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ON THE NON-VANISHING PROPERTY FOR REAL ANALYTIC SOLUTIONS OF THE p-LAPLACE EQUATION

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ABSTRACT. By using a nonassociative algebra argument, we prove that $u \equiv 0$ is the only cubic homogeneous polynomial solution to the p-Laplace equation $\operatorname{div}|Du|^{p-2}Du(x) = 0$ in $\mathbb{R}^n$ for any $n \geq 2$ and $p \notin \{1, 2\}$.

1. INTRODUCTION

In this paper, we continue the study applications of nonassociative algebras to elliptic PDEs started in [19], [16]. Let us consider the p-Laplace equation

$$\Delta_p u := |Du|^2 \Delta u + \tfrac{p-2}{2} \langle Du, D|Du|^2 \rangle = 0. \tag{1.1}$$

Here $u(x)$ is a function defined on a domain $E \subset \mathbb{R}^n$, Du is its gradient and $\langle , \rangle$ denotes the standard inner product in $\mathbb{R}^n$. It is well-known that for $p > 1$ and $p \neq 2$ a weak (in the distributional sense) solution to (1.1) is normally in the class $C^{1,\alpha}(E)$ [22], [21], [4], but need not to be a Hölder continuous or even continuous in a closed domain with nonregular boundary [12]. On the other hand, if $u(x)$ is a weak solution of (1.1) such that $\operatorname{ess\,sup} |Du(x)| > 0$ holds locally in a domain $E \subset \mathbb{R}^n$ then $u(x)$ is in fact a real analytic function in E [13].

An interesting problem is whether the converse non-vanishing property holds true. More precisely: is it true that any real analytic solution $u(x)$ to (1.1) for $p > 1$, $p \neq 2$, in a domain $E \subset \mathbb{R}^n$ with vanishing gradient $Du(x_0) = 0$ at some $x_0 \in E$ must be identically zero? Notice that the analyticity assumption is necessarily because for any $d \geq 2$ and $n \geq 2$ there exists plenty non-analytic $C^{d,\alpha}$-solutions $u(x) \not\equiv 0$ to (1.1) in $\mathbb{R}^n$ for which $Du(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, see [11], [2], [9], [23], [17].

The non-vanishing property was first considered and solved in affirmative in $\mathbb{R}^2$ by John L. Lewis in [14] as a corollary of the following crucial result (Lemma 2 in [14]): if $u(x)$ is a real homogeneous polynomial of degree $m = \deg u \geq 2$ in $\mathbb{R}^2$ and $\Delta_p u(x) = 0$ for $p > 1$, $p \neq 2$ then $u(x) \equiv 0$. Concerning the general case $n \geq 3$, it is not difficult to see (see also Remark 4 in [14]) that the non-vanishing property for real analytic solutions to (1.1) in $\mathbb{R}^n$ is equivalent to following conjecture.

Conjecture 1.1. Let $u(x)$ be a real homogeneous polynomial of degree $m = \deg u \geq 2$ in $\mathbb{R}^n$, $n \geq 3$. If $\Delta_p u(x) = 0$ for $p > 1$, $p \neq 2$ then $u(x) \equiv 0$.

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Notice that a simple analysis shows that Conjecture 1.1 is true for $m = 2$ and any dimension $n \geq 2$, therefore the only interesting case is when $m \geq 3$. In [14], Lewis mentioned that Conjecture 1.1 holds also true for $n = m = 3$ (unpublished). In 2011, J.L. Lewis asked the author whether Conjecture 1.1 remains true for any $n \geq 3$ and $m \geq 3$. In this paper we obtain the following partial result for the cubic polynomial case.

Theorem 1.2. *Conjecture 1.1 is true for $m = 3$ and any $n \geq 2$. More precisely, if $u(x)$ is a homogeneous degree three solution of (1.1) in $\mathbb{R}^n$, $n \geq 2$ and $p \notin \{1, 2\}$ then $u(x) \equiv 0$.*

It follows from the above discussion that the following property holds true.

