Supplementary Material for “On parametric lower bounds for discrete-time filtering”

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Abstract
This report contains supplementary material for the paper [1], and gives detailed proofs of all theorems and lemmas that could not be included into the paper due to space limitations.

Keywords: Parametric Cramér-Rao Lower Bounds, biased estimator, nonlinear filtering, state estimation
Supplementary Material for “On parametric lower bounds for discrete-time filtering”

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1 Proof of Theorem 1

The proof of Theorem 1 makes use of Lemma 2. Since we want to derive a parametric CRLB for unbiased estimators, the result of Lemma 2 simplifies to

\[
\mathcal{M}(\hat{x}_{0:k}(y_{0:k})|x_{0:k}) \geq [J_{0:k}(x_{0:k})]^{-1},
\]

(1)

where \(\hat{x}_{0:k}(y_{0:k}) = [\hat{x}_{0}^T(y_{0:k}), \ldots, \hat{x}_{k}^T(y_{0:k})]\) is any unbiased estimator of the state sequence \(x_{0:k}\), and \(J_{0:k}(x_{0:k})\) is the auxiliary Fisher information matrix of the state sequence \(x_{0:k}\), defined as

\[
J_{0:k}(x_{0:k}) = \mathbb{E}_{y_{0:k}, z_{1:k}} \{-\Delta x_{0:k} \log(p(y_{0:k}, z_{1:k}|x_{0:k}))|x_{0:k}\}.
\]

(2)

The above lemma is required to incorporate information from the deterministic state dynamics into the bound calculations. In particular, since the state sequence \(x_{0:k}\) is deterministic, it can be rewritten as a set of equality constraints: \(0 = x_{i+1} - f_i(x_i) - G_i w_i, i = 0, \ldots, k - 1\). These again can be interpreted as “perfect” measurements \(z_{i+1} = x_{i+1} - f_i(x_i) - G_i w_i\) with \(z_{i+1} = 0\) \(\forall i\). In order to stay in a probabilistic framework, we add to these equations zero-mean Gaussian noise with covariance \(M\), which is later on set to zero to recover the equality constraint. Thus, it is possible to establish the following recursion

\[
p(y_{0:k}, z_{1:k}|x_{0:k}) = p(y_{k}|x_{k})p(z_k|x_{k}, x_{k-1}) \times p(y_{0:k-1}, z_{1:k-1}|x_{0:k-1}).
\]

(3)

The auxiliary Fisher information matrix \(J_{0:k-1}(x_{0:k-1})\) can be then partitioned as follows:

\[
J_{0:k-1}(x_{0:k-1}) = -\mathbb{E}_{y_{0:k-1}, z_{1:k-1}} \left\{ \begin{array}{c}
\begin{bmatrix}
\Delta x_{0:k-2} & \Delta x_{0:k-2} \\
\Delta x_{0:k-2} & \Delta x_{0:k-2}
\end{bmatrix}
\Delta x_{0:k-2} \int \log(p(y_{0:k-1}, z_{1:k-1}|x_{0:k-1})) |x_{0:k-1}
\end{array} \right\}
\]

\[
= \begin{bmatrix}
A_{11}^{k-1} & A_{12}^{k-1} \\
A_{21}^{k-1} & A_{22}^{k-1}
\end{bmatrix}
\]

(4)

The auxiliary Fisher information submatrix \(J_{k-1}(x_{0:k-1})\) is computed as the inverse of the \(n \times n\) lower-right partition of \([J_{0:k-1}(x_{0:k-1})]^{-1}\), given by

\[
J_{k-1}(x_{0:k-1}) = A_{11}^{k-1} - (A_{12}^{k-1})\Delta A_{12}^{k-1} - A_{12}^{k-1}.
\]

(5)

Similarly, by making use of the recursion (3) it can be easily verified that the matrix \(J_{0:k}(x_{0:k})\) can be simplified as follows

\[
J_{0:k}(x_{0:k}) = -\mathbb{E}_{y_{0:k}, z_{1:k}} \left\{ \begin{array}{c}
\begin{bmatrix}
\Delta x_{0:k-2} & \Delta x_{0:k-2} \\
\Delta x_{0:k-2} & \Delta x_{0:k-2}
\end{bmatrix}
\Delta x_{0:k-2} \int \log(p(y_{0:k}, z_{1:k}|x_{0:k})) |x_{0:k}
\end{array} \right\}
\]

\[
= \begin{bmatrix}
A_{11}^{k-1} & A_{12}^{k-1} & 0 \\
A_{21}^{k-1} & A_{22}^{k-1} + L_{k}^{11} & L_{k}^{12} \\
0 & L_{k}^{21} & L_{k}^{22}
\end{bmatrix}
\]

(6)

with elements

\[
L_{k}^{11} = \mathbb{E}_{y_{0:k}, z_{1:k}} \{-\Delta x_{0:k-1} \log(p(z_{k}|x_{k}, x_{k-1}))))|x_{0:k}\},
\]

(7a)

\[
L_{k}^{12} = \mathbb{E}_{y_{0:k}, z_{1:k}} \{-\Delta x_{0:k-1} \log(p(z_{k}|x_{k}, x_{k-1}))))|x_{0:k}\},
\]

(7b)

\[
L_{k}^{22} = \mathbb{E}_{y_{0:k}, z_{1:k}} \{-\Delta x_{0:k} \log(p(z_{k}|x_{k}, x_{k-1})))|x_{0:k})
\]

\[
+ \mathbb{E}_{y_{0:k}, z_{1:k}} \{-\Delta x_{0:k} \log(p(y_{k}|x_{k}))))|x_{0:k}\}.
\]

(7c)
Finally setting $M = 0$ concludes the proof.

