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Discrete Taut Strings and Real Interpolation

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Abstract
Classical taut strings and their multidimensional generalizations appear in a broad range of applications. In this paper we suggest a general approach based on the $K$-functional of real interpolation that provides a unifying framework of existing theories and extend the range of applications of taut strings. More exactly, we introduce the notion of invariant $K$-minimal sets, explain their connection to taut strings and characterize all bounded, closed and convex sets in $\mathbb{R}^n$ that are invariant $K$-minimal with respect to the couple $(\ell^1, \ell^\infty)$.

Keywords: Taut strings, Real interpolation, Invariant $K$-minimal sets

2010 MSC: 46E30, 46N10

Introduction

Let $a = x_0 < x_1 < \ldots < x_n = b$ be $n + 1$ points on the real line, $n \geq 1$, given by

$$x_i = a + \frac{i(b-a)}{n}, i = 0, 1, \ldots, n.$$ 

Let us consider two continuous functions $F \leq G$ on the interval $[a, b]$ that are linear on the intervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$. We suppose that

$$F(a) = G(a), F(b) = G(b).$$
Let $\Gamma_{F,G}$ denote the set of all continuous piecewise linear functions $f$ on $[a,b]$ with nodes in $x_i$, $i = 0, 1, ..., n$, and which satisfy the inequalities $F \leq f \leq G$. A function $f^* \in \Gamma_{F,G}$ is called taut string if it has minimal length among all functions $f \in \Gamma_{F,G}$, i.e.

$$
\int_a^b \sqrt{1 + (f'(x))^2} \, dx = \inf_{f \in \Gamma_{F,G}} \int_a^b \sqrt{1 + (f'(x))^2} \, dx.
$$

For an illustration of the taut string, see Figure 1.

The notion of taut string was introduced by G.B. Dantzig in 1971, see [3]. Dantzig notes that he first presented taut strings in R. Bellman’s seminar at RAND Corporation in 1952 in connection with problems in optimal control. Later on taut strings and their one- and multidimensional generalizations have been used in different applied problems, in particular in statistics, see e.g. [1] and [12], and image processing, see [14]. Recently, new applications to stochastic processes, see [11], and communication theory, see [17] and [16], have been found.

Our interest in taut strings is based on the following connection to the theory of real interpolation. Consider the set

$$
\Omega := \left\{ u = (u_i) \in \mathbb{R}^n : u_i = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}, \ i = 1, ..., n, f \in \Gamma_{F,G} \right\}. \quad (1)
$$

It can be shown, see Theorem 5.1 below, that the element $u^*_s \in \Omega$ with $u^*_{s,i} = \frac{f_s(x_i) - f_s(x_{i-1})}{x_i - x_{i-1}}$, i.e. the vector with elements corresponding to the values of the
piecewise constant derivative \( f'_* \) of the taut string \( f_* \in \Gamma_{F,G} \), has minimal \( K \)-

functional in \( \Omega \) with respect to the couple \((\ell^1, \ell^\infty)\) on \( \mathbb{R}^n \) (everywhere below the spaces \( \ell^p \) are considered on \( \mathbb{R}^n \)). Hence, for all exact interpolation spaces of \((\ell^1, \ell^\infty)\) generated by the \( K \)-method, \( u_* \) has minimal norm in \( \Omega \). In particular, as \( \ell^2 \) is an exact interpolation space of \((\ell^1, \ell^\infty)\) generated by the \( K \)-method, we have

\[
\int_a^b (f'_*(x))^2 \, dx = \inf_{f \in \Gamma_{F,G}} \int_a^b (f'(x))^2 \, dx,
\]
i.e. the taut string has minimal energy among all functions in \( \Gamma_{F,G} \).

Recall that given a Banach couple \((X_0, X_1)\), the \( K \)-functional is defined for \( x \in X_0 + X_1 \) and \( t > 0 \) by

\[
K(t, x; X_0, X_1) := \inf_{x = x_0 + x_1} (\|x_0\|_{X_0} + t \|x_1\|_{X_1}).
\]

Further, an intermediate space \( X \) of \((X_0, X_1)\) is an exact interpolation space of \((X_0, X_1)\) if for every linear operator \( T : X_0 + X_1 \to X_0 + X_1 \) such that \( T : X_i \to X_i \) and \( \|T\|_{X_i \to X_i} < \infty \), \( i = 0, 1 \), we have \( T : X \to X \) and

\[
\|T\|_{X \to X} \leq \max_{i=0,1} \|T\|_{X_i \to X_i}.
\]

For these and other topics in the theory of real interpolation we refer to e.g. [2].

Suppose now we choose a non-uniform partition of the interval \([a, b]\) in the taut string problem, i.e. \( a = x_0 < x_1 < \ldots < x_n = b \) and for at least one \( i \in \{1, \ldots, n-1\} \) we have \( x_i \neq a + \frac{(b-a)}{n} \). Then it can be shown that the element \( u_* \) has minimal \( K \)-functional in \( \Omega \) with respect to the weighted couple \((\ell^1, \ell^\infty(w))\) where \( w = (w_i) \in \mathbb{R}^n \) with \( w_i = \frac{1}{x_i - x_{i-1}} \), \( i = 1, \ldots, n \). However, in this paper we will stick to the uniform partition.

One can further show that the set \( \Omega \) has the following property: given any element \( a \in \mathbb{R}^n \), there exists an element \( u_{*,a} \in \Omega \) such that \( u_{*,a} - a \) has minimal \( K \)-functional in the set \( \Omega_{a} = \Omega - a \) with respect to \((\ell^1, \ell^\infty)\). Hence, given \( a \in \mathbb{R}^n \) we can find \( u_{*,a} \in \Omega \) that is a nearest element of \( a \) in \( \Omega \) with respect to all exact interpolation spaces of \((\ell^1, \ell^\infty)\) generated by the \( K \)-method. With these properties of \( \Omega \) in mind, \( \Omega \) is denoted an invariant \( K \)-minimal set with respect to the couple \((\ell^1, \ell^\infty)\). For a general definition see Section [1]
It is natural to ask if there are other sets in \( \mathbb{R}^n \) with this property which leads to the problem of characterizing all invariant \( K \)-minimal sets in \( \mathbb{R}^n \) with respect to the couple \( (\ell^1, \ell^\infty) \). Let \( \{e_i\}_{i=1}^n \) denote the standard basis of \( \mathbb{R}^n \). The main result of this paper is a characterization of the bounded, closed and convex sets that are invariant \( K \)-minimal for \( (\ell^1, \ell^\infty) \) according to

**Theorem 0.1** A bounded, closed and convex set \( \Omega \subset \mathbb{R}^n \) is invariant \( K \)-minimal with respect to \( (\ell^1, \ell^\infty) \) on \( \mathbb{R}^n \) if and only if \( \Omega \) is a convex polytope where the affine hull of any face of \( \Omega \) is a shifted subspace of \( \mathbb{R}^n \) spanned by a basis consisting of vectors of the type \( e_i, e_i + e_j \) and \( e_i - e_j \).

Recall that a convex polytope is a bounded, closed and convex set which is the intersection of a finite number of closed half-spaces. Further, given a convex set \( S \subset \mathbb{R}^n \), a set \( F \subset S \) is a face of \( S \) if either \( F = \emptyset \) or \( F = S \) or if there exists a supporting hyperplane \( H \) of \( S \) such that \( F = S \cap H \). For a comprehensive treatment of convex polytopes, we refer to [5].

The proof of the necessity part of Theorem 0.1 is rather complicated. One of the reasons for this is the fact that the intersection of two invariant \( K \)-minimal sets need not to be invariant \( K \)-minimal.

Closely related to \( K \)-minimal sets with respect to \( (\ell^1, \ell^\infty) \) are so called \( \varphi \)-minimal sets. First we recall the classical majorization inequality due to Hardy, Littlewood and Pólya from 1929, see [6] and [7]. To formulate their result we need some notation. Given \( z = (z_1, ..., z_n) \in \mathbb{R}^n \), let \( z^\downarrow \in \mathbb{R}^n \) denote the vector with the elements of \( z \) sorted in decreasing order. Note that it is not the decreasing rearrangement of the modulus of the elements of \( z \). The result of Hardy-Littlewood-Pólya states that:

\[
\sum_{i=1}^{n} \varphi(x_i) \leq \sum_{i=1}^{n} \varphi(y_i)
\]

for \( x, y \in \mathbb{R}^n \) and all convex functions \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) if and only if

\[
\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow, \ k = 1, ..., n - 1,
\]

(2)
and

\[ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \]  \hspace{1cm} (3)

If (2) and (3) hold we say that \( x \) is majorized by \( y \). The Hardy-Littlewood-Pólya majorization inequality has found important applications in e.g. statistics, physics and economics. For a detailed survey of applications and extensions related to this result we refer to the book [13].

A set \( \Omega \subset \mathbb{R}^n \) will be denoted \( \varphi \)-minimal if there exists an element \( x^* \in \Omega \) such that

\[ \sum_{i=1}^{n} \varphi (x^*, i) \leq \sum_{i=1}^{n} \varphi (x_i), \forall x \in \Omega, \]

for all convex functions \( \varphi : \mathbb{R} \to \mathbb{R} \). That is, \( x^* \) is majorized by any other element \( x \in \Omega \). Analogously to invariant \( K \)-minimal sets, a set \( \Omega \subset \mathbb{R}^n \) is referred to as invariant \( \varphi \)-minimal if \( \Omega - a \) for any \( a \in \mathbb{R}^n \) is \( \varphi \)-minimal. Building on the characterization of invariant \( K \)-minimal sets with respect to \((\ell^1, \ell^\infty)\), we characterize all bounded, closed and convex sets \( \Omega \subset \mathbb{R}^n \) that are invariant \( \varphi \)-minimal according to

**Theorem 0.2** A bounded, closed and convex set \( \Omega \subset \mathbb{R}^n \) is invariant \( \varphi \)-minimal if and only if \( \Omega \) is a convex polytope where the affine hull of any face of \( \Omega \) is a shifted subspace of \( \mathbb{R}^n \) spanned by a basis consisting of vectors of the type \( e_i - e_j \).

The paper is organized into five sections. In Section 1 we will give a general definition of invariant \( K \)-minimal sets with respect to the Banach couple \((X_0, X_1)\). Next, in Section 2 we show that the lines \( L_{b,v} = \{ x \in \mathbb{R}^n : x = sv + b, s \in \mathbb{R} \} \),

where \( b, v \in \mathbb{R}^n \) are given, are invariant \( K \)-minimal sets for \((\ell^1, \ell^\infty)\) if \( v \) belongs to some special set of vectors which we denote special directions. In fact, this is also a necessary condition but this will be proved later. In Section 3 we introduce the notion of special cone property which builds upon the special directions. It is shown that the special cone property provides a necessary and sufficient condition for a bounded, closed and convex set in \( \mathbb{R}^n \) to be invariant
A $K$-minimal set with respect to $(\ell^1, \ell^\infty)$. Moreover, Theorem 0.1 is proved in this section. Next, in Section 4 we investigate the connection between invariant $K$-minimal sets with respect to $(\ell^1, \ell^\infty)$ and invariant $\varphi$-minimal sets. In particular, Theorem 0.2 is proved. Finally, in Section 5 we consider a generalized taut string problem where we allow for non-fixed ends, i.e. $F(a) < G(a)$ and/or $F(b) < G(b)$. It is shown that the set $\Omega$, cf. (1), also in this setting is an invariant $K$-minimal set with respect to $(\ell^1, \ell^\infty)$. We then briefly consider an application to a discrete version of the Wiener process where this generalized taut string problem appears.

In [10], we consider more applications and an algorithm for computing the element with minimal $K$-functional which is based on projections along special directions. Moreover, we establish in [10] a sufficient condition for sets $\Omega \subset L^1([0,1]^m)$ to be invariant $K$-minimal with respect to the couple $(L^1([0,1]^m), L^\infty([0,1]^m))$.

1. General definition

Let $(X_0, X_1)$ be a Banach couple and $x \in X_0 + X_1$. The $K$-method of the theory of real interpolation is based on Peetre’s $K$-functional

$$K(t,x; X_0, X_1) := \inf_{x = x_0 + x_1} (\|x_0\|_{X_0} + t \|x_1\|_{X_1}), t > 0.$$  

We note that the $K$-functional for given $x \in X_0 + X_1$ is a concave function on $\mathbb{R}^+$. The $K$-method generates, in particular, the family of interpolation spaces $X_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$, of $(X_0, X_1)$ according to

$$X_{\theta,q} := \left\{ x \in X_0 + X_1 : \|x\|_{X_{\theta,q}} < \infty \right\},$$

where

$$\|x\|_{X_{\theta,q}} := \left( \int_0^\infty \left( t^{-\theta} K(t,x; X_0, X_1) \right)^q dt \right)^{\frac{1}{q}}, 1 \leq q < \infty,$$

and

$$\|x\|_{X_{\theta,\infty}} := \sup_{t>0} t^{-\theta} K(t,x; X_0, X_1).$$

For a general introduction to the theory of real interpolation we refer to [2].

