for the general case. We also characterise many (Γ_1, Γ_2) -critical graphs. As special cases, we obtain complete characterisations of all (C_n, C_3) -critical graphs for $n \geq 5$, and all (C_n, C_5) -critical graphs for $n \geq 6$.

In Paper B, we study the combinatorics of certain partially ordered sets. These posets are unions of conjugacy classes of involutions in the symmetric group S_n , with the order induced by the Bruhat order on S_n . We obtain a complete characterisation of the posets that are graded. In particular, we prove that the set of involutions with exactly one fixed point is graded, which settles a conjecture of Hultman in the affirmative. When the posets are graded, we give their rank functions. We also give a short, new proof of the EL-shellability of the set of fixed-point-free involutions, recently proved by Can, Cherniavsky, and Twelbeck.

Populärvetenskaplig sammanfattning

Den här uppsatsen består av två artiklar inom olika områden av kombinatorik.

Ramseyteori, som är temat för den första artikeln, är ett klassiskt ämne inom grafteori. En *graf* består av ett antal hörn och ett antal kanter som förbinder vissa av hörnen. Den *kompletta grafen* har en kant mellan varje par av hörn, och en *röd-blå graf* är en komplett graf med varje kant färgad antingen röd eller blå. Det *generaliserade ramseytalet* $R(\mathcal{G}_1, \mathcal{G}_2)$ är det minsta positiva heltalet n , sådant att **varje** röd-blå graf på n hörn innehåller en röd kopia av en graf från mängden $\mathcal{G}_1$ eller en blå kopia av en graf från mängden $\mathcal{G}_2$. Ramseytalet beskriver alltså hur många hörn en röd-blå graf måste ha för att den garanterat ska innehålla någon av vissa specificerade delgrafer. Startpunkten för ramseyteori var Frank Ramseys bevis 1930 av att alla ramseytal existerar, dvs. att om bara grafen har tillräckligt många hörn, så kommer den alltid att innehålla någon av de specificerade delgraferna. Ämnet tog fart på allvar på 1950-talet och idag är det ett av de mest aktiva fälten inom grafteori.

Vi studerar generaliserade ramseytal $R(\mathcal{G}_1, \mathcal{G}_2)$ där mängderna $\mathcal{G}_1$ och $\mathcal{G}_2$ består av cykler (om man ritar upp grafen så ser den ut som en cykel). Detta problem löstes på 1970-talet då mängderna endast innehåller en cykel var, men det allmänna problemet har bara delvis studerats tidigare. Vi bestämmer alla sådana generaliserade ramseytal där $\mathcal{G}_1$ eller $\mathcal{G}_2$ (eventuellt båda) innehåller en cykel med högst 6 hörn.

Förutom att beräkna ramseytal, försöker matematiker att beskriva de grafer som inte innehåller någon av de specificerade delgraferna. Framför allt försöker man att *karaktärisera* de $(\mathcal{G}_1, \mathcal{G}_2)$ -*kritiska* graferna, dvs. ge en beskrivning av precis de grafer på $R(\mathcal{G}_1, \mathcal{G}_2) - 1$ hörn som varken innehåller en röd delgraf från $\mathcal{G}_1$ eller en blå delgraf från $\mathcal{G}_2$. Vi karaktäriserar alla (C_n, C_3) -kritiska grafer för $n \geq 5$, och alla (C_n, C_5) -kritiska grafer för $n \geq 6$.

I den andra artikeln studerar vi kombinatoriken hos vissa *partialordnade*

mängder (eller *pomängder*). Det är mängder där en del element är större än andra, men det finns i allmänhet också element som inte är jämförbara med varandra (inget av elementen är större än det andra). Den *symmetriska gruppen* S_n består av alla *permutationer* av talen $1, 2, \dots, n$, dvs. alla upp-räkningar av talen, till exempel $3, 2, 1, 5, 4$ eller $2, 3, 1, 5, 4$. Permutationerna kan ordnas med den så kallade *Bruhatordningen*, som har viktiga kopplingar till algebra och geometri. På så vis blir S_n en pomängd.