## 2 Proof of Lemma 1

Recall that the Kalman filter recursions are given by

\begin{align}
\hat{x}_{k+1|k} &= F\hat{x}_{k|k}, \\
P_{k+1|k} &= FP_{k|k}F^T + Q, \\
\tilde{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - H\hat{x}_{k|k-1}), \\
P_{k|k} &= P_{k|k-1} - K_k S_k K_k^T, \\
S_k &= HP_{k|k-1}H^T + R, \\
K_k &= P_{k|k-1}H^T S_k^{-1}.
\end{align}

Hence, the conditional bias $b_k(x_{0:k})$ can be rewritten as

\begin{align}
b_{k+1}(x_{0:k+1}) &\triangleq E_{y_{0:k+1}}[\tilde{x}_{k+1|k+1}|x_{0:k+1}] - x_{k+1} \\
&= E_{y_{0:k}}\left[(I_n - K_{k+1}H)\tilde{x}_{k|k} + K_{k+1}y_{k+1}|x_{0:k+1}\right] - x_{k+1} \\
&= (I_n - K_{k+1}H)F_{y_{0:k}}E_{y_{0:k}}[\tilde{x}_{k|k}|x_{0:k+1}] + K_{k+1}E_{y_{0:k}}[y_{k+1}|x_{0:k+1}] - x_{k+1} \\
&= (I_n - K_{k+1}H)F_{y_{0:k}}E_{y_{0:k}}[\tilde{x}_{k|k}|x_{0:k}] + K_{k+1}H\tilde{x}_{k+1} - x_{k+1} \\
&= (I_n - K_{k+1}H)F_{y_{0:k}}E_{y_{0:k}}[\hat{x}_{k|k}|x_{0:k}] - x_{k+1} + K_{k+1}H\tilde{x}_{k+1} - (I_n - K_{k+1}H)x_{k+1} \\
&= (I_n - K_{k+1}H)F_{y_{0:k}}E_{y_{0:k}}[\hat{x}_{k|k}|x_{0:k}] - x_{k+1} + K_{k+1}H\tilde{x}_{k+1} - (I_n - K_{k+1}H)x_{k+1} \\
&= (I_n - K_{k+1}H)F_{y_{0:k}}\left[b_k(x_{0:k}) + (I_n - K_{k+1}H)F\tilde{x}_{k+1}\right] - (I_n - K_{k+1}H)x_{k+1} \\
&= (I_n - K_{k+1}H)F_{y_{0:k}}\left[b_k(x_{0:k}) - (I_n - K_{k+1}H)(x_{k+1} - F\tilde{x}_{k+1})\right],
\end{align}

where $I_n$ denotes the identity matrix of size $n \times n$, and where we used the fact that the Kalman filter gain $K_k$ is independent of $y_k$ for all times. Hence, the recursive equation for the conditional bias is given as

\begin{align}
b_{k+1}(x_{0:k+1}) &= (I_n - K_{k+1}H)F_{y_{0:k}}\left[b_k(x_{0:k}) - (I_n - K_{k+1}H)(x_{k+1} - F\tilde{x}_{k+1})\right],
\end{align}
which can equivalently be written as

\[ b_{k+1}(x_{0:k+1}) = (I_n - K_{k+1}H)Fb_k(x_{0:k}) - (I_n - K_{k+1}H)w_k, \]  

where \( w_k \) denotes the specific process noise realization associated with the given state sequence \( x_{0:k} \). This concludes the proof.

\[ \square \]

### 3 Proof of Lemma 2

The proof of Lemma 2 is slightly non-standard. We omit the time dependency on the variables to enhance readability. In case the estimator \( \hat{x}(y) \) has a bias \( b(x) \) we can write

\[ \int (\hat{x}(y) - x)^T p(y|x) \, dy = b^T(x). \]  

(18)

The left hand side of the above equation can be further expanded to include the measurement \( z \)

\[ \int \int (\hat{x}(y) - x)^T p(y,z|x) \, dy \, dz = b^T(x). \]  

(19)

Taking the gradient \( \nabla_x = [\partial/\partial x_1, \ldots, \partial/\partial x_n]^T \) on both sides yields

\[ \int \int \nabla_x p(y,z|x) (\hat{x}(y) - x)^T \, dy \, dz = I + \nabla_x b^T(x), \]  

or equivalently

\[ \int \int \nabla_x \log(p(y,z|x)) (\hat{x}(y) - x)^T p(y,z|x) \, dy \, dz = I + \nabla_x b^T(x). \]  

(21)

Consider now the random vector

\[ \begin{bmatrix} \hat{x}(y) - x \\ \nabla_x \log(p(y,z|x)) \end{bmatrix}. \]  

(22)

The conditional mean of this vector is given by

\[ \mathbb{E}_{y,z} \left\{ \begin{bmatrix} \hat{x}(y) - x \\ \nabla_x \log(p(y,z|x)) \end{bmatrix} \left| x \right. \right\} = \begin{bmatrix} b^T(x) \\ 0 \end{bmatrix}. \]  

(23)

The conditional covariance matrix is positive semi-definite by construction and is given by

\[ \mathbb{E}_{y,z} \left\{ \begin{bmatrix} \hat{x}(y) - x \\ \nabla_x \log(p(y,z|x)) \end{bmatrix} \begin{bmatrix} \hat{x}(y) - x \\ \nabla_x \log(p(y,z|x)) \end{bmatrix}^T \left| x \right. \right\} = \begin{bmatrix} P & (I + \nabla_x b^T(x))^T \\ I + \nabla_x b^T(x) & J(x) \end{bmatrix}, \]  

(24)

where we have defined the conditional covariance

\[ P = \mathbb{E}_y \{ (\hat{x}(y) - x) (\hat{x}(y) - x)^T | x \} \]  

(25)

and the auxiliary Fisher information matrix

\[ J(x) = \mathbb{E}_{y,z} \{ \nabla_x \log(p(y,z|x)) \nabla_x^T \log(p(y,z|x)) | x \} \]

\[ = \mathbb{E}_{y,z} \{ -\Delta u^T \log(p(y,z|x)) | x \}, \]  

(26)

where \( \Delta u^T = \nabla_u [\nabla_y]^T \) holds. Since we are interested in a bound on \( P \), the Schur complement of the upper-right corner of (24) gives the desired result

\[ P \geq (I + \nabla_x b^T(x))^T [J(x)]^{-1} (I + \nabla_x b^T(x)). \]  

(27)

Since the estimator \( \hat{x}(y) \) is biased, the conditional MSE matrix can be further bounded from below as follows

\[ \mathcal{M}(\hat{x}(y)|x) = P + b(x)b^T(x) \geq (I + \nabla_x b^T(x))^T [J(x)]^{-1} (I + \nabla_x b^T(x)) + b(x)b^T(x), \]  

