Minimal eventually positive realizations of externally positive systems

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Minimal eventually positive realizations of externally positive systems

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Abstract

It is a well-know fact that externally positive linear systems may fail to have a minimal positive realization. In order to investigate these cases, we introduce the notion of minimal eventually positive realization, for which the state update matrix becomes positive after a certain power. Eventually positive realizations capture the idea that in the impulse response of an externally positive system the state of a minimal realization may fail to be positive, but only transiently. As a consequence, we show that in discrete-time it is possible to use downsampling to obtain minimal positive realizations matching decimated sequences of Markov coefficients of the impulse response. In continuous-time, instead, if the sampling time is chosen sufficiently long, a minimal eventually positive realization leads always to a sampled realization which is minimal and positive.

Key words: positive linear systems; minimal realization; eventually positive matrices; Perron-Frobenius theorem.

1 Introduction

The positive realization problem for an externally (i.e., input-output) positive linear system consists in finding a state space representation which is itself positive, i.e., a triple $\{A, b, c\}$ for which A, b and c are nonnegative. The problem has been investigated for several decades, see Benvenuti & Farina (2004); Farina & Rinaldi (2000) for an introduction and a survey of the main results.

Unlike existence, which is well-characterized (Ohta *et al.* , 1984; Maeda & Kodama, 1981; Farina & Benvenuti, 1995; Anderson *et al.* , 1996), the problem of constructing positive realizations of minimal order is a difficult one, far from being completely solved. The positivity constraints, in fact, imply that not all externally positive systems admit a realization which is both minimal and positive, i.e., which is simultaneously positive, controllable and observable. There is by now a consistent literature on the topic, notably dealing with conditions on the order of the achievable realizations (Hadjicostis, 1999; Nagy & Matolcsi, 2003), and developing algorithms to construct positive realizations in special cases (Benvenuti *et al.* , 2000; Benvenuti, 2013; Canto *et al.* , 2007; Kim, 2012; Nagy & Matolcsi, 2005). However, systematic procedures for obtaining minimal positive realizations are in general unknown.

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Rather than contributing to this search, the scope of this paper is to investigate the structure of the minimal realization of externally positive systems, and to suggest a class of minimal realizations capturing the gap between external and internal positivity. The starting point of our approach is the observation that the fundamental mathematical principle behind positivity is the Perron-Frobenius theorem. In essence, existence of a positive realization is associated to existence of a polyhedral cone which is A-invariant (Ohta *et al.* , 1984; Anderson *et al.* , 1996). Such cone contains the positive eigenvector associated to the dominant eigenvalue, and at least when dominance is strict and the input is vanishing, the evolution of the linear system tends to become aligned with that eigenspace. If we relax the assumption of positivity of A while maintaining the condition that the eigenvector must be contained in $\mathbb{R}^n_+$, then we still have that the free evolution of the state of a minimal realization becomes positive after a transient. Matrices A having both left and right dominant eigenvector in $\mathbb{R}^n_+$ form a special class of matrices called eventually positive, see Altafini & Lini (2015); Noutsos (2006). While these matrices can have negative entries (hence they do not correspond to positive realizations), they have the property that after a certain power they become positive matrices. Therefore in discrete-time this property naturally leads to free evolutions of the state variables that become nonnegative after a certain number of steps. If in addition the reachable cone associated to the realization is contained in $\mathbb{R}^n_+$, then the entire state vector must become positive after a transient. Notice that our approach is qualitatively different from Guidorzi (2014), where relaxing the positivity of A may lead to state trajectories which never become positive when initialized outside the reachable cone.

Formalizing our argument, we shown in the paper that a minimal realization in which A is eventually positive and b (resp. c) belongs to the corresponding A-invariant cone (resp. dual cone) is guaranteed to be externally positive. As for the converse, we provide constructive procedures to obtain a minimal eventually positive realization from a given externally positive system. While we do not have an explicit proof that every externally positive system admits a minimal realization of this type, it is tempting to conjecture that indeed it is so, at least in the case of a simple strictly dominant pole.

With respect to a preliminary version of this manuscript appearing as a conference paper (Altafini, 2015), the constructive procedure proposed here (Algorithm 1) is more general, and recovers the result of Altafini (2015) as a special case (Algorithm 2). Such special case is instrumental to show that a consequence of the existence of a minimal eventually positive realization is that the sequence of Markov parameters that compose the impulse response has decimated subsequences for which a minimal positive realization exists, and can be found downsampling the eventually positive realization. The number of steps it takes for A to become positive gives a lower bound on the sought decimation factor. In continuous-time, instead, provided the sampling time is chosen sufficiently high, the sampled system obtained from a minimal eventually positive realization is itself positive and minimal. Also in this case (which is not treated in Altafini (2015)), once an eventually positive realization is available, a lower bound on the sampling time leading to a minimal positive sampled realization is known. These results on sampled/downsampled systems can be interpreted as a dual of the usual Nyquist-Shannon theorem: instead of seeking for a sampling frequency sufficiently high so as to preserve all interesting frequencies of the system, if one selects a sampling frequency enough low it is possible to achieve an internal minimal representation of the system which remains positive, because it disregards the high frequency content. In externally positive systems with strict dominance of the real eigenvalue, as we consider here, these frequencies are associated to non-dominant modes, hence they are necessarily transient. Therefore the nonpositive entries of our eventually positive realizations and the violations of positivity in the state vectors they induce must necessarily be associated to the non-dominant modes.

2 Linear algebra background

For a matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, in this paper $A \geq 0$ means $a_{ij} \geq 0$ for any $i, j \in 1, \ldots, n$, and $A \neq 0$, while $A > 0$ means $a_{ij} > 0$ for all $i, j = 1, \ldots, n$. The matrix A is *nonnegative* (resp. *positive*) if $A \geq 0$ (resp. $A > 0$). This notation is used also for vectors. The spectrum of A is denoted $\mathrm{sp}(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$, where $\lambda_i(A)$, $i = 1, \ldots, n$, are the eigenvalues of A. The *spectral radius* of A, $\rho(A)$, is the smallest real positive number such that $\rho(A) \geq |\lambda_i(A)|$, $\forall i = 1, \ldots, n$.

2.1 Eventually positive matrices

Definition 1 *A matrix* $A \in \mathbb{R}^{n \times n}$ *has the* strong Perron-Frobenius property *if* $\rho(A)$ *is a simple positive eigenvalue of A s.t. $\rho(A) > |\lambda|$ for every $\lambda \in \mathrm{sp}(A)$, $\lambda \neq \rho(A)$, and v, the right eigenvector relative to $\rho(A)$, is positive.*

Denote $\mathscr{PF}_n$ the set of matrices in $\mathbb{R}^{n \times n}$ that possess the strong Perron-Frobenius property. These properties are associated to a class of matrices called *eventually positive* (Friedland, 1978; Johnson & Tarazaga, 2004; Noutsos, 2006; Elhashash & Szyld, 2008), class that is strictly bigger than that of positive matrices, in the sense that the matrices can contain negative entries.

Definition 2 *A real square matrix A is said to be* eventually positive *if $\exists \, \eta_o \in \mathbb{N}$ such that $A^\eta > 0$ for all $\eta \geq \eta_o$.*

The smallest integer η_o of Definition 2 is called the power index of A. Following Olesky *et al.* (2009), eventually positive matrices will be denoted $A \overset{\vee}{>} 0$. For eventually positive matrices a necessary and sufficient condition for the fulfillment of the strong Perron-Frobenius property is available.

