Dynamics of a Rolling and Sliding Disk in a Plane. Asymptotic Solutions, Stability and Numerical Simulations

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Abstract—We present a qualitative analysis of the dynamics of a rolling and sliding disk in a horizontal plane. It is based on using three classes of asymptotic solutions: straight-line rolling, spinning about a vertical diameter and tumbling solutions. Their linear stability analysis is given and it is complemented with computer simulations of solutions starting in the vicinity of the asymptotic solutions. The results on asymptotic solutions and their linear stability apply also to an annulus and to a hoop.

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1. INTRODUCTION

The problem of a disk rolling (without sliding) in a plane is integrable [1, 4, 9, 15]. It is described in the Euler angles by four dynamical equations and this system admits three additional integrals of motion. It can be in principle reduced to a single quadrature. The difficulty is, however, that only the energy integral is given explicitly, while two other integrals, of angular momentum type, are defined implicitly as constants of integrations for the Legendre equation for the dependence of ω_3 on $\cos \theta$ [22] and [9, §8]. References to original articles treating a rolling disk up to 1992 are contained in the bibliography of the book [18] and more recent results can be found, e.g., in [2, 19, 22].

Equations for a rolling and sliding disk have two more dynamical variables to describe the motion of the center of mass. These equations are dissipative, have monotonically decreasing energy and do not admit any known integrals of motion. A variety of friction models are discussed in the literature. The laws of friction that are used most often in mechanics are the dry Coulomb friction law when the friction force is proportional to the reaction force $g_n(t)$ and the law of viscous friction when the friction force is proportional to the sliding velocity of the rigid body. Coulomb friction considers two cases: rolling/sticking and sliding. The first one occurs when the sliding velocity is zero in the tangent plane of the contact frame and sliding friction occurs when the velocity is nonzero. The value of the sliding friction force is equal to $F_f = \mu g_n$ and for a rolling or sticking contact friction the Coulomb law is given by the inequality $F_f \leq \mu g_n$, where μ is the constant friction coefficient. The dry Coulomb friction law has been used to describe the dynamics of a disk on a plane, e.g., in [16, 17]. The viscous friction has been used for describing the motion of a disk in a plane in [20] and in Chapter 4, §4 of the book [18], see also [12] where a body of revolution was considered. We notice that [18, Ch. 4 §4] contains more general results concerning asymptotic motions of a solid body with an arbitrary shape moving on the horizontal plane and include results of Moshchuk. The value $\mu = 0$ corresponds to the case of an absolutely smooth surface and an appropriate limit $\mu \rightarrow \infty$ corresponds to the case of an absolutely rough surface, see, e.g., [21].

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In this paper we study the rolling and sliding motion of a disk under the assumption of viscous friction force $\mathbf{F}_f = -\mu g_n \mathbf{v}_A$ acting against the sliding velocity $\mathbf{v}_A$ and proportional to the value of the reaction force $g_n(t)$. This law of friction interpolates between the dry friction law $\mathbf{F}_f = -\mu g_n \frac{\mathbf{v}_A}{|\mathbf{v}_A|}$ and the viscous friction law $\mathbf{F}_f = -\mu \mathbf{v}_A$ and it is convenient for numerical simulation as the value of $g_n(t)$ is a dynamically determined variable, which, as it is easy to check, stays positive during numerical simulation. The viscous friction of this form has been used in several papers on the tippe top [8, 24]. The qualitative behavior of the tippe top obtained numerically using the viscous friction law [24] and that obtained from calculations with the Coulomb friction law [5] are very similar. The possibility of detachment of the body from the plane and aspects of the friction law were discussed in [3, 11]. There are also models of friction when the contact between the rigid body and the plane is no longer a point but a small region and the friction force is obtained by integration over the whole contact region, see, e.g., [6, 14, 26].

The purpose of this work is to study behavior of nonbouncing solutions of a sliding disk for the concrete model of friction force $\mathbf{F}_f = -\mu g_n \mathbf{v}_A$ and for initial conditions that correspond to typical situations of how a disk may be launched into motion. The main tool to understand dynamics is the study of asymptotic (stationary) solutions and their stability and the study of the behavior of solutions in some vicinity of the asymptotic solutions. Stability properties of asymptotic solutions provide a framework for interpreting results of numerical simulations.

For fixing notation we rederive the Euler equations of motion in Section 2. In Section 3 we solve stationary equations with friction and obtain three types of stationary solutions (rolling, spinning and tumbling) with vertical reaction force that are well known for the problem of a rolling disk [1]. They are in agreement with the results of a general study by N.K. Moschuk [20] of asymptotic motion for a rigid body moving in a plane under the action of a viscous friction force as presented in the book by A.P. Markeev [18]. The linear stability analysis of the three types of asymptotic solutions is given in Section 4. It provides only necessary conditions for stability as Jacobian matrices for linearization admit zero eigenvalues. In Section 5 we present results of numerical simulations. The starting point are two types of initial conditions (*IC*), called here reference *IC*, which differ from the rolling and spinning solution by a small tilt angle of the symmetry axis. We study them for the range of initial angular velocities, which guarantee positivity of the vertical reaction force $g_n > 0$ to avoid bouncing from the plane. The results of numerical study of stability for tested solutions are in agreement with the necessary conditions for linear stability, indicating that these conditions may also turn out to be sufficient.