(28)

which concludes the proof. \[ \square \]
4 Proof of Theorem 2

The joint covariance for conditionally biased estimates \( \hat{x}_{0:k}(y_{0:k}) \) of \( x_{0:k} \) is bounded by the parametric CRLB \( \tilde{C}_{0:k} \) as given below

\[
\text{Cov} \left( \hat{x}_{0:k}(y_{0:k}) | x_{0:k} \right) \geq \tilde{C}_{0:k} \triangleq \tilde{B}_k C_{0:k} \tilde{B}_k^T, 
\]

where

\[
\tilde{B}_k \triangleq I_{(k+1)n} + B_k 
\]

and \( C_{0:k} \) denotes the parametric CRLB for unbiased estimates given as

\[
C_{0:k} \triangleq J_{0:k}^{-1} 
\]

with

\[
J_{0:k} \triangleq E_{y_{0:k}, z_{1:k}} \left[ \nabla_{x_{0:k}} \log p(y_{0:k}, z_{1:k} | x_{0:k}) \nabla_{x_{0:k}}^T \log p(y_{0:k}, z_{1:k} | x_{0:k}) \right] x_{0:k} 
\]

denoting the auxiliary Fisher information matrix. The parametric CRLB \( \tilde{C}_k \) bounding the covariance of the biased estimate \( \hat{x}_k(y_{0:k}) \) can then be written as

\[
\tilde{C}_k \triangleq \left[ \tilde{B}_k \right]_{k:} J_{0:k}^{-1} \left[ \tilde{B}_k \right]_{k:}^T, 
\]

where the notation \([:\) \(k:` \) denotes the \(k\)th block row of the argument matrix. The parametric CRLB \( \tilde{C}_{k+1} \) for the covariance of the biased estimate \( \hat{x}_{k+1}(y_{0:k}) \) is given as

\[
\tilde{C}_{k+1} = \left[ \tilde{B}_{k+1} \right]_{k+1:} J_{0:k+1}^{-1} \left[ \tilde{B}_{k+1} \right]_{k+1:}^T = \left[ \tilde{B}_{k+1} \right]_{k+1:} C_{0:k+1} \left[ \tilde{B}_{k+1} \right]_{k+1:}^T. 
\]

In order to be able to find a recursive expression for \( \tilde{C}_{k+1} \), we have to express \( C_{0:k+1} \) in terms of \( C_{0:k} \) and \( \left[ \tilde{B}_{k+1} \right]_{k+1:} \), in terms of \( \left[ \tilde{B}_k \right]_{k:} \). We can do the latter easily as follows. Suppose we decompose the matrix into blocks \( \tilde{B}_{k}^{\ell m} \in \mathbb{R}^{n \times n}, 0 \leq \ell, m \leq k \). The blocks \( \tilde{B}_{k}^{\ell m}, 0 \leq \ell, m \leq k \) would be given as

\[
\tilde{B}_{k}^{\ell m} = \begin{cases} 
I_n + B_{k}^{\ell m}, & m = \ell, \\
B_{k}^{\ell m}, & \text{otherwise}.
\end{cases} 
\]

For the computation of \( \tilde{B}_{k}^{\ell m} \), we can make use of the following proposition.

**Proposition 1.** Suppose the matrix \( \left[ \tilde{B}_k \right]_{k:} \) can be decomposed into blocks \( \tilde{B}_{k}^{\ell m} \in \mathbb{R}^{n \times n}, 0 \leq \ell, m \leq k \). Then given the bias recursions (17), \( \tilde{B}_{k}^{\ell m} \) can be determined as follows

\[
\tilde{B}_{k}^{\ell m} = \begin{cases} 
\left( \prod_{i=1}^{m} (I_n - K_i H) F \left( \nabla_{x_{0:k}} b_{0}(x_0) + \mathbb{1}(\{ \ell \neq 0 \}) I_n \right) \right), & m = 0, \\
\left( \prod_{i=m+1}^\ell (I_n - K_i H) F \right) K_m H, & 1 \leq m \leq \ell, \\
0, & \ell < m \leq k.
\end{cases} 
\]

**Proof:** See Section 6. \( \square \)

Using the expression in (36), it is easy to show that

\[
\left[ \tilde{B}_{k+1} \right]_{k+1:-} = \left( I_n - K_{k+1} H \right) F \left[ \tilde{B}_k \right]_{k:-} K_{k+1} H
\]

holds. In a next step we express \( C_{0:k+1} \) in terms of \( C_{0:k} \). Suppose the state vector is decomposed as follows

\[
x_{0:k} = \left[ x_{0:k-1}, x_k^T \right]^T. \]

Then the corresponding auxiliary Fisher information matrix \( J_{0:k} \) can be decomposed accordingly

\[
J_{0:k} = \begin{bmatrix} A & B & C \\ B^T & \tilde{B}_k & \tilde{C}_k \end{bmatrix}. 
\]
Similarly, the decomposition of the state vector according to \( x_{0:k} = [x_{0:k-1}^T, x_k^T, x_{k+1}^T]^T \) yields the auxiliary Fisher information matrix

\[
J_{0:k+1} = \begin{bmatrix}
A & B \\
B^T & C + D_{11} & D_{12} \\
0 & D_{21} & D_{22}
\end{bmatrix},
\]

(39)

where \( D_{21} = D_{12}^T \). The inverse of \( J_{0:k} \), i.e., \( C_{0:k} \) is given as follows

\[
C_{0:k} = \begin{bmatrix}
M_k & N_k \\
N_k & C_k
\end{bmatrix},
\]

(40)

where \( C_k \) denotes the parametric CRLB for unbiased estimates \( \hat{x}_k(y_{0:k}) \). Note that using the matrix inversion formula in block form, the inverse of \( J_{0:k+1} \), i.e., \( C_{0:k+1} \) can be written as

\[
C_{0:k+1} = \begin{bmatrix}
M_{k+1} & N_{k+1} \\
N_{k+1} & C_{k+1}
\end{bmatrix},
\]

(41)

where

\[
M_{k+1} \triangleq \begin{bmatrix}
A & B \\
B^T & C + D_{11} - D_{12}D_{22}^{-1}D_{21}
\end{bmatrix}^{-1},
\]

(42)

\[
N_{k+1} \triangleq -M_{k+1} \begin{bmatrix}
0 \\
D_{12}
\end{bmatrix}D_{22}^{-1},
\]