Theorem 1 *(Noutsos (2006), Theorem 2.2) For $A \in \mathbb{R}^{n \times n}$ the following are equivalent:*

(1) Both $A, A^T \in \mathscr{PF}_n$;
(2) $A \overset{\vee}{>} 0$;
(3) $A^T \overset{\vee}{>} 0$.

A matrix A is said *exponentially positive* if $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} > 0 \; \forall t \geq 0$, and A is exponentially positive if and only if A is *Metzler*, i.e., $a_{ij} \geq 0 \; \forall i \neq j$ (Noutsos & Tsatsomeros, 2008).

Definition 3 *A matrix $A \in \mathbb{R}^{n \times n}$ is said* eventually exponentially positive *if $\exists \, t_o \in [0, \infty)$ such that $e^{At} > 0 \; \forall t \geq t_o$.*

We denote the smallest such t_o the exponential index of A. The relationship between eventual positivity and eventual exponential positivity is given in Noutsos & Tsatsomeros (2008).

Theorem 2 *(Noutsos & Tsatsomeros (2008), Theorem 3.3) Given $A \in \mathbb{R}^{n \times n}$, A is eventually exponentially positive if and only if $\exists \, d \geq 0$ such that $A + dI \overset{\vee}{>} 0$.*

2.2 Invariant cones and eventually positive matrices

A set $\mathscr{K} \subset \mathbb{R}^n$ is called a convex cone if $\alpha_1 x_1 + \alpha_2 x_2 \in \mathscr{K} \; \forall x_1, x_2 \in \mathscr{K}$, $\alpha_1, \alpha_2 \geq 0$. $\mathscr{K}$ is called solid if the interior of $\mathscr{K}$, $\mathrm{int}(\mathscr{K})$, is nonempty, and pointed if $\mathscr{K} \cap$

$(-\mathcal{K}) = \{0\}$. A proper cone is a closed, pointed, solid cone. A cone is polyhedral if it can be expressed as the nonnegative combination of a finite number of generating vectors $\omega_1, \ldots, \omega_\mu \in \mathbb{R}^n$:

$$\mathcal{K} = \text{cone}(\Omega) = \left\{ x = \Omega\alpha = \sum_{i=1}^{\mu} \alpha_i \omega_i, \ \alpha_i \geq 0 \right\}, \quad (1)$$

where $\Omega = \begin{bmatrix} \omega_1 \ldots \omega_\mu \end{bmatrix} \in \mathbb{R}^{n \times \mu}$, $\alpha = \begin{bmatrix} \alpha_1 \ldots \alpha_\mu \end{bmatrix}^T \in \mathbb{R}_+^\mu$. It is well-known that alternatively to the "vertices description" (1) for $\mathcal{K}$ one can use the "face description"

$$\mathcal{K} = \{ x \text{ s. t. } \Gamma x \geq 0 \}, \qquad \Gamma \in \mathbb{R}^{\mu \times n}.$$

The pair $\{\Omega, \Gamma\}$ forms a "double description pair" for $\mathcal{K}$. Let $\mathcal{K}^* = \{ y \in \mathbb{R}^n \text{ s. t. } y^T x \geq 0 \ \forall x \in \mathcal{K} \}$ be the dual cone of $\mathcal{K}$. In terms of the double description pair $\{\Omega, \Gamma\}$, we have:

$$\mathcal{K}^* = \{ y \text{ s. t. } y = \Gamma^T \beta, \ \beta \geq 0 \} = \{ y \text{ s. t. } \Omega^T y \geq 0 \},$$

i.e., $\{\Gamma^T, \Omega^T\}$ is a double description pair for $\mathcal{K}^*$.
Given $A \in \mathbb{R}^{n \times n}$, the cone $\mathcal{K}$ is said A-invariant if $A\mathcal{K} \subseteq \mathcal{K}$. For an A-invariant cone $\mathcal{K}$, A is said $\mathcal{K}$-positive if $A(\mathcal{K} \setminus \{0\}) \subseteq \text{int}(\mathcal{K})$, i.e., A maps any nonzero element of $\mathcal{K}$ into $\text{int}(\mathcal{K})$. Notice that if A is $\mathcal{K}$-positive then A is $\mathcal{K}$-irreducible, i.e., it does not leave any of the faces of $\mathcal{K}$ invariant (except for $\{0\}$ and $\mathcal{K}$ itself). Theorem 1.3.16 of Berman & Plemmons (1994) says that A that leaves $\mathcal{K}$ invariant is $\mathcal{K}$-irreducible if and only if A has exactly one (up to scalar multiples) eigenvector in $\mathcal{K}$, and this vector is in $\text{int}(\mathcal{K})$. A is $\mathcal{K}$-positive if and only if A^T is $\mathcal{K}^*$-positive Berman & Plemmons (1994).

The Perron-Frobenius theorem for invariant polyhedral cones can be found e.g. in Berman & Plemmons (1994) (Theorem 1.3.26) or Valcher & Farina (2000) (Theorem 3.3).

Theorem 3 *Given $A \in \mathbb{R}^{n \times n}$, the following are equivalent:*

(1) $\exists$ a proper A-invariant polyhedral cone $\mathcal{K} \in \mathbb{R}^n$ for which A is $\mathcal{K}$-positive;
(2) $\rho(A)$ is a simple positive eigenvalue in $\text{sp}(A)$, and for each $\lambda \in \text{sp}(A)$, $\lambda \neq \rho(A)$, $|\lambda| < \rho(A)$.

Furthermore, for the right eigenvector v relative to $\rho(A)$ it holds $v \in \text{int}(\mathcal{K})$.

The following theorem links eventually positive matrices with invariant cones. The notation $A^\eta \mathcal{K}$ stands for the η-th iterated cone of $\mathcal{K}$.

Theorem 4 *(Altafini & Lini (2015), Theorem 5) $A \overset{\vee}{>} 0$ if and only if $\exists$ a proper polyhedral A-invariant cone $\mathcal{K}$ for which A is $\mathcal{K}$-positive, and $\mathcal{K}$ is such that $A^\eta \mathcal{K} \subset \text{int}(\mathbb{R}_+^n) \cup \{0\}$, $(A^T)^\eta \mathcal{K}^* \subset \text{int}(\mathbb{R}_+^n) \cup \{0\} \ \forall \eta \geq \eta_o$.*

By construction, the cone $\mathcal{K}$ of Theorem 4 contains no other eigenvector of A than v. If instead of $A \overset{\vee}{>} 0$ we have the "one-sided" condition $A \in \mathscr{PF}_n$, then Theorem 4 can be replaced by the following corollary.

Corollary 1 *(Altafini & Lini (2015), Corollary 2) $A \in \mathscr{PF}_n$ if and only if $\exists$ a proper polyhedral A-invariant cone $\mathcal{K}$ for which A is $\mathcal{K}$-positive, and $\mathcal{K}$ is such that $A^\eta \mathcal{K} \subset \text{int}(\mathbb{R}_+^n) \cup \{0\} \ \forall \eta \geq \eta_o$.*

3 Discrete-time eventually positive realizations

The discrete-time SISO linear system

$$\begin{aligned} x(k+1) &= Ax(k) + bu(k) \qquad k = 0, 1, \ldots \\ y(k) &= cx(k) \end{aligned} \quad (2)$$

is said *externally positive* if for any nonnegative input sequence $\{u(k)\}_{k=0}^\infty$ the forced output (in correspondence of $x(0) = 0$) is nonnegative. A well-known necessary and sufficient condition for external positivity is that the impulse response $\{u(k) = \delta_k\}_{k=0}^\infty$ is nonnegative. The system (2) is said (internally, hereafter omitted) *positive* if for any nonnegative input sequence $\{u(k)\}_{k=0}^\infty$ and nonnegative initial condition $x(0)$, the state $x(k)$ and the output $y(k)$ are always nonnegative. A necessary and sufficient condition for positivity of (2) is that $A \geq 0$, $b \geq 0$ and $c \geq 0$, see Farina & Rinaldi (2000).

