A Stable and Accurate Davies-like Relaxation Procedure using Multiple Penalty Terms for Lateral Boundary Conditions

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Abstract

A lateral boundary treatment using summation-by-parts operators and simultaneous approximation terms is introduced. The method is similar to Davies relaxation technique used in the weather prediction community and have similar areas of application, but is also provably stable. In this paper, it is shown how this technique can be applied to the shallow water equations, and that it reduces the errors in the computational domain.

Keywords: Boundary conditions, Boundary layers, Summation-by-parts, Energy method, Convergence, Shallow water equations

1. Introduction

Accurate numerical calculations on large domains are often unfeasible. To reduce the computational effort, a coarse global mesh is often used with

Abbreviations:

Summation-by-parts (SBP)

Simultaneous approximations terms (SAT)

Multiple penalty technique (MPT)

4 inter-spaced local domains. The local domains are typically employed where
5 fine meshes are needed to capture phenomena with higher accuracy than in
6 the global domain. In the weather prediction community, local domains are
7 often used to model local weather phenomena.

8 The local domains need to be matched correctly to the global one. Davies
9 (1976) introduced a lateral boundary procedure for weather prediction mod-
10 els where the numerical results on the coarse grid are interpolated into the lo-
11 cal, fine grid domain. The method was later referred to as Davies relaxation.
12 Other types of interpolation methods are also used within the weather predic-
13 tion community, see Williamson and Browning (1974), Birchfield (1960) and
14 Hill (1968). In most applications, one aims for accurate results in the local
15 domains, while the accuracy in the global domain is considered given. There-
16 fore, the coupling between the domains is in most cases neglected. There are
17 methods, see Harrison and Elsberry (1972), where the coupling between the
18 meshes is considered, such that the results from the local domain influence
19 the global one. However, this coupling often introduces stability issues unless
20 optimally done, see for example Nordström et al. (2009b), Nordström et al.
21 (2009a). In this work, we will only consider the case where the domains are
22 uncoupled, which is known as one-way nesting.

23 Many existing lateral boundary schemes produce reflections at the lat-
24 eral boundaries and experience issues due to over-specification of boundary
25 data (Davies (1983)). These issues are commonly dealt with by choosing
26 the relaxation coefficients properly (Lehmann (1993)) and adding dissipa-
27 tion to the numerical scheme (Davies (1983)). However, altering the scheme
28 in these manners may result in an inaccurate and even unstable scheme.
29 The method introduced in this work can effectively reduce reflections at the
30 lateral boundaries (Nordström et al. (2014)), without ruining stability and
31 accuracy.

32 We demonstrate our technique on the shallow water equations, discretized
33 using operators with the summation-by-parts (SBP) property (Carpenter
34 et al. (1999), Strand (1994), Mattsson and Nordström (2004)), augmented
35 with simultaneous approximation terms (SAT) (Carpenter et al. (1994)).
36 This is often referred to as the SBP-SAT technique, and a comprehensive re-
37 view is given by Svård and Nordström (2014). If well-posed boundary condi-
38 tions are available, this technique yields stable numerical schemes (Carpenter
39 et al. (1999)).

40 Nordström et al. (2014) showed that if data is available inside the com-
41 putational domain, additional penalty terms can be applied without ruining

42 stability. This method, which we call the multiple penalty technique (MPT),
 43 can be used to assimilate global data into local area models, construct non-
 44 reflecting boundaries, improve accuracy of the scheme and increase the rate
 45 of convergence to steady state, see also Erickson and Nordström (2014). The
 46 MPT is similar to the Davies relaxation technique mentioned above, but can
 47 also be proven to be stable.

48 The rest of the paper will proceed as follows. In the following section,
 49 we define well-posedness of a problem and stability of a numerical scheme.
 50 Section 3 describes the SBP-SAT technique and MPT on a simple model
 51 problem in one space dimension. In section 4, we derive well-posed boundary
 52 conditions for the shallow water equations and discretize them by using the
 53 SBP-SAT technique. Stability conditions are derived and the MPT is applied
 54 such that stability is preserved. In section 5, numerical experiments are
 55 performed to illustrate the increased rate of convergence when using the
 56 MPT. The assessments of this paper are concluded in section 6.

57 2. Preliminaries

58 Before moving on to the problem under consideration, we will define
 59 well-posedness of an initial-boundary value problem (IBVP) and stability of
 60 a numerical approximation.

61 Consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= Hu + F, \quad \bar{\mathbf{r}} \in \Omega, t \geq 0 \\ u(\bar{\mathbf{r}}, 0) &= f(\bar{\mathbf{r}}) \\ Lu(\bar{\mathbf{r}}, t) &= g(t), \quad \bar{\mathbf{r}} \in \delta\Omega \end{aligned} \tag{1}$$

62 where H is a differential operator, Ω the spatial domain, $\delta\Omega$ the boundary
 63 to Ω , $\bar{\mathbf{r}}$ the position vector and F, f, g are known functions. In (1), L is a
 64 boundary operator. The problem (1) is well-posed if it has a unique solution
 65 that, for $g = F = 0$, satisfies

$$||u(\bar{\mathbf{r}}, t)||^2 \leq K_c^1 ||f||^2, \tag{2}$$

66 where K_c^1 is bounded for finite time and independent of f . Moreover, (1) is
 67 strongly well-posed if

$$||u(\bar{\mathbf{r}}, t)||^2 \leq K_c^2 [||f||^2 + \int_0^t (||F||^2 + ||g||^2) d\tau] \tag{3}$$

for non-zero g and F . In (2) and (3), $\|f\|$, $\|g\|$ and $\|F\|$ can be expressed in arbitrary norms. The function K_c^2 is bounded for finite time and does not depend on f, g and F .

Let a global semi-discrete approximation of (1) be

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} &= \mathbf{H}\mathbf{v} + \mathbf{F}, \quad t \geq 0 \\ \mathbf{v}(0) &= \mathbf{f} \\ \mathbf{L}\mathbf{v}(t) &= \mathbf{g}\end{aligned}\tag{4}$$

where $\mathbf{v}$ is the numerical approximation of u and $\mathbf{H}$ is a discrete operator that approximates H . The functions $\mathbf{F}, \mathbf{f}, \mathbf{g}$ are grid functions of F, f and g , i.e. the values are injected at the grid points. Analogously to (2), (4) is stable if

$$\|\mathbf{v}(t)\|^2 \leq K_d^1 \|\mathbf{f}\|^2 \tag{5}$$

for $\mathbf{F} = \mathbf{g} = 0$. In (5), K_d^1 is bounded for finite time and independent of $\mathbf{f}$ and the mesh size. For non-zero $\mathbf{g}$ and $\mathbf{F}$, we say that (4) is strongly stable if

$$\|\mathbf{v}(t)\|^2 \leq K_d^2 [\|\mathbf{f}\|^2 + \int_0^t (\|\mathbf{F}\|^2 + \|\mathbf{g}\|^2) d\tau] \tag{6}$$

where K_d^2 is bounded for finite time and independent of $\mathbf{f}, \mathbf{g}, \mathbf{F}$ and the mesh-size. As in the continuous estimate, $\|\mathbf{f}\|$, $\|\mathbf{g}\|$ and $\|\mathbf{F}\|$ may be given in arbitrary norms. For a detailed discussion of well-posedness and stability, the reader is referred to Svård and Nordström (2014).