Corollary 1.3. *Let $u(x) \not\equiv 0$ be a real analytic solution of (1.1) in a domain $E \subset \mathbb{R}^n$, $n \geq 2$ and $p \notin \{1, 2\}$. If $Du(x_0) = 0$ at some point $x_0 \in E$ then $D^3u(x_0) = 0$.*

Remark 1.4. Concerning the two exceptional cases in Theorem 1.2, notice that when $p = 2$ there is a reach class of homogeneous polynomial solutions of (1.1) of any degree $m \geq 1$. In the other exceptional case, $p = 1$, one easily verifies that $u_m(x) = (a_1x_1 + \ldots + a_nx_n)^m$ satisfy (1.1) for any $n \geq 1$ and $m \geq 1$. In fact, one can show that $u_3(x)$ are the only cubic homogeneous polynomial solutions of (1.1) in $\mathbb{R}^n$ for $p = 1$ and $n \geq 2$; this fact is essentially equivalent to Proposition 6.6.1 in [16], but see also a self-contained explanation in Remark 3.2 below.

Remark 1.5. In the limit case $p = \infty$, an elementary argument (see Proposition 4.1 below) yields the non-vanishing property for real analytic solutions of the ∞-Laplacian

$$\Delta_\infty u := \langle Du, D|Du|^2\rangle = 0. \tag{1.2}$$

On the other hand, it is interesting to note that, in contrast to the case $p \neq \infty$, the non-vanishing property holds still true for Hölder continuous ∞-harmonic functions. Namely, for C^2-solutions of (1.2) and $n = 2$ the non-vanishing property was established by G. Aronnson [1]. In any dimension $n \geq 2$ it was proved for C^4-solutions by L. Evans [5] and for C^2-solutions by Yifeng Yu [24]. The non-vanishing property for C^2-smooth ∞-harmonic maps was recently established by N. Katzourakis [8].

The proof of Theorem 1.2 is by contradiction and makes use of a nonassociative algebra argument which was earlier applied for an eiconal type equation in [18], [19] and study of Hsiang cubic minimal cones [16]. First, in section 2 we recall the definition of a metrised algebra and give some preparatory results. In particular, in Proposition 2.3 we reformulate the original PDE-problem for cubic polynomial solutions as the existence of a metrised non-associative algebra structure on $\mathbb{R}^n$ satisfying a certain fourth-order identity. Then in Proposition 3.1, we show that any such algebra must be zero, thus implying the claim of Theorem 1.2.

2. Preliminaries

2.1. **Metrised algebras.** By an algebra on a vector space V over a field $\mathbb{F}$ we mean an $\mathbb{F}$-bilinear form $(x, y) \to xy \in V$, $x, y \in V$, also called the multiplication and in what follows denoted by juxtaposition. An algebra V is called a zero algebra if $xy = 0$ for all $x, y \in V$.

Suppose that (V, Q) is an inner product vector space, i.e. a vector space V over a field $\mathbb{F}$ with a non-degenerate bilinear symmetric form $Q : V \otimes V \to \mathbb{F}$. The inner product Q on an algebra V is called associative (or invariant) [3], [10, p. 453] if

$$Q(xy, z) = Q(x, yz), \qquad \forall x, y, z \in V. \tag{2.1}$$

An algebra V with an associative inner product is called *metrised* [3], [16, Ch. 6].

In what follows, we assume that $\mathbb{F} = \mathbb{R}$ and that (V, Q) is a commutative, but may be non-associative metrised algebra. Let us consider the cubic form

$$u(x) := Q(x^2, x) : V \to \mathbb{R}.$$

Then it is easily verified that the multiplication $(x, y) \to xy$ is uniquely determined by the identity

$$Q(xy, z) = u(x; y; z), \tag{2.2}$$

where

$$u(x; y; z) := u(x + y + z) - u(x + y) - u(x + z) - u(y + z) + u(x) + u(y) + u(z)$$

is a symmetric trilinear form obtained by the linearization of u. For further use notice the following corollary of the homogeneity of $u(x)$:

$$u(x; x; y) = 2\partial_y u|_x. \tag{2.3}$$

In the converse direction, given a cubic form $u(x) : V \to \mathbb{R}$ on an inner product vector space (V, Q), (2.2) yields a non-associative commutative algebra structure on V called the Freudenthal-Springer algebra of the cubic form $u(x)$ and denoted by $V^{\mathrm{FS}}(Q, u)$, see for instance [16, Ch. 6]). According to the definition, $V^{\mathrm{FS}}(Q, u)$ is a metrised algebra with an associative inner product Q.

We point out that the multiplication operator $L_x : V \to V$ defined by $L_x y = xy$ is self-adjoint with respect to the inner product $\langle , \rangle$. Indeed, it follows from the symmetricity of $u(x, y, z)$ that

$$Q(L_x y, z) = Q(xy, z) = Q(y, xz) = Q(y, L_x z).$$

Furthermore, for $k \geq 1$ one defines the kth principal power of $x \in V$ by

$$x^k = L_x^{k-1} x = \underbrace{x(x(\cdots(xx)\cdots))}_{k \text{ copies of } x} \tag{2.4}$$

In particular, we write $x^2 = xx$ and $x^3 = xx^2$. Since V is non-associative, in general $x^k x^m \neq x^{k+m}$. However, one easily verifies that the latter power-associativity holds for $k + m \leq 3$.

