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Abstract

Most high order methods for solving conservation laws can be shown to satisfy a summation-by-parts rule. In this work we present a general framework for multi-block and multi-element summation-by-parts implementations in several dimensions that includes most, if not all of the previously known schemes on summation-by-parts form. This includes finite volume, spectral and nodal discontinuous Galerkin methods, as well as high order multi-block finite difference schemes on curvilinear domains. We use the framework to derive general conditions for conservation and stability, and formulate extended representations of conservative and energy stable couplings between completely general multi-block, multi-element or hybrid meshes.

1. Introduction

Summation-by-parts (SBP) operators were originally developed in order to improve the stability properties of High Order Finite Difference Methods (HOFDM) [1, 2]. Together with a suitable technique to impose boundary conditions, these operators enable a straightforward application of the energy method to prove stability for discretizations of well posed problems. Especially important for this development was the introduction of the Simultaneous-Approximation-Term (SAT) technique for imposing well posed boundary conditions weakly [3]. The combined SBP-SAT technique is an ideally suited framework for constructing proofs of numerical conservation and energy stability, and it has been applied to various high order methods in a wide range of applications. See [4, 5] for a comprehensive review of this development

including applications. The utility of HOFDM's on SBP-SAT form was significantly enhanced with the development of curvilinear and multi-domain techniques [6, 7, 8, 9, 10], including also the recent develoment of interpolation techniques for multi-domain formulations with non-conforming grid interfaces [11, 12, 13, 14].

Although traditionally applied to HOFDM's, the SBP-SAT technique has more recently been extended to include a number of other classes of methods as well. The node-centered finite volume method (FVM) for unstructured grids was augmented with SAT conditions in [15, 16], allowing for stable hybrid implementations with HOFDM and FVM [17, 18, 19]. In [20], the SBP-SAT technique was successfully extended to spectral Galerkin methods on arbitrary grids, and in [21] it was demonstrated that some nodal discontinuous Galerkin methods (dG) can also be included within the SBP-SAT framework. The element approach of such dG methods can be seen as another form of the multi-block HOFDM approach.

A generalized theory for one-dimensional SBP operators was presented in [22], incorporating such non-classical operators as those appearing in spectral Galerkin and dG methods, and also including the case of nodal distributions that do not conform to the element boundaries. Finally we highlight the recent extension of SBP-SAT schemes to the time domain [23, 24, 25, 26, 27, 28], which has been shown to incorporate many of the classical Runge-Kutta methods [29], as well as to allow for stable space-time implementations with moving curvilinear domains [7].

Introducing a general framework for multi-block and hybrid methods on SBP-SAT form offers a number of possible opportunities. Firstly, it allows for compact formulations of theoretical proofs involving concepts such as conservation, stability and accuracy for a large class of discretizations. Secondly, it formalizes the construction and implementation of stable interface couplings between general SBP-SAT discretzations. Thirdly, it could lead to new theoretical insights that may be utilized to incorporate further classes of methods in the SBP-SAT framework, or alternatively to aid in the development of entirely new types of methods.

In this work we introduce the concept of multi-domain SBP operators, and use this to formulate a general theory for multi-block (or equivalently, multi-element) SBP discretizations in several dimensions. This collects all of the above mentioned methods and techniques into a single compact framework. It also allows for several operators defined on different subdomains to be merged into a single one by the use of interface couplings. We demonstrate

the generality of the new framework with several explicit examples, including dG formulations and HOFDM operators on curvilinear grids. We identify the properties needed for conservative and energy stable implementations with the use of multi-domain SBP operators. In particular, we formulate a theoretical result showing that conservation is a necessary prerequisite to energy stability for a large class of high order accurate SBP discretizations. This result leads directly to an extended formulation of so called SBP preserving interpolation operators, a concept first introduced in [11] for cartesian SBP discretizations, which guarantees both conservation and energy stability. The framework presented herein thus provides a concise recipe for conservative and energy stable implementations on completely general multi-block and hybrid grids.

For clarity, we focus in this paper on hyperbolic problems with constant coefficients. The addition of parabolic terms can be done separately and independently of the hyperbolic treatment, and does not pose any additional difficulties.

2. One-dimensional discretizations

We begin with a review of the well-known properties of one-dimensional SBP operators and the corresponding SAT method of imposing boundary and interface conditions weakly. For simplicity of presentation, we avoid the generalized approach in [22] and assume that all grids conform to the domain boundaries.

Consider a scalar constant coefficient hyperbolic problem in one space dimension with unit wave speed, written as

$$\begin{aligned} u_t + u_x = 0, \qquad & x \in (\alpha, \beta) \subset \mathbb{R} \quad t > 0 \\ u = g(t), \quad & x = \alpha \qquad\qquad t > 0 \\ u = f(x), \quad & t = 0 \qquad\qquad x \in (\alpha, \beta). \end{aligned} \tag{1}$$

For functions ϕ and ψ defined on (α, β), we define the L_2 scalar product and norm as

$$(\phi, \psi) = \int_\alpha^\beta \phi\psi \; dx, \quad \|\phi\|^2 = \sqrt{(\phi, \phi)}. \tag{2}$$

Central to the continuous analysis of the model problem (1) is the Integration-By-Parts (IBP) rule in one dimension, given by

$$(\phi, \psi_x) = (\phi\psi)|_\beta^\alpha - (\phi_x, \psi). \tag{3}$$

This property can be used to prove well-posedness of (1) with the energy method. Indeed, we multiply (1) with u and integrate over the domain. The result can be written on the form

$$\|u\|_t^2 + (u, u_x) + (u_x, u) = 0.$$

After inserting the boundary conditions, the IBP formula (3) now leads directly to the energy estimate

$$\|u\|_t^2 = g(t)^2 - u(\beta, t)^2. \tag{4}$$

2.1. The discrete problem

Next, we discretize the domain (α, β) by introducing a discrete grid $X = (X_0, X_1, \ldots, X_N)^T$, where for simplicity we assume that $X_0 = \alpha$ and $X_N = \beta$, and a corresponding discrete solution vector U. We denote the discrete solution at the two boundary points with $U_\alpha = U_0$ and $U_\beta = U_N$, and introduce restriction operators in the form of two row vectors $e_\alpha = (1, 0, \ldots, 0)$ and $e_\beta = (0, \ldots, 0, 1)$, so that

$$U_\alpha = e_\alpha U, \quad U_\beta = e_\beta U. \tag{5}$$

We discretize (1) using the standard SBP-SAT technique. With the dual consistent choice of penalty coefficient $\sigma = -1$ (see [30, 31]), we get

$$U_t + P^{-1}QU = -P^{-1}e_\alpha^T(U_\alpha - g(t)), \tag{6}$$

where $P^{-1}Q$ is a discrete SBP first derivative operator. The penalty term on the right hand side in (6) enforces the boundary condition using the weak SAT penalty technique, while the column vector e_α^T acts to impose the condition on the row corresponding to the variable U_α in (6). The minus sign in front of the penalty term signifies that the point $x = \alpha$ is located at the left boundary of the domain.

By referring to $P^{-1}Q$ as an SBP operator, we state that it satisfies a discrete version of the IBP property (3), as defined in the following way. Firstly, we require that P is a symmetric and positive definite matrix, assuming the role of a discrete L_2 integration operator. Let Φ and Ψ be two vectors of the same dimension as X. The discrete L_2 scalar product and norm induced by P are then defined as

$$(\Phi, \Psi)_P = \Phi^T P \Psi, \quad \|\Phi\|_P^2 = \sqrt{(\Phi, \Phi)_P}. \tag{7}$$

Secondly, the discrete version of (3) is obtained by imposing the SBP property

$$Q + Q^T = e_\beta^T e_\beta - e_\alpha^T e_\alpha. \tag{8}$$

This leads directly to the following discrete relation, mimicking the IBP property (3)

$$(\Phi, P^{-1}Q\Psi)_P = \Phi_\beta\Psi_\beta - \Phi_\alpha\Psi_\alpha - (P^{-1}Q\Phi, \Psi)_P, \tag{9}$$

since

$$(\Phi, P^{-1}Q\Psi)_P = \Phi^T Q\Psi = \Phi^T(Q+Q^T-Q^T)\Psi = \Phi^T(Q+Q^T)\Psi-(P^{-1}Q\Phi, \Psi)_P.$$

To show that (6) is stable we apply the discrete energy method by multiplying with $U^T P$ from the left and then adding the transpose. We get, after adding and subtracting g^2,

$$\frac{d}{dt}\|U\|_P^2 = g^2 - U_\beta^2 - (U_\alpha - g)^2. \tag{10}$$

Clearly, (10) is an energy stable discrete approximation of the continuous estimate (4).

2.2. Example: Galerkin approximations

As was mentioned in the introduction, the SBP-SAT technique was extended to spectral Galerkin methods on arbitrary grids in [20]. Here we follow the approach taken in [32], and consider a Galerkin approximation of (1) using the discrete grid X as node distribution. We let v be a finite dimensional polynomial approximation of u, using the Lagrange basis on X, given by

$$v = L(x)^T U, \tag{11}$$

where U is the discrete solution in the node points, and $L(x) = (l_0(x), \dots, l_N(x))^T$ is the set of Lagrange polynomials on X.

We insert (11) into (1), multiply with L and then integrate in space to get

$$PU_t + QU = 0, \tag{12}$$

where

$$P = \int_\alpha^\beta LL^T\, dx, \quad Q = \int_\alpha^\beta LL_x^T\, dx. \tag{13}$$

We first note that P is a symmetric and positive definite matrix. Moreover, applying the continuous IBP rule (3) to the expression for Q, we get

$$Q + Q^T = \int_\alpha^\beta (LL_x^T + L_x L^T) dx = (LL^T)|_\alpha^\beta. \tag{14}$$

The Lagrange basis of polynomials satisfy the following conditions

$$\begin{aligned} l_i(\alpha) l_j(\alpha) &= 1, \quad \text{if } i = j = 0, \quad 0 \text{ otherwise.} \\ l_i(\beta) l_j(\beta) &= 1, \quad \text{if } i = j = N, \quad 0 \text{ otherwise.} \end{aligned}$$

The boundary terms in (14) can thus be written as

$$L(\alpha)L(\alpha)^T = e_\alpha^T e_\alpha, \quad L(\beta)L(\beta)^T = e_\beta^T e_\beta. \tag{15}$$

Thus, (14) becomes exactly the SBP property (8), and hence $P^{-1}Q$ is an SBP operator. After multiplying (12) with P^{-1} and adding weak SAT boundary conditions to the right hand side, we get a discretization on the SBP-SAT form (6).