En *involution* i S_n är en permutation som fås genom att bara talen inom vissa par byter plats med varandra. Till exempel är $3, 2, 1, 5, 4$ en involution (1 byter plats med 3 och 4 byter plats med 5), men $2, 3, 1, 5, 4$ är det inte. Det är naturligt att betrakta mängden I_n av alla involutioner som en pomängd, där ordningen bestäms av Bruhatordningen på S_n . Man kan också betrakta vissa delmängder av I_n som pomängder, till exempel F_n^0 , de involutioner som inte har några *fixpunkter* (ovan är 2 den enda fixpunkten, ty talet 2 står på plats 2). Vi karakteriserar de pomängder som är *graderade*. Det är en viktig egenskap som en pomängd kan ha, och betyder ungefär att elementen i pomängden kan delas in i olika "nivåer". Vi bestämmer också *rangfunktionen* av de graderade pomängderna. Den beskriver vilken "nivå" varje element tillhör. Vi ger också ett kort, nytt bevis av att F_n^0 är *EL-skalbar*. Det betyder att pomängden kan "märkas" på ett sätt som säger något om dess topologi.

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Part I

Introduction

Chapter 1

Ramsey theory and Bruhat order

For a non-negative integer n , let $[n] = \{1, 2, \dots, n\}$ (so $[0] = \emptyset$), and for integers $a \leq b$, let $[a, b] = \{a, a + 1, \dots, b\}$.

1.1 Ramsey theory

Ramsey theory is a classic topic in graph theory. In this section, we give its basic definitions and some of its classic results; we also mention a recent result that is of interest to us.

We consider only non-empty graphs that are finite, simple, and undirected. Also, (G_1, G_2) and $(\mathcal{G}_1, \mathcal{G}_2)$ denote a pair of graphs and a pair of non-empty sets of graphs, respectively. By a *red-blue graph* we shall mean a *complete* graph with each edge coloured either red or blue. Red will always be the first colour and blue will always be the second.

We have the following definition:

Definition 1.1.1. The *Ramsey number* $R(G_1, G_2)$ is the least positive integer n , such that each red-blue graph on n vertices contains G_1 as a red subgraph or G_2 as a blue subgraph; let $R(i, j) = R(K_i, K_j)$.

It was proved by Ramsey [13] in 1930 that $R(G_1, G_2)$ always exists. Clearly, $R(G_1, G_2) = R(G_2, G_1)$, $R(G, K_1) = 1$, and $R(G, K_2) = |V(G)|$. As a first non-trivial Ramsey number, let us prove that $R(3, 3) = 6$.

Proposition 1.1.2. $R(3, 3) = 6$.

Proof. $R(3, 3) \geq 6$: We have to demonstrate the existence of a red-blue graph on 5 vertices with no monochromatic 3-clique. This is done in Figure 1.1.1.

Figure 1.1.1: $R(3, 3) \geq 6$.

$R(3, 3) \leq 6$: We have to show that every red-blue graph on 6 vertices contains a monochromatic 3-clique. Let the vertices be v, a, b, c, d , and e . We may assume that va, vb , and vc are red. If ab, ac , or bc is red, then we have a red 3-clique; otherwise we have a blue 3-clique. $\square$

To compute Ramsey numbers is, in general, a very hard problem. For instance, the exact value of $R(i, j)$, where $i, j \geq 3$, is known only when $i = 3$ and $j \leq 9$, and for $(i, j) \in \{(4, 4), (4, 5)\}$. When G_1 and G_2 are not complete graphs, more is known. Most important to us is that the cycle versus cycle case has been completely solved. This was done independently by Rosta [15] and by Faudree and Schelp [4] in the 1970s. The formula given here is due to Schwenk (see Harary [6]).