We recall that an element $c \in V$ is called an *idempotent* if $c^2 = c$. By $\mathscr{I}(V)$ we denote the set of all non-zero idempotents of V.

Lemma 2.1. *Let (V, Q) be a non-zero commutative metrised algebra with positive definite inner product Q. Then $\mathscr{I}(V) \neq \emptyset$.*

Proof. First notice that the cubic form $u(x) := Q(x^2, x) \not\equiv 0$, because otherwise the linearization would yield $Q(xy, z) \equiv 0$ for all $x, y, z \in V$, implying $xy \equiv 0$, i.e. V is a zero algebra, a contradiction. Next notice that in virtue of the positive definiteness assumption, the unit sphere $S = \{x \in V : Q(x) = 1\}$ is compact in the standard Euclidean topology on V. Therefore as u is a continuous function on S, it attains its maximum value at some point $y \in S$, $Q(y) = 1$. Since $u \not\equiv 0$ is an odd function, the maximum value $u(y)$ must be strictly positive and the stationary equation $\partial_x u|_y = 0$ holds whenever $x \in V$ satisfies the tangential condition

$$Q(y; x) = 0. \tag{2.5}$$

Using (2.3) and (2.2) we have

$$0 = \partial_x u|_y = \frac{1}{2}u(y; y; x) = \tfrac{1}{2}Q(y^2; x)$$

which implies in virtue of the non-degeneracy of Q and (2.5) that $y^2 = ky$, for some $k \in \mathbb{R}^\times$. It follows that

$$kQ(y; y) = Q(y^2; y) = u(y) > 0,$$

which yields $k \neq 0$. Then setting $c = y/k$ we obtain $c^2 = c$, i.e. $c \in \mathscr{I}(V)$. □

Remark 2.2. In a general finite-dimensional non-associative algebra over $\mathbb{R}$, there exist either an idempotent or an absolute nilpotent, see a topological proof, for example, in [15].

2.2. **Preliminary reductions.** Now suppose that $V = \mathbb{R}^n$ is the Euclidean space endowed with the standard inner product $Q(x; y) = \langle x, y\rangle$. Let $u : V \to \mathbb{R}$ be a cubic homogeneous polynomial solution of (1.1) and let $V^{\mathrm{FS}}(u)$ denotes the corresponding Freudenthal-Springer algebra with multiplication xy uniquely defined by

$$\langle xy, z\rangle = u(x; y; z). \tag{2.6}$$

Then the homogeneity of $u(x)$ and (2.3) yield

$$\langle x^2, x\rangle = u(x; x; x) = 2\partial_x u|_x = 6u(x). \tag{2.7}$$

Similarly, it follows from (2.3) that

$$\langle x^2, y\rangle = u(x; x; y) = 2\partial_y u|_x = 2\langle Du(x), y\rangle \tag{2.8}$$

which yields the expression for the gradient of u as an element of the Freudenthal-Springer algebra:

$$Du(x) = \tfrac{1}{2}x^2. \tag{2.9}$$

A further polarization of (2.8) yields

$$\langle y, D^2u(x)\, z\rangle = u(x; y; z) = \langle y, L_x z\rangle,$$

where $L_x y = xy$ is the multiplication operator by x and $D^2u(x)$ is the Hessian operator of u. This implies

$$D^2u(x) = L_x, \tag{2.10}$$

Proposition 2.3. *A cubic form $u : V = \mathbb{R}^n \to \mathbb{R}$ satisfies (1.1) if and only if its Freudenthal-Springer algebra $V^{\mathrm{FS}}(u)$ satisfies the following identity:*

$$\langle b, x\rangle\langle x^2, x^2\rangle + (p-2)\langle x^2, x^3\rangle = 0 \tag{2.11}$$

where

$$b = b(V) := \sum_{i=1}^{n} e_i^2, \tag{2.12}$$

and $e_1, \ldots, e_n$ is an arbitrary orthonormal basis of $\mathbb{R}^n$.