It should also be noted that in practice, the matrix P appearing in Galerkin approximations such as (12) is typically approximated with a quadrature rule. If those quadrature points are collocated with the Lagrange interpolation points X, then P becomes a diagonal matrix with the quadrature weights positioned on the diagonal. Indeed, by approximating each entry in P (13) with a quadrature rule, we get

$$P_{ij} = \int_\alpha^\beta l_i(x) l_j(x)\, dx \approx \sum_{k=0}^{N} p_k l_i(x_k) l_j(x_k) = p_k \delta_{ij},$$

where δ_{ij} is the Kronecker delta function, and p_k denote the weights of the quadrature rule. The most common choices of such collocation points are those associated with Gauss-Legendre or Gauss-Legendre-Lobatto quadrature formulas [33]. In the latter case, the boundaries of the interval are included in the node set while in the former they are not. The Gauss-Legendre quadrature rules hence requires an extended definition of the boundary operators e_α and e_β based on extrapolation, see [22].

3. Multi-domain SBP operators in one dimension

In this section we extend the general theory for one-dimensional SBP operators outlined above to include multi-domain discretizations.

3.1. General formulation

We begin by considering the two-domain case, and thus divide the domain $[\alpha, \beta]$ into the two subintervals $\Omega_L = [\alpha, x_I]$ and $\Omega_R = [x_I, \beta]$, with a common interface at the point $x = x_I$. We introduce the corresponding discrete grids X^L and X^R and discrete solutions U^L and U^R, as well as two SBP operators $P_L^{-1}Q_L$ and $P_R^{-1}Q_R$ approximating $\frac{\partial}{\partial x}$ on the two grids respectively.

Recall the one-dimensional SBP property (8) for these two operators, given by

$$\begin{aligned} Q_L + Q_L^T &= e_{L,x_I}^T e_{L,x_I} - e_{L,\alpha}^T e_{L,\alpha} \\ Q_R + Q_R^T &= e_{R,\beta}^T e_{R,\beta} - e_{R,x_I}^T e_{R,x_I}. \end{aligned} \tag{16}$$

Now consider the following form of a two-domain SBP-SAT discretization, in which the interface penalty treatment is scaled by the two penalty coefficients σ_L and σ_R,

$$\begin{aligned} U_t^L + P_L^{-1}Q_L U^L &= P_L^{-1} e_{L,x_I}^T \sigma_L (U_{x_I}^L - U_{x_I}^R) - e_{L,\alpha}^T (U_\alpha^L - g(t)) \\ U_t^R + P_R^{-1}Q_R U^R &= P_R^{-1} e_{R,x_I}^T \sigma_R (U_{x_I}^R - U_{x_I}^L). \end{aligned} \tag{17}$$

Similar to the approach taken in the analysis of multi-domain SBP implementations in [6], we proceed by considering a more compact formulation.

The system in (17) can be written

$$U_t + P^{-1}QU = -P^{-1}e_\alpha^T (U_\alpha - g(t)), \tag{18}$$

where $U = ((U^L)^T, (U^R)^T)^T$ is the combined discrete solution from the two grids. Moreover, we have defined the restrictions to the physical boundaries of the whole domain as $U_\alpha = e_\alpha U$ and $U_\beta = e_\beta U$, where

$$e_\alpha = \begin{pmatrix} e_{L,\alpha} & 0 \end{pmatrix}, \quad e_\beta = \begin{pmatrix} 0 & e_{R,\beta} \end{pmatrix}. \tag{19}$$

Expanding the interface treatment in (17) on matrix form, we get

$$\begin{aligned} &\begin{pmatrix} e_{L,x_I}^T \sigma_L (U_{x_I}^L - U_{x_I}^R) \\ e_{R,x_I}^T \sigma_L (U_{x_I}^R - U_{x_I}^L) \end{pmatrix} = \begin{pmatrix} e_{L,x_I}^T \begin{pmatrix} \sigma_L & -\sigma_L \end{pmatrix} \\ e_{R,x_I}^T \begin{pmatrix} -\sigma_R & \sigma_R \end{pmatrix} \end{pmatrix} \begin{pmatrix} U_{x_I}^L \\ U_{x_I}^R \end{pmatrix} \\ =& \begin{pmatrix} e_{L,x_I}^T & 0 \\ 0 & e_{R,x_I}^T \end{pmatrix} \begin{pmatrix} \sigma_L & -\sigma_L \\ -\sigma_R & \sigma_R \end{pmatrix} \begin{pmatrix} e_{L,x_I} & 0 \\ 0 & e_{R,x_I} \end{pmatrix} U \\ =& E_{x_I}^T \Sigma_{x_I} E_{x_I} U. \end{aligned} \tag{20}$$

In (20), E_{x_I} is a 2-by-$N_L + N_R + 2$ rectangular matrix that restricts the solution to both interface points, and Σ_{x_I} is the corresponding penalty matrix, given by

$$E_{x_I} = \begin{pmatrix} e_{L,x_I} & 0 \\ 0 & e_{R,x_I} \end{pmatrix}, \quad \Sigma_{x_I} = \begin{pmatrix} \sigma_L & -\sigma_L \\ -\sigma_R & \sigma_R \end{pmatrix}. \tag{21}$$

Using these definitions, the differentiation operator $P^{-1}Q$ used in (18) to discretize the two-domain problem can now be expressed as

$$P = \begin{pmatrix} P_L & 0 \\ 0 & P_R \end{pmatrix}, \quad Q = \begin{pmatrix} Q_L & 0 \\ 0 & Q_R \end{pmatrix} - E_{x_I}^T \Sigma_{x_I} E_{x_I}. \tag{22}$$

The SBP property of the two-domain operator (22) becomes, after using (16) and the definitions in (19) and (21):

$$Q + Q^T = e_\beta^T e_\beta - e_\alpha^T e_\alpha + E_{x_I}^T M_{x_I} E_{x_I}, \tag{23}$$

where M_{x_I} represents the contribution from the interface, given by

$$M_{x_I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - (\Sigma_{x_I} + \Sigma_{x_I}^T). \tag{24}$$

Note that (23) is identical to (8) except for the added interface term M_{x_I}. The resulting discrete relation corresponding to (9) thus becomes

$$(\Phi, P^{-1}Q\Psi)_P = \Phi_\beta \Psi_\beta - \Phi_\alpha \Psi_\alpha - (P^{-1}Q\Phi, \Psi)_P + \Phi_{x_I}^T M_{x_I} \Psi_{x_I}. \tag{25}$$

We can already here make the observation that the additional term involving M_{x_I} in (25) leads to the addition of $-U_{x_I}^T M_{x_I} U_{x_I}$ to the right hand side of the energy estimate (10). This observation leads us to making the following definition.

Definition 1. *We say that the two-domain discretization (18) is energy stable if the symmetric matrix M_{x_I} in (24) is positive semi-definite.*

We conclude by noting that the two-domain SBP definition (23) can be extended to multi-domain operators. The general multi-domain problem may thus be considered as a recursive sequence of two domain implementations, each one involving a single interface. Following this approach, we can formulate stable couplings to any multi-block or multi-element grid in one dimension.

3.2. Example: nodal dG methods

Consider a Galerkin approximation of the two-domain problem using Lagrange bases, given by

$$\begin{aligned} P_L U_t^L + Q_L U^L &= 0 \\ P_R U_t^R + Q_R U^R &= 0, \end{aligned} \tag{26}$$

where P_L, P_R, Q_L and Q_R satisfy the definitions in (13) on the intervals $[\alpha, x_I]$ and $[x_I, \beta]$ respectively. Both $P_L^{-1}Q_L$ and $P_R^{-1}Q_R$ are SBP operators as shown in section 2.2. Thus, adding SAT terms to (26) (i.e. leading to (17)) leads to a family of stable methods on SBP-SAT form. Following [21], we will now demonstrate that adding SAT terms to (26) is equivalent to a nodal dG approach.

The weak form of a two-domain nodal dG scheme is obtained by applying IBP to (26) and replacing the boundary terms with numerical fluxes. In section 2.2 we showed that the continuous IBP rule (3) is equivalent to the SBP formula (8) for Galerkin approximations using Lagrange polynomial bases. The weak form of dG is thus obtained by substituting Q_L and Q_R in (26) using (16) and inserting numerical fluxes, which gives

$$\begin{aligned} P_L U_t^L - Q_L^T U^L &= e_{L,\alpha}^T \hat{U}_\alpha^L - e_{L,x_I}^T \hat{U}_{x_I}^L \\ P_R U_t^R - Q_R^T U^R &= e_{R,x_I}^T \hat{U}_{x_I}^R - e_{R,x_I}^T \hat{U}_\beta^R. \end{aligned} \tag{27}$$

The numerical fluxes inserted in (27) are defined by

$$\hat{U}_\alpha^L = g(t), \quad \hat{U}_{x_I}^L = \hat{F}_{x_I}^L, \quad -\hat{U}_{x_I}^R = \hat{F}_{x_I}^R, \quad \hat{U}_\beta^R = U_\beta^R, \tag{28}$$

where $\hat{F}_{x_I}^L$ and $\hat{F}_{x_I}^R$ are linear combinations the two variables $U_{x_I}^L$ and $U_{x_I}^R$. The negative sign in front of $\hat{U}_{x_I}^R$ in (28) signifies that x_I is an inflow boundary to the right domain. This notation will prove to be convenient for extending the formulation to multi-dimensional discretizations.

The strong formulation is obtained by applying IBP (or equivalently, SBP) again to substitute back from Q_L^T and Q_R^T to Q_L and Q_R in (27), giving

$$\begin{aligned} P_L U_t^L + Q_L U^L &= e_{L,x_I}^T (U_{x_I}^L - \hat{F}_{x_I}^L) - e_{L,\alpha}^T (U_\alpha^L - g(t)) \\ P_R U_t^R + Q_R U^R &= e_{R,x_I}^T (-U_{x_I}^R - \hat{F}_{x_I}^R). \end{aligned}$$

The terms on the right hand side of the strong dG formulation can be seen as a penalty treatment equivalent to the SAT method, and the standard form (17) of SBP-SAT is obtained with the substitution

$$\hat{F}_{x_I}^L = U_{x_I}^L - \sigma_L (U_{x_I}^L - U_{x_I}^R), \quad \hat{F}_{x_I}^R = -U_{x_I}^R - \sigma_R (U_{x_I}^R - U_{x_I}^L).$$

Conversely, we note that any two-domain SBP-SAT scheme (17) may equivalently be expressed on the flux form:

$$\begin{aligned} U_t^L + P_L^{-1} Q_L U^L &= P_L^{-1} e_{L,x_I}^T (U_{x_I}^L - \hat{F}_{x_I}^L) - e_{L,\alpha}^T (U_\alpha^L - g(t)) \\ U_t^R + P_R^{-1} Q_R U^R &= P_R^{-1} e_{R,x_I}^T (-U_{x_I}^R - \hat{F}_{x_I}^R). \end{aligned} \tag{29}$$

We will take advantage of this alternative SBP-SAT formulation later when extending the results derived for one-dimensional discretizations to the multi-dimensional setting in section 4.