Consider the strictly proper, rational transfer function of order n

$$H(z) = \frac{P(z)}{Q(z)} = \frac{p_1 z^{n-1} + p_2 z^{n-2} + \ldots + p_{n-1} z + p_n}{z^n + q_1 z^{n-1} + \ldots + q_{n-1} z + q_n} \quad (3)$$

where we assume that $P(z)$ and $Q(z)$ are coprime polynomials. Expanding (3) in terms of the Markov parameters $\{h(i)\}_{i=1}^\infty$ we can write

$$H(z) = \sum_{i=1}^{\infty} h(i) z^{-i} \quad (4)$$

where the recursive expression for the $\{h(i)\}_{i=1}^\infty$ is

$$h(i) = -\sum_{j=1}^{i-1} q_{i-j} h(j) + p_i. \quad (5)$$

A pole of $H(z)$ is called a dominating pole if its modulus is maximum among all poles of $H(z)$. It is called strictly dominating if it is dominating and all other poles of $H(z)$ have strictly smaller modulus. A triplet $\{A, b, c\}$ is a realization of the strictly proper rational transfer function $H(z)$ if $H(z) = c(zI - A)^{-1} b$. A realization is *minimal* if and only if $\{A, b\}$ is controllable and $\{A, c\}$ observable. In this case

n denotes both the dimension of A and the order of $H(z)$. A realization is *positive* if $\{A, b, c\}$ is a positive system.

Throughout this paper we will make the following assumption.

Assumption 1 $H(z)$ has a simple, strictly dominating real pole, denoted ρ, i.e., $Q(z)$ factorizes as $Q(z) = Q'(z)(z - \rho)$.

Under this assumption, any minimal realization $\{A, b, c\}$ has to have ρ as a simple, strictly dominating real eigenvalue, i.e., $\rho = \rho(A) > |\lambda| \ \forall \lambda \in \text{sp}\{A\}, \ \lambda \neq \rho(A)$, see Benvenuti & Farina (2004).

Necessary and sufficient conditions for existence of a positive realization of $H(z)$ are provided in Ohta *et al.* (1984); Maeda & Kodama (1981); Farina & Benvenuti (1995). The formulation in the following Theorem is taken from Anderson *et al.* (1996) (Theorems 2.1, 2.2 and Remark in between).

Theorem 5 *Let $H(z)$ be a rational transfer function with minimal realization $\{A, b, c\}$. Then $H(z)$ has a positive realization if and only if $\exists$ a proper polyhedral cone $\mathcal{K}$ such that*

(1) $A\mathcal{K} \subseteq \mathcal{K}$;
(2) $b \in \mathcal{K}$;
(3) $c \in \mathcal{K}^*$.

In particular, in the case of simple strictly dominant real pole, it is known that a positive realization of some dimension $\mu \geq n$ always exists (Theorem 31 of Farina & Rinaldi (2000)) hence from Theorem 5, a proper A-invariant polyhedral cone $\mathcal{K}$ for which $b \in \mathcal{K}$ and $c \in \mathcal{K}^*$ always exists. Once $\mathcal{K} = \text{cone}(\Omega)$ has been found, then $\Omega \in \mathbb{R}^{n \times \mu}$, $\mu \geq n$, can be used to form a μ-dimensional positive realization $\{A_p, b_p, v_p\}$, where

$$A\Omega = \Omega A_p, \quad b = \Omega b_p, \quad c_p = c\Omega.$$

When $\mu > n$, the realization $\{A_p, b_p, v_p\}$ is nonminimal.

In this paper we introduce a class of minimal realizations which can be used to fill the gap between external positivity and (internal) positivity, at least under Assumption 1. We call these realizations *eventually positive*.

Definition 4 *A realization $\{A_e, b_e, c_e\}$ is said eventually positive if $A_e \overset{\vee}{>} 0$ and $\exists$ an A_e-invariant cone $\mathcal{K}_e$ for which A_e is $\mathcal{K}_e$-positive and such that $b_e \in \mathcal{K}_e$, $c_e \in \mathcal{K}_e^*$.*

The eventually positive realizations we are interested in are minimal, i.e., $A_e \in R^{n \times n}$, $b_e, c_e^T \in \mathbb{R}^n$. Notice that from Theorem 1, $A_e \overset{\vee}{>} 0$ implies $\rho(A_e) > 0$ is a simple strictly dominating real eigenvalue, i.e., Assumption 1 holds. From Theorem 3, in this case, a proper polyhedral cone $\mathcal{K}_e$ for which A_e is $\mathcal{K}_e$-positive always exists.

3.1 A constructive procedure

From Definition 4 and Theorem 1, a necessary condition for a minimal realization $\{A, b, c\}$ of $H(z)$ to be eventually positive is that $v > 0$ and $w > 0$, where v and w are the right and left eigenvectors of A relative to ρ.

Recall that if M is an invertible matrix, the change of basis performed via M yields:

$$A' = M^{-1}AM, \qquad b' = M^{-1}b, \qquad c' = cM$$
$$v' = M^{-1}v, \qquad w' = M^T w. \tag{6}$$

In this section we give a procedure to construct a minimal eventually positive realization of $H(z)$ satisfying Assumption 1. Let us consider the realization $\{A_o, b_o, c_o\}$

$$A_o = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -q_n & -q_{n-1} & \dots & -q_1 \end{bmatrix}, \quad b_o = \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(n) \end{bmatrix}, \quad c_o = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T.$$

which is sometimes called Markov observability canonical form because of the Markov parameters appearing in b_o. In this case the polyhedral cone (hereafter $\mathcal{K}_o$) is called the Markov cone (Anderson *et al.* , 1996; Farina & Rinaldi, 2000) and it is generated by

$$\Omega_o = \begin{bmatrix} h(1) & h(2) & \dots & h(\mu) \\ \vdots & \vdots & & \vdots \\ h(n) & h(n+1) & \dots & h(\mu+n-1) \end{bmatrix}.$$

For the Markov observability canonical form, $v_o > 0$, while w_o has no fixed sign pattern. The condition $v_o > 0$ guarantees that $A_o \in \mathscr{P}\mathscr{F}_n$. From Corollary 1, it also guarantees that $A^\eta \mathcal{K}_o$ belongs to $\text{int}(\mathbb{R}_+^n)$ for sufficiently high $\eta \in \mathbb{N}$. It however does not imply that also $A_o^T \in \mathscr{P}\mathscr{F}_n$, nor that $(A^T)^\eta \mathcal{K}_o^* \subset \text{int}(\mathbb{R}_+^n)$ for any η. For that, one may have to "tilt" the $\{A_o, b_o, c_o\}$ realization, rendering w_o positive while not changing the sign of v_o. This can be done through operations with elementary matrices as in the following algorithm.

Algorithm 1.