(43)

\[
C_{k+1} \triangleq \left( D_{22} - \begin{bmatrix}
0 & D_{21}
\end{bmatrix} \begin{bmatrix}
A & B \\
B^T & C + D_{11}
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
D_{12}
\end{bmatrix} \right)^{-1}
\]

\( = \left( D_{22} - D_{21} (C + D_{11} - B^T A^{-1} B)^{-1} D_{12} \right)^{-1}
\]

\( = \left( D_{22} - D_{21} (J_k + D_{11})^{-1} D_{12} \right)^{-1},
\]

(44)

where \( J_k = C_k^{-1} \). We now consider the matrix \( M_{k+1} \) below

\[
M_{k+1} \triangleq \begin{bmatrix}
A & B \\
B^T & C + D_{11} - D_{12}D_{22}^{-1}D_{21}
\end{bmatrix}^{-1}
\]

\( = \left( J_{0:k} + \begin{bmatrix}
0 \\
0 & D_{11} - D_{12}D_{22}^{-1}D_{21}
\end{bmatrix} \right)^{-1}
\]

\( = \begin{bmatrix}
M_k & N_k \\
N_k & C_k
\end{bmatrix} - \begin{bmatrix}
N_k \\
C_k
\end{bmatrix} \left( C_k + (D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right)^{-1} \begin{bmatrix}
N_k^T & C_k
\end{bmatrix},
\]

(45)

where we used the following proposition to obtain the last equality.

**Proposition 2.** Consider the matrix defined as

\[
I \triangleq \begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]

(46)

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{m \times m} \). Consider the modification \( \tilde{I} \) of \( I \) defined as

\[
\tilde{I} \triangleq \begin{bmatrix}
A & B \\
C & D + E
\end{bmatrix},
\]

(47)

where \( E \in \mathbb{R}^{m \times m} \). We assume that both \( I \) and \( \tilde{I} \) are invertible. When the inverse of the matrix \( E \) exists, the inverse of \( \tilde{I} \) can be written in terms of inverse of \( I \) as follows

\[
\tilde{I}^{-1} = I^{-1} - \begin{bmatrix}
I^{-1_{11}} \\
I^{-1_{22}}
\end{bmatrix} \left( I^{-1_{11}} + E^{-1} \right)^{-1} \begin{bmatrix}
I^{-1_{21}} \\
I^{-1_{22}}
\end{bmatrix},
\]

(48)

where

- \( 0_{n \times m} \) denotes the zero matrix of size \( n \times m \);
- \( I_n \) denotes the identity matrix of size \( n \times n \);
The notation \([\Gamma^{-1}]_{ij}, 1 \leq i, j \leq 2\) denotes \(ij\)th block of \(\Gamma^{-1}\) defined as

\[
[\Gamma^{-1}]_{11} \triangleq \begin{bmatrix} I_n & 0_{n \times m} \\ 0_{m \times n} & I_m \end{bmatrix} \Gamma^{-1} \begin{bmatrix} I_n \\ 0_{m \times n} \end{bmatrix},
\]

(49)

\[
[\Gamma^{-1}]_{12} \triangleq \begin{bmatrix} I_n & 0_{n \times m} \\ 0_{m \times n} & I_m \end{bmatrix} \Gamma^{-1} \begin{bmatrix} I_n \\ I_m \end{bmatrix},
\]

(50)

\[
[\Gamma^{-1}]_{21} \triangleq \begin{bmatrix} 0_{m \times n} & I_m \end{bmatrix} \Gamma^{-1} \begin{bmatrix} I_n \\ 0_{m \times n} \end{bmatrix},
\]

(51)

\[
[\Gamma^{-1}]_{22} \triangleq \begin{bmatrix} 0_{m \times n} & I_m \end{bmatrix} \Gamma^{-1} \begin{bmatrix} 0_{n \times m} \\ I_m \end{bmatrix},
\]

(52)

which lead to the following partitioning of \(\Gamma^{-1}\)

\[
\Gamma^{-1} \triangleq \begin{bmatrix} [\Gamma^{-1}]_{11} & [\Gamma^{-1}]_{12} \\ [\Gamma^{-1}]_{21} & [\Gamma^{-1}]_{22} \end{bmatrix}.
\]

(53)

Proof: See Section 7.

As result, we have

\[
M_{k+1} = \begin{bmatrix} M_k & N_k \\ N_k^T & C_k \end{bmatrix} - \begin{bmatrix} N_k \\ C_k \end{bmatrix} \left( C_k + (D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right) \begin{bmatrix} N_k^T \\ C_k \end{bmatrix},
\]

(54a)

\[
= \begin{bmatrix} M_k & N_k \\ N_k^T & C_k \end{bmatrix} - \begin{bmatrix} N_k \\ C_k \end{bmatrix} J_k \left( C_k - (J_k + D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right) J_k \begin{bmatrix} N_k^T \\ C_k \end{bmatrix},
\]

(54b)

where the second form is to be used in the case the matrix \(D_{11} - D_{12}D_{22}^{-1}D_{21}\) is not invertible. Note that the lower-right block of \(M_{k+1}\) corresponds to \(\tilde{J}_k^{-1}\) and is given as

\[
\tilde{J}_k^{-1} = C_k - C_k \left( C_k + (D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right) \left( C_k + (D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right)^{-1} C_k
\]

\[
= (C_k^{-1} + D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} C_k
\]

\[
= (J_k + D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \left( C_k - (J_k + D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right) J_k \left( C_k - (J_k + D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right)^{-1} C_k
\]

(55)

Hence we have

\[
\tilde{J}_k = J_k + D_{11} - D_{12}D_{22}^{-1}D_{21}.
\]

(56)

We are going to shorten the expression for \(M_{k+1}\) as follows

\[
M_{k+1} = C_{0:k} - \begin{bmatrix} N_k \\ C_k \end{bmatrix} J_k \left( C_k - \tilde{J}_k^{-1} \right) J_k \begin{bmatrix} N_k^T \\ C_k \end{bmatrix}.
\]

(57)