Theorem 1.1.3. *Let $n \geq k \geq 3$ and define $f(x, y) = (2x - 1)(y - 2\lfloor y/2 \rfloor)$. Then*

$$R(C_n, C_k) = \max(6, n + k/2 - 1, f(k, n), f(n, k)).$$

Another problem in Ramsey theory is to describe (G_1, G_2) -avoiding graphs. These are graphs that contain neither a red copy of G_1 nor a blue copy of G_2 . Of particular interest is the characterisation of (G_1, G_2) -critical graphs, i.e., (G_1, G_2) -avoiding graphs on $R(G_1, G_2) - 1$ vertices. For instance, it is well known (and easy to show) that the graph in Figure 1.1.1 is the only $(3, 3)$ -critical graph (up to isomorphism). As for critical graphs for cycle versus cycle, Wu, Sun, and Radziszowski [18] recently characterised all (C_n, C_4) -critical graphs for $n \geq 6$.

For more known values and bounds of Ramsey numbers, as well as more on critical graphs, the reader is referred to Radziszowski [12].

There are many generalisations of Ramsey numbers, e.g., colouring complete bipartite graphs or colouring complete graphs with more than two

colours. In this thesis, we consider a generalisation that was first introduced by Erdős, Faudree, Rousseau, and Schelp [3, 5] in the 1970s:

Definition 1.1.4. The *generalised Ramsey number* $R(\mathcal{G}_1, \mathcal{G}_2)$ is the least positive integer n , such that each red-blue graph on n vertices contains a red subgraph from $\mathcal{G}_1$ or a blue subgraph from $\mathcal{G}_2$.

Observe that when $|\mathcal{G}_1| = |\mathcal{G}_2| = 1$, the generalised Ramsey number $R(\mathcal{G}_1, \mathcal{G}_2)$ reduces to the ordinary Ramsey number $R(G_1, G_2)$. It follows from the definition that $R(\mathcal{G}_1, \mathcal{G}_2) \leq R(\mathcal{H}_1, \mathcal{H}_2)$ if each $\mathcal{H}_i$ is a non-empty subset of $\mathcal{G}_i$. Thus

$$R(\mathcal{G}_1, \mathcal{G}_2) \leq R(G_1, G_2)$$

if each $G_i \in \mathcal{G}_i$. In particular, $R(\mathcal{G}_1, \mathcal{G}_2)$ always exists. As well as for ordinary Ramsey numbers, $R(\mathcal{G}_1, \mathcal{G}_2) = R(\mathcal{G}_2, \mathcal{G}_1)$.

The notions of $(\mathcal{G}_1, \mathcal{G}_2)$ -avoiding and $(\mathcal{G}_1, \mathcal{G}_2)$ -critical graphs are completely analogous to the situation when $|\mathcal{G}_1| = |\mathcal{G}_2| = 1$.

1.2 The Bruhat order and poset topology

Our goal of this section is to define Coxeter groups and the Bruhat order, as well as to give a very brief introduction to poset topology, in particular, the concept of shellability. Before any of this, however, we present some of the basic definitions about posets, in particular, that of a graded poset.

The definitions in Subsection 1.2.1 can be found in, e.g., [16, Chapter 3]. The results in Subsections 1.2.2 and 1.2.3 are all standard; general references are [1, Chapter 2] and [17, Chapter 3], respectively.

1.2.1 Posets

A *poset* (short for *partially ordered set*) is a set P together with a binary relation $\leq$ on P that is reflexive, antisymmetric, and transitive. Three examples of posets are the integers with the usual $\leq$ (a *totally ordered set*), the *boolean poset* B_n , which is the set of subsets of $[n]$ ordered by inclusion, and the *divisibility poset* D_n , which is the set of positive divisors of n ordered by divisibility. In the next subsection, we shall see how the symmetric group, and more generally, any Coxeter group, can be turned into a poset.

In a poset P , we say that y *covers* x if $x < y$ and there is no $z \in P$ such that $x < z < y$; we use the notation $x \triangleleft y$. Furthermore, a *chain* is a totally ordered subset, and a chain C is *saturated* if there is no $z \in P - C$ such that $C \cup \{z\}$ is a chain and $x < z < y$ for some $x, y \in C$, and *maximal* if it is not

contained in a larger chain. A finite chain $x_0 < x_1 < \cdots < x_k$ is saturated precisely when $x_{i-1} \triangleleft x_i$ for all $i \in [k]$. An x - y -chain is a *saturated* chain from x to y . We call P *bounded* if it has a minimum (denoted by $\hat{0}$) and a maximum (denoted by $\hat{1}$), which means that $\hat{0} \leq x$ and $\hat{1} \geq x$ for all $x \in P$. An element $x \in P$ is *minimal* if there is no $y \in P$ with $y < x$, and *maximal* if there is no $y \in P$ with $y > x$. Observe that a minimal (maximal) element need not be a minimum (maximum).