Input: $A = A_o$, $b = b_o$, $c = c_o$, $v = v_o$, $w = w_o$.
Step 0: To begin with, notice that possibly by multiplying by $-I_n$, we can always assume that w has at least $n/2$ (or $(n+1)/2$ if n is odd) nonnegative entries.
Step 1: Consider an index i for which $w_i \leq 0$. Two cases are possible:

4

Case 1: $\exists$ index $j \neq i$, $j \neq 1$, for which $w_j > 0$ and

$$0 \leq -\frac{w_i}{w_j} < \frac{v_j}{v_i} \qquad (7)$$

where by construction $-\frac{w_i}{w_j} \geq 0$ and $\frac{v_j}{v_i} > 0$. In this case, choosing $M^{(\xi)} = I_n + [m_{ji}^{(\xi)}]$, where $m_{ji}^{(\xi)} \in \left(-\frac{w_i}{w_j}, \frac{v_j}{v_i} \right) > 0$, one gets in (6):

$$
\begin{aligned}
w_i' &= w_i + m_{ji}^{(\xi)} w_j > 0, \quad w_j' = w_j > 0, \\
v_i' &= v_i > 0, \quad\quad\quad\quad v_j' = -m_{ji}^{(\xi)} v_i + v_j > 0,
\end{aligned}
\qquad (8)
$$

i.e., the sign of w_i has become positive.

Case 2: No index $j \neq i$ exists for which $w_j > 0$ and (7) holds. In this case, choosing any index $j \neq i$, $j \neq 1$ for which $w_j > 0$, it is possible to increase the corresponding v_j by combining the j-th row with any of the rows (indexed by k) having $w_k > 0$ and $c_k = 0$ (at least $n/2 - 1$ such rows exist by construction). In this case the elementary matrix that can be used is $M^{(\xi)} = I_n + [m_{jk}^{(\xi)}]$ with $-\frac{w_k}{w_j} < m_{jk}^{(\xi)} < 0$, which yields

$$
\begin{aligned}
w_j' &= w_j > 0, \quad\quad 0 < w_k' = m_{jk}^{(\xi)} w_j + w_k < w_k, \\
v_j' &= v_j - m_{jk}^{(\xi)} v_k > v_j > 0, \quad v_k' = v_k > 0.
\end{aligned}
\qquad (9)
$$

If v_j' is now sufficiently big that (7) holds, then Case 1) applies. If not, then another index k must be chosen and Case 2) iterated.

Step 2: Repeat until $w_j' > 0 \; \forall j = 1, \ldots, n$, or until Step 1 becomes unfeasible.

Step 3: If $w' > 0$ the Algorithm terminated successfully. In this case, if $M^{(\xi_1)}, \ldots, M^{(\xi_\psi)}$ are the ξ_ψ elementary matrices used in Step 1 above, then the change of basis $M = M^{(\xi_1)} \cdots M^{(\xi_\psi)}$ yields the sought realization $\{A_e, b_e, c_e\}$.

The following Proposition shows that indeed $\{A_e, b_e, c_e\}$ constructed in this way is a minimal eventually positive realization of $H(z)$.

Proposition 1 *Consider a strictly proper rational transfer function $H(z)$ obeying Assumption 1. Assume Algorithm 1 terminates successfully and consider the corresponding realization $\{A_e, b_e, c_e\}$. Then $\exists$ a proper polyhedral A_e-invariant cone $\mathcal{K}_e$ such that:*

(1) $A^\eta \mathcal{K}_e \subset \mathrm{int}(\mathbb{R}^n_+) \cup \{0\}$ for $\eta \geq \eta_o$;
(2) $(A^T)^\eta \mathcal{K}_e^* \subset \mathrm{int}(\mathbb{R}^n_+) \cup \{0\}$ for $\eta \geq \eta_o$;
(3) A_e is $\mathcal{K}_e$-positive;
(4) $b_e \in \mathcal{K}_e$;
(5) $c_e \in \mathcal{K}_e^*$.

Consequently, $\{A_e, b_e, c_e\}$ is a minimal eventually positive realization of $H(z)$.

Proof. Since $A_e \overset{\vee}{>} 0$, the first 3 items follow from Theorem 4. Consider the minimal realization in Markov observability canonical form $\{A_o, b_o c_o\}$. By construction $b_o \in \mathcal{K}_o$ and $c_o \in \mathcal{K}_o^*$. If Algorithm 1 terminates successfully then there exists a change of basis M such that $A_e = M^{-1} A_o M$, $b_e = M^{-1} b_o$, $c_e = c_o M$. Applying M to the double description pair (Ω_o, Γ_o) of the Markov cone: $\Omega_e = M^{-1}\Omega_o$ and $\Gamma_e = M^T \Gamma_o$. Since $b_o = \Omega \alpha$ for some $\alpha \in \mathbb{R}^\mu_+$, it is $b_e = M^{-1} \Omega_o \alpha = \Omega_e \alpha$, i.e., $b_e \in \mathcal{K}_e$. Similarly, from $c^T = \Gamma_o \beta$ for some $\beta \in \mathbb{R}^\mu_+$, it is $c_e^T = M^T \Gamma_o \beta = \Gamma_e \beta$ i.e., $c_e \in \mathcal{K}_e^*$. Therefore $\{A_e, b_e, c_e\}$ is an eventually positive realization of $H(z)$. Minimality of this realization follows from invariance of the minimality property to a change of basis. ∎

From Proposition 1 and from A_e-invariance of $\mathcal{K}_e$, for the reachability cone $\mathcal{R}_e = \mathrm{cone}(b_e, A_e b_e, A_e^2 b_e, \ldots)$ of the system $\{A_e, b_e, c_e\}$ it holds that $\mathcal{R}_e \subset \mathcal{K}_e$.

Recall that the solution of a linear discrete-time system can be split into free and forced evolution. For $\{A_e, b_e, c_e\}$ one gets:

$$x(k) = x_o(k) + x_f(k) = A_e^k x(0) + \sum_{j=0}^{k-1} A_e^{k-j-1} b_e u(j). \quad (10)$$

Proposition 2 *Consider a minimal eventually positive realization $\{A_e, b_e, c_e\}$ of $H(z)$. Then $\exists \eta_o \in \mathbb{N}$ such that $x_o(k) \geq 0 \; \forall k \geq \eta_o$, $\forall x(0) \in \mathbb{R}^n_+$. If in addition $\mathcal{R}_e \subset \mathbb{R}^n_+$, then $x(k) \geq 0 \; \forall k \geq \eta_o$, $\forall x(0) \in \mathbb{R}^n_+$, $\forall u(k) \in \mathbb{R}_+$.*

Proof. $A_e \overset{\vee}{>} 0$ implies $A_e^k > 0 \; \forall k \geq \eta_o$, hence $x_o(k) = A_e^k x(0) \geq 0 \; \forall x(0) \in \mathbb{R}^n_+$. Since b_e can have any sign, an analogous property does not hold in general for the forced evolution $x_f(k)$. However, $\mathcal{R}_e \subset \mathbb{R}^n_+$ implies $x_f(k) \geq 0 \; \forall k$. ∎

The following theorem summarizes the main result so far.