We can now rewrite \(N_{k+1}\) according to

\[
N_{k+1} \triangleq M_{k+1} \begin{bmatrix} 0 \\ D_{12} \end{bmatrix} D_{22}^{-1}
\]

\[
= \begin{bmatrix} N_k \\ C_k \end{bmatrix} D_{12}D_{22}^{-1} - \begin{bmatrix} N_k \\ C_k \end{bmatrix} \left( C_k + (D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right) C_k D_{12}D_{22}^{-1}
\]

\[
= - \begin{bmatrix} N_k \\ C_k \end{bmatrix} \left( I_n - \left( C_k + (D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right) C_k \right) D_{12}D_{22}^{-1}.
\]

(58)

Hence we have

\[
N_{k+1} = - \begin{bmatrix} N_k \\ C_k \end{bmatrix} \left( I_n - \left( C_k + (D_{11} - D_{12}D_{22}^{-1}D_{21})^{-1} \right) C_k \right) D_{12}D_{22}^{-1}
\]

(59a)

\[
= - \begin{bmatrix} N_k \\ C_k \end{bmatrix} J_k \left( J_k + D_{11} - D_{12}D_{22}^{-1}D_{21} \right)^{-1} D_{12}D_{22}^{-1},
\]

(59b)

where the second form is to be used in the case the matrix \(D_{11} - D_{12}D_{22}^{-1}D_{21}\) is not invertible. We are going to shorten the expressions for \(N_{k+1}\) as follows

\[
N_{k+1} = - \begin{bmatrix} N_k \\ C_k \end{bmatrix} J_k \tilde{J}_k^{-1} D_{12}D_{22}^{-1}.
\]

(60)
The expressions in (44), (54) and (59) give recursions so that the inverted matrix \( C_{0:k+1} = J_{0:k+1}^{-1} \) can be obtained directly from \( C_{0:k} = J_{0:k}^{-1} \) without making a direct inversion.

We are now in the position to derive a recursion for the parametric CRLB for biased estimates \( \tilde{C}_k \) as follows

\[
\tilde{C}_k = \left[ \tilde{B}_{k+1} \right]_{k+1:} \cdot \left[ J_{0:k+1}^{-1} \right]_{k+1:} \cdot \left[ \tilde{B}_{k+1} \right]_{k+1:}^T 
\]

\[
= \left[ (I_n - K_{k+1}H)F \left[ \tilde{B}_{k+1} \right]_{k+1:} \right] \cdot \left[ J_{k+1} \right]_{k+1:} \cdot \left[ M_{k+1} \right]_{k+1:} \cdot \left[ N_{k+1} \right]_{k+1:} \cdot \left[ C_{k+1} \right]_{k+1:} 
\]

\[
\times \left[ (I_n - K_{k+1}H)F \left[ \tilde{B}_{k+1} \right]_{k+1:} \right] \cdot \left[ K_{k+1} \right]_{k+1:} \cdot \left[ M_{k+1} \right]_{k+1:} \cdot \left[ N_{k+1} \right]_{k+1:} \cdot \left[ C_{k+1} \right]_{k+1:} 
\]

\[
= (I_n - K_{k+1}H)F \left[ \tilde{B}_{k+1} \right]_{k+1:} \cdot \left[ C_{0:k} \right]_{k+1:} \cdot \left[ \tilde{B}_{k+1} \right]_{k+1:} \cdot \left[ F^T \left( I_n - K_{k+1}H \right) \right]_{k+1:} 
\]

\[
= \left[ (I_n - K_{k+1}H)F \left[ \tilde{B}_{k+1} \right]_{k+1:} \right] \cdot \left[ C_{0:k} \right]_{k+1:} \cdot \left[ \tilde{B}_{k+1} \right]_{k+1:} \cdot \left[ F^T \left( I_n - K_{k+1}H \right) \right]_{k+1:} 
\]

\[
= \left[ (I_n - K_{k+1}H)F \left[ \tilde{B}_{k+1} \right]_{k+1:} \right] \cdot \left[ C_{0:k} \right]_{k+1:} \cdot \left[ \tilde{B}_{k+1} \right]_{k+1:} \cdot \left[ F^T \left( I_n - K_{k+1}H \right) \right]_{k+1:} 
\]

\[
= \left[ (I_n - K_{k+1}H)F \left[ \tilde{B}_{k+1} \right]_{k+1:} \right] \cdot \left[ C_{0:k} \right]_{k+1:} \cdot \left[ \tilde{B}_{k+1} \right]_{k+1:} \cdot \left[ F^T \left( I_n - K_{k+1}H \right) \right]_{k+1:} 
\]

where

\[
\psi_k \triangleq \left[ \tilde{B}_{k+1} \right]_{k+1:} \cdot \left[ C_{0:k} \right]_{k+1:} 
\]
We now find a recursion for $\Psi_k$ as follows

$$\Psi_{k+1} \triangleq \left[ \tilde{B}_{k+1} \right]_{k+1} \left[ \begin{array}{c} N_{k+1} \\ C_{k+1} \end{array} \right]$$

$$\triangleq (I_n - K_{k+1}H) F \left[ \tilde{B}_k \right]_{k} \left[ \begin{array}{c} N_{k+1} \\ C_{k+1} \end{array} \right]$$

$$= (I_n - K_{k+1}H) F \left[ \tilde{B}_k \right]_{k} \left[ N_{k+1} + K_{k+1}HC_{k+1} \right]$$

$$= -(I_n - K_{k+1}H) F \left[ \tilde{B}_k \right]_{k} \left[ \begin{array}{c} N_{k+1} \\ C_{k+1} \end{array} \right] J_k \tilde{J}_k^{-1} D_{12} D_{22}^{-1} + K_{k+1}HC_{k+1}$$

Hence the recursion for $\Psi_k$ is given as

$$\Psi_{k+1} = -(I_n - K_{k+1}H) F \tilde{J}_k \tilde{J}_k^{-1} D_{12} D_{22}^{-1} + K_{k+1}HC_{k+1}.$$  \hfill (73)

Before we continue to evaluate the information submatrices $D_{11}$, $D_{12}$, $D_{21}$ and $D_{22}$ for linear systems, we first summarize our findings. It is possible to calculate $\tilde{C}_k$ recursively without growing memory and computation requirements. The recursion for $\tilde{C}_k$ is given as

$$\tilde{C}_{k+1} = \left[ (I_n - K_{k+1}H) F \right] \left[ \begin{array}{c} \tilde{C}_k \end{array} \right] + (I_n - K_{k+1}H) F \tilde{J}_k \tilde{J}_k^{-1} D_{12} D_{22}^{-1} + K_{k+1}HC_{k+1}.$$  \hfill (74)