3.3. Conservation and energy stability

For non-linear conservation laws, it is well known that the numerical scheme must satisfy additional conservation properties in order to correctly capture non-smooth phenomena such as shocks, as described by the famous Lax-Wendroff theorem [34]. The close relation between numerical conservation and energy stability for linear problems was first studied in [8] for SBP-SAT interface problems, and later revisited in [9]. See also [35] for a slightly different example. The original proof in [8] showing that conservation is a necessary prerequisite to energy stability was based on a similarity transform of the interface matrix. Here, we will employ a slightly modified version of the proof that will later be used to extend the one-dimensional result to the general multi-dimensional case.

Consider the general (possibly non-linear) conservation law

$$u_t + \tilde{f}_x = 0.$$

The variational form of this conservation law is obtained by multiplying with a smooth function ϕ such that $\phi(\alpha) = \phi(\beta) = 0$. Integrating in both space and time and using the IBP rule (3) yields

$$\int_0^t ((\phi_t, u) + (\phi_x, \tilde{f}))d\tau = (\phi, u)|_0^t.$$

Note that this formulation allows for discontinuities in the solution, since all derivatives are transfered to the smooth test function ϕ.

We discretize the problem using the same type of two-domain first derivative operator as was used for the linear problem in (18), i.e.

$$U_t + P^{-1} Q \tilde{F} = 0,$$

where $P^{-1}Q$ is the two-domain SBP operator (22). By employing the discrete IBP property (25), we get the discrete variational equation

$$\int_0^t ((\Phi_t, U)_P + (P^{-1}Q\Phi, \tilde{F})_P)d\tau = (\Phi, U)_P|_0^t + \int_0^t \Phi_{x_I}^T M_{x_I} \tilde{F}_{x_I} d\tau,$$

where $\Phi_{x_I} = E_{x_I}\Phi$, $\tilde{F}_{x_I} = E_{x_I}\tilde{F}$.

Assume now that the operators $P^{-1}Q$ and P are accurate representations of the continuous operations $\partial/\partial x$ and $\int_\alpha^\beta$. Thus, if the discrete solution U converges, then provided that the additional term $\Phi_{x_I}^T M_{x_I} \tilde{F}_{x_I}$ vanishes for all $\tilde{F}_{x_I}$, then the discrete variational form approaches the continuous one. Note that any smooth function ϕ assumes the same value on both sides of the interface, i.e. we have $\Phi_{x_I} = \phi(x_I)1_{x_I}$, where $1_{x_I} = (1,1)^T$. Conservation thus follows from the condition

$$M_{x_I} 1_{x_I} = 0. \tag{30}$$

Note moreover that the penalty treatment is consistent in the sense that both rows in the penalty coefficient matrix Σ_{x_I} (21) sum to zero, i.e.

$$\Sigma_{x_I} 1_{x_I} = 0. \tag{31}$$

Using consistency (31), we can expand the expression in (30) as

$$M_{x_I} 1_{x_I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 1_{x_I} - \Sigma_{x_I} 1_{x_I} - \Sigma_{x_I}^T 1_{x_I} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \Sigma_{x_I}^T 1_{x_I}$$

The conservation condition (30) is thus equivalent to the following additional constraint on the penalty coefficient matrix

$$\Sigma_{x_I}^T 1_{x_I} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{32}$$

which in this case leads to the standard conservation condition $\sigma_L - \sigma_R = 1$, see [8].

Before stating the main result connecting conservation to energy stability, we introduce the following additional lemma.

Lemma 1. *Let A be a symmetric matrix, and let $\mathbf{x}$ be a vector such that the associated quadratic form is zero: $\mathbf{x}^T A\mathbf{x} = 0$. Then A is indefinite unless $A\mathbf{x} = 0$.*

Proof. For a proof, see Lemma 1 in [26]. □

Proposition 1. *The conservation condition (32) is necessary for energy stability of the scheme (18).*

Proof. We prove the result by showing that M_{x_I} must be indefinite unless (32) holds. By using consistency (31), we find

$$\begin{aligned} 1_{x_I}^T M_{x_I} 1_{x_I} &= 1_{x_I}^T \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - (\Sigma_{\Gamma_I} + \Sigma_{\Gamma_I}^T) \right) 1_{x_I} \\ &= -1_{x_I}^T (\Sigma_{x_I} 1_{x_I}) - (\Sigma_{x_I} 1_{x_I})^T 1_{x_I} = 0. \end{aligned}$$

Thus, by lemma 1 it now follows that M_{x_I} is indefinite unless (30) holds. We have already shown that (30) is equivalent to (32), due to (31). Thus, (32) is necessary for M_{x_I} to be positive. □

Remark 1. *Hence, for a consistent and energy stabile interface treatment in one spatial dimension, it is necessary to satisfy the conservation condition (32). As was mentioned earlier, the same result was derived already in [8], using a slightly different method of proof.*

The general solution to (32) can also be expressed by the parametrization $\sigma_L = \sigma + 1/2$ and $\sigma_R = \sigma - 1/2$, see [36]. Inserted into (21) and (24), this yields

$$\Sigma_{x_I} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} - \sigma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad M_{x_I} = 2\sigma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{33}$$

The matrix M_{x_I} thus becomes a positive semi-definite matrix scaled by the parameter σ. Hence, Σ_{x_I} in (33) is energy stable for $\sigma \geq 0$.

The added contribution to the energy estimate (10) due to this interface treatment can now be written as

$$-U_{x_I}^T M_{x_I} U_{x_I} = -2\sigma (U_{x_I}^R - U_{x_I}^L)^2.$$

By applying the energy method to (18), we thus obtain

$$\frac{d}{dt} \|U\|_P^2 = g^2 - U_\beta^2 - (U_\alpha - g)^2 - 2\sigma (U_{x_I}^R - U_{x_I}^L)^2, \tag{34}$$

i.e. we get the same type of estimate as (10) but with additional dissipation from the interface treatment if $\sigma > 0$.

Remark 2. *In the flux formulation of SBP-SAT (29), the conservative interface treatment (33) corresponds to using a combination of average and jump terms as numerical fluxes, i.e.*

$$\begin{aligned}\hat{F}^L_{\Gamma_I} &= \frac{1}{2}(U^L_{x_I} + U^R_{x_I}) + \sigma(U^L_{x_I} - U^R_{x_I}) \\ \hat{F}^R_{\Gamma_I} &= -\frac{1}{2}(U^R_{x_I} + U^L_{x_I}) + \sigma(U^R_{x_I} - U^L_{x_I}).\end{aligned} \tag{35}$$

4. Multi-dimensional discretizations

For general multi-dimensional discretizations we consider a $d-$dimensional region $x \in \Omega \subset \mathbb{R}^d$ in space with a boundary denoted by Γ. A scalar hyperbolic equation with constant coefficients, and boundary conditions specified at the inflow boundaries, can be formulated as

$$\begin{aligned} u_t + \mathbf{a}\cdot\nabla u = 0, \quad & x \in \Omega \subset \mathbb{R}^n, \quad t > 0 \\ u = g(x,t), \quad & x \in \Gamma^-, \quad t > 0 \\ u = f, \quad & x \in \Omega, \quad t = 0. \end{aligned} \tag{36}$$

In (36), we have divided the boundary into one inflow and one outflow part according to

$$\Gamma^+ = x \in \Gamma \ : \ \mathbf{a}\cdot\mathbf{n} \geq 0, \quad \Gamma^- = x \in \Gamma \ : \ \mathbf{a}\cdot\mathbf{n} < 0, \tag{37}$$

where $\mathbf{n}$ is the outward pointing normal to the surface Γ. The L_2 scalar product and norm on Ω is defined for functions ϕ and ψ as

$$(\phi,\psi) = \int_\Omega \phi\psi \ dV, \quad \|\phi\| = \sqrt{(\phi,\phi)}.$$

We consider the multi-dimensional IBP rule

$$(\phi, \mathbf{a}\cdot\nabla\psi) = \oint_\Gamma \phi\psi(\mathbf{a}\cdot\mathbf{n}) \ ds - (\mathbf{a}\cdot\nabla\phi, \psi). \tag{38}$$

The energy method applied to (36) now yields

$$\|u\|_t^2 + (u, \mathbf{a}\cdot\nabla u) + (\mathbf{a}\cdot\nabla u, u) = 0.$$

The IBP rule (38) leads directly to the estimate

$$\|u\|_t^2 = -\oint_\Gamma u^2(\mathbf{a}\cdot\mathbf{n}) \ ds. \tag{39}$$

We now split the integral in (39) into inflow and outflow parts as in (37), so that it defines two semi-norms on Γ^+ and Γ^- respectively, given by

$$\|u\|_{\text{out}}^2 = \oint_{\Gamma^+} u^2(\mathbf{a}\cdot\mathbf{n})\, ds, \quad \|u\|_{\text{in}}^2 = -\oint_{\Gamma^-} u^2(\mathbf{a}\cdot\mathbf{n})\, ds. \tag{40}$$

Note that $\|\cdot\|_{\text{out}}$ and $\|\cdot\|_{\text{in}}$ may not be proper norms on Γ^+ and Γ^- respectively, since the parameter $\mathbf{a}\cdot\mathbf{n}$ may be zero at some locations along the boundary. However, they can always be regarded as semi-norms. By inserting the boundary conditions and using the definitions in (40), we can rewrite (39) as

$$\|u\|_t^2 = \|g\|_{\text{in}}^2 - \|u\|_{\text{out}}^2. \tag{41}$$

4.1. The discrete problem

We assume that the domain Ω has a piecewise smooth boundary that can be decomposed into the n parts $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n$, each with a smooth outward pointing normal vector $\mathbf{n}_{\Gamma_i}$ for $i = 1, \ldots, n$. We introduce a discrete grid vector X and solution vector U for the domain Ω, as well as restriction operators in the form of rectangular matrices E_{Γ_i}. These are defined by the mapping $U_{\Gamma_i} = E_{\Gamma_i} U$, where U_{Γ_i} contains the discrete solution components that reside on the boundary Γ_i.