Now, we give the definition of $K$-minimal sets:
**Definition 1.1** A set $\Omega \subset X_0 + X_1$ will be called $K$-minimal, with respect to the couple $(X_0, X_1)$, if there exists an element $x_* \in \Omega$ such that

$$K(\cdot, x_*; X_0, X_1) \leq K(\cdot, x; X_0, X_1)$$

holds for all $x \in \Omega$.

It follows from the definition that the element $x_*$ is the nearest element of 0 in $\Omega$ with respect to all exact interpolation spaces of the couple $(X_0, X_1)$ generated by the $K$-method. In particular, $x_*$ is the nearest element of 0 in $\Omega$ for all spaces $X_{\theta,q}$. The nearest element of 0 in $\Omega$ is thus stable with respect to the norms $\|\cdot\|_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$.

Next, we give the definition of invariant $K$-minimal sets:

**Definition 1.2** The set $\Omega \subset X_0 + X_1$ will be called invariant $K$-minimal if for every $a \in X_0 + X_1$, the set $\Omega_a = \Omega - a$ is $K$-minimal, i.e. for every $a \in X_0 + X_1$ there exists an element $x_{*,a} \in \Omega$ such that

$$K(\cdot, x_{*,a} - a; X_0, X_1) \leq K(\cdot, x - a; X_0, X_1)$$

(4)

holds for all $x \in \Omega$.

Hence, $x_{*,a}$ is the nearest element of $a$ in $\Omega$ with respect to all exact interpolation spaces of $(X_0, X_1)$ generated by the $K$-method (it is not difficult to prove that this property is equivalent to $\Omega$ being an invariant $K$-minimal set with respect to $(X_0, X_1)$).

The stronger notion of invariant $K$-minimal sets, as compared to $K$-minimal sets, is important when considering applications where we want to approximate a general element outside $\Omega$.

### 2. Special directions

In this section we show that line segments in $\mathbb{R}^n$ with a particular class of direction vectors, so called special directions, are invariant $K$-minimal sets with respect to the couple $(\ell^1, \ell^\infty)$. It turns out that the special directions will be
instrumental for the subsequent characterization of invariant $K$-minimal sets with respect to $(\ell^1, \ell^\infty)$.

Let us now give the definition of special directions.

**Definition 2.1** The vectors

$$ v = \begin{cases} 
\pm e_k, & k \in \{1, \ldots, n\}, \\
\pm (e_k + e_l), & k, l \in \{1, \ldots, n\}, \; k \neq l, \\
\pm (e_k - e_l), & k, l \in \{1, \ldots, n\}, \; k \neq l,
\end{cases} $$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$, will be referred to as special directions in $\mathbb{R}^n$.

The following lemma will be needed for proving the forthcoming results.

**Lemma 2.1** Let $x \in \mathbb{R}^n$ be given. We have

$$ K(t, x; \ell^1, \ell^\infty) = \begin{cases} 
tx^*_1, & 0 < t \leq 1, \\
\sum_{i=1}^k x^*_i + (t - k) x^*_{k+1}, & k < t \leq k + 1, \; k = 1, \ldots, n - 1, \\
\sum_{i=1}^n x^*_i, & t > n
\end{cases} $$

where $x^* \in \mathbb{R}^n$ is the vector with elements of $(|x_1|, \ldots, |x_n|)$ sorted in decreasing order.

**Proof.** The lemma follows from the following well-known formula, see e.g. Theorem 5.1.6, p. 298 in [2], for the $K$-functional of the couple $(L^1, L^\infty)$:

$$ K(t, f; L^1, L^\infty) = \int_0^t f^*(s)ds $$

where $f^*$ denotes the decreasing rearrangement of $f$. \qed

We now prove the following sufficient condition in terms of the special directions that will be important in the sequel.

**Lemma 2.2** Let $v \in \mathbb{R}^n$ be a special direction. Then the line

$$ L_{b,v} = \{x \in \mathbb{R}^n : x = sv + b, s \in \mathbb{R}\} $$

through $b \in \mathbb{R}^n$ with direction $v$ is an invariant $K$-minimal set with respect to $(\ell^1, \ell^\infty)$. Moreover, any closed line segment of $L_{b,v}$ is an invariant $K$-minimal set with respect to $(\ell^1, \ell^\infty)$. 

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Proof. We need to prove that for any given \( a \in \mathbb{R}^n \) there exists \( x_{*,a} \in L_{b,v} \) such that (4) holds for all \( x \in L_{b,v} \).

1) Let \( v = e_k \), the case \( v = -e_k \) is treated analogously. Take \( x = b + s e_k \in L_{b,v} \). Then \( x_k - a_k = b_k + s - a_k \) is the only component of \( x - a \) which changes as we move along \( L_{b,v} \). We note that \( |b_k + s - a_k| \) is minimal when \( s = s_* = a_k - b_k \), monotonically decreasing on \((-\infty, s_*]\) and monotonically increasing on \([s_*, \infty)\). From the formula established in Lemma 2.1 it then follows that

\[
K(\cdot, x_{*,a} - a; \ell_1, \ell_\infty) \leq K(\cdot, x - a; \ell_1, \ell_\infty)
\]

for any \( x \in L_{b,v} \) where \( x_{*,a} = b + s_* v \). More precisely, \( K(\cdot, x - a; \ell_1, \ell_\infty) \) is monotonically increasing when moving away from \( x_{*,a} \) along \( L_{b,v} \). Hence, \( L_{b,v} \) is an invariant \( K \)-minimal set with respect to \((\ell_1, \ell_\infty)\).

2) Let next \( v = e_k + d e_l \), \( k \neq l \) and \( d = \pm 1 \). The case \( -e_k - e_l \) is treated analogously. It follows that \( x_k - a_k = b_k + s - a_k \) and \( x_l - a_l = b_l + sd - a_l \) are the only components of \( x - a \) which change as we move along \( L_{b,v} \). If there exists \( s = s_* \in \mathbb{R} \) that minimizes both

\[
g_1(s) := |b_k + s - a_k| + |b_l + sd - a_l|
\]

and

\[
g_\infty(s) := \max \{|b_k + s - a_k|, |b_l + sd - a_l|\}
\]

then the formula established in Lemma 2.1 shows that

\[
K(\cdot, x_{*,a} - a; \ell_1, \ell_\infty) \leq K(\cdot, x - a; \ell_1, \ell_\infty)
\]

for any \( x \in L_{b,v} \) where \( x_{*,a} = b + s_* v \). By simple geometric considerations in the plane it follows that for both \( v = e_k + e_l \) and \( v = e_k - e_l \) we can find a common minimizer \( s_* \) of \( g_1 \) and \( g_\infty \) that corresponds to the orthogonal projection of \( a \) onto \( L_{b,v} \). Hence, \( L_{b,v} \) is an invariant \( K \)-minimal set with respect to \((\ell_1, \ell_\infty)\). Moreover, we note that both \( g_1 \) and \( g_\infty \) are monotonically decreasing on \((-\infty, s_*] \) and monotonically increasing on \([s_*, \infty) \) which implies that \( K(\cdot, x - a; \ell_1, \ell_\infty) \) is monotonically increasing when moving away from \( x_{*,a} \) along \( L_{b,v} \). Hence, any closed line segment of \( L_{b,v} \) is an invariant \( K \)-minimal set with respect to \((\ell_1, \ell_\infty)\). \( \square \)
3. The special cone property

In this section we establish a necessary and sufficient condition, denoted the special cone property, for a bounded, closed and convex set $\Omega \subset \mathbb{R}^n$ to be an invariant $K$-minimal set with respect to $(\ell^1, \ell^\infty)$. This condition is formulated in terms of the special directions introduced in Section 2.

The section is organized as follows. First, in Subsection 3.1 we give the definition of the special cone property. Then in Subsection 3.2 the sufficiency of the special cone property is shown. Next, in Subsection 3.3 through the establishment of a series of intermediate results, a necessary condition in terms of the special cone property is shown. Finally, in Subsection 3.4 we combine the obtained results of Subsections 3.2 and 3.3 to characterize all bounded, closed and convex sets in $\mathbb{R}^n$ that are invariant $K$-minimal with respect to $(\ell^1, \ell^\infty)$.

3.1. Definition of special cone property

The definition of the special cone property is now given.

Definition 3.1 Let $\Omega \subset \mathbb{R}^n$ be closed and convex. For $x \in \Omega$, take all special directions $v$ such that $x + \beta v \in \Omega$ for sufficiently small $\beta > 0$. Let $S_x$ denote the set of all such special directions at $x \in \Omega$. Further, let $K_x = \{y \in \mathbb{R}^n : y = \sum_{v \in S_x} \alpha_v v, \alpha_v \geq 0\}$ be the convex cone generated by the special directions in $S_x$. We say that $\Omega$ has the special cone property if $(x + K_x) \cap \Omega = \Omega$ for each $x \in \Omega$.

3.2. Sufficiency of the special cone property

We now turn to the problem of showing sufficiency of the special property.

Theorem 3.1 Closed and convex sets $\Omega \subset \mathbb{R}^n$ with the special cone property are invariant $K$-minimal sets with respect to $(\ell^1, \ell^\infty)$.

Proof. Let $\Omega \subset \mathbb{R}^n$ be a nonempty closed and convex set with the special cone property. Fix $\tau > 0$ and $a \in \mathbb{R}^n$. There is at least one element $y$ of $\Omega$ for which

$$K(\tau, y - a; \ell^1, \ell^\infty) = \inf_{x \in \Omega} K(\tau, x - a; \ell^1, \ell^\infty)$$

(5)
holds since $\Omega$ is closed and convex and $K(\tau, x - a; \ell^1, \ell^\infty)$ is convex on $\Omega$. We collect these elements of $\Omega$ in

$$\Omega_{\text{min}} = \left\{ y \in \Omega : K(\tau, y - a; \ell^1, \ell^\infty) = \inf_{x \in \Omega} K(\tau, x - a; \ell^1, \ell^\infty) \right\}. \quad (6)$$

Since $\Omega_{\text{min}}$ is closed and convex there is a unique element of best approximation of $a$ in $\Omega_{\text{min}}$ with respect to the strictly convex $\ell^2$-norm. Denote this element by $x_{*, a}$. We will now show that $x_{*, a}$ in fact is the element of best approximation of $a$ in the whole set $\Omega$ with respect to the $\ell^2$-norm. As there is only one such element this gives that $x_{*, a}$ is independent of the choice of $\tau > 0$.