3. The SBP-SAT-MPT technique on a model problem

We start by illustrating the SBP-SAT-MPT technique on the advection equation,

$$u_t + au_x = 0. \tag{7}$$

In (7), $a > 0$ is a constant and the domain under consideration is $x \in [0, 1], t \geq 0$. Unless stated otherwise, we use subscripts to denote partial derivatives, i.e. $u_x = \partial u / \partial x$ and $u_t = \partial u / \partial t$ in (7).

For determining well-posed boundary conditions to (7), we multiply with u and integrate over the spatial domain,

$$\int_0^1 uu_t dx + a \int_0^1 uu_x dx = \frac{1}{2} \int_0^1 (u^2)_t dx + \frac{a}{2} \int_0^1 (u^2)_x dx = 0. \tag{8}$$

91 By defining the standard L_2 -norm $\|u\| = \sqrt{\int_0^1 u^2 dx}$ and imposing the bound-
 92 ary condition $u(0, t) = g(t)$, where $g(t)$ is known boundary data, (8) becomes
 93

$$(\|u\|^2)_t = ag^2(t) - au^2(1, t). \quad (9)$$

94 After integrating (9) in time, we obtain

$$\|u\|^2 + a \int_0^t u^2(1, \tau) d\tau = \|f\|^2 + a \int_0^t g^2(\tau) d\tau.$$

95 Hence, one can conclude that $\|u\|$ satisfy an estimate of the form (3), and (7)
 96 with the boundary condition $u(0, t) = g(t)$ is therefore strongly well-posed.

97 3.1. The semi-discrete problem

98 Equation (7) is discretized in space using the SBP finite difference oper-
 99 ator $\mathbf{D} = \mathbf{P}^{-1}\mathbf{Q}$:

$$\mathbf{v}_t + a\mathbf{P}^{-1}\mathbf{Q}\mathbf{v} = \alpha_0\mathbf{P}^{-1}(v_0 - g(t))\tilde{\mathbf{e}}_0, \quad (10)$$

100 where $\tilde{\mathbf{e}}_0 = (1, 0, \dots, 0)$, α_0 is a penalty coefficient to be determined and
 101 $\mathbf{v}$ is the numerical approximation of u . The matrix $\mathbf{P}$ is symmetric and
 102 positive definite, and the matrix $\mathbf{Q}$ satisfies the SBP property: $\mathbf{Q} + \mathbf{Q}^T =$
 103 $\text{diag}(-1, 0, \dots, 0, 1)$. The SAT-term on the right-hand side of (10) enforces
 104 the boundary condition $u(0, t) = g(t)$ weakly; see Carpenter et al. (1994) for
 105 more details of this technique.

106 By multiplying (10) with $\mathbf{v}^T\mathbf{P}$ from the left, one obtains

$$\mathbf{v}^T\mathbf{P}\mathbf{v}_t + a\mathbf{v}^T\mathbf{Q}\mathbf{v} = \alpha_0 v_0(v_0 - g(t)). \quad (11)$$

107 The multiplication with $\mathbf{v}^T\mathbf{P}$ from the left is analogous to multiplication of
 108 u and integration over the domain in the continuous setting. Adding the
 109 transpose of (11) to itself and using the SBP property of $\mathbf{Q}$ results in

$$(\|\mathbf{v}\|_{\mathbf{P}}^2)_t = ag^2(t) - av_N^2 - a(v_0 - g(t))^2, \quad (12)$$

110 where we have chosen $\alpha_0 = -a$ and defined $\|\mathbf{v}\|_{\mathbf{P}}^2 = \mathbf{v}^T\mathbf{P}\mathbf{v}$. After integrating
 111 (12) in time, one can clearly see that $\|\mathbf{v}\|$ satisfies an estimate of the form
 112 (6), and (10) is therefore strongly stable. Note also that (12) mimics (9) if
 113 $v_0 = g(t)$.

114 As the boundary condition is imposed weakly with the SAT, i.e. v_0 is
 115 allowed to slightly deviate from $g(t)$, additional damping is introduced that
 116 stabilizes the numerical scheme. For more details of the SBP finite difference
 117 operators, the reader is referred to Strand (1994) and Svärd and Nordström
 118 (2014).

Figure 1: Illustration of the mesh in one space dimension. The circles denotes the grid points and filled circles denotes grid points where SAT's are applied.

3.2. The multiple penalty technique

Consider the problem (7) and assume that data can be obtained at several grid points anywhere in the domain, see Figure 1 for an illustration. We denote the set of grid points where additional data is known as Ω_M .

By applying penalty terms (SAT's) wherever data is available, (10) can be extended to

$$\begin{aligned} \mathbf{v}_t + a\mathbf{P}^{-1}\mathbf{Q}\mathbf{v} &= \alpha_0\mathbf{P}^{-1}(v_0 - g(t))\tilde{\mathbf{e}}_0 \\ &+ \sum_{i \in \Omega_M} \alpha_i\mathbf{P}^{-1}(v_i - g_i(t))\tilde{\mathbf{e}}_i. \end{aligned} \quad (13)$$

In (13), α_i are additional penalty coefficients to be determined, $g_i(t)$ are known data, and $\tilde{\mathbf{e}}_i$ are projection vectors at grid point i . That is, all elements of $\tilde{\mathbf{e}}_i$ are zero, except the element i , which is one. As an example, consider $\tilde{\mathbf{e}}_3 = [0, 0, 1, 0, \dots, 0]^T$. The additional SAT's in (13) are referred to as *multiple penalty terms*.

Stability is obtained in the same way as in section 3.1, i.e. (13) is multiplied with $\mathbf{v}^T\mathbf{P}$ from the left, and the transpose of the outcome is added to the expression,

$$\begin{aligned} (||\mathbf{v}||_{\mathbf{P}}^2)_t &= ag^2(t) - a(v_0 - g(t))^2 - av_N^2 \\ &+ 2 \sum_{i \in \Omega_M} \alpha_i v_i (v_i - g_i(t)) \end{aligned} \quad (14)$$

where we have used $\alpha_0 = -a$. Equation (14) can be rearranged into

$$\begin{aligned} (||\mathbf{v}||_{\mathbf{P}}^2)_t &= ag^2(t) - a(v_0 - g(t))^2 - av_N^2 \\ &+ \sum_{i \in \Omega_M} \alpha_i (v_i - g_i)^2 + \alpha_i (v_i^2 - g_i^2). \end{aligned} \quad (15)$$

After integrating (15) in time, one can observe that $||\mathbf{v}||$ satisfies an estimate of the form (6) if $\alpha_i \leq 0$, and the scheme is therefore strongly stable for these values of α_i . Note also that if $g_i = v_i$, (15) will mimic the continuous

138 estimate (9). The parameters α_i can be used to alter the scheme in different
 139 ways (while maintaining stability). For example, one can use the MPT to
 140 construct buffer zones to reduce reflections, increase the order of accuracy
 141 or, as illustrated also in this paper, increase the rate of convergence, see
 142 Nordström et al. (2014).