The notion of a graded poset is the most important one of this subsection. There are a few different, although very similar, definitions. We use the following one:

Definition 1.2.1. A poset P is *graded of rank n* if every maximal chain consists of $n + 1$ elements. In this case, there is a unique *rank function* $\rho : P \rightarrow \{0, 1, \dots, n\}$ such that $\rho(x) = 0$ if x is a minimal element of P , and $\rho(y) = \rho(x) + 1$ if $x \triangleleft y$ in P ; x has *rank i* if $\rho(x) = i$.

1.2.2 Coxeter groups

Let S be a set. A *Coxeter matrix* is a function $m : S \times S \rightarrow \{1, 2, \dots, \infty\}$ such that $m(s, s') = m(s', s)$ for all $s, s' \in S$, and $m(s, s') = 1 \Leftrightarrow s = s'$. Equivalently, m can be represented by a *Coxeter graph* whose vertex set is S and where ss' is an edge if $m(s, s') \geq 3$, labelled by this number if it is greater than 3. Every Coxeter matrix m determines a group W with the presentation whose set of generators is S and whose relations are $(ss')^{m(s, s')} = e$ whenever $m(s, s') < \infty$. Since $m(s, s) = 1$, $s^2 = e$, so the generators are involutions. The pair (W, S) is called a *Coxeter system* and W is called a *Coxeter group*. The system is *irreducible* if its Coxeter graph is connected.

Consider now the symmetric group S_n , i.e., the set of permutations of $[n]$. For each $i \in [n - 1]$, let s_i denote the adjacent transposition $(i, i + 1)$, and let S be the set of all s_i . For any permutation $\pi = \pi(1) \cdots \pi(i) \pi(i + 1) \cdots \pi(n)$ (one line notation), $\pi s_i = \pi(1) \cdots \pi(i + 1) \pi(i) \cdots \pi(n)$. Hence, S generates S_n . Furthermore, $s_i s_j = s_j s_i$ if $|i - j| \geq 2$ and $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$. These facts make it conceivable that (S_n, S) is the Coxeter system whose Coxeter graph is the path

This also turns out to be the case (see, e.g., [1, Proposition 1.5.4]). Hence, the symmetric group is an example of a Coxeter group.

The Bruhat order

Let (W, S) be a Coxeter system. If $w \in W$ is written as a product $w = s_1 \cdots s_k$ of generators, and w cannot be written as a product of fewer generators, then k is the *length* of w , and the word $s_1 \cdots s_k$ is a *reduced word* for w . A *subword* of a word $s_1 \cdots s_q$ is a word of the form $s_{i_1} \cdots s_{i_m}$, where $1 \leq i_1 < \cdots < i_m \leq q$.

The *Bruhat order* is a partial order $\leq$ on W . It may be defined as follows (see, e.g., [1, Theorem 2.2.2]):

Definition 1.2.2. Let $u, w \in W$. Then $u \leq w$ if and only if every reduced word for w has a subword that is a reduced word for u .

There is a simple characterisation of the Bruhat order on S_n . For $\sigma \in S_n$ and $(k, l) \in [n]^2$, let $\sigma[k, l] = |\{i \leq k \mid \sigma(i) \geq l\}|$. We have the following (see, e.g., [1, Theorem 2.1.5]):

Theorem 1.2.3. Let $\sigma, \tau \in S_n$. Then $\sigma \leq \tau$ if and only if $\sigma[k, l] \leq \tau[k, l]$ for all $(k, l) \in [n]^2$.