Moreover, we divide each one of the boundaries into inflow and outflow parts in the same way as we did for the boundary in (37), i.e.

$$\Gamma_i^+ = \{x \in \Gamma_i \ : \ (\mathbf{a}\cdot\mathbf{n}_{\Gamma_i}) \geq 0\}, \quad \Gamma_i^- = \{x \in \Gamma_i \ : \ (\mathbf{a}\cdot\mathbf{n}_{\Gamma_i}) < 0\}. \tag{42}$$

For simplicity, we assume that the restriction operators E_{Γ_i} are arranged in such a way that they can be split according to the in- and outflow parts of Γ_i as

$$E_{\Gamma_i} = \begin{pmatrix} E_{\Gamma_i^+} \\ E_{\Gamma_i^-} \end{pmatrix}, \quad U_{\Gamma_i} = E_{\Gamma_i} U = \begin{pmatrix} E_{\Gamma_i^+} U \\ E_{\Gamma_i^-} U \end{pmatrix} = \begin{pmatrix} U_{\Gamma_i^+} \\ U_{\Gamma_i^-} \end{pmatrix}. \tag{43}$$

We can now define the SBP-SAT discretization of (36) by generalizing the one-dimensional formulation (6) into

$$U_t + P^{-1}QU = P^{-1} \sum_{i=1}^{n} E_{\Gamma_i^-}^T \Lambda_{\Gamma_i^-} P_{\Gamma_i^-} (U_{\Gamma_i^-} - G_{\Gamma_i}(t)), \tag{44}$$

where P is a symmetric and positive definite matrix approximating the continuous L_2 scalar product and norm over the domain. We define the corresponding scalar product and norm as

$$(\Phi, \Psi)_P = \Phi^T P \Psi, \quad \|\Phi\|_P = \sqrt{(\Phi, \Phi)}. \tag{45}$$

In (44), the boundary data is contained in the vectors $G_{\Gamma_i}(t) = g(X_{\Gamma_i^-}, t)$. Furthermore, we have introduced additional discrete norms P_{Γ_i} which approximate the boundary integrals and define the scalar products

$$(\Phi_{\Gamma_i}, \Psi_{\Gamma_i})_{P_{\Gamma_i}} = \Phi_{\Gamma_i}^T P_{\Gamma_i} \Psi_{\Gamma_i}, \tag{46}$$

where Φ_{Γ_i} and Ψ_{Γ_i} are grid functions on Γ_i. We also introduce the diagonal matrices Λ_{Γ_i} defined by

$$\Lambda_{\Gamma_i} = \mathrm{diag}(\mathbf{a} \cdot \mathbf{n}_{\Gamma_i}). \tag{47}$$

Finally, we have split both P_{Γ_i} and Λ_{Γ_i} in line with (42) as

$$P_{\Gamma_i} = \begin{pmatrix} P_{\Gamma_i^+} & 0 \\ 0 & P_{\Gamma_i^-} \end{pmatrix}, \quad \Lambda_{\Gamma_i} = \begin{pmatrix} \Lambda_{\Gamma_i^+} & 0 \\ 0 & \Lambda_{\Gamma_i^-} \end{pmatrix}. \tag{48}$$

Note that Λ_{Γ_i} above may include zeros along the diagonal, but the two matrices $\Lambda_{\Gamma_i^+} P_{\Gamma_i^+}$ and $-\Lambda_{\Gamma_i^-} P_{\Gamma_i^-}$ must be both symmetric and positive semi-definite. This automatically holds if either P_{Γ_i} is diagonal or if the outward pointing normal $\mathbf{n}_{\Gamma_i}$ is constant along the boundary Γ_i. As discrete analogues to the continuous semi-norms defined on the inflow and outflow boundaries in (40), we define the discrete semi-norms

$$\|U_{\Gamma_i^+}\|_{OUT}^2 = U_{\Gamma_i^+}^T \Lambda_{\Gamma_i^+} P_{\Gamma_i^+} U_{\Gamma_i^+}, \quad \|U_{\Gamma_i^-}\|_{IN}^2 = -U_{\Gamma_i^-}^T \Lambda_{\Gamma_i^-} P_{\Gamma_i^-} U_{\Gamma_i^-}. \tag{49}$$

A discrete version of the multi-dimensional IBP rule (38) can now be obtained if the operator $P^{-1}Q$ in (44) satisfies the SBP property

$$Q + Q^T = \sum_{i=1}^{n} E_{\Gamma_i}^T \Lambda_{\Gamma_i} P_{\Gamma_i} E_{\Gamma_i}. \tag{50}$$

Indeed, this leads directly to the following discrete relation, mimicking the IBP rule (38),

$$(\Phi, P^{-1}Q\Psi)_P = \sum_{i=1}^{n} \Phi_{\Gamma_i}^T \Lambda_{\Gamma_i} P_{\Gamma_i} \Psi_{\Gamma_i} - (P^{-1}Q\Phi, \Psi)_P, \tag{51}$$

since

$$(\Phi, P^{-1}Q\Psi)_P = \Phi^T Q\Psi = \Phi^T(Q+Q^T-Q^T)\Psi = \Phi^T(Q+Q^T)\Psi-(P^{-1}Q\Phi)^T P\Psi.$$

Next, we apply the discrete energy method by multiplying (44) with $U^T P$, and then adding the transpose. This leads to

$$\begin{aligned}(U,U_t)_P + (U_t,U)_P + (U,P^{-1}QU)_P + (P^{-1}QU,U)_P = \\ = \sum_{i=1}^{n}(U_{\Gamma_i^-}^T \Lambda_{\Gamma_i^-} P_{\Gamma_i^-}(U_{\Gamma_i^-} - G_{\Gamma_i}(t)) + (U_{\Gamma_i^-} - G_{\Gamma_i}(t))^T \Lambda_{\Gamma_i^-} P_{\Gamma_i^-} U_{\Gamma_i^-}).\end{aligned}$$

Using the SBP property (50), we get

$$\begin{aligned}\frac{d}{dt}\|U\|_P^2 + \sum_{i=1}^{n} U_{\Gamma_i}^T \Lambda_{\Gamma_i} P_{\Gamma_i} U_{\Gamma_i} = \sum_{i=1}^{n}(U_{\Gamma_i^-}^T \Lambda_{\Gamma_i^-} P_{\Gamma_i^-}(U_{\Gamma_i^-} - G_{\Gamma_i}(t)) \\ +(U_{\Gamma_i^-} - G_{\Gamma_i}(t))^T \Lambda_{\Gamma_i^-} P_{\Gamma_i^-} U_{\Gamma_i^-}).\end{aligned}$$

After adding and subtracting $\sum_{i=1}^{n} G_{\Gamma_i}^T \Lambda_{\Gamma_i^-} P_{\Gamma_i^-} G_{\Gamma_i}$, and using the definitions in (49), we finally obtain the energy estimate

$$\frac{d}{dt}\|U\|_P^2 = \sum_{i=1}^{n} \|G_{\Gamma_i}(t)\|_{IN}^2 - \|U_{\Gamma_i^+}\|_{OUT}^2 - \|U_{\Gamma_i^-} - G_{\Gamma_i}(t)\|_{IN}^2. \tag{52}$$

Note that (52) closely mimics the continuous estimate (41), with added extra damping due to the weak imposition of the boundary conditions.

Remark 3. *The relation between (52) and (41) is the multi-dimensional version of the relation between (10) and (4) in one dimension.*

4.2. Example: Tensor product formulations on cartesian grids

To illustrate the multi-dimensional SBP definition introduced in the previous section with a first example, we consider (36) on a two-dimensional cartesian domain Ω, i.e.

$$u_t + au_x + bu_y = 0, \tag{53}$$

where the outward pointing normal vectors to the four boundaries of Ω are given by

$$\begin{aligned}\mathbf{n}_{\Gamma_1} &= \begin{pmatrix}-1, & 0\end{pmatrix}^T, \quad \mathbf{n}_{\Gamma_3} &= \begin{pmatrix}1, & 0\end{pmatrix}^T \\ \mathbf{n}_{\Gamma_2} &= \begin{pmatrix}0, & -1\end{pmatrix}^T, \quad \mathbf{n}_{\Gamma_4} &= \begin{pmatrix}0, & 1\end{pmatrix}^T.\end{aligned}$$

An SBP operator approximating the continuous hyperbolic operator $a\frac{\partial}{\partial x}+b\frac{\partial}{\partial x}$ on this domain can be constructed using one-dimensional operators through the tensor product formulation

$$P = (P_x \otimes P_y), \quad Q = a(Q_x \otimes P_y) + b(P_x \otimes Q_y), \tag{54}$$

where $P_x^{-1}Q_x$ and $P_y^{-1}Q_y$ are SBP operators acting on the one-dimensional domains $x \in (0,1)$ and $y \in (0,1)$, as defined in section 2. In (54), $\otimes$ denotes the Kronecker product.

We denote the boundary restriction operators associated with $P_x^{-1}Q_x$ and $P_y^{-1}Q_y$ with $e_{x,0}$, $e_{x,1}$, $e_{y,0}$ and $e_{y,1}$. The restriction operators associated with the two-domain operator $P^{-1}Q$ can now be defined as

$$\begin{aligned} E_{\Gamma_1} &= e_{x,0} \otimes I_y, \quad E_{\Gamma_3} = e_{x,1} \otimes I_y \\ E_{\Gamma_2} &= I_x \otimes e_{y,0}, \quad E_{\Gamma_4} = I_x \otimes e_{y,1}. \end{aligned}$$

This gives, using (8),

$$\begin{aligned} Q + Q^T &= a((e_{x,1}^T e_{x,1} - e_{x,0}^T e_{x,0}) \otimes P_y) + b(P_x \otimes (e_{y,1}^T e_{y,1} - e_{y,0}^T e_{y,0})) \\ &= a(E_{\Gamma_3}^T P_y E_{\Gamma_3} - E_{\Gamma_1} P_y E_{\Gamma_1}^T) + b(E_{\Gamma_4} P_x E_{\Gamma_4}^T - E_{\Gamma_2} P_x E_{\Gamma_2}^T) \\ &= \sum_{i=1}^{4} E_{\Gamma_i} \Lambda_{\Gamma_i} P_{\Gamma_i} E_{\Gamma_i}^T, \end{aligned} \tag{55}$$

where we have denoted $P_{\Gamma_1} = P_{\Gamma_3} = P_y$ and $P_{\Gamma_2} = P_{\Gamma_4} = P_x$, giving

$$\begin{aligned} \Lambda_{\Gamma_1} P_{\Gamma_1} &= -aP_y, \quad \Lambda_{\Gamma_3} P_{\Gamma_3} = aP_y \\ \Lambda_{\Gamma_2} P_{\Gamma_2} &= -bP_x, \quad \Lambda_{\Gamma_4} P_{\Gamma_4} = bP_x. \end{aligned}$$

This shows that $P^{-1}Q$ in (54) is an SBP operator, satisfying the general property (50).