143 3.3. A numerical example

144 Consider a slightly altered version of (7),

$$\begin{aligned} u_t + au_x &= F(x, t) \\ u(0, t) &= g(t) \\ u(x, 0) &= 0 \end{aligned} \tag{16}$$

145 where $F(x, t)$ is a known forcing function and $a = 1$. By choosing $F(x, t) =$
 146 $-2\pi\sin(2\pi x)$ and $g(t) = 1$, the exact steady-state solution to (16) becomes
 147 $u_{SS}(x) = \cos(2\pi x)$. This technique is known as the method of manufactured
 148 solutions (Lindström and Nordström (2010)). In this test case, we assume
 149 that we know the solution in the domains $x \in [0, 0.2]$ and $x \in [0.8, 1]$. We can
 150 then apply the MPT by setting the data $g_i = u_{SS}(x_i)$ at the corresponding
 151 grid points. The aim is to get a better approximation in the region $x \in$
 152 $[0.2, 0.8]$, where the solution is unknown.

153 We now employ the SBP-SAT technique described above to (16) and mea-
 154 sure the deviation from the steady-state solution, u_{SS} . Additional penalty
 155 terms are applied in the domains $x \in [0, 0.2]$ and $x \in [0.8, 1]$ and the number
 156 of grid points is $N = 40$. For simplicity, we choose $\alpha_i = -1$. A fourth order
 157 explicit Runge Kutta is used to integrate in time and the error at the final
 158 time $T = 5$ is calculated as $\|\mathbf{e}\|_{\mathbf{P}}^2 = (\mathbf{v} - \mathbf{u}_{SS})^T \mathbf{P} (\mathbf{v} - \mathbf{u}_{SS})$. However, the
 159 following results would be similar in any L_2 -equivalent norm. As illustrated
 160 in Figure 2, the error decrease as the MPT is applied, also at grid points in
 161 the domain $x \in [0.2, 0.8]$, where no data is available.

162 4. The shallow water equations

163 Consider the inviscid shallow water equations,

$$\begin{aligned} \bar{\mathbf{V}}_t + \bar{\mathbf{V}} \nabla \cdot \bar{\mathbf{V}} + f \mathbf{e}_3 \times \bar{\mathbf{V}} + g \nabla h &= 0 \\ h_t + \bar{\mathbf{V}} \cdot \nabla h + h \nabla \cdot \bar{\mathbf{V}} &= 0 \end{aligned} \tag{17}$$

Figure 2: The error at $T = 5$ when using standard and multiple penalties. $N = 40$

164 where $\bar{\mathbf{V}}$ is the velocity vector, f the Coriolis parameter, h the height and
 165 g the gravitational constant. The vector $\mathbf{e}_3 = [0, 0, 1]^T$ is the unit vector in
 166 the z -direction. In this work, we consider (17) linearized around a reference
 167 state $\mathbf{u}_0 = [\bar{u}, \bar{v}, H]^T$, in which H is the reference height and $\bar{u}, \bar{v}$ reference
 168 state velocities. More details about linearizing and symmetrizing the shallow
 169 water equations is provided by Ghader and Nordström (2014).

170 Consider the linearized and symmetrized problem

$$\begin{aligned}
 \mathbf{u}_t + \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y + \mathbf{C}\mathbf{u} &= \mathbf{F}(x, y, t) \\
 \mathbf{L}_w \mathbf{u}(0, y, t) &= \mathbf{g}_w(y, t) \\
 \mathbf{L}_e \mathbf{u}(1, y, t) &= \mathbf{g}_e(y, t) \\
 \mathbf{L}_s \mathbf{u}(x, 0, t) &= \mathbf{g}_s(x, t) \\
 \mathbf{L}_n \mathbf{u}(x, 1, t) &= \mathbf{g}_n(x, t) \\
 \mathbf{u}(x, y, 0) &= \mathbf{f}(x, y),
 \end{aligned} \tag{18}$$

171 where the domain under consideration is $x, y \in [0, 1], t \geq 0$. This domain
 172 constitutes a local domain inserted in a global one, and the boundary data
 173 has been obtained from calculations using the global mesh, see Figure 3 for

Figure 3: An illustration of the problem set-up. The calculations are performed in the central domain where boundary data has been interpolated from the coarse, global domain. The notations w, e, n, s refers to the west, east, north and south boundary, respectively. The central domain has its corners at $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$.

174 an illustration. The solution to (18) is

$$\mathbf{u} = [u', v', \frac{gh'}{\bar{c}}]^T$$

175 where g is the gravitational constant and u', v' are the deviation from $\bar{u}, \bar{v}$,
176 respectively. The variable h' denotes the deviation from H , and $\mathbf{g}_{w,e,s,n}, \mathbf{f}$
177 are given boundary and initial data. For completeness, a forcing function
178 $\mathbf{F}(x, y, t)$ is added to the right-hand side. In (18),

$$\mathbf{A} = \begin{bmatrix} \bar{u} & 0 & \bar{c} \\ 0 & \bar{u} & 0 \\ \bar{c} & 0 & \bar{u} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \bar{v} & 0 & 0 \\ 0 & \bar{v} & \bar{c} \\ 0 & \bar{c} & \bar{v} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

179 where $\bar{c} = \sqrt{gH}$ is the gravity wave speed and the matrix $\mathbf{C}$ describes the
180 Coriolis force acting on the system. In this work, we focus on sub-critical
181 flows, $\bar{u}^2 + \bar{v}^2 < \bar{c}^2$.

182 *4.1. Well-posedness*

183 The boundary operators $\mathbf{L}_{\mathbf{w},\mathbf{e},\mathbf{s},\mathbf{n}}$ must be determined such that (18) is
 184 well-posed. For clarity, the forcing function $\mathbf{F}(x, y, t)$ is neglected in the
 185 following analysis.

186 The technique in section 3.1 is applied to problem (18); that is, (18) is
 187 multiplied with $\mathbf{u}^T$ from the left and integrated over the spatial domain. One
 188 obtains,

$$(\|\mathbf{u}\|^2)_t = - \int_{\partial\Omega} \mathbf{u}^T (\alpha \mathbf{A} + \beta \mathbf{B}) \mathbf{u} ds, \quad (19)$$

189 where we have employed Greens theorem and defined $\|\mathbf{u}\|^2 = \int_{\Omega} \mathbf{u}^T \mathbf{u} d\Omega$. In
 190 (19), $\hat{\mathbf{n}} = [dy, -dx]^T / \sqrt{dx^2 + dy^2} = [\alpha, \beta]$ is the outward pointing normal to
 191 the domain. The term $ds = \sqrt{dx^2 + dy^2}$ is the differential along the bound-
 192 ary $\partial\Omega$ to the domain Ω . Since $\mathbf{C}$ is skew-symmetric, the Coriolis force does
 193 not affect the estimate (19).