Theorem 6 *Consider a strictly proper rational transfer function $H(z)$ obeying Assumption 1. If $H(z)$ admits a minimal eventually positive realization then it is externally positive. Conversely, if $H(z)$ is externally positive and Algorithm 1 terminates successfully, then $H(z)$ has a minimal eventually positive realization.*

Proof. Consider an eventually positive realization and the corresponding A_e-invariant polyhedral cone $\mathcal{K}_e$. From $b_e \in \mathcal{K}_e$ and $c_e \in \mathcal{K}_e^*$, it follows that $c_e b_e \geq 0$. A_e-invariance then implies that $A_e^k b_e \in \mathcal{K}_e \; \forall k = 1, 2, \ldots$ and therefore that $c_e A_e^k b_e \geq 0$. Conversely, Proposition 1 implies that success in Algorithm 1 yields an eventually positive realization $\{A_e, b_e, c_e\}$. Minimality also follows from Proposition 1. ∎

According to Definition 4, in an eventually positive realization no sign constraint is imposed on b_e and c_e. It is however

5

easy to modify Algorithm 1 adding as constraints $b_e \geq 0$ and $c_e \geq 0$.

Algorithm 2.

Input: $A = A_o$, $b = b_o$, $c = c_o$, $v = v_o$, $w = w_o$.
Step 0: same as in Algorithm 1.
Step 1: Case 1: similar to Algorithm 1, but replacing (7) with

$$0 \leq -\frac{w_i}{w_j} < \min\left(\frac{b_j}{b_i}, \frac{v_j}{v_i}\right), \qquad (11)$$

where by construction $\frac{b_j}{b_i} > 0$ when $b_i \neq 0$ ($\frac{b_j}{b_i} = +\infty$ otherwise, also admissible). In this case, $M^{(\xi)} = I_n + [m_{ji}^{(\xi)}]$ can be chosen as $m_{ji}^{(\xi)} \in \left(-\frac{w_i}{w_j}, \min\left(\frac{b_j}{b_i}, \frac{v_j}{v_i}\right)\right) > 0$,
Case 2: same as Algorithm 1.
Step 2 & 3: same as Algorithm 1.

Proposition 3 *Consider a strictly proper rational transfer function $H(z)$ obeying Assumption 1. Assume Algorithm 2 terminates successfully and consider the corresponding realization $\{A_e, b_e, c_e\}$. Then Proposition 1 holds, and in addition:*

(1) $b_e \geq 0$;
(2) $c_e \geq 0$.

Proof. If Algorithm 2 is successful then so is Algorithm 1, hence Proposition 1 holds. Since b_o contains Markov coefficients, $b_o \geq 0$ and $c_o \geq 0$. By construction, the condition (11) guarantees that in Case 1, when applying $M^{(\xi)} = I_n + [m_{ji}^{(\xi)}]$, alongside (8) one gets $b_i' = b_i > 0$, $b_j' = -m_{ji}^{(\xi)}b_i + b_j > 0$, $c_i' = c_i$, $c_j' = c_j = 0$, i.e., the sign of w_i has become positive while the sign of b and c is preserved at each iteration. Similarly, in Case 2, since $m_{jk}^{(\xi)} < 0$, in addition to (9), $M^{(\xi)} = I_n + [m_{jk}^{(\xi)}]$ yields $b_j' = b_j - m_{jk}^{(\xi)}b_k > b_j > 0$, $b_k' = b_k > 0$, $c_j' = c_j = 0$, $c_k' = c_k = 0$, i.e., the sign of b and c is preserved. ∎

Remark 1 Notice that $b_e \geq 0 \not\Rightarrow \mathcal{R}_e \subset \mathbb{R}_+^n$, and hence $\not\Rightarrow x(k) \geq 0$ for $k \geq \eta_o$. It follows however from Proposition 2 that the impulse response of a minimal eventually positive realization with $b_e \geq 0$ and $c_e \geq 0$ is such that $x(t) \geq 0$ $\forall k \geq \eta_o$, $\forall x(0) \in \mathbb{R}_+^n$. In other words, while the impulse response completely characterizes external positivity it is not enough to characterize internal positivity, as expected.

In Guidorzi (2014) the author considers a class of minimal realizations of externally positive systems which he calls "quasi-positive", meaning realizations $\{A_q, b_q, c_q\}$ such that $b_q \geq 0$, $c_q \geq 0$ and $\mathcal{R}_q \subseteq \mathbb{R}_+^n$. It follows that for such realization $x_f(k) \geq 0$ $\forall k \geq 0$, while the sign of $x_o(k)$ cannot be

guaranteed, not even asymptotically (unless, trivially, also $x(0) \in \mathcal{R}_q$), i.e., a "quasi-positive" realization may have state variables which never become positive even if $x(0) \geq 0$. As an example, consider the Markov observability canonical form $\{A_o, b_o, c_o\}$ which by construction is always one such realization for any externally positive system. For it $v_o > 0$, hence $A_o \in \mathscr{P}\mathscr{F}_n$ when Assumption 1 holds. Therefore Corollary 1 holds but not necessarily Theorem 4, as the sign of w_o is not fixed a priori. As a consequence, asymptotically, the free evolution $\lim_{k \to \infty} x_o(k) = \frac{v_o w_o^T x(0)}{w_o^T v_o}$ can assume any sign even if $x(0) \geq 0$.

For eventually positive realizations, instead, the following stronger characterization holds.

Proposition 4 *Consider a minimal eventually positive realization $\{A_e, b_e, c_e\}$ of $H(z)$ such that $b_e \geq 0$ and $c_e \geq 0$. Then $\exists \eta_o \in \mathbb{N}$ such that $x(k) \geq 0$ $\forall k \geq \eta_o$, $\forall x(0) \in \mathbb{R}_+^n$. If in addition $\mathcal{R}_e \subset \mathbb{R}_+^n$, then $x_f(k) \geq 0$ $\forall k \geq 0$, $\forall u(k) \in \mathbb{R}_+$.*

Proof. The proof is analogous to that of Proposition 2, with the extra condition $b_e \geq 0$ guaranteeing now that $x_f(k) \geq 0$ $\forall k \geq \eta_o$. When in addition $\mathcal{R}_e \subset \mathbb{R}_+^n$, then obviously $x_f(k) \geq 0$ $\forall k \geq 0$, $\forall u(k) \in \mathbb{R}_+$. ∎

3.2 Examples

Example 1 Consider the following transfer function

$$H(z) = \frac{0.1048z^3 + 0.1312z^2 - 0.02171z - 0.01499}{z^5 - 0.9631z^4 - 0.05818z^3 + 0.03503z^2 - 0.01051z - 0.003276}$$

As $\rho = 1$, the Markov parameters are all nonnegative and "stabilize" at $\lim_{k \to \infty} h(k) = 0.193$, we can conclude that $H(z)$ is externally positive. The Markov observability canonical form $\{A_o, b_o, c_o\}$ has $v_o = \mathbf{1}$ and

$$w_o = \begin{bmatrix} 0.0033 & 0.0138 & -0.0212 & 0.0369 & 0.9990 \end{bmatrix}^T,$$

hence the dual cone $(A^T)^\eta \mathscr{K}_e^*$ is not contained in $\mathbb{R}_+^5$ for any η. However, transforming this canonical form as in (6) with $M_1 = I_5 + [m_{43}]$, $m_{43} = 0.6076$, one gets

$$A_{e_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.6076 & 1 & 0 \\ 0 & 0 & -0.3692 & -0.6076 & 1 \\ 0.0033 & 0.0105 & 0.0003 & 0.0582 & 0.9631 \end{bmatrix}$$

6

which has positive right and left dominant eigenvectors

$$v_{e_1} = \begin{bmatrix} 1 & 1 & 1 & 0.3924 & 1 \end{bmatrix}^T$$
$$w_{e_1} = \begin{bmatrix} 0.0033 & 0.0138 & 0.0012 & 0.0369 & 1 \end{bmatrix}^T.$$