4.3. Example: curvilinear domains

Now we instead consider (53) posed on a stretched domain $(x,y) \in \Omega$ obtained with a smooth transformation $x = x(\xi,\eta)$, $y = (\xi,\eta)$ from the cartesian domain $(\xi,\eta) \in \hat{\Omega}$. Equation (53) can be written on conservative form in the cartesian coordinate system as

$$Ju_t + (\hat{a}u)_\xi + (\hat{b}u)_\eta,$$

where $J = x_\xi y_\eta - x_\eta y_\xi$ is the Jacobian determinant of the transformation, and $\hat{a} = a y_\eta - b x_\eta$, $\hat{b} = -a y_\xi + b x_\xi$. See e.g. [6, 7], for more details on curvilinear grids. The outward pointing normals expressed in the cartesian and stretched coordinate systems are now given by

$$\begin{aligned}
\hat{\mathbf{n}}_{\Gamma_1} &= \begin{pmatrix} -1, & 0 \end{pmatrix}^T, & \mathbf{n}_{\Gamma_1} &= \begin{pmatrix} -y_\eta, & x_\eta \end{pmatrix}^T / \sqrt{x_\eta^2 + y_\eta^2} \\
\hat{\mathbf{n}}_{\Gamma_2} &= \begin{pmatrix} 0, & -1 \end{pmatrix}^T, & \mathbf{n}_{\Gamma_2} &= \begin{pmatrix} -y_\xi, & x_\xi \end{pmatrix}^T / \sqrt{x_\xi^2 + y_\xi^2} \\
\hat{\mathbf{n}}_{\Gamma_3} &= \begin{pmatrix} 1, & 0 \end{pmatrix}^T, & \mathbf{n}_{\Gamma_3} &= \begin{pmatrix} y_\eta, & -x_\eta \end{pmatrix}^T / \sqrt{x_\eta^2 + y_\eta^2} \\
\hat{\mathbf{n}}_{\Gamma_4} &= \begin{pmatrix} 0, & 1 \end{pmatrix}^T, & \mathbf{n}_{\Gamma_4} &= \begin{pmatrix} y_\xi, & -x_\xi \end{pmatrix}^T / \sqrt{x_\xi^2 + y_\xi^2}.
\end{aligned} \tag{56}$$

See Figure 1 for a sketch of the two domains $\hat{\Omega}$ and Ω including the outward

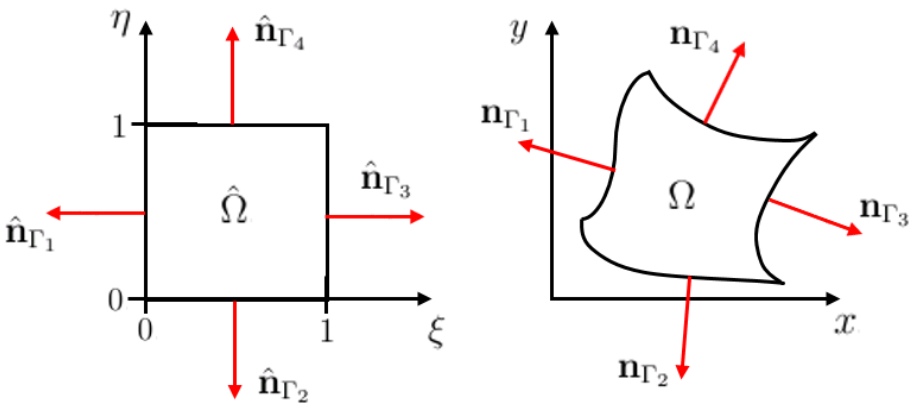

Figure 1: A sketch of the cartesian and the corresponding stretched domain.

pointing normal vectors.

Before proceeding, we derive a relation between the two variables $(a, b) \cdot \mathbf{n}_{\Gamma_i}$ and $(\hat{a}, \hat{b}) \cdot \hat{\mathbf{n}}_{\Gamma_i}$. Let $\hat{s} = \eta$ for $i = 1, 3$ and $\hat{s} = \xi$ for $i = 2, 4$ denote the parametrization of the four boundaries in cartesian coordinates. The corresponding parameter s in stretched coordinates satisfies $ds = \sqrt{dx^2 + dy^2}$, which leads to

$$\frac{ds}{d\hat{s}} = \sqrt{x_\eta^2 + y_\eta^2}, \quad i = 1, 3, \qquad \frac{ds}{d\hat{s}} = \sqrt{x_\xi^2 + y_\xi^2}, \quad i = 2, 4.$$

By comparing these results to (56), we find

$$(\hat{a},\hat{b})\cdot\hat{\mathbf{n}}_{\Gamma_i}=\frac{ds}{d\hat{s}}(a,b)\cdot\mathbf{n}_{\Gamma_i},\quad i=1,\dots,4, \tag{57}$$

a formula which we will take advantage of later.

By writing the discrete hyperbolic operator on split form (after having used the geometric conservation law $\hat{a}_\xi+\hat{b}_\eta=0$, see [7] for details), the operator $P^{-1}Q$ approximating $a\frac{\partial}{\partial x}+b\frac{\partial}{\partial x}$ can be defined as

$$\begin{aligned}P&=\mathcal{J}(P_\xi\otimes P_\eta)\\ Q&=\frac{1}{2}[(Q_\xi\otimes P_\eta)\hat{A}+\hat{A}(Q_\xi\otimes P_\eta)+(P_\xi\otimes Q_\eta)\hat{B}+\hat{B}(P_\xi\otimes Q_\eta)],\end{aligned} \tag{58}$$

where $P_\xi^{-1}Q_\xi$ and $P_\eta^{-1}Q_\eta$ are one-dimensional operators in the ξ and η directions. Moreover, $\mathcal{J}=\mathrm{diag}(J)$, $\hat{A}=\mathrm{diag}(\hat{a})$ and $\hat{B}=\mathrm{diag}(\hat{b})$. Finally we assume that the norms P_ξ and P_η are diagonal.

The restriction operators are given by

$$\begin{aligned}E_{\Gamma_1}&=e_{\xi,0}\otimes I_\eta,\quad E_{\Gamma_3}&&=e_{\xi,1}\otimes I_\eta\\ E_{\Gamma_2}&=I_\xi\otimes e_{\eta,0},\quad E_{\Gamma_4}&&=I_\xi\otimes e_{\eta,1}.\end{aligned}$$

Since P is a diagonal matrix, it commutes with both $\hat{A}$ and $\hat{B}$, which gives

$$Q+Q^T=((e_{\xi,1}^Te_{\xi,1}-e_{\xi,0}^Te_{\xi,0})\otimes P_\eta)\hat{A}+(P_\xi\otimes(e_{\eta,1}^Te_{\eta,1}-e_{\eta,0}^Te_{\eta,0}))\hat{B}.$$

With analogous manipulations to those applied in (55), we get

$$Q+Q^T=\sum_{i=1}^4E_{\Gamma_i}^T\hat{\Lambda}_{\Gamma_i}\hat{P}_{\Gamma_i}E_{\Gamma_i}, \tag{59}$$

where $\hat{P}_{\Gamma_i}=P_\eta$ for $i=1,3$, and $\hat{P}_{\Gamma_i}=P_\xi$ for $i=2,4$. The diagonal matrices $\hat{\Lambda}_{\Gamma_i}=\mathrm{diag}((\hat{a},\hat{b})\cdot\hat{\mathbf{n}}_{\Gamma_i})$ in (59) are given by

$$\begin{aligned}\hat{\Lambda}_{\Gamma_1}&=-\hat{A},\quad \hat{\Lambda}_{\Gamma_3}=\hat{A},\\ \hat{\Lambda}_{\Gamma_2}&=-\hat{B},\quad \hat{\Lambda}_{\Gamma_4}=\hat{B}.\end{aligned}$$

Using (57), we may now rewrite (59) into the standard SBP formula (50):

$$Q+Q^T=\sum_{i=1}^4E_{\Gamma_i}^T\Lambda_{\Gamma_i}P_{\Gamma_i}E_{\Gamma_i}, \tag{60}$$

where $P_{\Gamma_i} = \mathrm{diag}(\frac{ds}{d\hat{s}})\hat{P}_{\Gamma_i}$ are symmetric and positive definite matrices.

Note that the two ways (59) and (60) of expressing the SBP property in either of the domains $\hat{\Omega}$ or Ω correspond to the continuous transformation

$$\begin{aligned}\int_{\Gamma_i} \phi\psi((\hat{a},\hat{b})\cdot\hat{\mathbf{n}}_{\Gamma_i})\ d\hat{s} &= \int_{\Gamma_i} \phi\psi\frac{ds}{d\hat{s}}((a,b)\cdot\mathbf{n}_{\Gamma_i})\ d\hat{s}\\ &= \int_{\Gamma_i} \phi\psi((a,b)\cdot\mathbf{n}_{\Gamma_i})\ ds,\end{aligned}$$

where we have used (57) and the standard substitution rule for integrals. In particular, (60) shows that $P^{-1}Q$ is an SBP operator in the general sense (50) on the domain Ω.

5. Multi-domain SBP operators in several dimensions

In this section we extend the one-dimensional multi-domain formulation in section 3 to the general multi-dimensional setting.

5.1. General formulation

In analogy with the approach in section 3, we begin by dividing Ω into two subdomains $\Omega = \Omega_L \cup \Omega_R$ with a common interface denoted by Γ_I. We introduce the corresponding discrete grids and solution vectors X^L, X^R, U^L and U^R, and let $P_L^{-1}Q_L$ and $P_R^{-1}Q_R$ be two SBP operators approximating the hyperbolic operator $\mathbf{a}\cdot\nabla$ on the two domains, respectively.