194 By considering the domain shown in Figure 3, (19) can be written out
 195 explicitly for each separate part of the boundary,

$$\begin{aligned} \|\mathbf{u}\|_t^2 = & - \oint_{\partial\Omega} [\mathbf{u}^T \mathbf{A} \mathbf{u}, \mathbf{u}^T \mathbf{B} \mathbf{u}] \cdot \hat{\mathbf{n}} ds = \int_{x=0}^1 \mathbf{u}(x, 0, t)^T \mathbf{B} \mathbf{u}(x, 0, t) dx \\ & - \int_{x=0}^1 \mathbf{u}(x, 1, t)^T \mathbf{B} \mathbf{u}(x, 1, t) dx + \int_{y=0}^1 \mathbf{u}(0, y, t)^T \mathbf{A} \mathbf{u}(0, y, t) dy \\ & - \int_{y=0}^1 \mathbf{u}(1, y, t)^T \mathbf{A} \mathbf{u}(1, y, t) dy. \end{aligned} \quad (20)$$

196 All boundary operators that bound the right-hand side of (20) result in an
 197 energy estimate, and well-posedness follows if the correct number of boundary
 198 conditions is used (Svrd and Nordstrm (2014)).

199 The boundary operators used in this work are

$$\begin{aligned} \mathbf{L}_{\mathbf{w}} &= \mathbf{S}_{\mathbf{A}} \mathbf{\Lambda}_{\mathbf{A}}^+ \mathbf{S}_{\mathbf{A}}^T = \mathbf{A}^+ \\ \mathbf{L}_{\mathbf{s}} &= \mathbf{S}_{\mathbf{B}} \mathbf{\Lambda}_{\mathbf{B}}^+ \mathbf{S}_{\mathbf{B}}^T = \mathbf{B}^+ \\ \mathbf{L}_{\mathbf{e}} &= \mathbf{S}_{\mathbf{A}} \mathbf{\Lambda}_{\mathbf{A}}^- \mathbf{S}_{\mathbf{A}}^T = \mathbf{A}^- \\ \mathbf{L}_{\mathbf{n}} &= \mathbf{S}_{\mathbf{B}} \mathbf{\Lambda}_{\mathbf{B}}^- \mathbf{S}_{\mathbf{B}}^T = \mathbf{B}^-, \end{aligned} \quad (21)$$

200 and $\mathbf{\Lambda}_{\mathbf{A},\mathbf{B}}^{\pm}$ are the positive and negative part of the eigenvalue matrices, re-

201 spectively. The matrices

$$\mathbf{S}_\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{S}_\mathbf{B} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

202 are the similarity transformations that diagonalizes $\mathbf{A}, \mathbf{B}$. We can of course
 203 handle any type of flow field, but to exemplify we consider the case $\bar{u} > \bar{v} > 0$
 204 and $\bar{u}^2 + \bar{v}^2 < \bar{c}^2$. Under these circumstances, the operators in (21) are

$$\begin{aligned} \mathbf{L}_\mathbf{w} = \mathbf{A}^+ &= \frac{1}{2} \begin{bmatrix} \bar{u} + \bar{c} & 0 & \bar{u} + \bar{c} \\ 0 & 2\bar{u} & 0 \\ \bar{u} + \bar{c} & 0 & \bar{u} + \bar{c} \end{bmatrix}, & \mathbf{L}_\mathbf{s} = \mathbf{B}^+ &= \frac{1}{2} \begin{bmatrix} 2\bar{v} & 0 & 0 \\ 0 & \bar{v} + \bar{c} & \bar{v} + \bar{c} \\ 0 & \bar{v} + \bar{c} & \bar{v} + \bar{c} \end{bmatrix} \\ \mathbf{L}_\mathbf{e} = \mathbf{A}^- &= \frac{1}{2} \begin{bmatrix} \bar{u} - \bar{c} & 0 & \bar{c} - \bar{u} \\ 0 & 0 & 0 \\ \bar{c} - \bar{u} & 0 & \bar{u} - \bar{c} \end{bmatrix}, & \mathbf{L}_\mathbf{n} = \mathbf{B}^- &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{v} - \bar{c} & \bar{c} - \bar{v} \\ 0 & \bar{c} - \bar{v} & \bar{v} - \bar{c} \end{bmatrix}. \end{aligned}$$

205 By using the operators (21), the boundary conditions of (18) becomes

$$\begin{aligned} \mathbf{A}^+ \mathbf{u}(0, y, t) &= \mathbf{g}_\mathbf{w}(y, t) \\ \mathbf{A}^- \mathbf{u}(1, y, t) &= \mathbf{g}_\mathbf{e}(y, t) \\ \mathbf{B}^+ \mathbf{u}(x, 0, t) &= \mathbf{g}_\mathbf{s}(x, t) \\ \mathbf{B}^- \mathbf{u}(x, 1, t) &= \mathbf{g}_\mathbf{n}(x, t). \end{aligned} \tag{22}$$

206 The boundary conditions (22) inserted into (20), leads to

$$\begin{aligned} \|\mathbf{u}\|_t^2 &= \int_{x=0}^1 \mathbf{g}_\mathbf{s}^T ([\mathbf{B}^{-1}]_+) \mathbf{g}_\mathbf{s} dx + \int_{x=0}^1 \mathbf{u}(x, 0, t)^T \mathbf{B}^- \mathbf{u}(x, 0, t) dx \\ &\quad - \int_{x=0}^1 \mathbf{g}_\mathbf{n}^T ([\mathbf{B}^{-1}]_-) \mathbf{g}_\mathbf{n} dx - \int_{x=0}^1 \mathbf{u}(x, 1, t)^T \mathbf{B}^+ \mathbf{u}(x, 1, t) dx \\ &\quad + \int_{y=0}^1 \mathbf{g}_\mathbf{w}^T ([\mathbf{A}^{-1}]_+) \mathbf{g}_\mathbf{w} dy + \int_{y=0}^1 \mathbf{u}(0, y, t)^T \mathbf{A}^- \mathbf{u}(0, y, t) dx \\ &\quad - \int_{y=0}^1 \mathbf{g}_\mathbf{e}^T ([\mathbf{A}^{-1}]_-) \mathbf{g}_\mathbf{e} dy - \int_{y=0}^1 \mathbf{u}(1, y, t)^T \mathbf{A}^+ \mathbf{u}(1, y, t) dx \end{aligned} \tag{23}$$

207 where $[\mathbf{A}^{-1}]_\pm$ and $[\mathbf{B}^{-1}]_\pm$ denotes the positive and negative part of the inverse
 208 of $\mathbf{A}$ and $\mathbf{B}$, respectively. Since $\mathbf{A}^+, \mathbf{B}^+ \geq 0$ and $\mathbf{A}^-, \mathbf{B}^- \leq 0$, (23) states
 209 that $\|\mathbf{u}\|_t$ is bounded by data. After integrating (23) in time, $\|\mathbf{u}\|$ will

Table 1: The number of boundary conditions at each side of the unit square for the shallow water equations when considering the case $\bar{u} > 0, \bar{v} > 0$ and $\bar{u}^2 + \bar{v}^2 < \bar{c}$.