Proof. Using (2.9) and (2.10), one obtains

$$\langle Du, D^2u|_x Du\rangle = \tfrac{1}{4}\langle L_x x^2, x^2\rangle = \tfrac{1}{4}\langle x^3, x^2\rangle,$$

and similarly,

$$\Delta u(x) = \operatorname{tr} D^2u|_x = \operatorname{tr} L_x = \sum_{i=1}^{n}\langle L_x e_i, e_i\rangle = \sum_{i=1}^{n}\langle e_i^2, x\rangle = \langle b(V), x\rangle, \tag{2.13}$$

where b is defined by (2.12). Inserting the found relations into (1.1) yields (2.11). In the converse direction, if V is a metrised algebra satisfying (2.11) then $u(x)$ defined by (2.7) is easily seen to satisfy (1.1). □

3. Proof of Theorem 1.2

Using the introduced above definitions and Proposition 2.3, one easily sees that the following property is equivalent to Theorem 1.2.

Proposition 3.1. *A commutative metrised algebra* (V, Q) *with* $\dim V \geq 2$ *and satisfying (2.13) with* $p \notin \{1, 2\}$*, is a zero algebra.*

Proof. We argue by contradiction and assume that $(V, \langle,\rangle)$ is a non-zero commutative metrised algebra satisfying (2.11). Since $p \neq 2$, this identity is equivalent to

$$\langle q, x\rangle\langle x^2, x^2\rangle + \langle x^2, x^3\rangle = 0, \tag{3.1}$$

where

$$q = \frac{1}{p-2} b(V) \in V. \tag{3.2}$$

Polarizing (3.1) we obtain in virtue of

$$\partial_y x^3 = \partial_y(x(xx)) = yx^2 + 2x(xy)$$

and the associativity of the inner product that

$$\langle q, y\rangle\langle x^2, x^2\rangle + 4\langle q, x\rangle\langle xy, x^2\rangle + 4\langle xy, x^3\rangle + \langle x^2, yx^2\rangle = 0,$$

implying by the arbitrariness of y that

$$\langle x^2, x^2\rangle q + 4\langle q, x\rangle x^3 + 4x^4 + x^2x^2 = 0, \tag{3.3}$$

we according to (2.4) $x^4 = xx^3$. A further polarization of (3.3) yields

$$4\langle x^2, xy\rangle q + 4\langle q, y\rangle x^3 + 4\langle q, x\rangle(yx^2 + 2x(xy)) + 4yx^3 + 4x(yx^2 + 2x(xy)) + 4x^2(xy) = 0,$$

which implies an operator identity

$$2L_x^3 + L_{x^3} + \langle q, x\rangle(L_{x^2} + 2L_x^2) + L_x L_{x^2} + L_{x^2}L_x + (q \otimes x^3 + x^3 \otimes q) = 0. \tag{3.4}$$

Here $a \otimes b$ denotes the rank one operator acting by $(a \otimes b)y = a\langle b, y\rangle$.

Now, notice that by our assumption and Lemma 2.1, $\mathscr{I}(V) \neq \emptyset$. Let $c \in \mathscr{I}(V)$ be an arbitrary idempotent. Then setting $x = c$ in (3.1) we find

$$|c|^2 q + (4\langle q, c\rangle + 5)c = 0.$$

Taking the scalar product of the latter identity with c yields

$$\langle q, c\rangle = -1, \qquad q = -\frac{1}{|c|^2} c \tag{3.5}$$

in particular $q \neq 0$. Furthermore, setting $x = c$ in (3.4) and applying (3.5) yields

$$2L_c^3 + L_c + \langle q, c\rangle(L_c + 2L_c) + 2L_c^2 + (q \otimes c + c \otimes q) = 2L_c^3 - \frac{2}{|c|^2} c \otimes c = 0,$$

therefore

$$L_c^3 = \frac{1}{|c|^2} c \otimes c. \tag{3.6}$$

The latter identity, in particular, implies that

$$L_c = 0 \text{ on } c^\perp := \{x \in V : \langle c, x\rangle = 0\}, \tag{3.7}$$

where by the assumption $\dim c^\perp = \dim V - 1 \geq 1$.

We claim that $c^\perp$ is a zero subalgebra of V. Indeed, if $x, y \in c^\perp$ then by the associativity of the inner product and (3.7) we have

$$\langle xy, c\rangle = \langle x, cy\rangle = \langle x, L_c y\rangle = 0,$$

hence $xy \in c^\perp$ which implies that $c^\perp$ is a subalgebra (in fact, an ideal) of V. Suppose that $c^\perp$ is a non-zero subalgebra, then it follows by Lemma 2.1 that there is a nontrivial idempotent in $c^\perp$, say w. Then by the second identity in (3.5) we have $\langle w, q\rangle = 0$, therefore (3.1) yields

$$\langle w^2, w^3\rangle = |w|^2 = 0. \tag{3.8}$$

The obtained contradiction proves our claim.