As an illustration of Theorem 1.2.3, observe that $34251 > 21435$, whereas $\sigma = 34251$ and $\tau = 21534$ are incomparable in the Bruhat order (since $\sigma[1, 3] > \tau[1, 3]$ and $\sigma[3, 5] < \tau[3, 5]$).

The Bruhat order turns any Coxeter group into a graded poset whose rank function is the length function. The length of a permutation $\sigma \in S_n$ is the number of inversions of σ , where an *inversion* is a pair $(i, j) \in [n]^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

We end this subsection by briefly describing a classic motivation for studying the Bruhat order. Denote by Fl_n the set of all sequences $0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbf{C}^n$ of subspaces; the sequences are called *flags*. There is a bijection between Fl_n and G/B , where $G = \text{GL}_n(\mathbf{C})$ and B is the subgroup of upper triangular matrices. It turns out that G/B is the disjoint union

$$G/B = \bigsqcup_{w \in S_n} BwB/B,$$

where we identify a permutation $w \in S_n$ with the corresponding permutation matrix. We call $C_w = BwB/B$ a *Schubert cell* (or a *Bruhat cell*); the word “cell” is due to the fact that one may define a topology on Fl_n under which each C_w is a cell (i.e., homeomorphic to an open ball). The topological closure of a Schubert cell C_w is called a *Schubert variety*; we use the notation $\overline{C_w}$. It turns out that

$$\overline{C_w} = \bigcup_{v \leq w} C_v,$$

where $\leq$ is the Bruhat order on S_n . Hence, $\overline{C_u} \subseteq \overline{C_w}$ precisely when $u \leq w$.

1.2.3 Shellability

Let us first define simplicial complexes. Note that we assume these to be finite.

Definition 1.2.4. A *d-dimensional geometric simplex* (or just a *d-simplex*) σ is the convex hull of $d+1$ affinely independent points (called the *vertices* of σ) in $\mathbf{R}^n$ for some $n \geq d$. The convex hull of k vertices of σ , where $0 \leq k \leq d+1$, is also a simplex, called a *face* of σ ; its *dimension* is $k-1$. The maximal faces are called *facets*.

A *geometric simplicial complex* (or just a *simplicial complex*) Δ is a finite collection of simplices such that (i) if $\tau \in \Delta$ and σ is a face of τ , then $\sigma \in \Delta$, and (ii) if $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of both σ and τ . We call F a *face* of Δ if F is a face of some $\sigma \in \Delta$. The *dimension* of Δ , denoted by $\dim \Delta$, is the maximum of the dimensions of the facets of Δ . We call Δ *pure* if every facet has the same dimension. We view Δ as a topological space, with the subspace topology induced by the usual topology on $\mathbf{R}^n$, by identifying Δ with the union of its simplices.

Every finite poset defines a simplicial complex, as follows:

Definition 1.2.5. Given a finite poset P , identify its elements with $|P|$ affinely independent points in $\mathbf{R}^n$ for some $n \geq |P| - 1$. The *order complex* of P is the simplicial complex $\Delta(P)$ whose vertices are the affinely independent points in $\mathbf{R}^n$, and whose faces are the convex hulls of the points that correspond to each chain of P .

Note that the facets of $\Delta(P)$ are the maximal chains of P . Hence, $\Delta(P)$ is pure if and only if P is graded.

We shall consider a property that a pure simplicial complex may have, namely, being shellable.

Definition 1.2.6. Given a pure simplicial complex Δ , let $\mathcal{F} = \{F_1, \dots, F_m\}$ be its set of facets. A *shelling order* of Δ is a total order on $\mathcal{F}$, say that $F_1 < \dots < F_m$, such that

$$F_k \cap \left(\bigcup_{i=1}^{k-1} F_i \right)$$

is a pure subcomplex of Δ of codimension 1 (i.e., of dimension $\dim \Delta - 1$) for all $k \in [2, m]$. If Δ has a shelling order, then Δ is *shellable*.