Suppose we can find $\beta > 0$ such that $x_{*, a} + \lambda v_i \in \Omega$ for $0 \leq \lambda \leq \beta$ where $v_i$ is some special direction. From Lemma 2.2 follows that the line segment

$$\{ x \in \mathbb{R}^n : x = x_{*, a} + \lambda v_i, \ 0 \leq \lambda \leq \beta \}$$

is a $K$-minimal set with respect to $(\ell^1, \ell^\infty)$ which in turn gives

$$\|x_{*, a} + \lambda v_i - a\|_{\ell^2}^2 = \langle x_{*, a} + \lambda v_i - a, x_{*, a} + \lambda v_i - a \rangle =$$

$$= \langle x_{*, a} - a, x_{*, a} - a \rangle + 2\lambda \langle x_{*, a} - a, v_i \rangle + \lambda^2 \langle v_i, v_i \rangle \geq$$

$$\geq \langle x_{*, a} - a, x_{*, a} - a \rangle = \|x_{*, a} - a\|_{\ell^2}^2.$$

So, we have

$$2\lambda \langle x_{*, a} - a, v_i \rangle + \lambda^2 \langle v_i, v_i \rangle \geq 0.$$

Since $\lambda$ can be arbitrary close to 0 it follows that

$$\langle x_{*, a} - a, v_i \rangle \geq 0. \quad (7)$$

Let now $v_1, ..., v_k$ be all special directions for which $x_{*, a} + \lambda v_i \in \Omega$, $0 \leq \lambda \leq \beta$, for sufficiently small $\beta > 0$. Consider an arbitrary element $x$ of the form

$$x = x_{*, a} + \sum_{i=1}^k \lambda_i v_i$$
with $0 \leq \lambda_i \leq \beta$, $i = 1, \ldots, k$. We have

$$\|x - a\|_2^2 = \left\| x_{*,a} + \sum_{i=1}^{k} \lambda_i v_i - a \right\|_2^2 =$$

$$= \left( x_{*,a} + \sum_{i=1}^{k} \lambda_i v_i - a, x_{*,a} + \sum_{i=1}^{k} \lambda_i v_i - a \right) =$$

$$= \langle x_{*,a} - a, x_{*,a} - a \rangle + 2 \sum_{i=1}^{k} \lambda_i \langle x_{*,a} - a, v_i \rangle + \sum_{i,j=1}^{k} \lambda_i \lambda_j \langle v_i, v_j \rangle.$$

From (7) it follows that

$$\sum_{i=1}^{k} \lambda_i \langle x_{*,a} - a, v_i \rangle \geq 0. \quad (8)$$

Further, $\sum_{i,j=1}^{k} \lambda_i \lambda_j \langle v_i, v_j \rangle$ is the quadratic form $\lambda^T M \lambda$ where $\lambda = (\lambda_i) \in \mathbb{R}^k$ and $M = ((v_i, v_j)) \in \mathbb{R}^{k \times k}$. As $M$ is a Gram matrix and therefore positive semidefinite, see Theorem 7.2.10 in [8], we have

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \langle v_i, v_j \rangle \geq 0. \quad (9)$$

Taking into account (8) and (9) it follows that

$$\|x - a\|_2 \geq \|x_{*,a} - a\|_2 \quad (10)$$

for any $x = x_{*,a} + \sum_{i=1}^{k} \lambda_i v_i$ with $0 \leq \lambda_i \leq \beta$, $i = 1, \ldots, k$.

Assume now that there exists $z \in \Omega$ such that

$$\|z - a\|_2 < \|x_{*,a} - a\|_2.$$  

As $\Omega$ has the special cone property it follows that $z \in x_{*,a} + K_{x_{*,a}}$. So,

$$z = x_{*,a} + \sum_{i=1}^{k} \mu_i v_i$$

for some $\mu_i \geq 0$. Let $y_\alpha = \alpha z + (1 - \alpha)x_{*,a}$ for $\alpha \in [0,1]$. It follows that

$$y_\alpha = x_{*,a} + \sum_{i=1}^{k} \alpha \mu_i v_i.$$
For any \( \alpha \in (0,1] \) we have
\[
\|y_\alpha - a\|_{\ell^2} = \|\alpha z + (1 - \alpha)x_{*,a} - a\|_{\ell^2} \leq \\
\leq \alpha \|z - a\|_{\ell^2} + (1 - \alpha)\|x_{*,a} - a\|_{\ell^2} < \\
< \alpha \|x_{*,a} - a\|_{\ell^2} + (1 - \alpha)\|x_{*,a} - a\|_{\ell^2} = \|x_{*,a} - a\|_{\ell^2}.
\]
By choosing \( \alpha \) close enough to 0 the bounds \( 0 \leq \alpha \mu_i \leq \beta, \> i = 1, \ldots, k \), hold. Hence, (11) contradicts (10). So, \( x_{*,a} \) is the unique element of best approximation of \( a \) in the whole set \( \Omega \) with respect to the strictly convex \( \ell^2 \)-norm. Note that this holds true for any choice of \( \tau > 0 \). Therefore must \( x_{*,a} \) satisfy (5) for any \( \tau > 0 \), i.e.
\[
K(\tau,x_{*,a} - a;\ell^1,\ell^\infty) \leq K(\tau,x - a;\ell^1,\ell^\infty)
\]
holds for every \( x \in \Omega \) and every \( \tau > 0 \). Hence, \( \Omega \) is an invariant \( K \)-minimal set with respect to \( (\ell^1,\ell^\infty) \).

### Remark 3.1
Assume the set \( \Omega \) of Theorem 3.1 in addition satisfies \( \sum_{i=1}^{n} x_i = C \), where \( C \) is some fixed real number, for every \( x \in \Omega \). Then \( \Omega \) is in addition an invariant \( \varphi \)-minimal set (recall Definition 4.2).

**Proof.** In (5) and (6), instead of considering \( K(\tau,x - a;\ell^1,\ell^\infty) \) and \( K(\tau,y - a;\ell^1,\ell^\infty) \) for some particular choice of \( \tau > 0 \), we consider the functionals \( \sum_{i=1}^{n} \varphi(x_i - a_i) \) and \( \sum_{i=1}^{n} \varphi(y_i - a_i) \) for some particular choice of convex function \( \varphi : \mathbb{R} \to \mathbb{R} \). Then from the same arguments as in the proof of Theorem 3.1 it follows that the element \( x_{*,a} \) is the \( \ell^2 \)-minimizer of \( a \) in the whole set \( \Omega \). This will be true for any choice of \( \varphi \). Hence,
\[
\sum_{i=1}^{n} \varphi(x_{*,a,i} - a_i) \leq \sum_{i=1}^{n} \varphi(x_i - a_i)
\]
holds for every \( x \in \Omega \) and any choice of \( \varphi \) which implies that \( \Omega \) is invariant \( \varphi \)-minimal.

### 3.3. Necessity of the special cone property
The next problem is the necessity of the special cone property. The proof is rather complicated and established through a sequence of lemmas. Besides
special directions, the notion of convex polytopes is frequently used in the for-

mulation of the lemmas and their proofs. Therefore, before stating the first

lemma, some basic notion connected with convex polytopes will be introduced.

The notion and facts of convex polytopes used in this paper can be found in
e.g. Chapters 2 and 3 in [5].

Let \( S \subset \mathbb{R}^n \). By \( \text{aff}(S) \) we denote the affine hull of \( S \). The dimension of

\( S \) is defined as the dimension of the subspace parallel to \( \text{aff}(S) \). Further, let \( \text{relint}(S) \) denote the relative interior of \( S \), i.e.

\[
\text{relint}(S) := \{ y \in \text{aff}(S) : \exists \varepsilon > 0, B(y, \varepsilon) \cap \text{aff}(S) \subset S \}.
\]

It can be shown that if \( S \neq \emptyset \) then \( \text{relint}(S) \neq \emptyset \). The relative boundary of \( S \), \( \text{relbd}(S) \), is defined through \( \text{relbd}(S) := \text{cl}(S) \setminus \text{relint}(S) \). The reason of intro-
ducing \( \text{relint} \) and \( \text{relbd} \) is that these correspond to the interior and boundary of \( S \) when \( S \) is regarded as a subset of \( \text{aff}(S) \) where it has full dimension.

Assume now in addition that \( S \subset \mathbb{R}^n \) is convex. A set \( F \subset S \) is a face of \( S \) if either \( F = \emptyset \) or \( F = S \) or if there exists a supporting hyperplane \( H \) of \( S \) such that \( F = S \cap H \). Faces \( F \) of \( S \) such that \( F \neq S \) and \( F \neq \emptyset \) will be denoted proper faces. A convex polytope \( S \) in \( \mathbb{R}^n \) is a bounded, closed and convex set which is the intersection of a finite number of closed half-spaces. Faces of a convex polytope of dimension 0, 1 and \( n - 1 \) are called vertices, edges and facets respectively.

We also introduce some additional notion that will be useful in the proof of Lemma 3.1 and Lemma 3.3 below. Let \( I := \{1, ..., n\} \). Given a subspace \( V \subset \mathbb{R}^n \), let \( I_0 := \{ i \in I : e_i \in V \} \) and \( \tilde{I} := I \setminus I_0 \). We introduce a binary relation \( \sim \) on \( \tilde{I} \) according to the following. If \( i_1, i_2 \in \tilde{I} \) are such that \( e_{i_1} - e_{i_2} \in V \) then \( i_1 \sim i_2 \). It is easy to check that \( \sim \) is an equivalence relation. Hence, \( \sim \) gives a partition of \( \tilde{I} \) into subsets \( I_1, I_2, ..., I_N \) consisting of equivalent indices. We stress that \( i, j \in I_m \) for some \( m \in \{1, ..., N\} \) if and only if \( i, j \in \tilde{I} \) and \( e_i - e_j \in V \).

We say that two subsets \( I_m \) and \( I_l \) are associated if \( e_i + e_j \in V \) for some \( i \in I_m \) and \( j \in I_l \). Note that the association does not depend on the particular choice of \( i \in I_m \) and \( j \in I_l \). We write the association as \( I_m \simeq I_l \). Note that it
is possible that a set \( I_m \) is not associated with any other set \( I_l \). We then make
the convention that \( I_m \) is associated with the empty set \( \emptyset \) in order to facilitate
notation. Moreover, a set \( I_m \) can be associated with at most another set \( I_l \).
Let us show why. Suppose that \( I_m \) is associated both with \( I_{l_1} \) and \( I_{l_2} \), \( l_1 \neq l_2 \).
Then \( e_{i_1} + e_{j_1}, e_{i_2} + e_{j_2} \) are both in \( V \), where \( i_1, i_2 \in I_m, j_1 \in I_{l_1} \) and \( j_2 \in I_{l_2} \).
Since \( e_{i_1} - e_{i_2} \in V \) it follows that \( e_{j_1} - e_{j_2} \in V \) and therefore \( j_2 \in I_{l_1} \), which is
a contradiction.

Let \( U^\perp \) denote the orthogonal complement of a subspace \( U \in \mathbb{R}^n \). We are
now ready to formulate and prove the first lemma in this section.

**Lemma 3.1** Let \( V \) be a subspace of \( \mathbb{R}^n \). Suppose \( V \) is not spanned by special
directions. Then it is possible to find \( w \in V^\perp \) and \( u \in V \) such that
\[
\|w - u\|_\infty < \|w\|_\infty.
\]

**Proof.** Let \( v_1, \ldots, v_k \) denote all special directions in \( V \) and suppose
\( \text{span}\{v_1, \ldots, v_k\} \neq V \). Recall the notion of \( I, I_0 \) and \( \tilde{I} \) from above. By as-
sumption \( \tilde{I} \neq \emptyset \) since otherwise \( V = \text{span}\{e_i : i \in I\} = \mathbb{R}^n \), i.e. \( V \) would be
spanned by special directions. Second, note that we cannot have \( \tilde{I} = \{j\} \) for
some \( j \in I \) because this implies \( V = \text{span}\{e_i : i \neq j\} \), i.e. \( V \) is spanned by
special directions. Hence, \( |\tilde{I}| \geq 2 \).

Let \( V_0 := \text{span}\{e_i : i \in I_0\} \) and \( V_{lm} := \text{span}\{e_i : i \in I_l \cup I_m, I_l \cong I_m\} \). Note
that \( V_{l_1,m_1} \perp V_{l_2,m_2} \). For \( V_{lm} \), we define the subset
\[
V_{lm,0} := \text{span}\{e_i - e_j : i, j \in I_m\} \cup \text{span}\{e_i - e_j : i, j \in I_l\} \cup
\text{span}\{e_i + e_j : i \in I_m, j \in I_l\}.
\]
Now, we can express
\[
\mathbb{R}^n = V_0 \oplus_{l,m: l \cong I_m} V_{lm}
\]
and
\[
\text{span}\{v_1, \ldots, v_k\} = V_0 \oplus_{l,m: l \cong I_m} V_{lm,0}.
\]
As \( V_{lm,0} \subset V_{lm} \) it follows that
\[
\text{span}\{v_1, \ldots, v_k\}^\perp = \oplus_{l,m: l \cong I_m} V_{lm,0}^\perp \cap V_{lm,0}^\perp \cup V_{lm,0}^\perp \cap V_{lm,0}^\perp
\]
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where $\perp_{V_{lm}}$ denotes the orthogonal complement restricted to the space $V_{lm}$.

Take $\hat{v} \in V_{lm,0}$. We have

$$\hat{v} = \sum_{i \in I_l} \lambda_i e_i + \sum_{j \in I_m} \mu_j e_j.$$  

As $\hat{v} \cdot (e_i - e_j) = 0$ for $i, j \in I_l$ it follows that $\lambda_i = \lambda, \forall i \in I_l$. Analogously, we obtain $\mu_i = \mu, \forall i \in I_m$. Further, from $\hat{v} \cdot (e_i + e_j) = 0$ for $i \in I_l$ and $j \in I_m$ we obtain $\lambda = -\mu$. Hence,

$$\dim \left( V_{lm,0} \right) = 1.$$  

This gives that $V$ cannot contain any element $\hat{v} \in V_{lm,0}$ since then $V_{lm} \subset V$ and therefore $e_i \in V, \forall i \in I_l \cup I_m$, which is in contradiction with the construction of the sets $I_j, j = 0, 1, ..., N$. Hence, elements $u \in V \cap \operatorname{span} \{v_1, ..., v_k\} \perp \{0\}$ must have non-zero components for indices in at least two subsets $I_{l_1}$ and $I_{l_2}$ of $\tilde{I}$ with $I_{l_1} \neq I_{l_2}$, otherwise $u$ is proportional to $\hat{v}$.