We initialize $\tilde{C}_0$ using the parametric CRLB for unbiased estimates as follows

$$\tilde{C}_0 = \left[ I_n + \nabla^T_{x_0} b_0(x_0) \right] C_0 \left[ I_n + \nabla^T_{x_0} b_0(x_0) \right]^T.$$  \hfill (75)

The intermediate matrix $\Psi_k$ has the recursion

$$\Psi_{k+1} = -(I_n - K_{k+1}H) F \tilde{J}_k \tilde{J}_k^{-1} D_{12} D_{22}^{-1} + K_{k+1}HC_{k+1},$$  \hfill (76)

which is initialized as

$$\Psi_0 = \left[ I_n + \nabla^T_{x_0} b_0(x_0) \right] C_0.$$  \hfill (77)

The parametric CRLB $C_k$ for unbiased estimates has the following recursion

$$C_{k+1} = \left( D_{22} - D_{21} (J_k + D_{11})^{-1} D_{12} \right)^{-1},$$  \hfill (78)

where $J_k = C_k^{-1}$ is the auxiliary FIM. The recursion (78) is initialized with the true initial covariance of the initial estimate $\hat{x}_0(y_0)$ (not with true MSE matrix!). The quantity $\tilde{J}_k^{-1}$ is evaluated from

$$\tilde{J}_k^{-1} = \left( J_k + D_{11} - D_{12} D_{22}^{-1} D_{21} \right)^{-1}.$$  \hfill (79)

For the calculation of the parametric CRLB for a linear system, we have the information submatrices $D_{11}$, $D_{12}$, $D_{21}$ and $D_{22}$ given as

$$D_{11} = F^T Q^{-1} F,$$  \hfill (80)

$$D_{12} = F^T Q^{-1},$$  \hfill (81)

$$D_{21} = Q^{-1} F,$$  \hfill (82)

$$D_{22} = Q^{-1} + H^T R^{-1} H,$$  \hfill (83)

with $Q \rightarrow 0$. In this case, first calculating $D_{11}$, $D_{12}$, $D_{21}$ and $D_{22}$ and then obtaining the relations $D_{12} D_{22}^{-1}$, $D_{11} - D_{12} D_{22}^{-1} D_{21}$ and $D_{22} - D_{21} (I_{k|k} + D_{11})^{-1} D_{12}$ required in (74) to (79) is not possible.
Equivalent expressions for \( D_{12}D_{22}^{-1}, D_{11} - D_{12}D_{22}^{-1}D_{21} \) and \( D_{22} - D_{21} (I_{k|k} + D_{11})^{-1} D_{12} \) which do not require \( Q^{-1} \) are given below

\[
D_{12}D_{22}^{-1} = F^T Q^{-1} (Q^{-1} + H^T R^{-1} H) \\
= F^T Q^{-1} (Q^{-1} + H^T R^{-1} H) Q^{-1} Q \\
= F^T (Q^{-1} (Q^{-1} + H^T R^{-1} H) (Q^{-1} - Q^{-1} + Q^{-1}) Q \\
= F^T \left( Q^{-1} - \left( Q + (H^T R^{-1} H)^{-1} \right)^{-1} \right) Q \\
= F^T \left( I_n - \left( Q + (H^T R^{-1} H)^{-1} \right)^{-1} Q \right), \tag{84}
\]

\[
D_{11} - D_{12}D_{22}^{-1}D_{21} = F^T Q^{-1} F - F^T Q^{-1} (Q^{-1} + H^T R^{-1} H)^{-1} F Q^{-1} \\
= F^T \left( Q^{-1} - Q^{-1} (Q^{-1} + H^T R^{-1} H)^{-1} Q^{-1} \right) F \\
= F^T \left( Q + (H^T R^{-1} H)^{-1} \right)^{-1} F, \tag{85}
\]

\[
D_{22} - D_{21} (J_k + D_{11})^{-1} D_{12} = Q^{-1} + H^T R^{-1} H - Q^{-1} F (J_k + D_{11})^{-1} F^T Q^{-1} \\
= H^T R^{-1} H + Q^{-1} - Q^{-1} F (J_k + D_{11})^{-1} F^T Q^{-1} \\
= (FC_k F^T + Q)^{-1} + H^T R^{-1} H. \tag{86}
\]

Therefore, when \( Q \to 0 \), the expressions \( D_{12}D_{22}^{-1}, D_{11} - D_{12}D_{22}^{-1}D_{21} \) and \( D_{22} - D_{21} (I_{k|k} + D_{11})^{-1} D_{12} \) are given as

\[
D_{12}D_{22}^{-1} = F^T, \tag{87}
\]

\[
D_{11} - D_{12}D_{22}^{-1}D_{21} = F^T H^T R^{-1} H F, \tag{88}
\]

\[
D_{22} - D_{21} (J_k + D_{11})^{-1} D_{12} = (FC_k F^T)^{-1} + H^T R^{-1} H. \tag{89}
\]

Substituting these results into (74) and noting that \( C_k = J_k^{-1} \) finally gives

\[
\hat{C}_{k+1} = \begin{bmatrix} (I_n - K_{k+1} H) F & K_{k+1} H \\ \end{bmatrix} \\
\times \begin{bmatrix} \hat{C}_k - \Psi_k J_k \left( J_k^{-1} - \tilde{J}_k^{-1} \right) J_k \Psi_k^T & -\Psi_k J_k \tilde{J}_k^{-1} F^T \\ -F \tilde{J}_k^{-1} J_k \Psi_k^T & J_k^{-1} \end{bmatrix} \\
\times \begin{bmatrix} (I_n - K_{k+1} H) F & K_{k+1} H \\ \end{bmatrix}^T \tag{90}
\]

with

\[
\Psi_{k+1} = -(I_n - K_{k+1} H) F \Psi_k J_k \tilde{J}_k^{-1} F^T + K_{k+1} H J_k^{-1}, \tag{91}
\]

\[
\tilde{J}_k^{-1} = (J_k + F^T H^T R^{-1} H F)^{-1}, \tag{92}
\]

\[
J_k^{-1} = (F J_k^{-1} F^T)^{-1} + H^T R^{-1} H, \tag{93}
\]

which concludes the proof. \( \square \)