Boundary	South	East	North	West
Boundary conditions	2	1	1	2

210 satisfy an estimate of the form (3). Since we have used the correct number
 211 of boundary conditions, we can summarize the results of this section in the
 212 following proposition.

213 **Proposition 1.** *The problem (18) with $\bar{u} > 0, \bar{v} > 0$ and $\bar{u}^2 + \bar{v}^2 < \bar{c}$ is*
 214 *strongly well-posed if*

$$\mathbf{L}_{\mathbf{w},\mathbf{e}} = \mathbf{A}^\pm, \quad \mathbf{L}_{\mathbf{s},\mathbf{n}} = \mathbf{B}^\pm.$$

215 **Remark 1.** *The number of boundary conditions at each boundary is equal*
 216 *to the rank of the corresponding boundary operator. For example, at the*
 217 *northern boundary, we have*

$$\mathbf{B}^- \mathbf{u}(x, 1, t) = \mathbf{g}_n \iff (\bar{v} - \bar{c})v' + (\bar{c} - \bar{v})\frac{gh'}{\bar{c}} = g_n(x, t) \quad (24)$$

218 where $g_n(x, t)$ is the boundary data. According to (24), one boundary condi-
 219 tion is applied at $y = 1$. The number of boundary conditions at each boundary
 220 is summarized in Table 1.

221 **Remark 2.** *With different signs of $\bar{u}, \bar{v}$, the boundary operators will change.*
 222 *For example, if $\bar{u} < 0$, we need $\mathbf{L}_{\mathbf{w}} = \mathbf{A}^-$, $\mathbf{L}_{\mathbf{e}} = \mathbf{A}^+$.*

223 4.2. The semi-discrete formulation

224 The semi-discrete approximation of (18) with the boundary operators
 225 (21) using the SBP-SAT technique is

$$\begin{aligned} & \mathbf{v}_t + (\mathbf{P}_{\mathbf{x}}^{-1} \mathbf{Q}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}} \otimes \mathbf{A}) \mathbf{v} + (\mathbf{I}_{\mathbf{x}} \otimes \mathbf{P}_{\mathbf{y}}^{-1} \mathbf{Q}_{\mathbf{y}} \otimes \mathbf{B}) \mathbf{v} + (\mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}} \otimes \mathbf{C}) \mathbf{v} \\ &= (\mathbf{P}_{\mathbf{x}}^{-1} \mathbf{E}_0 \otimes \mathbf{P}_{\mathbf{y}}^{-1} \otimes \mathbf{I}_3) \Sigma_w (\bar{\mathbf{A}}^+ \mathbf{v} - \mathbf{g}_w) \\ &+ (\mathbf{P}_{\mathbf{x}}^{-1} \mathbf{E}_N \otimes \mathbf{P}_{\mathbf{y}}^{-1} \otimes \mathbf{I}_3) \Sigma_e (\bar{\mathbf{A}}^- \mathbf{v} - \mathbf{g}_e) \\ &+ (\mathbf{P}_{\mathbf{x}}^{-1} \otimes \mathbf{P}_{\mathbf{y}}^{-1} \mathbf{E}_0 \otimes \mathbf{I}_3) \Sigma_s (\bar{\mathbf{B}}^+ \mathbf{v} - \mathbf{g}_s) \\ &+ (\mathbf{P}_{\mathbf{x}}^{-1} \otimes \mathbf{P}_{\mathbf{y}}^{-1} \mathbf{E}_N \otimes \mathbf{I}_3) \Sigma_n (\bar{\mathbf{B}}^- \mathbf{v} - \mathbf{g}_n) \end{aligned} \quad (25)$$

where $\mathbf{P}_{\mathbf{x},\mathbf{y}}$ are symmetric and positive definite, and $\mathbf{Q}_{\mathbf{x},\mathbf{y}}$ satisfies the SBP-property,

$$\mathbf{Q}_{\mathbf{x},\mathbf{y}} + \mathbf{Q}_{\mathbf{x},\mathbf{y}}^T = \mathbf{B}_{\mathbf{x},\mathbf{y}} = -\mathbf{E}_0 + \mathbf{E}_N = \text{diag}(-1, 0, \dots, 0, 1).$$

The subscripts on $\mathbf{Q}$ and $\mathbf{P}$ in (25) indicate what derivative that is approximated. In (25), $\mathbf{I}_{\mathbf{x},\mathbf{y}}$ are identity matrices with the same size as $\mathbf{P}_{\mathbf{x},\mathbf{y}}$, respectively, $\mathbf{I}_3$ is a 3×3 identity matrix and $\mathbf{g}_{\mathbf{e},\mathbf{w},\mathbf{s},\mathbf{n}}$ are the boundary data.

The symbol $\otimes$ in (25) denotes the Kronecker product, which for two arbitrary matrices $\mathbf{C}$ and $\mathbf{D}$ is defined as

$$\mathbf{C} \otimes \mathbf{D} = \begin{bmatrix} c_{11}\mathbf{D} & c_{12}\mathbf{D} & \dots & c_{1m}\mathbf{D} \\ c_{21}\mathbf{D} & c_{22}\mathbf{D} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1}\mathbf{D} & \dots & \dots & c_{nm}\mathbf{D} \end{bmatrix}.$$

where c_{ij} are the elements of $\mathbf{C}$.

In (25), $\bar{\mathbf{A}}^\pm = \mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}} \otimes \mathbf{A}^\pm$ and $\bar{\mathbf{B}}^\pm = \mathbf{I}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}} \otimes \mathbf{B}^\pm$ and the matrices $\mathbf{E}_{0,N}$ are projection matrices, such that $(\mathbf{E}_0)_{11} = (\mathbf{E}_N)_{NN} = 1$ and zero otherwise. The penalty matrices $\Sigma_{w,s,e,n}$ will be determined such that stability is obtained and $\mathbf{g}_{\mathbf{w},\mathbf{e},\mathbf{s},\mathbf{n}}$ are the boundary data.