A_{e_1} is eventually positive, with a power index $\eta_o = 8$, hence the $\{A_{e_1}, b_{e_1}, c_{e_1}\}$ realization is eventually positive. For it

$$b_{e_1} = \begin{bmatrix} 0 & 0.1048 & 0.2321 & 0.0669 & 0.1951 \end{bmatrix}^T \qquad c_{e_1} = c_o,$$

while in the Markov observability canonical form it was:

$$b_o = \begin{bmatrix} 0 & 0.1048 & 0.2321 & 0.2079 & 0.1951 \end{bmatrix}^T.$$

In this case, altering (lowering) one of the Markov parameters has been enough to "compensate" the negative entry in the left eigenvector of A_o. A simulation of the impulse response for this eventually positive realization is shown in Fig. 1 left (dashed lines). It shows the transient violation of positivity of the state vector, which is not "visible" at the output. Since $b_{e_1} \geq 0$ and $A_{e_1}^k b_{e_1} \geq 0 \; \forall k$, in this case strict dominance implies that the violation of positivity must necessarily be transient for any nonnegative $\{u(k)\}_{k=0}^{\infty}$.

Example 2 It is worth observing that the power index of an eventually positive realization may vary among different eventually positive realizations of the same externally positive transfer function. Consider again Example 1. If instead of using the change of basis M_1 we use the following $M_2 = I_5 + [m_{53}]$ with $m_{53} = 0.1032$, then we get another eventually positive realization, $\{A_{e_2}, b_{e_2}, c_{e_2}\}$, whose impulse response is shown in Fig.1, right. Also this time "tilting" one of the coordinates is enough to achieve positivity of the left dominant eigenvector. However by acting on the fifth state component (instead of the fourth of Example 1) we obtain an eventually positive realization whose power index is 5 ($A_{e_2}^k \geq 0$ already at $k = 2$, then we must take further powers to get all positive entries).

4 Recovering positivity through downsampling

The extra conditions $b_e \geq 0$ and $c_e \geq 0$ imposed in Algorithm 2 can be used to obtain minimal positive realizations for decimated subsequences of the Markov parameters.

Theorem 7 *Consider a strictly proper rational transfer function $H(z)$ satisfying Assumption 1 and the corresponding sequence of Markov parameters $\{h(k)\}_{k=1}^{\infty}$. Assume $\exists \; r \in \mathbb{N}$, $r \neq 0$, such that $r\mathrm{Im}[\lambda_i - \lambda_j] \neq 2\pi\xi$, $\xi = \pm 1, \pm 2, \ldots$, for all distinct poles λ_i and λ_j of $H(z)$ having $\mathrm{Re}[\lambda_i - \lambda_j] = 0$. If $H(z)$ admits a minimal eventually positive realization $\{A_e, b_e, c_e\}$ such that $b_e \geq 0$ and $c_e \geq 0$, then $\exists \; \sigma \in \mathbb{N}$ such that the decimated sequence of Markov*

parameters $\{h_s(k) = h((k-1)s+1)\}_{k=1}^{\infty}$ admits a minimal positive realization $\{A_s, b_e, c_e\}$ with $A_s = A_e^{\sigma}$.

Proof. Consider a minimal eventually positive realization $\{A_e, b_e, c_e\}$ of $H(z)$. If η_o is the power index of A_e, $A_e^{\sigma} > 0$ $\forall \sigma \in \mathbb{N}$, $\sigma \geq \eta_o$. Then each decimated sequence $\{h_s(k)\}_{k=1}^{\infty}$ admits a positive realization $\{A_s, b_e, c_e\}$, where $A_s = A_e^{\sigma}$. In fact, from $h(k) = c_e A_e^{k-1} b_e$, one gets $h_s(k) = h((k-1)\sigma + 1) = c_e A_e^{(k-1)\sigma} b_e = c_e A_s^{k-1} b_e$. In order for $\{A_s, b_e, c_e\}$ to be minimal, it must be $\{A_s, b_e\}$ controllable, and $\{A_s, c_e\}$ observable. Since the system is SISO and $\{A_e, b_e\}$ controllable, $\{A_e, c_e\}$ observable by construction, the geometric multiplicity of each distinct eigenvalue has to be 1 (i.e., there is a single Jordan block associated to each distinct eigenvalue). Denote $\lambda_1, \ldots, \lambda_m$, $m \leq n$, the distinct eigenvalues of A_e, of multiplicities $\zeta_1, \ldots, \zeta_m$. Since the eigenvalues of A_s are λ_i^{σ}, $i = 1, \ldots, m$, whenever a merge happens, i.e., $\lambda_i^{\sigma} = \lambda_j^{\sigma}$ for $i, j \in \{1, \ldots, m\}$, $i \neq j$, then controllability and observability of $\{A_s, b_e, c_e\}$ are lost, i.e., $\{A_s, b_e, c_e\}$ is a nonminimal realization of $\{h_s(k)\}_{k=1}^{\infty}$ (the Markov subsequence "looses rank" because of the decimation). However, if $\exists$ nonzero $r \in \mathbb{N}$ such that $r\mathrm{Im}[\lambda_i - \lambda_j] \neq 2\pi\xi$ $\xi = \pm 1, \pm 2, \ldots$, for all distinct eigenvalues of A_e such that $\mathrm{Re}[\lambda_i - \lambda_j] = 0$, then $h(k)$ and $h_s(k)$ admit minimal realizations of the same dimension. In this case, an admissible decimation factor is $\sigma = \min_{k \in \mathbb{N}}$ s.t. $rk \geq \eta_o$ and $rk\mathrm{Im}[\lambda_i - \lambda_j] \neq 2\pi\xi$, $\xi = \pm 1, \pm 2, \ldots$. ∎

Example 3 Consider again Example 1 and the $\{A_{e_1}, b_{e_1}, c_{e_1}\}$ eventually positive realization. Since the power index is 8, $A_{s_1} = A_{e_1}^8$, and $\{A_{s_1}, b_{e_1}, c_{e_1}\}$ is a minimal positive realization of the $\{h_8(k) = h((8k-7)\}_{k=1}^{\infty}$ subsequence of Markov parameters. By plotting the resulting dynamics in correspondence of the same nonzero initial condition plus impulse input of Example 1, we clearly see in Fig. 1 that the downsampling "neglects" the transient response and hence avoids one or more of its states to become temporarily negative. Analogous considerations hold for the realization $\{A_{e_2}, b_{e_2}, c_{e_2}\}$ (the decimation factor used in Fig. 1 is $\sigma = 2$).

In Theorem 7 the Markov parameters $h_s(\cdot)$ are a decimated sequence of the original Markov parameters $h(\cdot)$. This automatically means that the downsampled impulse is represented as $u(k) = \delta(k)$. It is in principle possible to downsample using other input interpolation techniques such as for example using a Zero-Order Hold (ZOH) methodology. The following remark and corollary show however that such an approach requires extra assumptions to be well-posed.