5 Proof of Lemma 3

The conditional MSE matrix for a Kalman filter given a specific state sequence is defined as follows

\[
\mathcal{M}(\hat{x}_{k|k}(y_{0:k})|x_{0:k}) = \mathbb{E}_{y_{0:k}} \left[ \begin{bmatrix} \hat{x}_{k|k}(y_{0:k}) - x_k \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k}(y_{0:k}) - x_k \end{bmatrix}^T \right] \tag{94}
\]

Similar to the bias case, we call the quantity \( \mathcal{M}(\hat{x}_{k|k}(y_{0:k})|x_{0:k}) \) as the conditional MSE of the Kalman filter. In the following, we omit the dependency of the estimator \( \hat{x}_{k|k} \) on the measurement sequence \( y_{0:k} \) to enhance readability. Notice that the conditional MSE matrix can be decomposed into two terms as follows

\[
\mathcal{M}(\hat{x}_{k|k}|x_{0:k}) = \text{Cov}_k(\hat{x}_{k|k}|x_{0:k}) + b_k(x_{0:k}) b_k^T(x_{0:k}), \tag{95}
\]

where the quantity \( \text{Cov}_k(\hat{x}_{k|k}|x_{0:k}) \) defined as

\[
\text{Cov}_k(\hat{x}_{k|k}|x_{0:k}) \triangleq \mathbb{E}_{y_{0:k}} \left[ \begin{bmatrix} \hat{x}_{k|k} - E_{y_{0:k}} \left[ \hat{x}_{k|k} \right] \end{bmatrix} \begin{bmatrix} \hat{x}_{k|k} - E_{y_{0:k}} \left[ \hat{x}_{k|k} \right] \end{bmatrix}^T \right] x_{0:k} \tag{96}
\]
is called as the conditional covariance matrix. In the following we find a recursive expression for the conditional MSE matrix

\[(\hat{x}_{k+1|k+1} - x_{k+1}) = (I_n - K_{k+1}H)F\hat{x}_{k|k} + K_{k+1}v_{k+1} - x_{k+1}\]

\[= (I_n - K_{k+1}H)F\hat{x}_{k|k} + K_{k+1}Hx_{k+1} + K_{k+1}v_{k+1} - x_{k+1}\]

\[= (I_n - K_{k+1}H)F\hat{x}_{k|k} - (I_n - K_{k+1}H)x_{k+1} + K_{k+1}v_{k+1}\]

\[= (I_n - K_{k+1}H)F\hat{x}_{k|k} - (I_n - K_{k+1}H)Fx_k - (I_n - K_{k+1}H)(x_{k+1} - Fx_k) + K_{k+1}v_{k+1}\]

Hence,

\[= (I_n - K_{k+1}H)F(\hat{x}_{k|k} - x_k) - (I_n - K_{k+1}H)(x_{k+1} - Fx_k) + K_{k+1}v_{k+1}.\] (97)

Now taking the expectation of both sides with respect to \(y_{0:k+1}\) (i.e., with respect to \(v_{0:k+1}\)) given \(x_{0:k+1}\) would give the bias recursion we have found in Lemma 1. In order to find the conditional MSE, we can now take the dyadic product of both sides with their transpose and then take expected value of both sides with respect to \(y_{0:k+1}\) (i.e., with respect to \(v_{0:k+1}\)) given \(x_{0:k+1}\) which gives

\[M(\hat{x}_{k+1|k+1}|x_{0:k+1}) \triangleq E_{y_{0:k+1}}[(\hat{x}_{k+1|k+1} - x_{k+1})(\hat{x}_{k+1|k+1} - x_{k+1})^T|x_{0:k+1}]\]

\[= (I_n - K_{k+1}H)E_{y_{0:k+1}}[(F(\hat{x}_{k|k} - x_k) - (x_{k+1} - Fx_k))^T]x_{0:k+1}\]

\[\quad \times (I_n - K_{k+1}H)^T + K_{k+1}R_K^T\]

\[= (I_n - K_{k+1}H)FM(\hat{x}_{k|0})F^T - Fb_k(x_{0:k})Fx_k^T\]

\[\quad + (x_{k+1} - Fx_k)b_k^T(x_{0:k})F^T + (x_{k+1} - Fx_k)(x_{k+1} - Fx_k)^T(I_n - K_{k+1}H)^T\]

\[\quad + K_{k+1}R_K^T,\] (98)

which is the recursion for the conditional MSE matrix. Note that by using the system dynamics, we can equivalently write (98) as

\[M(\hat{x}_{k+1|k+1}|x_{0:k+1}) = (I_n - K_{k+1}H)[FM(\hat{x}_{k|0})F^T - Fb_k(x_{0:k})w_k^T - w_kb_k^T(x_{0:k})F^T\]

\[\quad + w_kw_k^T(I_n - K_{k+1}H)^T + K_{k+1}R_K^T],\] (99)

where \(w_k\) denotes the specific process noise realization associated with the given state sequence \(x_{0:k}\). This concludes the proof. 

6 Proof of Proposition 1

Considering the recursion (17), we can find an explicit formula for the conditional bias

\[b_k(x_{0:k}) = (\prod_{i=1}^k(I_n - K_iH)F)b_0(x_0) - \sum_{i=1}^k \left(\prod_{j=i+1}^k(I_n - K_jH)F\right)(I_n - K_iH)(x_i - Fx_{i-1}).\] (100)

We now define the conditional bias Jacobian matrix \(B_k\) as follows

\[B_k \triangleq \nabla^T_{x_{0:k}} \left[ b_k^T(x_0) \ b_k^T(x_{0:1}) \ldots b_k^T(x_{0:k}) \right]^T = [B_k^m] \in \mathbb{R}^{(k+1)n \times (k+1)n},\] (101)

where \(B_k^m \in \mathbb{R}^{n \times n}, 0 \leq \ell, m \leq k\) is defined as

\[B_k^m \triangleq \nabla^T_{x_{m}} b_r(x_{0:\ell}).\] (102)