By multiplying (25) with $\mathbf{v}^T \mathbf{P}_{\mathbf{x}} \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{I}_3$ from the left and choosing $\Sigma_{w,e} = \alpha_{1,2} \mathbf{I}_{\mathbf{x}} \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{I}_3$, $\Sigma_{s,n} = \alpha_{3,4} \mathbf{P}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}} \otimes \mathbf{I}_3$ with $\alpha_{1,3} = -1$, $\alpha_{2,4} = 1$, one obtains

$$\begin{aligned} (||\mathbf{v}||_{\mathbf{P}_{\mathbf{x}} \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{I}_3}^2)_t = & -\mathbf{v}^T (\mathbf{E}_0 \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{I}_3) \bar{\mathbf{A}}^+ \mathbf{v} \\ & + 2\mathbf{v}^T (\mathbf{E}_0 \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{I}_3) \mathbf{g}_{\mathbf{w}} + \mathbf{v}^T (\mathbf{E}_0 \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{A}^-) \mathbf{v} \\ & + \mathbf{v}^T (\mathbf{E}_N \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{I}_3) \bar{\mathbf{A}}^- \mathbf{v} \\ & - 2\mathbf{v}^T (\mathbf{E}_N \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{I}_3) \mathbf{g}_{\mathbf{e}} - \mathbf{v}^T (\mathbf{E}_N \otimes \mathbf{P}_{\mathbf{y}} \otimes \mathbf{A}^+) \mathbf{v} \\ & - \mathbf{v}^T (\mathbf{P}_{\mathbf{x}} \otimes \mathbf{E}_0 \otimes \mathbf{I}_3) \bar{\mathbf{B}}^+ \mathbf{v} \\ & + 2\mathbf{v}^T (\mathbf{P}_{\mathbf{x}} \otimes \mathbf{E}_0 \otimes \mathbf{I}_3) \mathbf{g}_{\mathbf{s}} + \mathbf{v}^T (\mathbf{P}_{\mathbf{x}} \otimes \mathbf{E}_0 \otimes \mathbf{B}^-) \mathbf{v} \\ & + \mathbf{v}^T (\mathbf{P}_{\mathbf{x}} \otimes \mathbf{E}_N \otimes \mathbf{I}_3) \bar{\mathbf{B}}^- \mathbf{v} \\ & - 2\mathbf{v}^T (\mathbf{P}_{\mathbf{x}} \otimes \mathbf{E}_0 \otimes \mathbf{I}_4) \mathbf{g}_{\mathbf{n}} - \mathbf{v}^T (\mathbf{P}_{\mathbf{x}} \otimes \mathbf{E}_N \otimes \mathbf{B}^+) \mathbf{v}. \end{aligned} \quad (26)$$

Note that the Coriolis term vanish since $\mathbf{C}$ is skew-symmetric, as in the

242 continuous case. By completing the squares, (26) finally becomes

$$\begin{aligned}
& (\|\mathbf{v}\|_{\mathbf{P}_x \otimes \mathbf{P}_y \otimes \mathbf{I}_3}^2)_t = \\
& + \mathbf{g}_w^T (\mathbf{E}_0 \otimes \mathbf{P}_y \otimes [\mathbf{A}^{-1}]_+) \mathbf{g}_w + \mathbf{v}^T (\mathbf{E}_0 \otimes \mathbf{P}_y \otimes \mathbf{A}^-) \mathbf{v} \\
& - \mathbf{g}_e^T (\mathbf{E}_N \otimes \mathbf{P}_y \otimes [\mathbf{A}^{-1}]_-) \mathbf{g}_e - \mathbf{v}^T (\mathbf{E}_N \otimes \mathbf{P}_y \otimes \mathbf{A}^+) \mathbf{v} \\
& + \mathbf{g}_s^T (\mathbf{P}_x \otimes \mathbf{E}_0 \otimes [\mathbf{B}^{-1}]_+) \mathbf{g}_s + \mathbf{v}^T (\mathbf{P}_x \otimes \mathbf{E}_0 \otimes \mathbf{B}^-) \mathbf{v} \\
& - \mathbf{g}_n^T (\mathbf{P}_x \otimes \mathbf{E}_N \otimes [\mathbf{B}^{-1}]_-) \mathbf{g}_n - \mathbf{v}^T (\mathbf{P}_x \otimes \mathbf{E}_N \otimes \mathbf{B}^+) \mathbf{v} \quad (27) \\
& - (\bar{\mathbf{A}}^+ \mathbf{v} - \mathbf{g}_w)^T (\mathbf{E}_0 \otimes \mathbf{P}_y \otimes [\mathbf{A}^{-1}]_+) (\bar{\mathbf{A}}^+ \mathbf{v} - \mathbf{g}_w) \\
& + (\bar{\mathbf{A}}^- \mathbf{v} - \mathbf{g}_e)^T (\mathbf{E}_N \otimes \mathbf{P}_y \otimes [\mathbf{A}^{-1}]_-) (\bar{\mathbf{A}}^- \mathbf{v} - \mathbf{g}_e) \\
& - (\bar{\mathbf{B}}^+ \mathbf{v} - \mathbf{g}_s)^T (\mathbf{P}_x \otimes \mathbf{E}_0 \otimes [\mathbf{B}^{-1}]_+) (\bar{\mathbf{B}}^+ \mathbf{v} - \mathbf{g}_s) \\
& + (\bar{\mathbf{B}}^- \mathbf{v} - \mathbf{g}_n)^T (\mathbf{P}_x \otimes \mathbf{E}_N \otimes [\mathbf{B}^{-1}]_-) (\bar{\mathbf{B}}^- \mathbf{v} - \mathbf{g}_n).
\end{aligned}$$

243 In (27), $[\mathbf{A}^{-1}]_\pm$ and $[\mathbf{B}^{-1}]_\pm$ refers to the negative and positive parts of the
 244 inverses of $\mathbf{A}$ and $\mathbf{B}$, respectively.

245 The matrices $[\mathbf{A}^{-1}]_+$, $[\mathbf{B}^{-1}]_+$, $\mathbf{A}^+$ and $\mathbf{B}^+$ are all positive semi definite,
 246 while the matrices $[\mathbf{A}^{-1}]_-$, $[\mathbf{B}^{-1}]_-$, $\mathbf{A}^-$ and $\mathbf{B}^-$ are all negative semi definite.
 247 This implies that the only terms that are non-negative are the ones contain-
 248 ing data. Hence, the growth of $\mathbf{v}$ is bounded by applied data. After time-
 249 integration, (27) will satisfy an estimate of the form (6), and (25) is therefore
 250 strongly stable. Note that the terms $(\mathbf{g}_{w,e})^T (\mathbf{E}_{0,N} \otimes \mathbf{P}_y \otimes [\mathbf{A}^{-1}]_\pm) \mathbf{g}_{w,e}$ and
 251 the terms $(\mathbf{g}_{s,n})^T (\mathbf{P}_x \otimes \mathbf{E}_{0,N} \otimes [\mathbf{A}^{-1}]_\pm) \mathbf{g}_{s,n}$ in (27) mimics the integrals in
 252 (23) containing the boundary data, and $\mathbf{v}^T (\mathbf{E}_{0,N} \otimes \mathbf{P}_y \otimes \mathbf{A}^\pm) \mathbf{v}$ and $\mathbf{v}^T (\mathbf{P}_x \otimes \mathbf{E}_{0,N} \otimes \mathbf{B}^\pm) \mathbf{v}$
 253 mimics the remaining integrals. Hence, (27) mimics the continuous energy
 254 estimate, given by (23), if the small damping terms are neglected.