Apart from the common interface Γ_I, we assume that there are n_L and n_R exterior boundaries to Ω_L and Ω_R respectively. Since Γ_I is a boundary to both of the two subdomains, the multi-dimensional SBP property (50) for the two operators can be written as

$$\begin{aligned}Q_L + Q_L^T &= E_{L,\Gamma_I}^T \Lambda_{L,\Gamma_I} P_{L,\Gamma_I} E_{L,\Gamma_I} + \sum_{i=1}^{n_L} E_{L,\Gamma_i}^T \Lambda_{L,\Gamma_i} P_{L,\Gamma_i} E_{L,\Gamma_i}\\ Q_R + Q_R^T &= E_{R,\Gamma_I}^T \Lambda_{R,\Gamma_I} P_{R,\Gamma_I} E_{R,\Gamma_I} + \sum_{i=1}^{n_R} E_{R,\Gamma_i}^T \Lambda_{R,\Gamma_i} P_{R,\Gamma_i} E_{R,\Gamma_i}.\end{aligned} \tag{61}$$

Remark 4. *Note that the outward pointing normal vectors at the interface Γ_I have opposite signs depending on which of the two subdomains is considered. For continuous contour integrals, we thus have the identity*

$$\oint_{\Gamma_I} \phi\psi(\mathbf{a}\cdot\mathbf{n}_{L,\Gamma_I})ds = -\oint_{\Gamma_I} \phi\psi(\mathbf{a}\cdot\mathbf{n}_{R,\Gamma_I})ds. \tag{62}$$

For the corresponding discrete operations, this identity is approximated as

$$(\Phi^L_{\Gamma_I})^T P_{L,\Gamma_I} \Lambda_{L,\Gamma_I} \Psi^L_{\Gamma_I} \approx -(\Phi^R_{\Gamma_I})^T P_{R,\Gamma_I} \Lambda_{R,\Gamma_I} \Psi^R_{\Gamma_I} \tag{63}$$

We write the two-domain SBP-SAT discretization of (36) on the form

$$\begin{aligned} U^L_t + P_L^{-1} Q_L U^L &= P_L^{-1}(BC^L + IC^L) \\ U^R_t + P_R^{-1} Q_R U^R &= P_R^{-1}(BC^R + IC^R). \end{aligned} \tag{64}$$

where the exterior boundary treatment in (64) is represented by the penalty terms

$$\begin{aligned} BC^L &= \sum_{i=1}^{n_L} E^T_{L,\Gamma_i^-} \Lambda_{L,\Gamma_i^-} P_{L,\Gamma_i^-} (U^L_{\Gamma_i^-} - G_{\Gamma_i}(t)) \\ BC^R &= \sum_{i=1}^{n_R} E^T_{R,\Gamma_i^-} \Lambda_{R,\Gamma_i^-} P_{R,\Gamma_i^-} (U^R_{\Gamma_i^-} - G_{\Gamma_i}(t)). \end{aligned} \tag{65}$$

The weak interface conditions in (64) consist of general linear combinations of all solution values at the interface, expressed by

$$\begin{aligned} IC^L &= E^T_{L,\Gamma_I} (\Sigma_{LL} U^L_{\Gamma_I} + \Sigma_{LR} U^R_{\Gamma_I}) \\ IC^R &= E^T_{R,\Gamma_I} (\Sigma_{RL} U^L_{\Gamma_I} + \Sigma_{RR} U^R_{\Gamma_I}). \end{aligned} \tag{66}$$

We can expand the interface terms in (66) into a matrix form as

$$\begin{aligned} &\begin{pmatrix} E^T_{L,\Gamma_I}(\Sigma_{LL} U^L_{\Gamma_I} + \Sigma_{LR} U^R_{\Gamma_I}) \\ E^T_{R,\Gamma_I}(\Sigma_{RL} U^L_{\Gamma_I} + \Sigma_{RR} U^R_{\Gamma_I}) \end{pmatrix} = \begin{pmatrix} E^T_{L,\Gamma_I} \begin{pmatrix} \Sigma_{LL} & \Sigma_{LR} \end{pmatrix} \\ E^T_{R,\Gamma_I} \begin{pmatrix} \Sigma_{RL} & \Sigma_{RR} \end{pmatrix} \end{pmatrix} \begin{pmatrix} E_{L,\Gamma_I} & 0 \\ 0 & E_{R,\Gamma_I} \end{pmatrix} U \\ &= \begin{pmatrix} E^T_{L,\Gamma_I} & 0 \\ 0 & E^T_{R,\Gamma_I} \end{pmatrix} \begin{pmatrix} \Sigma_{LL} & \Sigma_{LR} \\ \Sigma_{RL} & \Sigma_{RR} \end{pmatrix} \begin{pmatrix} E_{L,\Gamma_I} & 0 \\ 0 & E_{R,\Gamma_I} \end{pmatrix} U \\ &= E^T_{\Gamma_I} \Sigma_{\Gamma_I} E_{\Gamma_I} U, \end{aligned}$$

just as in the one-dimensional case (20). This leads us to the combined interface restriction operator and the corresponding penalty matrix

$$E_{\Gamma_I} = \begin{pmatrix} E_{L,\Gamma_I} & 0 \\ 0 & E_{R,\Gamma_I} \end{pmatrix}, \quad \Sigma_{\Gamma_I} = \begin{pmatrix} \Sigma_{LL} & \Sigma_{LR} \\ \Sigma_{RL} & \Sigma_{RR} \end{pmatrix}. \tag{67}$$

Note that Σ_{Γ_I} in (67) is a square matrix of the same dimension as the number of rows in E_{Γ_I}.

Next, we denote the number of exterior boundaries to the whole domain with $n = n_L + n_R$. For each of these, we define

$$\begin{array}{l} E_{\Gamma_i} = \begin{pmatrix} E_{L,\Gamma_i} & 0 \end{pmatrix}, \quad P_{\Gamma_i} = P_{L,\Gamma_j}, \quad \Lambda_{\Gamma_i} = \Lambda_{L,\Gamma_i}, \quad i \leq n_L \\ E_{\Gamma_i} = \begin{pmatrix} 0 & E_{R,\Gamma_j} \end{pmatrix}, \quad P_{\Gamma_i} = P_{R,\Gamma_j}, \quad \Lambda_{\Gamma_i} = \Lambda_{R,\Gamma_j}, \quad i = n_L + j \leq n, \end{array} \tag{68}$$

which yields the set of matrices E_{Γ_i}, P_{Γ_i} and Λ_{Γ_i} for $i = 1, \ldots, n$. These new matrices can clearly be decomposed into inflow and outflow parts corresponding to (43) and (48).

The two-domain discretization can now be written on a compact form corresponding to (18) in the one-dimenional case. Using (68) together with (43) and (48), we can write (64) as

$$U_t + P^{-1}QU = P^{-1} \sum_{i=1}^{n} E_{\Gamma_i^-}^T P_{\Gamma_i^-} \Lambda_{\Gamma_i^-} (U_{\Gamma_i^-} - G_{\Gamma_i}(t)), \tag{69}$$

The operator $P^{-1}Q$ in (69) is given by

$$P = \begin{pmatrix} P_L & 0 \\ 0 & P_R \end{pmatrix} \quad Q = \begin{pmatrix} Q_L & 0 \\ 0 & Q_R \end{pmatrix} - E_{\Gamma_I}^T \Sigma_{\Gamma_I} E_{\Gamma_I}. \tag{70}$$

Note that the right hand side of (69) represents the exterior boundary treatment previously included in BC^L and BC^R in (64). The specific interface treatment is included in the SBP operator itself.

By using the SBP properties (61) of the one-domain operators and applying the definitions in (67) and (68), the SBP property for the combined operator (70) becomes

$$Q + Q^T = \sum_{i=1}^{n} E_{\Gamma_i}^T P_{\Gamma_i} \Lambda_{\Gamma_i} E_{\Gamma_i} + E_{\Gamma_I}^T M_{\Gamma_I} E_{\Gamma_I}, \tag{71}$$

where

$$M_{\Gamma_I} = P_{\Gamma_I} \Lambda_{\Gamma_I} - (\Sigma_{\Gamma_I} + \Sigma_{\Gamma_I}^T), \tag{72}$$

and

$$P_{\Gamma_I} = \begin{pmatrix} P_{L,\Gamma_I} & 0 \\ 0 & P_{R,\Gamma_I} \end{pmatrix}, \quad \Lambda_{\Gamma_I} = \begin{pmatrix} \Lambda_{L,\Gamma_I} & 0 \\ 0 & \Lambda_{R,\Gamma_I} \end{pmatrix}. \tag{73}$$

This generalizes the one-domain SBP property (50), and leads to the addition of an interface term given by $\Phi^T E_{\Gamma_I}^T M_{\Gamma_I} E_{\Gamma_I} \Psi = \Phi_{\Gamma_I}^T M_{\Gamma_I} \Psi_{\Gamma_I}$ to the discrete

IBP rule (51). We find

$$(\Phi, P^{-1}Q\Psi)_P = \sum_{i=1}^{n} \Phi_{\Gamma_i}^T \Lambda_{\Gamma_i} P_{\Gamma_i} \Psi_{\Gamma_i} - (P^{-1}Q\Phi, \Psi)_P + \Phi_{\Gamma_I}^T M_{\Gamma_I} \Psi_{\Gamma_I}. \quad (74)$$

Note that the additional term involving M_{Γ_I} in (74) above also leads to the addition of $-U_{\Gamma_I}^T M_{\Gamma_I} U_{\Gamma_I}$ to the right hand side of the energy estimate (52). This observation leads us to the following multi-dimensional extension of Definition 1.

Definition 2. *The two-domain discretization (69) is energy stable if the symmetric matrix M_{Γ_I} defined in (72) is positive semi-definite.*

We conclude by noting that the two-domain SBP property (71) can be extended to multi-domain operators in a straightforward way. The general multi-domain problem may thus be considered as a recursive tree of two-grid implementations, each one involving just a single interface. This approach leads to stable implementations on completely general multi-block, multi-element or hybrid grids.

5.2. Example: nodal dG methods

Consider the finite dimensional approximation $L(x)^T U$ of u using Lagrange polynomials. The discrete variational form of equation (36) on a single dG element, without boundary and interface conditions, can be written as

$$PU_t + QU = 0, \quad (75)$$

where

$$P = \int_\Omega LL^T \, dV, \quad Q = \int_\Omega L(\mathbf{a} \cdot \nabla L)^T \, dV.$$

Applying the IBP formula (38) to each component in Q, we get

$$Q_{i,j} = \int_\Omega l_i(\mathbf{a} \cdot \nabla l_j) \, dV = -\int_\Omega (\mathbf{a} \cdot \nabla l_i) l_j \, dV + \oint_\Gamma l_i l_j (\mathbf{a} \cdot \mathbf{n}) \, ds,$$

which leads to

$$Q + Q^T = \oint_\Gamma LL^T (\mathbf{a} \cdot \mathbf{n}) \, ds. \quad (76)$$

With all element edges being straight (i.e. possesing a constant outward pointing normal), we can write (76) as

$$Q + Q^T = \sum_{i=1}^{n} (\mathbf{a} \cdot \mathbf{n}_{\Gamma_i}) \oint_{\Gamma_i} LL^T \, ds. \tag{77}$$

The Lagrange polynomial $l_j(x)$ is zero everywhere on Γ_i unless the point x_j itself resides on Γ_i. Thus, we may rewrite (77) into the general SBP formula (50), with

$$P_{\Gamma_i} = \oint_{\Gamma_i} L_{\Gamma_i} L_{\Gamma_i}^T \, ds,$$

where the vector L_{Γ_i} contains all Lagrange polynomials $l_j(x)$ for which x_j resides on Γ_i. We conclude that for the discrete variational equation (75), the continuous IBP formula (38) is equivalent to the SBP property (50).