Let us describe the structure of $u \in V \cap \operatorname{span} \{v_1, ..., v_k\} \perp \{0\}$ further. Let $I_l$ be an arbitrary subset in the partition of $\tilde{I}$. As $u \in \operatorname{span} \{v_1, ..., v_k\}$ we have $u \cdot (e_i - e_j) = 0$, i.e. $u_i = u_j$, for $i, j \in I_l$. Suppose there is a subset $I_m$ of $\tilde{I}$ with $I_l \simeq I_m$. Then we have $u \cdot (e_i + e_j) = 0$, i.e. $u_i = -u_j$, for $i \in I_l$ and $j \in I_m$. We conclude that an element $u \in V \cap \operatorname{span} \{v_1, ..., v_k\} \perp \{0\}$ has equal components on $I_l$ and, moreover, its components on the associated set $I_m$ have equal modulus but opposite sign with respect to the components on $I_l$. In short, we can represent $u \in V \cap \operatorname{span} \{v_1, ..., v_k\} \perp \{0\}$ according to

$$u = \sum_{l, m: I_l \simeq I_m} a_{lm} \left( \sum_{i \in I_l} e_i - \sum_{j \in I_m} e_j \right)$$  

where $a_{lm} \in \mathbb{R}$ and at least two coefficients $a_{lm}$ are non-zero.

We now turn to the construction of $w \in V$. As $V \subset \operatorname{span} \{v_1, ..., v_k\}$ the structure of $w$ is analogous to the structure of $u \in V \cap \operatorname{span} \{v_1, ..., v_k\} \perp \{0\}$, i.e.

$$w = \sum_{l, m: I_l \simeq I_m} b_{lm} \left( \sum_{i \in I_l} e_i - \sum_{j \in I_m} e_j \right)$$  

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where \( b_{lm} \in \mathbb{R} \). It will be shown that it is possible to construct \( w \) such that for some pair of associated sets \( I_l \) and \( I_m \), where an element \( u \in V \cap \text{span} \{v_1, ..., v_k\}^\perp \setminus \{0\} \) has non-zero components, the components of \( w \) of maximal modulus are restricted only to indices of \( I_l \cup I_m \). Taking into account the structure of (12) and (13) this will be sufficient for proving the lemma.

First, if for every \( u \in V \cap \text{span} \{v_1, ..., v_k\}^\perp \setminus \{0\} \) we have \( u_i = 0 \) for \( i \in I_l \) we also let \( w_i = 0 \) for \( i \in I_l \). Note that this will not violate the restriction \( w \in V^\perp \). Such sets will be excluded from the subsequent discussion, i.e. below it is assumed that for each set of the type \( I_j \) there exists at least one element \( u \in V \cap \text{span} \{v_1, ..., v_k\}^\perp \setminus \{0\} \) with non-zero components at the corresponding indices in \( I_j \). Note that with such exclusion we are still guaranteed to find pairs of non-associated subsets of \( ˜I \) by recalling from above that the elements in \( V \cap \text{span} \{v_1, ..., v_k\}^\perp \setminus \{0\} \) have at least two non-associated sets with non-zero components.

Take two subsets \( I_{l_1} \) and \( I_{l_2} \) of \( ˜I \), \( I_{l_1} \not\sim I_{l_2} \). Let \( \in I_{l_1} \) and \( j \in I_{l_2} \). Then \( e_i \pm e_j \not\in V \). We claim it is possible to choose \( w \) such that \( w \cdot (e_i - e_j) \neq 0 \). Suppose not. Then we have \( V^\perp \subset H_{e_i - e_j} \) where

\[
H_{e_i - e_j} := \{ y \in \mathbb{R}^n : \langle y, e_i - e_j \rangle = 0 \}.
\]

But then \( H_{e_i - e_j}^\perp \subset V \) which implies that \( e_i - e_j \in V \), a contradiction. We can in addition impose that \( w \) simultaneously satisfies \( w \cdot (e_i + e_j) \neq 0 \) by assuming that \( w \) avoids the hyperplane

\[
H_{e_i + e_j} := \{ y \in \mathbb{R}^n : \langle y, e_i + e_j \rangle = 0 \}.
\]

Taken together, this gives \(|w_i| = \delta_i \neq \delta_j = |w_j| \) for every \( i \in I_{l_1} \) and \( j \in I_{l_2} \).

Since we have a finite number of different pairs of non-associated sets, we can find \( w \in V^\perp \) such that \(|w_i| = \delta_i \neq \delta_i = |w_j| \) for every \( i \in I_{l_1} \) and \( j \in I_{l_2} \) for any pair of non-associated sets \( I_{l_1} \) and \( I_{l_2} \). Hence, such \( w \) will have components of maximal modulus occurring only in one pair of associated sets \( I_l \) and \( I_m \). At the same time, there exists an element \( u \in V \cap \text{span} \{v_1, ..., v_k\}^\perp \setminus \{0\} \) with
non-zero components in $I_l \cup I_m$. We therefore obtain

$$\|w - \varepsilon u\|_{\ell_\infty} < \|w\|_{\ell_\infty}$$

for $\varepsilon$ with proper sign and small enough modulus which proves the lemma. □

We get the following characterization of affine subspaces of $\mathbb{R}^n$ that are invariant $K$-minimal with respect to $(\ell^1,\ell^\infty)$.

**Corollary 3.1** Let $V$ be a subspace of $\mathbb{R}^n$. The affine subspaces $V_a := V + a$, $a \in \mathbb{R}^n$, of $\mathbb{R}^n$ are invariant $K$-minimal sets with respect to $(\ell^1,\ell^\infty)$ if and only if $V$ is spanned by special directions.

**Proof.** If $V$ is spanned by special directions it is clear that $V$ has the special cone property and therefore by Theorem 3.1 is an invariant $K$-minimal set with respect to $(\ell^1,\ell^\infty)$. By shifting $V$, it follows that also $V_a$ is an invariant $K$-minimal set with respect to $(\ell^1,\ell^\infty)$.

Assume now that $V$ is not spanned by special directions. Then by Lemma 3.1 we can find $w \in V^\perp$ and $u \in V$ such that $\|w - u\|_{\ell_\infty} < \|w\|_{\ell_\infty}$. On the other hand, $\|w - u\|_{\ell_2} > \|w\|_{\ell_2}$ and we conclude that $V$ cannot be an invariant $K$-minimal set with respect to $(\ell^1,\ell^\infty)$ in this case. □

**Remark 3.2** From Corollary 3.1 follows the necessity of $v$ being a special direction in order for the line $L_{b,v}$ to be invariant $K$-minimal with respect to $(\ell^1,\ell^\infty)$. Recall that Lemma 2.2 gave the sufficiency of this condition.

Now, our next result reads

**Lemma 3.2** Let $\Omega \subset \mathbb{R}^n$ be a bounded, closed and convex set that is invariant $K$-minimal with respect to $(\ell^1,\ell^\infty)$. Then $\Omega$ is a convex polytope where the affine hull of its proper faces of largest dimension are shifts of subspaces of $\mathbb{R}^n$ spanned by special directions.

**Remark 3.3** Note that when $\dim(\Omega) = 1$, i.e. $\Omega$ is a line or a line segment, the proper faces of $\Omega$ are just single points.
Proof. Let us first consider the case when $\Omega$ has full dimension in $\mathbb{R}^n$.

From Theorem V.9.8, p. 450, in [4] follows that points with tangent, i.e. unique supporting, hyperplanes are dense in the boundary $\text{bd}(\Omega)$. By possibly shifting $\Omega$, we can suppose that $0 \in \text{bd}(\Omega)$ and that $\Omega$ has a tangent hyperplane $H$ at $0$. Let $\{v_1, ..., v_k\}$ denote the set of special directions in $H$. Suppose $H \neq \text{span} \{v_1, ..., v_k\}$. From Lemma 3.1, it follows that we then can find $w \in H^\perp$ and $u \in H$ such that $\|w - u\|_{\ell^\infty} < \|w\|_{\ell^\infty}$. Consider the differentiable curve $r : (-\delta, \delta) \to \text{bd}(\Omega)$, $\delta > 0$, which goes through $0$ (corresponding to parameter $t = 0 \in \mathbb{R}$) and have tangent $u$ there. So, we have

$$r(t) = r(0) + r'(0)t + o(t) = ut + o(t).$$

It follows that for $t > 0$

$$\|w - r(t)\|_{\ell^\infty} = \|w - ut + o(t)\|_{\ell^\infty} \leq (1 - t) \|w\|_{\ell^\infty} + t \|w - u\|_{\ell^\infty} + \|o(t)\|_{\ell^\infty}.$$

(14)

For $t > 0$ small enough we have

$$\|o(t)\|_{\ell^\infty} < t(\|w\|_{\ell^\infty} - \|w - u\|_{\ell^\infty}).$$

(15)

Combining (14) and (15) gives

$$\|w - r(t)\|_{\ell^\infty} < \|w\|_{\ell^\infty}$$

for $t > 0$ small enough. So, it is possible to find a better approximation in $\ell^\infty$-norm of $w$ in $\Omega$ when moving away from $0$ along the curve $r$. As $0$ is the element of best approximation of $w$ in $\ell^2$-norm in $\Omega$, it follows that $\Omega$ is not an invariant $K$-minimal set with respect to $(\ell^1, \ell^\infty)$ which is in contradiction with the assumption. So, we must have $H = \text{span} \{v_1, ..., v_k\}$.

It follows, by shifting, that for a tangent hyperplane $H_a$ at $a \in \text{bd}(\Omega)$ we have $H_a = a + \text{span} \{v_{a_1}, ..., v_{a_m}\}$ for some special directions $v_{a_1}, ..., v_{a_m}$. The number of tangent hyperplanes of $\Omega$ is finite as we have a finite number of special directions and a particular subset of special vectors can be used in the construction of maximum two different tangent hyperplanes. By Theorem 18.8
in [14], a closed convex set of dimension $d$ in $\mathbb{R}^d$ is the intersection of the closed half-spaces tangent to it, i.e. the closed half-spaces with boundary hyperplanes being tangent hyperplanes of the set. It follows that $\Omega$ is the intersection of a finite number of closed half-spaces and therefore a convex polytope. As the affine hulls of the facets of $\Omega$, i.e. the faces of $\Omega$ of dimension $n-1$, correspond to tangent hyperplanes of $\Omega$ the lemma is established.

Now, consider the case $\dim(\Omega) < n$. An analogous proof applies for this case since the cited results of [4] and [14] are valid also when $\Omega$ is not of full dimension, the notion of tangent hyperplanes here being referred to the setting of $\text{aff}(\Omega)$.

\[ \square \]

We refine the description of the class of sets $\Omega$ in Lemma 3.2 further.

**Lemma 3.3** Let $\Omega$ be a convex polytope in $\mathbb{R}^n$ that is an invariant $K$-minimal set with respect to $\ell^1, \ell^\infty$. Then the affine hull of any edge, i.e. any one-dimensional face, of $\Omega$ is a shifted one-dimensional subspace of $\mathbb{R}^n$ spanned by a special direction.

**Proof.** Consider first the case $\dim(\Omega) = 1$, i.e. $\Omega$ itself is an edge. By possibly shifting $\Omega$, we can assume that $\Omega$ contains 0 in its relative interior. Suppose that $\text{aff}(\Omega)$ is not spanned by a special direction. Then it follows from Lemma 3.1 that we can find an element $w \in \text{aff}(\Omega)^\perp$ and $u \in \text{aff}(\Omega)$ such that $\|w - u\|_\ell^\infty < \|w\|_\ell^\infty$. By possibly scaling, we can assume that $u \in \Omega$. As $\|w\|_\ell^2 < \|w - x\|_\ell^2$ for any $x \in \text{aff}(\Omega) \setminus \{0\}$ we arrive at a contradiction to the assumption of $\Omega$ being an invariant $K$-minimal set with respect to $(\ell^1, \ell^\infty)$. Hence, $\text{aff}(\Omega)$ must be spanned by a special direction.