255 We summarize the results of this section in the following proposition.

256 **Proposition 2.** *The semi-discrete approximation (25) with*

$$\begin{aligned}
\Sigma_{w,e} &= \alpha_{1,2} \mathbf{I}_x \otimes \mathbf{P}_y \otimes \mathbf{I}_3 \\
\Sigma_{s,n} &= \alpha_{3,4} \mathbf{P}_x \otimes \mathbf{I}_y \otimes \mathbf{I}_3
\end{aligned}$$

257 *is strongly stable if $\alpha_{1,3} = -1$ and $\alpha_{2,4} = 1$.*

258 4.3. The multiple penalty technique

259 Next, we move on to the MPT where additional SAT's are applied at
 260 several grid points in the domain. We denote the set of these grid points

261 Ω_M . To ease the derivations, we consider the error equation corresponding
 262 to (25) with additional penalty terms.

263 By inserting the analytic solution, $\mathbf{u}$, into (25) and subtracting the result
 264 from (25) with the numerical solution $\mathbf{v}$, one obtains the error equation

$$\begin{aligned}
 & \mathbf{e}_t + (\mathbf{P}_x^{-1} \mathbf{Q}_x \otimes \mathbf{I}_y \otimes \mathbf{A}) \mathbf{e} + (\mathbf{I}_x \otimes \mathbf{P}_y^{-1} \mathbf{Q}_y \otimes \mathbf{B}) \mathbf{e} = \\
 & \quad - (\mathbf{P}_x^{-1} \mathbf{E}_0 \otimes \mathbf{I}_y \otimes \mathbf{I}_3) \bar{\mathbf{A}}^+ \mathbf{e} \\
 & \quad + (\mathbf{P}_x^{-1} \mathbf{E}_N \otimes \mathbf{I}_y \otimes \mathbf{I}_3) \bar{\mathbf{A}}^- \mathbf{e} \\
 & \quad - (\mathbf{I}_x \otimes \mathbf{P}_y^{-1} \mathbf{E}_0 \otimes \mathbf{I}_3) \bar{\mathbf{B}}^+ \mathbf{e} \\
 & \quad + (\mathbf{I}_x \otimes \mathbf{P}_y^{-1} \mathbf{E}_N \otimes \mathbf{I}_3) \bar{\mathbf{B}}^- \mathbf{e} + \\
 & \quad + \sum_{i \in \Omega_M} (\mathbf{P}_x^{-1} \mathbf{E}_i \otimes \mathbf{I}_y \otimes \Sigma^i) \mathbf{L}_i^{\text{MP}} \mathbf{e} + \mathbf{T} \mathbf{e},
 \end{aligned} \tag{28}$$

265 where $\mathbf{e} = \mathbf{v} - \mathbf{u}$ is the error and $\mathbf{T} \mathbf{e}$ is the truncation error. The terms
 266 $(\mathbf{P}_x^{-1} \mathbf{E}_i \otimes \mathbf{I}_y \otimes \Sigma^i) \mathbf{L}_i^{\text{MP}} \mathbf{e}$ are the MPT's in Ω_M . The elements of $\mathbf{E}_i$ are
 267 $(\mathbf{E}_i)_{ii} = 1$ and zero otherwise. In (28), we have used the penalty matrices in
 268 Proposition 2 at the physical boundaries. The operators $\mathbf{L}_i^{\text{MP}}$ are arbitrary
 269 boundary operators in Ω_M , and Σ^i are penalty matrices to be determined
 270 such that the scheme is stable. To ease the derivations, we neglect the trun-
 271 cation error $\mathbf{T} \mathbf{e}$. It can be shown that neglecting the truncation error will
 272 not affect the following results.

273 By multiplying (28) with $\mathbf{e}^T (\mathbf{P}_x \otimes \mathbf{P}_y \otimes \mathbf{I}_3)$ from the left and adding the
 274 transpose of the outcome to itself, one arrives at

$$\|\mathbf{e}\|_t^2 \leq \sum_{i \in \Omega_M} \mathbf{e}^T \mathbf{E}_i \otimes \mathbf{P}_y \otimes [\Sigma^i \mathbf{L}_i^{\text{MP}} + (\Sigma^i \mathbf{L}_i^{\text{MP}})^T] \mathbf{e}, \tag{29}$$

275 since the remaining terms only contribute with a decay of energy, as con-
 276 cluded in section 4.2. Every choice of Σ^i that makes the right-hand side of
 277 (29) non-positive results in a decay of the error, and a stable scheme. Since
 278 $\mathbf{P}_y$ is positive definite and $\mathbf{E}_i$ is positive semi-definite, the right-hand side of
 279 (29) is non-positive if we choose Σ^i such that $\Sigma^i \mathbf{L}_i^{\text{MP}} + (\Sigma^i \mathbf{L}_i^{\text{MP}})^T \leq 0$.

280 We summarize the results in the following proposition.

281 **Proposition 3.** *The MPT applied in (28) preserves stability if*

$$\Sigma^i \mathbf{L}_i^{\text{MP}} + (\Sigma^i \mathbf{L}_i^{\text{MP}})^T \leq 0 \tag{30}$$

282 where $\mathbf{L}_i^{\text{MP}}$ are arbitrarily chosen boundary operators.

283 In this work, we apply the MPT close to the boundaries and use the operators
 284 $\mathbf{L}_{\mathbf{w},\mathbf{e}}^{\mathbf{MP}} = \bar{\mathbf{A}}^\pm$ and $\mathbf{L}_{\mathbf{s},\mathbf{n}}^{\mathbf{MP}} = \bar{\mathbf{B}}^\pm$, where w, e, s, n denotes the boundary where the
 285 MPT is applied. The penalty matrices are $\Sigma_{e,w}^i = \pm \mathbf{I}_3$ and $\Sigma_{n,s}^i = \pm \mathbf{I}_3$. One
 286 can easily confirm that these choices satisfy (30).

287 In (28), we have assumed that data is available in the domain Ω_M . When
 288 considering local area models, as illustrated in Figure 3, the data in Ω_M is
 289 obtained by interpolating the results from a global calculation on a coarse
 290 mesh. In the general case, however, the required data can be obtained in
 291 several ways, for example by measurements.

292 **Remark 3.** *It is easy to show that (30) is satisfied if $\Sigma^i = \gamma(\mathbf{L}_i^{\mathbf{MP}})^\mathbf{T}$ for*
 293 *any constant $\gamma \leq 0$. Hence, one can always find penalty matrices that fulfills*
 294 *Proposition 3.*

295 **Remark 4.** *As illustrated in section 3 and (29), the MPT contributes with*
 296 *additional damping terms. This accelerates the rate of convergence, even at*
 297 *grid points where no data is available.*

298 5. Numerical results

299 In our first steady-state test case, we consider (18) with the reference
 300 state $\bar{u} = \bar{v} = 1 \text{ ms}^{-1}$, $\bar{c} = 2 \text{ ms}^{-1}$ and $g = 9.82 \text{ ms}^{-2}$. The following
 301 results are obtained for sub-critical flows. Similar results are obtained for
 302 super-critical flows, as illustrated in Appendix A.