A topological consequence of shellability is the following (see, e.g., [17, Theorem 3.1.3]):

Theorem 1.2.7. *If a pure simplicial complex Δ is shellable, then Δ is homotopy equivalent to a wedge of spheres of dimension $\dim \Delta$.*

We next present a condition on a poset that implies shellability of its order complex. To this end, let P be a finite, bounded, and graded poset. An *edge-labelling* of P is a function $\lambda : \{(x, y) \in P^2 \mid x \triangleleft y\} \rightarrow Q$, where Q is a totally ordered set. If λ is an edge-labelling of P and $x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_k$ is a saturated chain, let $\lambda(x_0, x_1, \dots, x_k) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \dots, \lambda(x_{k-1}, x_k))$. The chain is called *increasing* if $\lambda(x_{i-1}, x_i) \leq \lambda(x_i, x_{i+1})$ for all $i \in [k-1]$.

Definition 1.2.8. An edge-labelling λ of P is an *EL-labelling* if, for all $x < y$ in P , there is exactly one increasing x - y -chain, say $x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_k$, and this chain is *lexicographically minimal* among the x - y -chains in P (i.e., if $y_0 \triangleleft y_1 \triangleleft \cdots \triangleleft y_k$ is any other x - y -chain, then $\lambda(x_{j-1}, x_j) < \lambda(y_{j-1}, y_j)$, where $j = \min\{i \in [k] \mid \lambda(x_{i-1}, x_i) \neq \lambda(y_{i-1}, y_i)\}$). If P has an EL-labelling, P is said to be *EL-shellable*.

The reason for this terminology is the following result (see, e.g., [17, Theorem 3.2.2]):

Theorem 1.2.9. *Let P be a finite, bounded, and graded poset. If P is EL-shellable, then its order complex $\Delta(P)$ is shellable.*

1.2.4 The Bruhat order on involutions

It is natural to view the set of involutions in a Coxeter group W as a poset, with the order induced by the Bruhat order on W . It turns out that this poset is always graded. This was proved in the case of the symmetric group S_n by Richardson and Springer [14] in 1990, and in the 2000s by Incitti [9, 11] (when W is finite of type A , B , or D) and by Hultman [7] (for general W). The corresponding rank functions were also described in the same papers. That the rank function of the set I_n of involutions in S_n can be expressed as the average of the number of inversions and the number of exceedances was proved by Incitti [10]; an element $i \in [n]$ is an *exceedance* of $\sigma \in S_n$ if $\sigma(i) > i$. In [10], Incitti also proved that I_n is EL-shellable.

One may also view certain subsets of I_n as posets, again with the order induced by the Bruhat order on S_n . For instance, Richardson and Springer [14] proved that the set F_n^0 of fixed-point-free involutions is graded. Hultman [8] studied (in the more general setting of twisted identities, which we describe in Paper B) F_n^0 and F_n^1 , the set of involutions with exactly one fixed point. It follows that F_n^0 is graded and Hultman conjectured that the same is true for F_n^1 . Recently, Can, Cherniavsky, and Twelbeck [2] proved that F_n^0 is EL-shellable.

Chapter 2

Summary of papers

Paper A: Generalised Ramsey numbers for two sets of cycles

We study the generalised Ramsey numbers of Definition 1.1.4, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are sets of cycles, which we denote by Γ_1 and Γ_2 . As our first main result, we determine, in particular, all generalised Ramsey numbers $R(\Gamma_1, \Gamma_2)$ such that Γ_1 or Γ_2 contains a cycle of length at most 6, or the shortest cycle in each set is even. This generalises Theorem 1.1.3, as well as some other results. An interesting corollary of our result is that $R(C_n, C_4) = R(\{C_n\}, \{C_3, C_4\})$ whenever $n \geq 6$; in other words, having both C_3 and C_4 in one of the sets, makes no difference from having only C_4 . We also give a conjecture for the value of $R(\Gamma_1, \Gamma_2)$, for general (Γ_1, Γ_2) .

In the second part of the paper, we prove that many (Γ_1, Γ_2) -avoiding graphs have to be bipartite (when one only considers either the red or the blue edges), even if the number of vertices is only about half the Ramsey number. As our second main result, we characterise most (Γ_1, Γ_2) -critical graphs such that Γ_1 or Γ_2 contains a cycle of length at most 5. For length 4, this extends the result of Wu, Sun, and Radziszowski [18]. As special cases for lengths 3 and 5, we obtain complete characterisations of all (C_n, C_3) -critical graphs for $n \geq 5$, and all (C_n, C_5) -critical graphs for $n \geq 6$.