To finish the proof, we consider an arbitrary orthonormal basis $\{e_i\}_{1\leq i\leq n}$ of V with $e_n = c/|c|$. Then $e_i \in c^\perp$ for all $1 \leq i \leq n-1$, hence by the above zero-algebra property we have $e_i^2 = 0$. Applying (2.12) we get

$$(p-2)q = b(V) = \sum_{i=1}^{n} e_i^2 = \frac{c}{|c|^2} = -q,$$

which yields in virtue of $q \neq 0$ that $p = 1$, a contradiction finishes the proof. □

Remark 3.2. In fact, it was established in the course of the proof that in the case $p = 1$, V decomposes in the orthogonal sum $\mathbb{R}c \oplus c^\perp$, $c^\perp$ being a zero algebra. Notice that the idempotent c is uniquely determined by the second identity in (3.5), and the latter decomposition immediately implies that V is rank 1 algebra, i.e. $\dim VV = 1$. Being translated on the functional level, this means that the only cubic solutions of (1.1) for $p = 1$ are the cubic polynomials $u(x) = k(c_1x_1 + \ldots + c_nx_n)^3$, where $c = (c_1, \ldots, c_n)$ and $k \in \mathbb{R}$ is an arbitrary real constant. This classifies all cubic polynomial solutions in the exceptional case $p = 1$.

4. Concluding remarks

We notice that the appearance of non-associative algebras in the above analysis of the p-Laplace equation is not an accident and becomes more substantial if one considers the following eigenfunction problem

$$\Delta_p u(x) = \lambda |x|^2 u(x), \quad \lambda \in \mathbb{R}, \quad p \neq 2, \tag{4.1}$$

with $u(x)$ being a cubic homogeneous polynomial. Notice that (1.1) correspond to $\lambda = 0$ in (4.1). The problem (4.1) for $p = 1$ has first appeared in Hsiang's study of cubic minimal cones in $\mathbb{R}^n$ [7]. In fact, it follows from recent results in [16, Ch. 6] that any cubic polynomial solution of (4.1) is necessarily harmonic, and thus satisfies (4.1) for *any* $p \neq 2$! The zero-locus of any such solution is an algebraic minimal cone in $\mathbb{R}^n$ [7]. Furthermore, it was shown in [16] that (4.1) has a large class of non-trivial cubic solutions for $p = 1$ (and thus for any $p \neq 2$) sporadically distributed over dimensions $n \geq 2$. It turns out that these solutions have a deep relation to rank 3 formally real Jordan algebras and their classification requires a mush more delicate analysis by using nonassociative algebras, we refer to [20] for more examples of solutions to (4.1) and their classification.

Finally, below we give an elementary proof of the non-vanishing property for real analytic ∞-harmonic functions. This result is not essentially new and goes back to an earlier result of B. Fuglede, see the example at the end of [6].

Proposition 4.1. *If $v(x)$ is a real analytic solution of the (1.2) in a domain $E \subset \mathbb{R}^n$ and $Dv(x_0) = 0$ for some $x_0 \in D \subset \mathbb{R}^n$ then $v(x) \equiv v(x_0)$.*

Proof. Indeed, we may assume that $x_0 = 0$ and suppose by contradiction that $v(x) \not\equiv v(0)$. Then a direct generalization of Lewis' argument given in Lemma 1 in [14] easily yields the existence of a real homogeneous polynomial $u(x) \not\equiv 0$ of order $\deg u = k \geq 2$ which also is a solution to (1.2). Notice that $u(x)$ attains its maximum value on the unit sphere $S = \{x \in \mathbb{R}^n : |x| = 1\}$ at some point y. The stationary equation yields $Du(y) = \lambda y$ for some real λ and by Euler's homogeneous function theorem

$$ku(y) = \langle y, Du(y)\rangle = \lambda |y|^2 = \lambda$$

and

$$\langle Du(y), D|Du|^2(y)\rangle = \lambda(2k-2)|Du|^2(y) = 2(k-1)\lambda^3,$$

which yields by (1.2) that $u(y) = 0$, hence

$$\max_{x\in S} u(x) = \frac{\lambda}{k} = 0.$$

A similar argument applied to the minimum value implies $\min_{x\in S} u(x) = 0$, a contradiction with $u \not\equiv 0$ follows. $\square$

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