Now, consider the two-domain discrete variational formulation

$$\begin{aligned} P_L U_t^L + Q_L U^L &= 0 \\ P_R U_t^R + Q_R U^R &= 0. \end{aligned}$$

By applying IBP (or equivalently, the SBP property (61)) and inserting numerical fluxes, we get

$$\begin{aligned} P_L U_t^L - Q_L^T U^L &= -\sum_{i=1}^{n_L} E_{L,\Gamma_i}^T P_{L,\Gamma_i} \Lambda_{L,\Gamma_i} \hat{U}_{\Gamma_i}^L - E_{L,\Gamma_I}^T P_{L,\Gamma_I} \Lambda_{L,\Gamma_I} \hat{U}_{\Gamma_I}^L \\ P_R U_t^R - Q_R^T U^R &= -\sum_{i=1}^{n_R} E_{R,\Gamma_i}^T P_{R,\Gamma_i} \Lambda_{R,\Gamma_i} \hat{U}_{\Gamma_i}^R - E_{R,\Gamma_I}^T P_{R,\Gamma_I} \Lambda_{R,\Gamma_I} \hat{U}_{\Gamma_I}^R. \end{aligned} \tag{78}$$

Recall that the two outward pointing normal vectors $\mathbf{n}_{L,\Gamma_I}$ and $\mathbf{n}_{R,\Gamma_I}$ used to define Λ_{L,Γ_I} and Λ_{R,Γ_I} point in the opposite directions (see also remark 4).

We can generalize the fluxes in (28) of the one-dimensional dG method by

$$\hat{U}_{\Gamma_i^-} = G_{\Gamma_i}(t), \quad \Lambda_{L,\Gamma_I} \hat{U}_{\Gamma_I}^L = \hat{F}_{\Gamma_I}^L, \quad \Lambda_{R,\Gamma_I} \hat{U}_{\Gamma_I}^R = \hat{F}_{\Gamma_I}^R, \quad \hat{U}_{\Gamma_i^+} = U_{\Gamma_i^+}.$$

By inserting these fluxes into (78), the strong form is finally obtained by applying IBP (or equivalenty, SBP (61)) again and rearranging the terms, which yields

$$\begin{aligned} P_L U_t^L + Q_L U^L &= BC^L + E_{L,\Gamma_I}^T P_{L,\Gamma_I} (\Lambda_{L,\Gamma_I} U_{\Gamma_I}^L - \hat{F}_{\Gamma_I}^L) \\ P_R U_t^R + Q_R U^R &= BC^R + E_{R,\Gamma_I}^T P_{R,\Gamma_I} (\Lambda_{R,\Gamma_I} U_{\Gamma_I}^R - \hat{F}_{\Gamma_I}^R), \end{aligned}$$

where BC^L and BC^R are defined as in (65). The right hand side to the strong dG formulation can be seen as SAT penalty terms, and the standard form of SBP-SAT (64) together with (66) is obtained with the substitution

$$\begin{aligned}\hat{F}^L_{\Gamma_I} &= \Lambda_{L,\Gamma_I} U^L_{\Gamma_I} - (\Sigma_{LL} U^L_{\Gamma_I} + \Sigma_{LR} U^R_{\Gamma_I}) \\ \hat{F}^R_{\Gamma_I} &= \Lambda_{R,\Gamma_I} U^R_{\Gamma_I} - (\Sigma_{RR} U^R_{\Gamma_I} + \Sigma_{RL} U^L_{\Gamma_I}).\end{aligned}$$

Conversely, it follows from this that any SBP-SAT discretization (64) can be expressed using interface conditions on flux form given by

$$\begin{aligned}IC^L &= E^T_{L,\Gamma_I} P_{L,\Gamma_I} (\Lambda_{L,\Gamma_I} U^L_{\Gamma_I} - \hat{F}^L_{\Gamma_I}) \\ IC^R &= E^T_{R,\Gamma_I} P_{R,\Gamma_I} (\Lambda_{R,\Gamma_I} U^R_{\Gamma_I} - \hat{F}^R_{\Gamma_I}).\end{aligned} \tag{79}$$

5.3. Conservation and energy stability

Consider the general conservation law on the domain Ω with boundary Γ, given by

$$u_t + \mathbf{a} \cdot \nabla \tilde{f} = 0.$$

We obtain the weak form by multiplying with a smooth function ϕ such that $\phi(\Gamma) = 0$, and then integrating in time and space. Using the IBP rule (38), we get

$$\int_0^t ((\phi_t, u) + (\mathbf{a} \cdot \nabla \phi, \tilde{f}))dx = (\phi, u)|_0^t.$$

Next, we apply the multi-domain operator defined in (70) to discretize the problem. The discretization becomes, ignoring exterior boundary conditions,

$$U_t + P^{-1} Q \tilde{F} = 0. \tag{80}$$

The two-domain discrete IBP rule (51) now leads to

$$\int_0^t ((\Phi_t, U)_P + (P^{-1} Q \Phi, \tilde{F})_P)dx = (\Phi, U)_P|_0^t + \int_0^t \Phi^T_{\Gamma_I} M_{\Gamma_I} \tilde{F}_{\Gamma_I},$$

where $\Phi_{\Gamma_I} = E_{\Gamma_I} \Phi$, $\tilde{F}_{\Gamma_I} = E_{\Gamma_I} \tilde{F}$.

Exact conservation is not always possible to achieve in this general setting, unlike in the one-dimensional case, since $X^L_{\Gamma_I}$ may contain different nodes than $X^R_{\Gamma_I}$, and thus $\phi(X^L_{\Gamma_I}) \neq \phi(X^R_{\Gamma_I})$. For the discrete weak form to converge however, it is sufficient that the term $\Phi^T_{\Gamma_I} M_{\Gamma_I} \tilde{F}_{\Gamma_I} = (M_{\Gamma_I} \Phi_{\Gamma_I})^T \tilde{F}_{\Gamma_I}$ approaches zero as the mesh is refined. Under certain reasonable accuracy

assumptions involving exact representations of grid polynomials, we can extend the result in Proposition 1 stating that conservation in necessary for energy stability. To clarify, we need

Definition 3. *The discrete conservation law (80) is conservative to order p, if*

$$M_{\Gamma_I}\Phi_{\Gamma_I} = 0, \tag{81}$$

for all grid polynomials $\Phi_{\Gamma_I} = \phi(X_{\Gamma_I})$ *of order* p *or less.*

We further consider interface penalty treatments that satisfy the same type of accuracy conditions as in (81). Thus, let

$$\Sigma_{\Gamma_I}\Phi_{\Gamma_I} = 0, \tag{82}$$

for all grid polynomials $\Phi_{\Gamma_I} = \phi(X_{\Gamma_I})$ of order p or less. By using (82), and using the definition of M_{Γ_I} in (72), we can write

$$M_{\Gamma_I}\Phi_{\Gamma_I} = P_{\Gamma_I}\Lambda_{\Gamma_I}\Phi_{\Gamma_I} - (\Sigma_{\Gamma_I}\Phi_{\Gamma_I} + \Sigma_{\Gamma_I}^T)\Phi_{\Gamma_I} = P_{\Gamma_I}\Lambda_{\Gamma_I}\Phi_{\Gamma_I} - \Sigma_{\Gamma_I}^T\Phi_{\Gamma_I}.$$

The conservation condition (81) is thus equivalent to the following relation

$$\Sigma_{\Gamma_I}^T\Phi_{\Gamma_I} = P_{\Gamma_I}\Lambda_{\Gamma_I}\Phi_{\Gamma_I}. \tag{83}$$

In order to derive a formal proof connecting conservation to energy stability for general interfaces, we first need an additional assumption involving the accuracy of discrete integration over the interface.

Assumption 1. *Assume that the discrete integral property (63) is satisfied exactly for all grid polynomials of order p or less. Especially, for* $\phi = \psi$*, this yields*

$$(\Phi_{\Gamma_I}^L)^T P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}\Phi_{\Gamma_I}^L = -(\Phi_{\Gamma_I}^R)^T P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}\Phi_{\Gamma_I}^R, \tag{84}$$

for all grid polynomials $\Phi_{\Gamma_I}^L = \phi(X_{\Gamma_I}^L)$ *and* $\Phi_{\Gamma_I}^R = \phi(X_{\Gamma_I}^R)$ *of order* p *or less.*

As an example of (84), consider the case of tensor product extensions based on one-dimensional SBP operators with diagonal norms. If the differentiation operators involved are exact for polynomials of order p or less, then it can be shown that also the discrete integration operators are exact for grid polynomials of order p or less [37]. This implies that (84) holds since the two continuous integrals have opposite signs due to the opposite orientation of the normal vectors, see also remark 4.