Assume now that $\dim(\Omega) > 1$. Consider an edge $F$ of $\Omega$. By possibly shifting $\Omega$, we can assume that $F$ contains 0 in its relative interior. Let $v$ denote the spanning vector of $\text{aff}(F)$. Assume that $v$ is not a special direction. We will show that it is then possible to construct an element $u \in \mathbb{R}^n$ such that 0 is the element of best approximation of $u$ in $\Omega$ with respect to $\ell^2$-norm but not with respect to $\ell^\infty$-norm.
Now, we have \( F = \cap_{j=1}^{m} F_j \) where \( F_j, i = 1, \ldots, m \) for some \( m \in \mathbb{N} \), are proper faces of highest dimension of \( \Omega \), see [5]. From Lemma 3.2 follows that \( \text{aff}(F_j), j = 1, \ldots, m \), are spanned by special directions.

For \( \text{aff}(F_j) \), let \( I_{0,j} := \{ i \in I : e_i \in \text{aff}(F_j) \} \) and \( \tilde{I}_j := I \setminus I_{0,j} \), cf. the notation introduced before Lemma 3.1. Then, since \( \text{aff}(F_j) \) is spanned by special directions we have the following description for an element \( z \) in \( \text{aff}(F_j) \)

\[
z = \sum_{k_j, l_j : I_{k_j} \supseteq I_{l_j}} c_{k_j l_j} \left( \sum_{i \in I_{k_j}} e_i - \sum_{j \in I_{l_j}} e_j \right),
\]

with \( I_{k_j}, I_{l_j} \subset \tilde{I}_j \) and \( c_{k_j l_j} \in \mathbb{R} \). Now, for each associated pair \( I_{k_j}, I_{l_j} \subset \tilde{I}_j \) we will take the element according to

\[
\sum_{i \in I_{k_j}} e_i - \sum_{j \in I_{l_j}} e_j \in \text{aff}(F_j)^\perp,
\]

where the order of \( I_{k_j} \) and \( I_{l_j} \) is such that

\[
\left\langle \sum_{i \in I_{k_j}} e_i - \sum_{j \in I_{l_j}} e_j, x \right\rangle \leq 0
\]

for all \( x \in \Omega \), i.e. the vector \( \sum_{i \in I_{k_j}} e_i - \sum_{j \in I_{l_j}} e_j \) is pointing outward from \( \Omega \). This is done for each \( F_j, j = 1, \ldots, m \). We denote the chosen vectors as \( n_1, \ldots, n_m \). Through these vectors we define the hyperplanes

\[
H_j := \{ x \in \mathbb{R}^n : \langle n_j, x \rangle = 0 \}.
\]

It follows that \( \text{span} \{ v \} = \text{aff}(F) = \cap_{j=1}^{m} H_j \). We then introduce the cone \( K_0 \)

\[
K_0 := \sum_{j=1}^{m} \alpha_j n_j, \alpha_j \geq 0.
\]

For \( x \in \Omega \setminus \{0\} \) and \( u \in K_0 \) we have

\[
\|u - x\|_{l^2}^2 = \langle u - x, u - x \rangle = \langle u, u \rangle - 2 \langle u, x \rangle + \langle x, x \rangle =
\]

\[
= \langle u, u \rangle - 2 \sum_{j=1}^{m} \alpha_j \langle n_j, x \rangle + \langle x, x \rangle \geq \langle u, u \rangle + \langle x, x \rangle > \langle u, u \rangle = \|u\|_{l^2}^2
\]

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Hence, 0 is the unique element of best approximation in Ω of \( u \in K_0 \). We will next show that we can find \( u \in K_0 \) with a unique component of maximal modulus.

We calculate how many vectors \( n_j \) that have the component 1 at the index \( i \), i.e. \( n_{j,i} = 1, i = 1, \ldots, n \), and denote this number by \( c_i \). Similarly, we calculate how many vectors \( n_j \) that have the component \( -1 \) at the index \( i \), i.e. \( n_{j,i} = -1, i = 1, \ldots, n \), and denote this number by \( d_i \). Let \( A = \max_{1 \leq i \leq n} \{ c_i, d_i \} \) and \( B = \min_{i: \max \{ c_i, d_i \} = A} \{ c_i, d_i \} \). Without loss of generality, we can assume that \((c_1, d_1) = (A, B)\).

We choose the coefficients \( \alpha_j \) such that

\[
 u = u(\varepsilon, \delta) = \sum_{j: n_{j,1} = 1} n_j + \varepsilon \sum_{j: n_{j,1} = -1} n_j + \delta \sum_{j: n_{j,1} = 0} n_j
\]

(16)

where it is assumed that \( \delta > \varepsilon > 0 \). Choosing \( \delta \) rather small together with the assumptions on \( c_1 \) and \( d_1 \) ensure that \( |u_1| = u_1 = \max_{1 \leq i \leq n} |u_i| \). To show that \( u_1 \) is the unique component of maximal modulus several special cases need to be considered.

If both \( c_i < c_1, i \neq 1 \), and \( d_i < c_1, i = 1, \ldots, n \), hold and \( \delta \), and thereby \( \varepsilon \), is chosen small enough then it is clear that \( u_1 \) is the unique component of maximal modulus of \( u \). Suppose now that there exists an index \( k \neq 1 \) such that either \( c_k = c_1 \) and \( d_k > d_1 \) or \( d_k = c_1 \) and \( c_k > d_1 \). Then it follows, again choosing \( \delta \) small enough, that \( u_1 \) is the unique component of maximal modulus of \( u \).

Next, consider the case when index \( k \neq 1 \) fulfills \( c_k = c_1, d_k = d_1 \) and \( d_1 \neq c_1 \). If \( |u_k| = u_1 \) then it is necessary that \( n_{j,k} = 1 \) for all \( j \) such that \( n_{j,1} = 1 \) as \( \delta \) can be chosen arbitrary close to 0. Also, if \( |u_k| = u_1 \) holds, we must have \( n_{j,k} = -1 \) for all \( j \) such that \( n_{j,1} = -1 \) because of \( \delta > \varepsilon \). Hence, it follows that \( n_{j,1} = n_{j,k} \), for all \( j = 1, \ldots, m \) since if \( n_{j,k} = 1 \) for some \( j \) where \( n_{j,1} = 0 \) we would have contradiction to the assumption of maximality of \( c_1 \) and if \( n_{j,k} = -1 \) for some \( j \) where \( n_{j,1} = 0 \) we would have contradiction to the assumption of minimality of \( d_k \). Then \( e_1 - e_k \perp n_j, j = 1, \ldots, m \) and
therefore \( \text{span} e_1 - e_k = \text{aff}(F) \), a contradiction to the assumption of \( v \) being a non-special direction.

The remaining cases (i) \( d_k = c_1 \), \( c_k = d_1 \) and \( c_1 \neq d_1 \) and (ii) \( c_1 = d_1 = c_k = d_k \) can be considered with analogous arguments which give that \( v \) must be a special direction in order for this to be possible.

We conclude that when \( v \) is a non-special direction it is possible to construct \( u \), by choosing \( \delta \) and \( \varepsilon \) properly, with an unique component of maximal modulus. With the construction of \( u \) established we will now consider two cases that cover all possible choices of non-special direction vector \( v \) of \( F \).

1) Assume that the direction vector \( v \) fulfills \( v_i \neq 0, \ i = 1, \ldots, n \). Since \( u \) has a unique coordinate of maximal modulus and \( v_i \neq 0, \ i = 1, \ldots, n \), we have

\[
||u - \mu v||_{\ell^\infty} < ||u||_{\ell^\infty}
\]

for \( \mu \in \mathbb{R} \) chosen with proper sign and small enough modulus. Note that \( \lambda v \in \Omega \) for \( \lambda \in \mathbb{R} \) with small enough modulus since 0 is an interior point of the edge \( F \). Hence, we arrive at a contradiction to the assumption of \( \Omega \) being an invariant \( K \)-minimal set with respect to \( (\ell^1, \ell^\infty) \).

2) Assume that the direction vector \( v \) have some indices \( i \in \{1, \ldots, n\} \) with \( v_i = 0 \). The problem here is that the above construction of \( u \) can give that the component of maximal modulus of \( u \) occur for an index \( i \) where \( v_i = 0 \). However, we will show that it is possible to construct \( u \) such that it has a unique component of maximal modulus for indices corresponding to non-zero components of \( v \). From this property we will then derive a contradiction to the assumption of \( v \) being a non-special direction.

Without loss of generality we can assume that \( v = (v_1, \ldots, v_k, 0, \ldots, 0) \) where \( v_i \neq 0, \ i = 1, \ldots, k \). Let \( P \) denote the orthogonal projection onto the \( k \)-dimensional subspace of \( \mathbb{R}^n \) spanned by \( e_i, \ i = 1, \ldots, k \). Note that \( v \perp n_j, \ j = 1, \ldots, m \), is equivalent to \( v \perp Pn_j, \ j = 1, \ldots, m \), from the assumption on \( v \). This gives that at least for one \( j \in \{1, \ldots, m\} \) we must have \( Pn_j \neq 0 \) because otherwise \( \text{span} \{e_1, \ldots, e_k\} \perp Pn_j, \ j = 1, \ldots, m \) which gives a contradiction to \( \text{span} \{v\} = \cap_{j=1}^m H_j \). With these observations at hand we can now repeat
the construction of $u$ but now with attention to the indices $i = 1,\ldots,k$. That is, $u$ is constructed such that it has a unique component of maximal modulus when restricting to the indices $1,\ldots,k$ and we do not care about what resulting components of $u$ we get for indices $k+1,\ldots,n$.

We claim that this construction of $u$ implies

$$K(t,u - \varepsilon v; \ell^1, \ell^\infty) < K(t,u; \ell^1, \ell^\infty) \quad \text{(17)}$$

for particular $t > 0$ by choosing $\varepsilon$ with proper sign and small enough modulus. Let us explain why. By $p$ we denote the index in $\{1,\ldots,k\}$ where $u$ has component of maximal modulus. Let $u^*$ and $(u - \varepsilon v)^*$ denote the decreasing rearrangements of $u$ and $u - \varepsilon v$ respectively. Choosing $\varepsilon$ with proper sign and small enough modulus, we obtain either

$$(u - \varepsilon v)^*_1 = |u_p - \varepsilon v_p| < |u_p| = u^*_1, \quad \text{(18)}$$

or

$$u^*_i = (u - \varepsilon v)^*_i, \ i = 1,\ldots,l - 1$$

$$(u - \varepsilon v)^*_l = |u_p - \varepsilon v_p| < |u_p| = u^*_l, \quad \text{(19)}$$

for some index $l \in \{2,\ldots,n\}$. Then (17) holds for $t \in (0,1]$ in the case of (18) and for $t \in (l-1,l]$ in the case of (19). Recall that 0 is the element of best approximation of $u$ in $\Omega$ with respect to the $\ell^2$-norm. Hence, $\Omega$ cannot be an invariant $K$-minimal set with respect to $(\ell^1, \ell^\infty)$ and we arrive at a contradiction.

Combining 1) and 2), which cover all configurations of non-special vectors, we conclude that $v$ must be a special direction. □

Our next lemma reads

**Lemma 3.4** Let $\Omega$ be a convex polytope in $\mathbb{R}^n$. Suppose the affine hull of any edge of $\Omega$ is a shifted one-dimensional subspace of $\mathbb{R}^n$ spanned by a special direction. Then the affine hull of any face of $\Omega$ is a shifted subspace of $\mathbb{R}^n$ spanned by special directions.
Proof. A face $F$ of $\Omega$ of dimension $k$ has $k+1$ affinely independent vertices \( \{x_0, x_1, ..., x_k\} \). We can by possibly shifting $\Omega$ assume that $x_0 = 0$. From the vertices we obtain $k$ linearly independent vectors $x_i - x_0 = x_i$, $i = 1, ..., k$. So, \( \{x_i : i = 1, ..., k\} \) is a basis of $\operatorname{aff}(F)$. Suppose that we can find $j \in \{1, ..., k\}$ for which $x_j$ is not a direction vector of some edge of $F$. The edge-vertex graph of a convex polytope is connected, see [5], so we can find a path $\tilde{P}$ between the vertices 0 and $x_j$ that consists of edges of $F$. Some edge $E$ of $\tilde{P}$ must then have a direction vector $v$ that is not in $\operatorname{span}\{x_i : i \neq j\}$, otherwise $x_j \in \operatorname{span}\{x_i : i \neq j\}$. By assumption $v$ is a special direction. We replace $x_j$ with $v$ and obtain a new basis $\{x_i : i \neq j\} \cup \{v\}$ of $\operatorname{aff}(F)$. Repeating this procedure for each vector $x_i$ that is not the direction vector of an edge of $\Omega$, we construct a basis of $\operatorname{aff}(F)$ only consisting of special directions. As the dimension of $F$ was arbitrarily chosen, the lemma follows. \( \square \)

By combining Lemmas 3.2, 3.3 and 3.4 with the next result, necessity of the special cone property will follow. See the proof of Theorem 3.2.