Hence, \(B_k^m\) is the block of the Jacobian matrix \(B_k\) corresponding to the Jacobian of the conditional bias vector \(b_r(x_{0:\ell})\) with respect to \(x_m\). The blocks \(B_k^m, 0 \leq \ell, m \leq k\) can be calculated as follows. Since the conditional bias \(b_r(x_{0:\ell})\) is only dependent on \(x_{0:\ell}\) and independent of \(x_i, i > \ell\), the matrix \(B_k\) is lower block diagonal, i.e., its blocks above the main block diagonal are all zero. Hence we have

\[B_k^m = 0\] (103)
for $m > \ell$. The blocks can be calculated as follows

\[
B_k^{\ell m} = \begin{cases}
(\prod_{j=1}^\ell (I_n - K_j H)F) (\nabla_{x_0}^T b_0(x_0) + \mathbb{I}(\{\ell \neq 0\})I_n), & m = 0, \\
(\prod_{j=m+1}^\ell (I_n - K_j H)F) K_m H, & 1 \leq m \leq \ell - 1, \\
-(I_n - K_\ell H), & m = \ell, \\
0, & \ell < m \leq k,
\end{cases}
\]  

(104)

where the notation $\mathbb{I}(\cdot)$ denotes the indicator function of the argument event defined as

\[
\mathbb{I}(E) = \begin{cases}
1, & E \text{ is true,} \\
0, & E \text{ is false.}
\end{cases}
\]  

(105)

This concludes the proof. \(\square\)

7 Proof of Proposition 2

By definition, we have

\[
\tilde{I} = I + \begin{bmatrix} 0_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & E \end{bmatrix}.
\]  

(106)

Hence, we can write the following

\[
\tilde{I}^{-1} = \lim_{\ell \to 0} \left( I + \begin{bmatrix} \ell I_n & 0_{n \times m} \\ 0_{m \times n} & E \end{bmatrix} \right)^{-1}.
\]  

(107)

Using the formula for the inversion of the sum of matrices given as

\[
(A + B)^{-1} = A^{-1} - A^{-1}(A^{-1} + B^{-1})^{-1}A^{-1},
\]  

(108)

we get

\[
\tilde{I}^{-1} = \lim_{\ell \to 0} I^{-1} - I^{-1} \left( I^{-1} + \begin{bmatrix} \frac{1}{\ell} I_n & 0_{n \times m} \\ 0_{m \times n} & E^{-1} \end{bmatrix} \right)^{-1} I^{-1}.
\]  

(109)

Substituting the partitioned form of $I^{-1}$ given in (53) into (109), we get

\[
\tilde{I}^{-1} = \lim_{\ell \to 0} I^{-1} - I^{-1} \left( \begin{bmatrix} I^{-1} & 0 \\ 0 & I^{-1} \end{bmatrix} \frac{1}{\ell} I_n + \begin{bmatrix} \frac{1}{\ell} I_n & 0_{n \times m} \\ 0_{m \times n} & E^{-1} \end{bmatrix} \right)^{-1} I^{-1},
\]  

(110)

\[
= \lim_{\ell \to 0} I^{-1} - I^{-1} \left[ \begin{bmatrix} I^{-1} & 0 \\ 0 & I^{-1} \end{bmatrix} \frac{1}{\ell} I_n + \begin{bmatrix} \frac{1}{\ell} I_n & 0_{n \times m} \\ 0_{m \times n} & E^{-1} \end{bmatrix} \right]^{-1} I^{-1}.
\]  

(111)

We now consider the matrix inversion in block form given below

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix},
\]  

(112)

where the matrices $\Delta_A$ and $\Delta_D$ defined as

\[
\Delta_A = A - BD^{-1}C,
\]  

\[
\Delta_D = D - CA^{-1}B,
\]  

(113)

(114)

denote the Schur complements of $A$ and $D$ respectively. Applying the matrix inversion in block form given above to take the block matrix inverse on the right hand side of (111) we get

\[
\tilde{I}^{-1} = \lim_{\ell \to 0} I^{-1}
\]  

\[-I^{-1} \left[ -\left((I^{-1})_{22} + E^{-1}\right)^{-1}(I^{-1})_{21} \Delta_{11}^{-1} -\Delta_{11}^{-1}(I^{-1})_{12} \left((I^{-1})_{22} + E^{-1}\right)^{-1} \right]^{-1} I^{-1},
\]  

(115)
where
\[
\Delta_{11} \triangleq [I^{-1}]_{11} + \frac{1}{\epsilon} I_n - [I^{-1}]_{12} \left( [I^{-1}]_{22} + E^{-1} \right)^{-1} [I^{-1}]_{21},
\]
(116)
\[
\Delta_{22} \triangleq [I^{-1}]_{22} + E^{-1} - [I^{-1}]_{21} \left( [I^{-1}]_{11} + \frac{1}{\epsilon} I_n \right)^{-1} [I^{-1}]_{12}.
\]
(117)

Noting that we have
\[
\lim_{\epsilon \to 0} \Delta_{11}^{-1} = 0_{n \times n},
\]
(118)
\[
\lim_{\epsilon \to 0} \Delta_{22}^{-1} = ( [I^{-1}]_{22} + E^{-1} )^{-1},
\]
(119)
we obtain
\[
I^{-1} = I^{-1} - I^{-1} \left[ \begin{array}{cc} 0_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & ([I^{-1}]_{22} + E^{-1})^{-1} \end{array} \right] I^{-1}
\]
(120)
\[
= I^{-1} - \left[ \begin{array}{cc} [I^{-1}]_{11} & [I^{-1}]_{12} \\ [I^{-1}]_{21} & [I^{-1}]_{22} \end{array} \right] \left[ \begin{array}{cc} 0_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & ([I^{-1}]_{22} + E^{-1})^{-1} \end{array} \right] \left[ \begin{array}{cc} [I^{-1}]_{11} & [I^{-1}]_{12} \\ [I^{-1}]_{21} & [I^{-1}]_{22} \end{array} \right]
\]
(121)
\[
= I^{-1} - \left[ \begin{array}{c} [I^{-1}]_{12} \\ [I^{-1}]_{22} \end{array} \right] ( [I^{-1}]_{22} + E^{-1} )^{-1} \left[ \begin{array}{c} [I^{-1}]_{21} \\ [I^{-1}]_{22} \end{array} \right].
\]
(122)

This concludes the proof. \(\square\)

References

Supplementary Material for “On parametric lower bounds for discrete-time filtering”

This report contains supplementary material for the paper [1], and gives detailed proofs of all theorems and lemmas that could not be included into the paper due to space limitations.