We can now prove

Proposition 2. *If the accuracy conditions (82) and (84) hold, then the conservation condition (83) is necessary for energy stability.*

Proof. We prove the result by showing that M_{Γ_I} must be indefinite unless (32) holds. By using (82), we find

$$\begin{aligned}\Phi_{\Gamma_I}^T M_{\Gamma_I}\Phi_{\Gamma_I} &= \Phi_{\Gamma_I}^T(P_{\Gamma_I}\Lambda_{\Gamma_I} - (\Sigma_{\Gamma_I} + \Sigma_{\Gamma_I}^T))\Phi_{\Gamma_I} \\ &= \Phi_{\Gamma_I}^T P_{\Gamma_I}\Lambda_{\Gamma_I}\Phi_{\Gamma_I} - \Phi_{\Gamma_I}^T(\Sigma_{\Gamma_I}\Phi_{\Gamma_I}) - (\Sigma_{\Gamma_I}\Phi_{\Gamma_I})^T\Phi_{\Gamma_I} \\ &= \Phi_{L,\Gamma_I}^T P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}\Phi_{L,\Gamma_I} + \Phi_{R,\Gamma_I}^T P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}\Phi_{R,\Gamma_I} = 0.\end{aligned}$$

Thus, by lemma 1 it follows that M_{Γ_I} is indefinite unless (81) holds. Moreover, we have shown that (82) implies that (81) is equivalent to (83). We conclude that (83) is necessary for M_{Γ_I} to be positive semi-definite. □

5.4. Conforming interfaces

We postpone the discussion of general interfaces to the next section, and consider the special case where both the grids and the discrete boundary integrals in (73) conform at the common interface:

$$\Lambda_{R,\Gamma_I} = -\Lambda_{L,\Gamma_I}, \quad P_{R,\Gamma_I} = P_{L,\Gamma_I}. \tag{85}$$

We also require that the matrices in (85) commute, i.e.

$$\Lambda_{L,\Gamma_I}P_{L,\Gamma_I} = P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}, \quad \Lambda_{R,\Gamma_I}P_{R,\Gamma_I} = P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}. \tag{86}$$

Note that (86) is automatically satisfied if either $P_{L,\Gamma_I} = P_{R,\Gamma_I}$ is a diagonal matrix, or if the outward pointing normal $\mathbf{n}_{\Gamma_I}$ is constant along Γ_I.

Consider the flux formulation (79) of the interface treatment. A straightforward generalization of the numerical fluxes used in the one-dimensional discretization (35) is

$$\begin{aligned}\hat{F}_{\Gamma_I}^L &= \frac{1}{2}\Lambda_{L,\Gamma_I}(U_{\Gamma_I}^L + U_{\Gamma_I}^R) + \sigma\Lambda_{L,\Gamma_I}^2(U_{\Gamma_I}^L - U_{\Gamma_I}^R) \\ \hat{F}_{\Gamma_I}^R &= \frac{1}{2}\Lambda_{R,\Gamma_I}(U_{\Gamma_I}^R + U_{\Gamma_I}^L) + \sigma\Lambda_{R,\Gamma_I}^2(U_{\Gamma_I}^R - U_{\Gamma_I}^L).\end{aligned} \tag{87}$$

On matrix form, (87) corresponds to choosing Σ_{Γ_I} in (70) as

$$\Sigma_{\Gamma_I} = \frac{1}{2}\begin{pmatrix} P_{L,\Gamma_I}\Lambda_{L,\Gamma_I} & -P_{L,\Gamma_I}\Lambda_{L,\Gamma_I} \\ -P_{R,\Gamma_I}\Lambda_{R,\Gamma_I} & P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}\end{pmatrix} - \sigma\begin{pmatrix} P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}^2 & -P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}^2 \\ -P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}^2 & P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}^2\end{pmatrix}. \tag{88}$$

By using (85) and (86), we find that (88) can be written as (see (33))

$$\Sigma_{\Gamma_I} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \otimes P_{L,\Gamma_I}\Lambda_{L,\Gamma_I} - \sigma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \Lambda_{L,\Gamma_I} P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}. \tag{89}$$

Both (82) and (83) are now automatically satisfied for all smooth grid functions, since smoothness implies that $\Phi^L_{\Gamma_I} = \Phi^R_{\Gamma_I}$. With this choice, the contribution M_{Γ_I} (72) from the interface treatment in the SBP formula (71) becomes

$$M_{\Gamma_I} = P_{\Gamma_I}\Lambda_{\Gamma_I} - (\Sigma_{\Gamma_I} + \Sigma^T_{\Gamma_I}) = 2\sigma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \Lambda_{L,\Gamma_I} P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}, \tag{90}$$

which is a positive semi-definite matrix if $\sigma \geq 0$. Especially, if $\sigma = 0$, then M_{Γ_I} vanishes and the two-grid operator $P^{-1}Q$ satisfies the regular SBP property (50). Compare (89) and (90) with the one-dimensional case (33).

By applying the energy method to (69) and using the extended SBP property (71), we find

$$\begin{aligned} \frac{d}{dt}\|U\|^2_P = \sum_{i=1}^{n} [\|G_{\Gamma_i}(t)\|^2_{IN} - \|U_{\Gamma_i^-}\|^2_{OUT} - \|U_{\Gamma_i^-} - G_{\Gamma_i}(t)\|^2_{IN}] \\ -2\sigma\|\Lambda_{L,\Gamma_I}[U^L_{\Gamma_I} - U^R_{\Gamma_I})\|^2_{P_{L,\Gamma_I}}. \end{aligned} \tag{91}$$

For $\sigma > 0$, we thus get an added dissipation from the interface, as in the one-dimensional case (34).

5.5. Non-conforming interfaces

For general non-conforming interfaces, i.e. for discretizations where (85) is not identically satisfied, additional interpolation operators are required to approximate fluxes across the common interface. In [11], a conservative and energy stable interface treatment for two-dimensional cartesian discretizations was derived based on so-called SBP preserving interpolation operators. In this section we show that the conservation condition (83) leads in a natural way to a generalized definition of SBP preserving interpolation operators.

Consider an approximation of (87) with $\sigma = 0$

$$\begin{aligned} \hat{F}^L_{\Gamma_I} &= \frac{1}{2}(\Lambda_{L,\Gamma_I} U^L_{\Gamma_I} + \tilde{\mathcal{P}}\Lambda_{R,\Gamma_I} U^R_{\Gamma_I}) \\ \hat{F}^R_{\Gamma_I} &= \frac{1}{2}(\Lambda_{R,\Gamma_I} U^R_{\Gamma_I} + \mathcal{P}\Lambda_{L,\Gamma_I} U^L_{\Gamma_I}), \end{aligned} \tag{92}$$

where $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are interpolation operators. These numerical fluxes correspond to a penalty matrix of the form (67) given by

$$\Sigma_{\Gamma_I} = \frac{1}{2}\begin{pmatrix} P_{L,\Gamma_I}\Lambda_{L,\Gamma_I} & P_{L,\Gamma_I}\tilde{\mathcal{P}}\Lambda_{R,\Gamma_I} \\ P_{R,\Gamma_I}\mathcal{P}\Lambda_{L,\Gamma_I} & P_{R,\Gamma_I}\Lambda_{R,\Gamma_I} \end{pmatrix}. \tag{93}$$

With this choice, the conservation condition (83) can be written as

$$\begin{aligned} \frac{1}{2}(P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}\Phi_{\Gamma_I^L} + \Lambda_{L,\Gamma_I}\mathcal{P}^T P_{R,\Gamma_I}\Phi_{\Gamma_I^R}) &= P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}\Phi_{\Gamma_I^L} \\ \frac{1}{2}(\Lambda_{R,\Gamma_I}\tilde{\mathcal{P}}^T P_{L,\Gamma_I}\Phi_{\Gamma_I^L} + P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}\Phi_{\Gamma_I^R}) &= P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}\Phi_{\Gamma_I^R}. \end{aligned}$$

This yields a set of constraints needed for conservation, given by

$$\begin{aligned} \Lambda_{L,\Gamma_I}\mathcal{P}^T P_{R,\Gamma_I}\Phi^R_{\Gamma_I} &= P_{L,\Gamma_I}\Lambda_{L,\Gamma_I}\Phi^L_{\Gamma_I} \\ \Lambda_{R,\Gamma_I}\tilde{\mathcal{P}}^T P_{L,\Gamma_I}\Phi^L_{\Gamma_I} &= P_{R,\Gamma_I}\Lambda_{R,\Gamma_I}\Phi^R_{\Gamma_I}, \end{aligned} \tag{94}$$

where $\Phi^L_{\Gamma_I}$ and $\Phi^R_{\Gamma_I}$ are grid polynomials of order p or less. Now consider the property

$$\Lambda_{L,\Gamma_I}\mathcal{P}^T P_{R,\Gamma_I} = -P_{L,\Gamma_I}\tilde{\mathcal{P}}\Lambda_{R,\Gamma_I}. \tag{95}$$

If the interpolation operators are constructed such that (95) holds, then the conservation conditions (94) reduce to a set of accuracy relations given by

$$\tilde{\mathcal{P}}\Lambda_{R,\Gamma_I}\Phi^R_{\Gamma_I} = -\Lambda_{L,\Gamma_I}\Phi^L_{\Gamma_I} \quad , \mathcal{P}\Lambda_{L,\Gamma_I}\Phi^L_{\Gamma_I} = -\Lambda_{R,\Gamma_I}\Phi^R_{\Gamma_I}, \tag{96}$$

for grid polynomials $\Phi^L_{\Gamma_I}$ and $\Phi^R_{\Gamma_I}$ of order p or less. The converse is of course also true. Thus , for a set of accurate interpolation operators (96) satisfying the property (95), conservation (94) is guaranteed. The additional property (95) can be seen as a direct extension to the general framework of the SBP preserving property derived in [11] for cartesian coordinates in $2D$.

For SBP preserving interpolation operators (95), the penalty matrix (93) can be written as

$$\Sigma_{\Gamma_I} = \frac{1}{2}\begin{pmatrix} P_{L,\Gamma_I}\Lambda_{L,\Gamma_I} & \Lambda_{L,\Gamma_I}\mathcal{P}^T P_{R,\Gamma_I} \\ P_{R,\Gamma_I}\mathcal{P}\Lambda_{L,\Gamma_I} & P_{R,\Gamma_I}\Lambda_{R,\Gamma_I} \end{pmatrix}.$$

The contribution to the SBP formula (71) from the interface becomes, using the definition in (72),

$$M_{\Gamma_I} = P_{\Gamma_I}\Lambda_{\Gamma_I} - (\Sigma_{\Gamma_I} + \Sigma^T_{\Gamma_I}) = 0.$$

We see that M_{Γ_I} vanishes for SBP preserving operators (95), so the two-grid operator $P^{-1}Q$ satisfies the regular SBP property (50) without dissipation with this choice. Applying the energy method to (69) and using the extended SBP property (71) thus leads to the estimate (52).

6. Conclusions

We have presented a new general framework for multi-block and multi-element summation-by-parts discretizations in multiple dimensions. The validity of the new formulation has been demonstrated for nodal dG methods in several dimensions, as well as for classical HOFDM schemes on curvilinear domains.

A large set of existing methods have thus been unified through a set of common theoretical results involving accuracy, conservation and energy stability. We also provide the tools necessary to couple different summation-by-parts methods on completely general multi-block, multi-element or hybrid grids.

We have also successfully formulated both conservative and energy stable interface procedures within this general framework, based on a generalized description of SBP preserving interpolation for arbitrary interfaces.

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