Paper B: The Bruhat order on conjugation-invariant sets of involutions in the symmetric group

We study conjugacy classes, and more generally, unions of conjugacy classes, of involutions in the symmetric group S_n . Since two permutations are conjugate if and only if, for each $i \geq 1$, they have the same number of disjoint cycles of length i , two involutions in S_n are conjugate precisely when they

have the same number of fixed points. Denote by F_n^a the conjugacy class in S_n consisting of the involutions with a fixed points, where $a \in \{0, 1, \dots, n\}$, and for $A \subseteq \{0, 1, \dots, n\}$, let

$$F_n^A = \bigcup_{a \in A} F_n^a.$$

We regard F_n^A as a poset, with the order induced by the Bruhat order on S_n .

Our main result is a complete characterisation of the sets A for which F_n^A is graded. Informally, F_n^A is graded precisely when $A - \{n\}$ is empty or an “interval,” which may consist of a single element if it is 0, 1, or $n - 2$. Hence, it follows that F_n^1 , the set of involutions with exactly one fixed point, is graded, thereby settling the conjecture of Hultman [8] in the affirmative. As steps in the proof, we show that F_n^A always has a maximum and describe its minimal elements. We also give the rank function of F_n^A when it exists; it turns out to be very similar to the rank function of I_n . Finally, we give a short, new proof of the EL-shellability of F_n^0 , the set of fixed-point-free involutions, recently proved by Can, Cherniavsky, and Twelbeck [2].

Bibliography

- [1] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.
- [2] M. B. Can, Y. Cherniavsky, and T. Twelbeck, *Lexicographic shellability of the Bruhat-Chevalley order on fixed-point-free involutions*, Israel J. Math. **207** (2015), 281–299.
- [3] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *On cycle-complete graph Ramsey numbers*, J. Graph Theory **2** (1978), 53–64.
- [4] R. J. Faudree and R. H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Math. **8** (1974), 313–329.
- [5] ———, *Some problems in Ramsey theory*, Theory and applications of graphs (Proc. Internat. Conf., Western Mich. Univ., Kalamazoo, Mich., 1976), Lecture Notes in Math., vol. 642, Springer, Berlin, 1978, pp. 500–515.
- [6] F. Harary, *A survey of generalized Ramsey theory*, Graphs and combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973), Lecture Notes in Math., vol. 406, Springer, Berlin, 1974, pp. 10–17.
- [7] A. Hultman, *Fixed points of involutive automorphisms of the Bruhat order*, Adv. Math. **195** (2005), 283–296.
- [8] ———, *Twisted identities in Coxeter groups*, J. Algebraic Combin. **28** (2008), 313–332.
- [9] F. Incitti, *The Bruhat order on the involutions of the hyperoctahedral group*, European J. Combin. **24** (2003), 825–848.
- [10] ———, *The Bruhat order on the involutions of the symmetric group*, J. Algebraic Combin. **20** (2004), 243–261.

- [11] ———, *Bruhat order on the involutions of classical Weyl groups*, Adv. in Appl. Math. **37** (2006), 68–111.
- [12] S. P. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin. Dynamic Survey **1** (2014), 94 pp. (electronic).
- [13] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1930), 264–286.
- [14] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata **35** (1990), 389–436.
- [15] V. Rosta, *On a Ramsey-type problem of J. A. Bondy and P. Erdős. I & II*, J. Combin. Theory Ser. B **15** (1973), 94–120.
- [16] R. P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.
- [17] M. L. Wachs, *Poset topology: tools and applications*, Geometric combinatorics, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 497–615.
- [18] Y. Wu, Y. Sun, and S. P. Radziszowski, *Wheel and star-critical Ramsey numbers for quadrilateral*, Discrete Appl. Math. **186** (2015), 260–271.

Part II

Papers

Papers

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