Remark 2 If $\{A_e, b_e, c_e\}$ is a minimal eventually positive realization of $H(z)$, then a ZOH downsampling of (10) by a factor $\sigma \in \mathbb{N}$ yields

$$A_{\mathrm{ZOH},s} = A_e^{\sigma}, \quad b_{\mathrm{ZOH},s} = \sum_{j=0}^{\sigma-1} A_e^{\sigma-j-1} b_e, \quad c_{\mathrm{ZOH},s} = c_e. \quad (12)$$
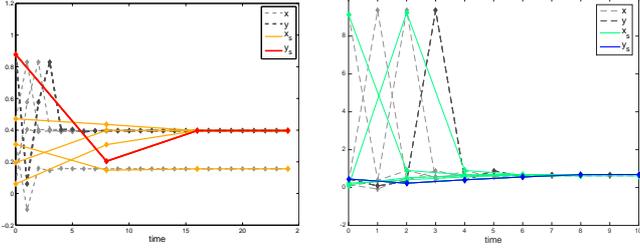
Fig. 1. Examples 1–3. Impulse response of the eventually positive system of Examples 1 (left, dashed) and 2 (right, dashed) and of the corresponding downsampled positive counterpart from the same initial condition (Example 3). The impulse response yields a positive output in all cases. In both realizations $\{A_{e_i}, b_{e_i}, c_{e_i}\}$ one of the states has a transient that violates positivity. The state and output of the downsampled positive realization $\{A_{s_i}, b_{e_i}, c_{e_i}\}$ are denoted resp. x_s and y_s. The state x_s is always positive, but its transient contains less information than the original x.

The realization $\{A_{\mathrm{ZOH},s}, b_{\mathrm{ZOH},s}, c_{\mathrm{ZOH},s}\}$ need not be positive. For it however, the Markov parameters are

$$h_{\mathrm{ZOH},s}(k) = c_{\mathrm{ZOH},s} A_{\mathrm{ZOH},s}^{k-1} b_{\mathrm{ZOH},s} = c_e A_e^{(k-1)\sigma} \sum_{j=0}^{\sigma-1} A_e^{\sigma-j-1} b_e$$

i.e., they are not a subsequence of the original $c_e A_e^{k-1} b_e$.

When $b_{\mathrm{ZOH},s} \geq 0$ we have the following straightforward corollary of Theorem 7.

Corollary 2 *Consider a minimal eventually positive realization $\{A_e, b_e, c_e\}$ of $H(z)$ such that $b_e \geq 0$ and $c_e \geq 0$. Assume $\exists\, r \in \mathbb{N}$, $r \neq 0$, such that $r\mathrm{Im}[\lambda_i - \lambda_j] \neq 2\pi\xi$, $\xi = \pm 1, \pm 2, \ldots$, for all distinct eigenvalues λ_i and λ_j of A_e having $\mathrm{Re}[\lambda_i - \lambda_j] = 0$. If $\mathscr{R}_e \subset \mathbb{R}_+^n$, then for $\sigma \geq \eta_o$ the ZOH downsampled realization (12) is positive and minimal.*

5 Continuous time eventually positive realizations

In a continuous-time linear system, external positivity of $H(s)$ corresponds to the impulse response function $h(t)$ being nonnegative: $h(t) \geq 0\ \forall t \geq 0$. For what concerns internal positivity, the matrix A has to be modified to satisfy the Metzler condition. A necessary and sufficient condition for a realization $\{A, b, c\}$ to be positive is in fact that A Metzler, $b \geq 0$ and $c \geq 0$, see Farina & Rinaldi (2000). For a transfer function $H(s)$, existence of a positive realization is given by the analogue of Theorem 5, which can be easily obtained from Theorem 4 of Ohta *et al.* (1984).

Theorem 8 *Let $H(s)$ be a strictly proper rational transfer function with minimal realization $\{A, b, c\}$. Then $H(s)$ has a positive realization if and only if*

(1) $(A + dI)\mathscr{K} \subseteq \mathscr{K}$ for some $d \geq 0$;
(2) $b \in \mathscr{K}$;

(3) $c \in \mathscr{K}^*$.

In this case, if $\mathscr{K} = \mathrm{cone}(\Omega)$, from $(A + dI)\Omega = \Omega A_+$, with $A_+ \geq 0$, one gets that the positive realization is $\{A_p = A_+ - dI, b_p, c_p\}$, where

$$A\Omega = \Omega A_p, \qquad b = \Omega b_p, \qquad c_p = c\Omega.$$

Starting from a realization $\{A, b, c\}$ of $H(s)$, denote $H_d(s - d) = c(sI - A - dI)^{-1}b = c((s-d)I - A)^{-1}b$ for some $d \geq 0$. If $h(t) = ce^{At}b \geq 0$, then also $ce^{(A+dI)t}b = ce^{At}e^{dt}b \geq 0$ for any $t \geq 0$, i.e., the extra factor dI does not alter the external positivity of the transfer function. Another necessary and sufficient condition for existence of a positive realization in continuous-time is then the following.

Theorem 9 *(Anderson et al. (1996), Theorem 5.1) A strictly proper rational transfer function $H(s)$ has a positive realization if and only if*

(1) $\exists\, d \geq 0$ such that $H_d(s - d)$ has nonnegative Markov parameters, i.e., if $H_d(s - d) = \sum_{j=1}^{\infty} h_d(j)(s - d)^{-j}$, then $h_d(j) \geq 0\ \forall j = 1, 2, \ldots$;
(2) there is a unique (possibly multiple) pole of $H(s)$ with maximal real part and the pole is real.

The class of $H(s)$ can be narrowed down in a manner similar to the discrete-time case by restricting to $H(s)$ with a simple dominant pole.

Assumption 2 *$H(a)$ has a simple real pole with real part strictly bigger than all other poles of $H(s)$.*

From Theorem 9, under Assumption 2, existence of a positive realization $\{A, b, c\}$ amounts to nonnegativity of the Markov parameters $h_d(k) = c(A - dI)^{k-1}b$ for a suitable $d \geq 0$. Also eventual positivity is affected by presence of the Metzler condition.

Definition 5 *A (continuous-time) realization $\{A_e, b_e, c_e\}$ is said eventually positive if $\exists\, d \geq 0$ such that $(A_e + dI) \overset{\vee}{>} 0$ and $\exists$ an A_e-invariant cone $\mathscr{K}_e$ for which A_e is $\mathscr{K}_e$-positive and such that $b_e \in \mathscr{K}_e$, $c_e \in \mathscr{K}_e^*$.*

Notice that $\mathscr{K}_e$ is A_e-invariant if and only if it is $A_e + dI$-invariant, and that A_e is $\mathscr{K}_e$-positive if and only if $A_e + dI$ is $\mathscr{K}_e$-positive for $d \geq 0$. Armed with this definition we can obtain the continuous-time analogue of Theorem 6.

Theorem 10 *Consider a strictly proper rational transfer function $H(s)$ obeying Assumption 2. If $H(s)$ admits a minimal eventually positive realization then it is externally positive. Conversely, if $H(s)$ is externally positive and $\exists\, d \geq 0$ such that for $H_d(s - d)$ Algorithm 1 terminates successfully, then $H(s)$ has a minimal eventually positive realization.*

Proof. Consider an eventually positive realization and the corresponding cone $\mathscr{K}_e$. Since $b_e \in \mathscr{K}_e$ and $c_e \in \mathscr{K}_e^*$, it is $c_e b_e \geq 0$. Since A_e-invariance implies $A_e + dI$-invariance of $\mathscr{K}_e$, then $(A_e + dI)^k b_e \in \mathscr{K}_e$ $\forall k = 1, 2, \ldots$, $\forall d \geq 0$. Hence $c_e(A_e + dI)^k b_e \geq 0$ $\forall k = 0, 1, 2, \ldots$, i.e., $H_d(s - d)$ is externally positive and so is $H(s)$. Conversely, if Algorithm 1 is successful for $H_d(s - d)$, then the triplet $\{A_e + dI, b_e, c_e\}$ is such that $(A_e + dI) \overset{\vee}{>} 0$ and Proposition 1 holds for it. Since the corresponding cone $\mathscr{K}_e$ is also A_e-invariant, $\{A_e, b_e, c_e\}$ is an eventually positive realization of $H(s)$. Minimality also follows from Proposition 1. ∎

5.1 A Nyquist-Shannon theorem for positivity of sampled systems

Recall that for a continuous-time linear system

$$\dot{x} = Ax + bu, \qquad y = cx,$$

the ZOH discretization is given by

$$x(k+1) = A_\delta x(k) + b_\delta u(k), \qquad y(k) = c_\delta x(k), \quad (13)$$

where

$$A_\delta = e^{AT}, \qquad b_\delta = \int_0^T e^{A\tau} b\, d\tau, \qquad c_\delta = c. \quad (14)$$

The following theorem says that whenever a continuous-time system has an eventually positive realization, then provided the sampling time is sufficiently long, its ZOH discretization will be positive.