Lemma 3.5 Let $\Omega$ be a convex polytope where the affine hull of any face of $\Omega$ is a shifted subspace of $\mathbb{R}^n$ spanned by special directions. Then all faces of $\Omega$, in particular $\Omega$ itself, have the special cone property.

Proof. Let $F$ be an one-dimensional face of $\Omega$, i.e. an edge. Then the affine hull of $F$ by assumption is a shifted one-dimensional subspace of $\mathbb{R}^n$ spanned by a special direction $v$. Take any element $x \in \operatorname{relint}(F)$. Then

$$K_x := \{y \in \mathbb{R}^n : y = \alpha_1 v + \alpha_2 (-v), \alpha_i \geq 0\}$$

gives that $(x + K_x) \cap F = F$. For an element $x \in \operatorname{relbd}(F)$, i.e. $x$ is one of the two endpoints of the edge, we have $(x + K_x) \cap F = F$ where either

$$K_x := \{y \in \mathbb{R}^n : y = \alpha v, \alpha \geq 0\}$$

or

$$K_x := \{y \in \mathbb{R}^n : y = \alpha (-v), \alpha \geq 0\}.$$
Hence, $F$ has the special cone property.

We now assume that all faces of dimension $k-1$ have the special cone property. The problem is then to show that faces of dimension $k$ also have the special cone property. Below we will only consider $x=0$ since the set $\Omega$ can be shifted.

Let $F$ be a face of dimension $k$. By assumption, $\text{aff}(F)$ is spanned by some special directions $v_1, \ldots, v_m$. Suppose $0 \in \text{relint}(F)$. Then $\lambda v_i \in F$ for $|\lambda| \leq \varepsilon$, $i = 1, \ldots, m$, for some $\varepsilon > 0$. Take

$$K_0 := \left\{ y \in \mathbb{R}^n : y = \sum_{i=1}^{m} (\alpha_i v_i + \beta_i (-v_i)), \alpha_i \geq 0, \beta_i \geq 0 \right\}.$$  

Then it follows that $(0 + K_0) \cap F = F$.

Assume now that $0 \notin \text{relint}(F)$ so we have $0 \in \text{relbd}(F)$ since $F$ is closed. Restrict to $\text{aff}(F)$ where $F$ is a convex polytope of full dimension. The boundary of $F$, interpreted in $\text{aff}(F)$, is then $\text{relbd}(F)$. From Theorem 3, p. 27 in [5], follows that the boundary of a polyhedral convex set of full dimension is the union of its facets. Hence, we have $0 \in F_1 \cap F_2 \cap \ldots \cap F_l$ where the faces $F_i$, $i = 1, \ldots, l$, of $\Omega$ are facets of $F$ when restricting to $\text{aff}(F)$. From the induction assumption follows that there exist cones $K_0^{F_i}$ corresponding to the faces $F_i$, $i = 1, \ldots, l$, such that $K_0^{F_i} \cap F_i = F_i$.

If $l = 1$ then $0 \in \text{relint}(F_1)$. Take the cone $K_0^{F_1}$ which is generated by the special directions $\{v_1, \ldots, v_N\}$. These special directions constitute a basis of $\text{aff}(F_1)$. Since $\text{aff}(F)$ is spanned by special directions we can find an additional special direction $v \notin \text{span} \{v_1, \ldots, v_N\}$ for which we have $\lambda v \in F$ for $0 \leq \lambda \leq \varepsilon$ for some $\varepsilon > 0$. Then for the cone $K_0 := K_0^{F_1} + \alpha v$, $\alpha \geq 0$, we have $K_0 \cap F = F$ and we conclude that $F$ has the special cone property in this case.

Suppose now $l \geq 2$. In this case we will show that the cone $K_0$ defined by

$$K_0 := \sum_{i=1}^{l} K_0^{F_i}$$

is what is needed to show that $F$ has the special cone property.
Let \( n_i \in \text{aff}(F_i)^\perp \) such that \( n_i \in \text{aff}(F) \). Take
\[
\tilde{F} := \{ y \in \text{aff}(F) : \langle n_i, y \rangle \leq 0, i = 1, \ldots, l \}.
\]
We have \( F \subset \tilde{F} \) since \( F \) is defined through the intersection of more half-spaces in \( \text{aff}(F) \) than \( \tilde{F} \), i.e., for \( F \) we might need to include half-spaces in \( \text{aff}(F) \) not going through 0 in the intersection. However, we have
\[
F \cap B(0, \varepsilon) = \tilde{F} \cap B(0, \varepsilon)
\]
for small \( \varepsilon > 0 \) since \( F \) is the intersection of finitely many half-spaces and by choosing \( \varepsilon > 0 \) small enough we can avoid intersecting half-spaces that are only defining \( F \).

Let
\[
H := \{ y \in \text{aff}(F) : \langle n_1 + \ldots + n_l, y \rangle = 0 \}.
\]
Now, \( n_1 + \ldots + n_l \neq 0 \). Suppose not. Then for any \( y \in F \) we have both \( \langle n_i, y \rangle \leq 0, i = 1, \ldots, l \) and \( \langle n_1 + \ldots + n_l, y \rangle = 0 \) which together imply \( \langle n_i, y \rangle = 0, i = 1, \ldots, l \).

Hence, \( y \in F_i, i = 1, \ldots, l \) and we cannot have \( \text{dim}(F) = k \).

Since \( \text{dim(aff}(F_1) \cap \ldots \cap \text{aff}(F_l)) \leq k - 2 \) and \( \text{dim}(H) = k - 1 \) we can find a non-zero element \( u \in H \setminus (\text{aff}(F_1) \cap \ldots \cap \text{aff}(F_l)) \). For this \( u \) there must exist some \( i, j \in \{1, \ldots, l\} \) such that
\[
\langle n_i, u \rangle > 0
\]
and
\[
\langle n_j, u \rangle < 0
\]
respectively. Now, take \( z \in F \setminus \bigcup_{i=1}^l F_i \). We then have
\[
\langle n_i, z \rangle < 0,
\]
for \( i = 1, \ldots, l \). It follows by moving from \( z \) along the direction given by \( u \) that there exists an index \( i_0 \in \{1, \ldots, l\} \) and corresponding constant \( \beta > 0 \) such that
\[
\langle n_{i_0}, z + \beta u \rangle = 0
\]
and
\[
\langle n_i, z + \beta u \rangle \leq 0,
\]
for \( i \in \{1, \ldots, l \} \setminus \{i_0 \} \). So, \( z + \beta u \in \tilde{F} \cap \mathrm{aff}(F_{i_0}) \). Then for the convex combination \( \alpha(z + \beta u) + (1 - \alpha)0 = \alpha(z + \beta u), \alpha \in [0, 1] \), we have

\[
\langle n_{i_0}, \alpha(z + \beta u) \rangle = 0
\]

and

\[
\langle n_i, \alpha(z + \beta u) \rangle \leq 0,
\]

\( i \in \{1, \ldots, l \} \setminus \{i_0 \} \). So, \( \alpha(z + \beta u) \in \tilde{F} \cap \mathrm{aff}(F_{i_0}) \) for \( \alpha \in [0, 1] \). For small \( \alpha > 0 \), recalling (20), it follows that \( \alpha(z + \beta u) \in F_{i_0} \subset F \). On the facet \( F_{i_0} \) we have the special cone property by the induction assumption, i.e. \( F_{i_0} = K_{F_{i_0}}^{F_{i_0}} \cap F_{i_0} \) where

\[
K_{F_{i_0}}^{F_{i_0}} = \left\{ y \in \mathbb{R}^n : y = \sum_{j=1}^{m_{i_0}} \alpha_j v_{i_0}^j \right\},
\]  

(21)

\( \alpha_j \geq 0 \), for some special directions \( v_{i_0}^1, \ldots, v_{i_0}^{m_{i_0}} \). So, \( \alpha(z + \beta u) \in K_{F_{i_0}}^{F_{i_0}} \). Since \( K_{F_{i_0}}^{F_{i_0}} \) is a cone it follows that also \( z + \beta u \in K_{0}^{F_{i_0}} \). Repeating the same procedure for the vector \( -u \), we obtain a real number \( \gamma > 0 \) such that \( z + \gamma(-u) \in K_{0}^{F_{i_1}} \) where \( K_{0}^{F_{i_1}} \) is a cone generated by special directions, analogous in structure to (21), corresponding to the facet \( F_{i_1} \) and fulfilling \( F_{i_1} = F_{i_1} \cap K_{0}^{F_{i_1}} \). As \( z \) is on the line segment between \( z + \beta u \) and \( z + \gamma(-u) \) it follows that \( z \) is in the sum of the cones \( K_{0}^{F_{i_0}} \) and \( K_{0}^{F_{i_1}} \).

Recall the set \( K_0 \):

\[
K_0 := \sum_{i=1}^{l} K_{F_i}.
\]

By construction \( K_0 \) is a cone generated by special directions \( v_1, \ldots, v_m \) where \( \lambda v_i \in F \) for \( 0 \leq \lambda \leq \varepsilon \) with \( \varepsilon > 0 \) small enough. As \( z \in F \setminus \bigcup_{i=1}^{l} F_i \) was arbitrary we conclude that \( F \setminus \bigcup_{i=1}^{l} F_i \subset K_0 \). Further, from the definition of \( K_0 \) follows that \( \bigcup_{i=1}^{l} F_i \subset K_0 \). Hence, \( K_0 \cap F = F \) and we conclude that \( F \) has the special cone property. By induction the lemma follows. \( \Box \)

3.4. Characterizations

We now characterize all bounded, closed and convex sets in \( \mathbb{R}^n \) that are invariant \( K \)-minimal with respect to \( (\ell^1, \ell^\infty) \).
The first characterization is

**Theorem 3.2** A bounded, closed and convex set \( \Omega \subset \mathbb{R}^n \) is invariant \( K \)-minimal with respect to \((\ell^1, \ell^\infty)\) if and only if it has the special cone property.

**Proof.** The sufficiency of the special cone property follows from Theorem 3.1.

In the necessity direction, Lemma 3.2 gives that \( \Omega \) is a convex polytope. Next, combining Lemma 3.3 and Lemma 3.4 give that the affine hull of any face of \( \Omega \) is a shifted subspace of \( \mathbb{R}^n \) spanned by special directions. From Lemma 3.5 then follows that \( \Omega \) have the special cone property. \( \square \)

The second characterization is in the form of Theorem 0.1 which we state once again and now prove:

**Theorem 3.3** A bounded, closed and convex set \( \Omega \subset \mathbb{R}^n \) is invariant \( K \)-minimal with respect to \((\ell^1, \ell^\infty)\) if and only if \( \Omega \) is a convex polytope where the affine hull of any face of \( \Omega \) is a shifted subspace of \( \mathbb{R}^n \) spanned by special directions.

**Proof.** Suppose that the affine hull of any face of \( \Omega \) is a shifted subspace of \( \mathbb{R}^n \) spanned by special directions. It follows by Lemma 3.5 that \( \Omega \) has the special cone property. By Theorem 3.1 it then follows that \( \Omega \) is an invariant \( K \)-minimal set with respect to \((\ell^1, \ell^\infty)\).

In the other direction, we assume that \( \Omega \) is invariant \( K \)-minimal with respect to \((\ell^1, \ell^\infty)\). From Lemma 3.2 follows that \( \Omega \) is a convex polytope. Lemma 3.4 then gives that any face of \( \Omega \) is a shifted subspace of \( \mathbb{R}^n \) spanned by special directions. \( \square \)

**Remark 3.4** It is possible to formulate the characterization of Theorem 3.3 only in terms of affine hulls of edges. This follows by combining Lemma 3.4 and Theorem 3.3. The reason for the chosen formulation of Theorem 3.3 is that we think an analogous result is true even for unbounded sets, i.e. polyhedrons, which might contain no edges.
4. Hardy-Littlewood-Pólya inequality and phi-minimal sets

In this section we investigate connections between invariant $K$-minimal sets with respect to $(\ell^1, \ell^\infty)$ and the Hardy-Littlewood-Pólya majorization inequality. First, some general notion and definitions are introduced in Subsection 4.1. Then in Subsection 4.2 we show a characterization of $K$-minimal sets with respect to $(\ell^1, \ell^\infty)$ in terms of an inequality related to the Hardy-Littlewood-Pólya majorization inequality. Finally, in Subsection 4.3 we give two characterizations of invariant $\varphi$-minimal sets, one in terms of invariant $K$-minimal sets with respect to $(\ell^1, \ell^\infty)$ contained in a certain class of hyperplanes and one in the form of Theorem 0.2 given in the Introduction.