303 The local area calculation is modeled by considering a standing wave of
 304 the form $\mathbf{u}(x, y) = \sin(2\pi x)[1, 1, 1]^T$ that we consider coming from a global
 305 calculation. This solution is obtained by choosing $\mathbf{F}(x, y, t)$ and $\mathbf{g}_{\mathbf{w},\mathbf{e},\mathbf{s},\mathbf{n}}$ ac-
 306 cordingly in (18) and adding random errors to it. At $t = 0$, we consider the
 307 wave to be poorly resolved, as shown to the right in Figure 4, and we wish
 308 to get a better resolution by applying the MPT close to the boundaries.

309 Figures 5-7 show the deviation from the steady-state solution using $N_{MP} =$
 310 rN (r is a number between zero and one) additional penalty terms in the
 311 vicinity of each boundary with $N = 20, 40, 80$ grid points in each space direc-
 312 tion. At grid point i, j in the penalty domains, we use data from the steady-
 313 state solution, i.e. $g(x_i, y_j) = \sin(2\pi x_i)[1, 1, 1]^T$, when applying the MPT.
 314 For the spatial discretization, we use a third order accurate SBP scheme.
 315 The lateral boundary operators used are $\mathbf{L}_{\mathbf{w},\mathbf{e}}^{\mathbf{MP}} = \bar{\mathbf{A}}^\pm$ and $\mathbf{L}_{\mathbf{s},\mathbf{n}}^{\mathbf{MP}} = \bar{\mathbf{B}}^\pm$ where
 316 w, e, s, n denotes the boundary where the MPT is applied. We use the penalty

Figure 4: Right: the unresolved solution at $t = 0$ modeled by adding random errors at each grid point. Left: the solution at steady-state.

317 matrices $\Sigma_{e,w}^i = \pm \mathbf{I}_3$ and $\Sigma_{n,s}^i = \pm \mathbf{I}_3$ to preserve stability. As one can see,
 318 the error levels are greatly reduced when the MPT is applied.

319 5.1. Time-dependent solutions

320 For the second time-dependent test case, we let the disturbance in height,
 321 h' , vary as an oscillating pulse of the form

$$h' = \exp(-10r^2)\sin[2\pi(r^2 - t)]$$

322 where $r^2 = (x - 0.5)^2 + (y - 0.5)^2$. The oscillating pulse is manufactured
 323 by choosing the boundary data and $\mathbf{F}(x, y, t)$ in (18) accordingly. We ap-
 324 ply the MPT at $r = N_{MP}/N$ additional grid points in the vicinity of each
 325 boundary. The applied data in the penalty regions is $g(x_i, y_j) = h'(x_i, y_j) =$
 326 $\exp(-10r_{i,j}^2)\sin[2\pi(r_{i,j}^2 - t)]$, where $r_{i,j}^2 = (x_i - 0.5)^2 + (y_j - 0.5)^2$. The simu-
 327 lation starts from the initial condition $\mathbf{u} = 0$, and is then integrated in time
 328 using a fourth order explicit Runge-Kutta method. The solution at different
 329 time levels is displayed in Figure 8, and the deviation (or error) from the ex-
 330 act solution as a function of time is shown in Figure 9-11. As one can see, the
 331 MPT speeds up the rate of convergence also for time-dependent problems.

Figure 5: The error as function of time for the shallow water equations. $N = 20$ and $r = N_{MP}/N$.

Figure 6: The error as function of time for the shallow water equations. $N = 40$ and $r = N_{MP}/N$.

Figure 7: The error as function of time for the shallow water equations. $N = 80$ and $r = N_{MP}/N$.

Figure 8: The computed time-dependent solution of h' at different time levels. Upper left: $t = 0$. Upper right: $t = 0.5$. Lower left: $t = 0.75$. Lower right: $t = 1.15$.

Figure 9: The error of h' as function of time for the shallow water equations using time-dependent boundary data. $N = 20$ and $r = N_{MP}/N$.

Figure 10: The error of h' as function of time for the shallow water equations using time-dependent boundary data. $N = 40$ and $r = N_{MP}/N$.

Figure 11: The error of h' as function of time for the shallow water equations using time-dependent boundary data. $N = 80$ and $r = N_{MP}/N$.

332 **Remark 5.** *All numerical experiments in this work are performed using a*
333 *third order accurate SBP scheme. A higher order of accuracy has only a*
334 *small effect on the rate of convergence. The increased rate of convergence is*
335 *mainly due to the additional damping terms in (29).*

336 6. Summary and conclusions

337 A new lateral boundary procedure (MPT) similar to the one introduced
338 by Davies (1976) has been applied to the linearized shallow water equations.
339 The technique builds on a generalization of the SBP-SAT technique.

340 If well-posed boundary conditions are available, the shallow water equa-
341 tions can be discretized using the SBP-SAT technique, such that the resulting
342 scheme is stable. It was shown how additional SAT's can be applied inside
343 the domain, while maintaining stability. The form of the additional SAT's
344 is completely general, and stability can always be obtained by appropriate
345 choices of penalty matrices. All theoretical results of this paper are also gen-
346 eral, and can easily be extended to other partial differential equations, such
347 as the Euler and Navier-Stokes equations.

348 The technique is straight-forward to implement, very general and stability

is easily shown by energy methods. Existing techniques building on Davies relaxation can easily be made stable using the MPT technique.

The additional penalty terms increases the rate of convergence to the correct solution for both steady-state and time-dependent problems. In the cases considered in this work, the increase of the rate of convergence is significant.

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Appendix A.

Consider (18) with the reference state $\bar{u} = \bar{v} = 3/2 \text{ ms}^{-1}$, $\bar{c} = 2 \text{ ms}^{-1}$ and $g = 9.82 \text{ ms}^{-2}$, such that $\bar{u}^2 + \bar{v}^2 > \bar{c}^2$. As in Section 5, we construct a steady-state solution $\mathbf{u}(x, y) = \sin(2\pi x)[1, 1, 1]^T$ by choosing the forcing function and boundary data accordingly. The MPT is applied at $N_{MP} = rN$ grid points in the vicinity of each boundary. A SBP scheme with third order overall accuracy is used for the space discretization and a fourth order implicit Runge-Kutta is used to integrate in time. The simulation starts from a poorly resolved wave, as illustrated in Figure 4, and the error as function of time is displayed in Figure A.12. As one can see, the results are similar to the ones obtained in Section 5, where sub-critical flows were considered.

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Figure A.12: The error as function of time for the shallow water equations with supercritical flows. $N = 40$ and $r = N_{MP}/N$.

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