Theorem 11 *Consider a strictly proper rational transfer function $H(s)$ satisfying Assumption 2. Assume $\exists\, r \in \mathbb{N}$, $r \neq 0$, such that $r\mathrm{Im}[\lambda_i - \lambda_j] \neq 2\pi\xi$, $\xi = \pm 1, \pm 2, \ldots$, for all distinct poles λ_i and λ_j of $H(s)$ having $\mathrm{Re}[\lambda_i - \lambda_j] = 0$. If $H(s)$ admits a minimal eventually positive realization $\{A_e, b_e, c_e\}$ such that $b_e \geq 0$ and $c_e \geq 0$ and $\mathscr{R}_e \subset \mathbb{R}_+^n$, then $\exists\, T_o$ such that $\forall$ sample times $T \geq T_o$ the ZOH sampled realization $\{A_\delta, b_\delta, c_\delta\}$ obtained from it is minimal and positive.*

Proof. $\{A_e, b_e, c_e\}$ eventually positive realization of $H(s)$ means that $\exists\, d \geq 0$ such that $(A_e + dI) \overset{\vee}{>} 0$, hence from Theorem 2 A_e is eventually exponentially positive, i.e., $\exists\, t_o \geq 0$ such that $e^{A_e t} > 0$ $\forall t \geq t_o$. Consequently in (14) $A_\delta > 0$ if $T \geq t_o$. In addition, from $e^{A_e \tau} b_e = \sum_{j=0}^{\infty} \frac{A_e^j \tau^j}{j!} b_e$, if $\mathscr{R}_e \subset \mathbb{R}_+^n$ then $e^{A_e \tau} b_e \geq 0$ $\forall \tau \geq 0$, meaning that also $b_\delta \geq 0$. Therefore $\{A_\delta, b_\delta, c_\delta\}$ is a positive realization of the ZOH sampled system (13) when $T \geq t_o$. Analogously to the proof of Theorem 7, the condition on r prevents losses of controllability/observability due to merging of eigenvalues with identical real part. Hence $\exists\, T_o \geq t_o$ such that when the sampling time is $T \geq T_o$, the realization $\{A_\delta, b_\delta, c_\delta\}$ is also minimal. ∎

Recall that when sampling a continuous-time linear system, if $\omega_s = \frac{2\pi}{T}$ is the sampling frequency, then the Nyquist frequency $\omega_N = \frac{\omega_s}{2} = \frac{\pi}{T}$ delimits the spectral bandwidth of the output that is univocally represented in the sampled signal. The Nyquist sampling theorem is normally invoked to determine a ω_N sufficiently high, so that the interesting frequencies can be correctly reconstructed from the sampled signal.

The result in Theorem 11 can be interpreted as a "dual" to the Nyquist sampling theorem: provided ω_N is chosen sufficiently small (i.e., from Theorem 11, $\omega_N < \frac{2\pi}{T_o}$), it is possible to guarantee the existence of minimal discrete-time realizations whose internal state remains positive for all inputs and for all initial conditions. For externally positive systems, eigenvalue dominance conditions such as Assumption 2 imply that the system shows at most damped oscillations, and a high sampling period relies on this fact to eliminate undesired transients, like those violating positivity.

Example 4 Consider the following transfer function

$$H(s) = \frac{8.843s^2 + 11.86s + 8.95}{s^3 + 1.361s^2 + 0.9938s}.$$

Since the poles are $\lambda = \{0, -0.6803 \pm 0.7286i\}$, $H(s)$ satisfies Assumption 2. The impulse response of $H(s)$, shown in Fig. 2 (grey dashed curves), implies that $H(s)$ is externally positive. Existence of a minimal positive realization is hard to verify even for such a simple system. The following is a minimal eventually positive realization

$$A_e = \begin{bmatrix} -0.3513 & -0.6949 & 0.3599 \\ 0.8229 & -0.8314 & 0.2594 \\ -0.0797 & 0.4214 & -0.1780 \end{bmatrix},$$

$$b_e = \begin{bmatrix} 3.3472 \\ 4.6663 \\ 10.1462 \end{bmatrix}, \quad c_e = \begin{bmatrix} 0.1621 & 0.0272 & 0.8056 \end{bmatrix},$$

for which $d = 0.9889$. The effect of the damped oscillatory modes is to induce a transient violation of internal positivity, see Fig. 2. Since $t_o = 4.84$, choosing a sample time $T = 5$ one gets the positive ZOH minimal realization

$$A_\delta = \begin{bmatrix} 0.0041 & 0.0447 & 0.1170 \\ 0.1140 & 0.0699 & 0.4012 \\ 0.2965 & 0.2044 & 0.8673 \end{bmatrix}, \; b_\delta = \begin{bmatrix} 9.2081 \\ 25.4148 \\ 52.6768 \end{bmatrix}, \; c_\delta = c_e$$

whose state and output impulse response is shown in Fig. 2 (blue and red curves). In this case, the sampling limits the

bandwidth of the system without altering significantly the Bode plot for the amplitudes. As a consequence of the long sampling time needed to have positivity, continuous-time and discrete-time output trajectories differ considerably.
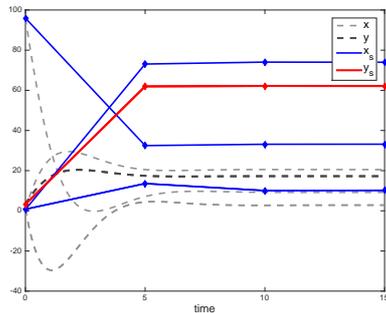

Fig. 2. Example 4. Impulse response of the continuous-time system (black, with state variables in gray) and of the ZOH sampled system (red, with state variables in blue). The continuous-time state violates positivity, while the ZOH state does not.

6 Conclusion

In an attempt to fill the gap between externally and internally positive linear systems, a novel class of realizations in introduced in this paper. These correspond to state update matrices that are eventually positive, i.e., that become positive after a certain power. The constructive procedure we provide in the paper to obtain such minimal eventually positive realizations is easy and quite general and, in our experience, always terminating with success when applied to externally positive systems having a single strictly dominating real eigenvalue of multiplicity 1. When the multiplicity is higher or when dominance is not strict, then the situation is more intricate and it is not clear whether eventual positivity can still play a key role. When the method is applicable, the insight into the structure of the systems that one gets through eventually positive realizations is considerable, as seen for example in our downsampling (or sampling, in continuous-time) theorems.

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