4.1. Definitions

Given the vector $z = (z_1, ..., z_n) \in \mathbb{R}^n$, let $z^\downarrow \in \mathbb{R}^n$ denote the vector with the elements of $z$ sorted in decreasing order, for example if $z = (1, -2, 4)$ then $z^\downarrow = (4, 1, -2)$. Now, let $x, y \in \mathbb{R}^n$. The result of Hardy-Littlewood-Pólya states that:

$$\sum_{i=1}^n \varphi(x_i) \leq \sum_{i=1}^n \varphi(y_i)$$

for all convex functions $\varphi: \mathbb{R} \to \mathbb{R}$ if and only if

$$\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow, \quad k = 1, ..., n - 1,$$

(22)

and

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$  

(23)

If (22) and (23) hold for $x, y \in \mathbb{R}^n$ we say that $x$ is majorized by $y$.

With the Hardy-Littlewood-Pólya inequality in mind we introduce the following notion:

**Definition 4.1** A set $\Omega \subset \mathbb{R}^n$ is called $\varphi$-minimal if there exists an element $x_\ast \in \Omega$ such that

$$\sum_{i=1}^n \varphi(x_\ast, i) \leq \sum_{i=1}^n \varphi(x_i)$$

for all $x \in \Omega$ and all convex functions $\varphi: \mathbb{R} \to \mathbb{R}$. 
Hence, $x_*$ is majorized by any other element $x \in \Omega$.

Analogously to invariant $K$-minimal sets, we introduce the notion of invariant $\varphi$-minimal sets:

**Definition 4.2** A set $\Omega \subset \mathbb{R}^n$ is called invariant $\varphi$-minimal if for any $a \in \mathbb{R}^n$, the set $\Omega - a$ is $\varphi$-minimal.

### 4.2. Characterization of $K$-minimal sets with respect to $(\ell^1, \ell^\infty)$

We have the following characterization of $K$-minimal sets with respect to the couple $(\ell^1, \ell^\infty)$:

**Theorem 4.1** The set $\Omega \subset \mathbb{R}^n$ is a $K$-minimal set with respect to $(\ell^1, \ell^\infty)$ if and only if there exists an element $x_* \in \Omega$ satisfying

$$
\sum_{i=1}^n \varphi(x_{*,i}) \leq \sum_{i=1}^n \varphi(x_i),
$$

for all $x \in \Omega$ and all even and convex functions $\varphi : \mathbb{R} \to \mathbb{R}$.

**Proof.** First, recall some notion from the theory of real interpolation. Let $(X_0, X_1)$ be a general Banach couple, $f \in X_0 + X_1$ and $t > 0$. The $E$-functional of $f$ is defined as

$$
E(t, f; X_0, X_1) := \inf_{\|h\|_{X_1} \leq t} \|f - h\|_{X_0}.
$$

The $K$-functional can be given in terms of the $E$-functional according to

$$
K(t, f; X_0, X_1) = \inf_{f = f_0 + f_1} \left( \|f_0\|_{X_0} + t \|f_1\|_{X_1} \right) = \inf_{s > 0, \|f_1\|_{X_1} = s} \left( \|f - f_1\|_{X_0} + ts \right) = \inf_{s > 0} \left( E(s, f; X_0, X_1) + ts \right). \tag{24}
$$

Now, the $E$-functional for $x \in \mathbb{R}^n$ with respect to the couple $(\ell^1, \ell^\infty)$ can be expressed as

$$
E(t, x; \ell^1, \ell^\infty) = \inf_{\|y\|_{\ell^\infty} \leq t} \|x - y\|_{\ell^1} = \inf_{\max_{1 \leq k \leq n} |y_k| \leq t} \sum_{i=1}^n |x_i - y_i| = \sum_{i=1}^n \inf_{|y_i| \leq t} |x_i - y_i| = \sum_{i=1}^n \max \{ |x_i| - t, 0 \} = \sum_{i=1}^n \psi_t(x_i)
$$

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where \( \psi_t(y) := \max \{|y| - t, 0\} \).

Suppose now that there exists an element \( x_* \in \Omega \) such that
\[
\sum_{i=1}^{n} \varphi(x_{*,i}) \leq \sum_{i=1}^{n} \varphi(x) \tag{25}
\]
for all \( x \in \Omega \) and all convex and even \( \varphi : \mathbb{R} \to \mathbb{R} \). Since \( \psi_t \) is convex and even for \( t > 0 \) we have from (25) that
\[
E(\cdot, x_*; \ell^1, \ell^\infty) \leq E(\cdot, x; \ell^1, \ell^\infty) \tag{26}
\]
for all \( x \in \Omega \). From (24) and (26) follows in turn that
\[
K(\cdot, x_*; \ell^1, \ell^\infty) \leq K(\cdot, x; \ell^1, \ell^\infty) \tag{27}
\]
for all \( x \in \Omega \). Hence, \( \Omega \) is a \( K \)-minimal set with respect to \( (\ell^1, \ell^\infty) \).

Assume now that \( \Omega \) is a \( K \)-minimal set with respect to \( (\ell^1, \ell^\infty) \). Then there is an element \( x_* \in \Omega \) such that
\[
K(\cdot, x_*; \ell^1, \ell^\infty) \leq K(\cdot, x; \ell^1, \ell^\infty) \tag{27}
\]
for all \( x \in \Omega \). Moreover, for a general Banach couple \((X_0, X_1)\) we have
\[
E(t, f; X_0, X_1) = \sup_{s > 0} (K(s, f; X_0, X_1) - st). \tag{28}
\]
From (27) and (28) then follows that
\[
E(\cdot, x_*; \ell^1, \ell^\infty) \leq E(\cdot, x; \ell^1, \ell^\infty)
\]
for all \( x \in \Omega \), or equivalently in terms of \( \psi_t \):
\[
\sum_{i=1}^{n} \psi_t(x_{*,i}) \leq \sum_{i=1}^{n} \psi_t(x_i) \tag{29}
\]
for all \( x \in \Omega \) and all \( t > 0 \).

Take an arbitrary convex and even function \( \varphi : \mathbb{R} \to \mathbb{R} \) that satisfies \( \varphi(0) = 0 \). Let \( \{f_m\}_{m \in \mathbb{N}} \) denote the sequence of piecewise linear functions which have knots in \((j/m, \varphi(j/m))\), \( j = 0, \pm 1, \pm 2, \ldots \), i.e. \( f_m(j/m) = \varphi(j/m) \). Now, \( f_m \) can be expressed in terms of the functions \( \psi_t \). Namely, we have
\[
f_m = \sum_{j=0}^{\infty} f_{m,j} \tag{30}
\]
where
\[ f_{m,j} = \gamma_{m,j} \psi_{\frac{m}{m}} \]
with
\[ \gamma_{m,j} = \frac{\varphi\left(\frac{j+1}{m}\right) - \varphi\left(\frac{j}{m}\right)}{\frac{1}{m}} - \frac{\varphi\left(\frac{j}{m}\right) - \varphi\left(\frac{j-1}{m}\right)}{\frac{1}{m}}. \]

Here \( \frac{\varphi\left(\frac{j}{m}\right) - \varphi\left(\frac{j-1}{m}\right)}{\frac{1}{m}} := \varphi(0) = 0 \) if \( j = 0 \). Note that \( \gamma_{m,j} \geq 0 \) as \( \varphi \) is convex and increasing on \([0, \infty)\).

Let next \( y \in \mathbb{R} \) be given. We then have \( y \in \left[ k_{\frac{m}{m}}, k_{\frac{m+1}{m}} \right) \) for some \( k_{\frac{m}{m}} \in \mathbb{Z} \) and therefore \( f_{m}(y) = \alpha_{m} \varphi\left(k_{\frac{m}{m}}\right) + (1 - \alpha_{m}) \varphi\left(k_{\frac{m+1}{m}}\right) \) for some \( \alpha_{m} \in [0, 1] \). We then estimate
\[
|\varphi(y) - f_{m}(y)| = \left| \varphi(y) - \left( \alpha_{m} \varphi\left(k_{\frac{m}{m}}\right) + (1 - \alpha_{m}) \varphi\left(k_{\frac{m+1}{m}}\right) \right) \right| \leq \\
\leq \alpha_{m} \left| \varphi(y) - \varphi\left(k_{\frac{m}{m}}\right) \right| + (1 - \alpha_{m}) \left| \varphi(y) - \varphi\left(k_{\frac{m+1}{m}}\right) \right| \leq (31)
\]

Since \( k_{\frac{m+1}{m}} - k_{\frac{m}{m}} = \frac{1}{m} \to 0 \) as \( m \to \infty \) and \( \varphi \) is continuous on \( \mathbb{R} \) it follows from the estimates (31) that
\[
\lim_{m \to \infty} f_{m}(y) = \varphi(y), \ \forall y \in \mathbb{R}. \hspace{1cm} (32)
\]

With (29), (30) and (32) we can derive
\[
\sum_{i=1}^{n} \varphi(x_{i},i) = \lim_{m \to \infty} \sum_{i=1}^{n} f_{m}(x_{i},i) = \lim_{m \to \infty} \sum_{i=1}^{n} \sum_{j=0}^{\infty} \gamma_{m,j} \psi_{\frac{m}{m}}(x_{i},i) = \\
\leq \lim_{m \to \infty} \sum_{i=1}^{n} \sum_{j=0}^{\infty} \gamma_{m,j} \psi_{\frac{m}{m}}(x_{i}) \leq \lim_{m \to \infty} \sum_{i=1}^{n} \sum_{j=0}^{\infty} \gamma_{m,j} \psi_{\frac{m}{m}}(x_{i}) = \\
= \lim_{m \to \infty} \sum_{i=1}^{n} \sum_{j=0}^{\infty} \gamma_{m,j} \psi_{\frac{m}{m}}(x_{i}) = \lim_{m \to \infty} \sum_{i=1}^{n} f_{m}(x_{i}) = \sum_{i=1}^{n} \varphi(x_{i}).
\]

Now, we can add any constant \( C \in \mathbb{R} \) to \( \varphi \) and have from the above inequality
\[
\sum_{i=1}^{n} (\varphi(x_{i},i) + C) \leq \sum_{i=1}^{n} (\varphi(x_{i}) + C).
\]

As \( \phi = \psi(y) = \varphi(y) + C \) is convex and even on \( \mathbb{R} \) it follows that we can drop the assumption \( \varphi(0) = 0 \). The theorem is thereby established. \( \square \)
4.3. Characterizations of invariant \( \phi \)-minimal sets

With Theorem 4.1 at our disposal we show the following characterization of bounded, closed and convex sets in \( \mathbb{R}^n \) that are invariant \( \phi \)-minimal.

**Theorem 4.2** Let \( \Omega \subset \mathbb{R}^n \) be a bounded, closed and convex set. Then \( \Omega \) is invariant \( \phi \)-minimal if and only if (i) \( \Omega \subset H = \{ y \in \mathbb{R}^n : \sum_{i=1}^{n} y_i = C \} \) for some \( C \in \mathbb{R} \) and (ii) \( \Omega \) is invariant \( K \)-minimal with respect to \( (\ell^1, \ell^\infty) \).

**Proof.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded, closed and convex set. Suppose first that \( \Omega \) is invariant \( \phi \)-minimal. Then it is clear that \( \Omega \subset H = \{ y \in \mathbb{R}^n : \sum_{i=1}^{n} y_i = C \} \), from the Hardy-Littlewood-Pólya majorization inequality. Next, from Theorem 4.1 follows that \( \Omega \) is invariant \( K \)-minimal with respect to \( (\ell^1, \ell^\infty) \). Hence, one direction in the characterization is established.

Suppose now that \( \Omega \) is invariant \( K \)-minimal with respect to \( (\ell^1, \ell^\infty) \) and

\[
\Omega \subset H = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{n} y_i = C \right\}
\]

for some \( C \in \mathbb{R} \). Let \( \mathbb{R}^n_+ = \{ y \in \mathbb{R}^n : y_i \geq 0, i = 1, \ldots, n \} \). Since \( \Omega \) is bounded we can find \( D = \{ d, \ldots, d \} \in \mathbb{R}^n \) such that \( \Omega + D \in \mathbb{R}^n_+ \). As \( \Omega \) is invariant \( K \)-minimal with respect to \( (\ell^1, \ell^\infty) \) there exists \( x^D_+ \in \Omega + D \) such that

\[
K(\cdot, x^D_+; \ell^1, \ell^\infty) \leq K(\cdot, x; \ell^1, \ell^\infty)
\]

for every \( x \in \Omega + D \). Moreover, for \( x \in \Omega + D \subset \mathbb{R}^n_+ \) we have

\[
K(t, x; \ell^1, \ell^\infty) = \begin{cases} 
  tx^1_+, & 0 < t \leq 1, \\
  \sum_{i=1}^{k} x^1_i + (t-k) x^1_{k+1}, & k < t \leq k+1, k = 0, 1, \ldots, n-1, \\
  \sum_{i=1}^{n} x^1_i, & t > n 
\end{cases}
\]

From the Hardy-Littlewood-Pólya majorization inequality then follows

\[
\sum_{i=1}^{n} \phi(x^D_{i+1}) \leq \sum_{i=1}^{n} \phi(x_i)
\]  

for every \( x \in \Omega + D \) and every convex function \( \phi : \mathbb{R} \rightarrow \mathbb{R} \). As the mapping \( \tilde{\phi}(\cdot) := \phi(\cdot + d) \) from the set of all real-valued convex functions on \( \mathbb{R} \) into itself
is a bijection it follows from (33) that for $x^* = x^D - D \in \Omega$ we have

$$\sum_{i=1}^{n} \varphi(x^*_{i}) \leq \sum_{i=1}^{n} \varphi(x_{i})$$

for every $x \in \Omega$ and every convex function $\varphi : \mathbb{R} \to \mathbb{R}$. So, $\Omega$ is $\varphi$-minimal.

For any given $a \in \mathbb{R}^n$ we can repeat the procedure and conclude that $\Omega - a$ is $\varphi$-minimal. Hence, $\Omega$ is invariant $\varphi$-minimal. □

To conclude this section, we state Theorem 0.2 again and give a proof of it.

**Theorem 4.3** A bounded, closed and convex set $\Omega \subset \mathbb{R}^n$ is invariant $\varphi$-minimal if and only if $\Omega$ is a convex polytope where the affine hull of any face of $\Omega$ is a shifted subspace of $\mathbb{R}^n$ spanned by special directions of the type $e_i - e_j$.

**Proof.** Let $\Omega \subset \mathbb{R}^n$ be bounded, closed and convex. Suppose $\Omega$ is invariant $\varphi$-minimal. This is equivalent, by Theorem 4.2, to $\Omega$ being invariant $K$-minimal with respect to $(\ell^1, \ell^\infty)$ and lying in some hyperplane. That $\Omega$ is invariant $K$-minimal with respect to $(\ell^1, \ell^\infty)$ is by Theorem 3.3 equivalent to $\Omega$ being a convex polytope where the affine hull of any face of $\Omega$ is a shifted subspace of $\mathbb{R}^n$ spanned by special directions. Finally, the condition of $\Omega$ lying in a hyperplane gives that the special directions are of the type $e_i - e_j$. □

**Remark 4.1** Analogous to Theorem 3.3 it is possible to restrict only to affine hulls of edges in the formulation of Theorem 4.3.

5. Applications

Taut strings and, more generally, invariant $K$-minimal sets with respect to the couple $(\ell^1, \ell^\infty)$ are connected with a broad range of applications. In this section we give an example of application where the setting of classical taut strings is too restrictive but where the more general notion of invariant $K$-minimal sets with respect to the couple $(\ell^1, \ell^\infty)$ is applicable.
5.1. Taut string with free ends

Recall the classical taut string problem, first given in the Introduction:

**Problem 5.1** Find the function $f_* \in \Gamma_{F,G}$ that satisfies
\[
\int_a^b \sqrt{1 + (f'_*(x))^2} \, dx = \inf_{f \in \Gamma_{F,G}} \int_a^b \sqrt{1 + (f'(x))^2} \, dx.
\]

The connection between the classical taut string problem and invariant $\varphi$-minimal sets is given by:

**Theorem 5.1** The set
\[
\Omega = \{ u \in \mathbb{R}^n : u_i = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}, i = 1, \ldots, n, f \in \Gamma_{F,G} \}
\]

is an invariant $\varphi$-minimal set.

**Proof.** Recall that $x_i - x_{i-1} = (b - a)/n$. We therefore have for $u \in \Omega$
\[
\sum_{i=1}^n u_i = \frac{f(x_n) - f(x_0)}{(b - a)/n} = C
\]
for some $C \in \mathbb{R}$ since $f(x_0) = F(a)$ and $f(x_n) = F(b)$.

Next, we can represent $u \in \Omega$ according to
\[
u = -\frac{f(x_0)}{(b - a)/n} e_1 + \sum_{i=1}^{n-1} \frac{f(x_i)}{(b - a)/n} (e_i - e_{i+1}) + \frac{f(x_n)}{(b - a)/n} e_n
\]
for some $f(x_i) \in [F(x_i), G(x_i)]$, $i = 1, \ldots, n - 1$. The set $S_u$ of generating special directions of the cone $K_u$ is then determined according to the following. Consider $i = 1, \ldots, n - 1$. If $f(x_i) = F(x_i)$ then $e_i - e_{i+1} \in S_u$, if $G(x_i) > f(x_i) > F(x_i)$ then $\pm (e_i - e_{i+1}) \in S_u$ and finally if $f(x_i) = G(x_i)$ then $- (e_i - e_{i+1}) \in S_u$. By construction it follows that $(u + K_u) \cap \Omega = \Omega$ and that $u + \beta v \in \Omega$, $\forall v \in S_u$, for small enough $\beta > 0$. As $u \in \Omega$ was arbitrary chosen we conclude that $\Omega$ has the special cone property which is equivalent to $\Omega$ being invariant $K$-minimal with respect to $(l^1, l^\infty)$.

From Theorem 4.2 now follows that $\Omega$ is invariant $\varphi$-minimal. \hfill \Box

So, the taut string is a minimizer for any choice of convex function $\varphi : \mathbb{R} \to \mathbb{R}$ in Problem 5.1.
The fixed end point conditions

\[ F(a) = G(a), F(b) = G(b) \]  \hspace{1cm} (34)

of classical taut strings are rather restrictive. Suppose that the conditions in (34) are relaxed, i.e. we allow for \( F(a) < G(a) \) and \( F(b) < G(b) \). The problem under study can still be formulated as

**Problem 5.2** Find the function \( f^* \in \Gamma_{F,G} \) that satisfies

\[ \int_a^b \sqrt{1 + (f'(x))^2} \, dx = \inf_{f \in \Gamma_{F,G}} \int_a^b \sqrt{1 + (f'(x))^2} \, dx. \]

Now, if \( F(a) < G(a) \) and/or \( F(b) < G(b) \) then

\[ \Omega = \left\{ u \in \mathbb{R}^n : u_i = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}, i = 1, ..., n, f \in \Gamma_{F,G} \right\} \]

is not an invariant \( \varphi \)-minimal set any more. This follows since there is no constant \( C \in \mathbb{R}^n \) such that \( \sum_{i=1}^n u_i = C \) for every \( u \in \Omega \). However, we still have

**Theorem 5.2** The set \( \Omega \) is an invariant \( K \)-minimal set with respect to \((l^1, l^\infty)\).

**Proof.** The proof is similar to the second part of the proof of Theorem 5.1. Besides what is outlined there we need to include additional generating special directions if \( F(x_0) < G(x_0) \) and/or \( F(x_n) < G(x_n) \). Suppose \( F(x_0) < G(x_0) \). If \( f(x_0) = F(x_0) \) then \( e_1 \in S_u \), if \( F(x_0) < f(x_0) < G(x_0) \) then \( \pm e_1 \in S_u \), and finally if \( f(x_0) = G(x_0) \) then \( -e_1 \in S_0 \). Similar considerations are done if \( F(x_n) < G(x_n) \) to determine if \( e_n \in S_u \) and \( -e_n \in S_u \).

---

5.2. Taut strings and Wiener process

In this section we briefly consider an application of taut strings with a free end to the Wiener process (standard one-dimensional Brownian motion). For an introduction to the Wiener process, we refer to Chapter 2 of [9]. Take a uniform piecewise linear approximation of the trajectory of the Wiener process.
on the interval \([0, T]\), i.e. \(W\) is evaluated at \(i \frac{T}{N}\), \(i = 0, 1, \ldots, N\), for some \(N \in \mathbb{N}\) and interpolated linearly in between these nodes. Given \(r, T > 0\), let \(F\) and \(G\) be the continuous piecewise linear functions on \([0, T]\) with nodes in \(i \frac{T}{N}\), \(i = 0, 1, \ldots, N\), where \(F(0) = G(0) = W(0) = 0\), \(F\left(i \frac{T}{N}\right) = W\left(i \frac{T}{N}\right) - r\) and \(G\left(i \frac{T}{N}\right) = W\left(i \frac{T}{N}\right) + r\), \(i = 1, \ldots, N\). In agreement with previous notation, let \(\Gamma_{F,G}\) denote the set of all continuous piecewise linear functions \(f\) on \([0, T]\) with nodes in \(i \frac{T}{N}\), \(i = 0, 1, \ldots, N\), and which satisfies \(F \leq f \leq G\). Hence, a function \(f \in \Gamma_{F,G}\) will start at 0, similar to the trajectory of the Wiener process, and for \(t \in (0, T]\) be at most a distance \(r\) from the approximate trajectory of the Wiener process.

Consider the problem

**Problem 5.3** Find the function \(f_* \in \Gamma_{F,G}\) that satisfies

\[
\int_0^T \sqrt{1 + (f'_*(t))^2} \, dt = \inf_{f \in \Gamma_{F,G}} \int_0^T \sqrt{1 + (f'(t))^2} \, dt.
\]

This is a taut string problem with a free end at \(t = T\). The set

\[
\Omega := \left\{ u \in \mathbb{R}^N : u_i = \frac{f\left(i \frac{T}{N}\right) - f\left((i-1) \frac{T}{N}\right)}{iT/N}, f \in \Gamma_{F,G} \right\}
\]

is therefore by Theorem 5.2 an invariant \(K\)-minimal set with respect to \((\ell^1, \ell^\infty)\). Hence, from Theorem 4.1 follows that the taut string \(f_*\) in addition also satisfies

\[
\int_0^T \varphi(f'_*(t)) \, dt = \inf_{f \in \Gamma_{F,G}} \int_0^T \varphi(f'(t)) \, dt
\]

for any convex and even function \(\varphi : \mathbb{R} \rightarrow \mathbb{R}\). Moreover, the algorithm for constructing the element \(u_* \in \Omega\) with minimal \(K\)-functional can be applied to compute the taut string \(f_*\).

Take now \(\varphi(x) = x^2\) and consider the quantity

\[
r \left( \frac{1}{T} \int_0^T (f'_*(t))^2 \, dt \right)^{1/2},
\]

i.e. the distance \(r\) times the square root of the average power of the taut string. Interestingly, numerical studies suggests that the mean of this quantity, for a
sample of independent trajectories of the Wiener process, converges towards a constant $C \approx 0.63$ as $T \to \infty$ with step size $T/N$ small enough. This numerical result has a rigorous interpretation in the continuous setting that we now describe.

Let $W$ be the Wiener process. Given $r, T > 0$ let

$$\Lambda_{r,T} := \left\{ f \in AC[0,T] : \sup_{t \in [0,T]} |f(t) - W(t)| \leq r, f(0) = W(0) = 0 \right\}$$

where $AC[0,T]$ denotes the space of absolutely continuous functions on $[0,T]$. So, $\Lambda_{r,T}$ is the tube of radius $r$ around the trajectory of the Wiener process on the interval $[0,T]$. In [11] it is established that there is a constant $C \in (0,\infty)$ such that for any fixed $r > 0$

$$\inf_{f \in \Lambda_{r,T}} r \left( \frac{1}{T} \int_0^T f'(t)^2 \, dt \right)^{1/2} \to C$$

both in $q$th mean, for any $q > 0$, and almost surely as $T \to \infty$. Hence, an absolutely continuous function must asymptotically spend on average $C^2/r^2$ amount of energy per unit of time if it is constrained to stay within the distance $r$ from the almost surely non-differentiable trajectory of $W$. The exact value of $C$ is unknown at present. In [11], a lower bound of approximately 0.38 and an upper bound of $\frac{\pi}{2}$ of $C$ is proved. Recall that the numerical simulations of the
discrete problem suggests $C \approx 0.63$ which is within the limits of the lower and upper bounds.

References