In the asymptotic behavior the tumbling solutions play a central role as they are attractors for solutions starting far away from the stable rolling and spinning solutions and for solutions with low initial energy. The relation between initial conditions and the final asymptotic solutions is difficult to predict and it shows great sensitivity to small changes of initial conditions for certain ranges of initial velocities.

2. EQUATIONS OF MOTION OF A DISK

In this paper we consider a thin disk of radius R and mass m having a single point of contact A with a horizontal plane. We assume that the mass distribution in the disk is homogeneous and that the center of mass CM coincides with its geometric center.

To describe the motion of the disk, we use three reference frames, see Fig. 1. The first one is a fixed inertial Cartesian frame $\mathbf{K}_0 = (\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}})$ with basis vectors $\hat{\mathbf{X}}, \hat{\mathbf{Y}}$ in the horizontal plane and $\hat{\mathbf{Z}}$ being perpendicular to this plane. The second reference frame $\tilde{\mathbf{K}} = (\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$, with origin at CM , has the basis vector $\hat{\mathbf{1}}$ is directed to the contact point A of the disk with the plane, the basis vector $\hat{\mathbf{3}}$ is perpendicular to the surface of the disk and is a symmetry axis of the disk. Therefore, $\hat{\mathbf{2}} = \hat{\mathbf{3}} \times \hat{\mathbf{1}}$ rests in the plane of the disk and is always parallel to the plane of support. We set a third coordinate frame $\mathbf{K} = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ with origin at the contact point A so that $\hat{\mathbf{z}} = \hat{\mathbf{Z}}, \hat{\mathbf{y}} = \hat{\mathbf{2}}$ and $\hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$.

Although the frame $\tilde{\mathbf{K}}$ is not a fixed principal reference frame, the symmetry axis $\hat{\mathbf{3}}$ is a main body axis and by symmetry of the disk the inertia tensor is diagonal, $\mathbb{I} = \text{diag}(I_1, I_1, I_3)$, where

$$I_1 := I_{11} = I_{22} = kmR^2, \quad I_3 = I_{33} = 2kmR^2. \quad (2.1)$$

Fig. 1. Geometry of a sliding and rolling disk.

The value $k = 1/4$ corresponds to a uniform distribution of the mass of the disk and $k = 1/2$ is taken for a disk with mass concentrated at the edge.

We denote by $\mathbf{s}$ the position of CM with respect to the origin of the frame $(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}})$ and by $\mathbf{a} = R\hat{\mathbf{1}}$ the vector connecting CM to the point A of contact with the horizontal plane.

Two external forces act on the disk: the gravity force $-mg\hat{\mathbf{z}}$ applied to the center of mass and the friction-reaction force $\mathbf{F}_R + \mathbf{F}_f$ applied at the point of contact A . It consists of the reaction force $\mathbf{F}_R = g_n\hat{\mathbf{z}}$, where we shall assume $g_n = g_n(\mathbf{L}, \dot{\mathbf{s}}, \mathbf{s}, \hat{\mathbf{3}}) \geq 0$ and the friction force $\mathbf{F}_f = -\mu g_n \mathbf{v}_A$ acting against the sliding velocity

$$\mathbf{v}_A = \dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a}, \quad (2.2)$$

with the friction coefficient $\mu = \mu(\mathbf{L}, \dot{\mathbf{s}}, \mathbf{s}, \hat{\mathbf{3}}, t) \geq 0$, which we later shall consider to be constant. Thus, in our considerations only the sliding friction is taken into account, while the rolling and the spinning friction is neglected.

The motion of the disk is described by a Newton equation for the linear momentum $m\dot{\mathbf{s}}$, an equation for the angular momentum $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$ and a kinematic equation describing the motion of the symmetry axis $\hat{\mathbf{3}}$ with respect to the fixed inertial frame $\mathbf{K}_0$. In vector form the equations are

$$m\dot{\mathbf{s}} = \mathbf{F}_R + \mathbf{F}_f - mg\hat{\mathbf{z}}, \quad \dot{\mathbf{L}} = \mathbf{a} \times (\mathbf{F}_R + \mathbf{F}_f), \quad \dot{\hat{\mathbf{3}}} = \boldsymbol{\omega} \times \hat{\mathbf{3}} = \frac{1}{I_1} (\mathbf{L} \times \hat{\mathbf{3}}). \quad (2.3)$$

The energy of the system is the sum of the kinetic energy of translational motion of the mass center E_{trans} , the rotational energy of the rigid body E_{rot} and of the potential energy of the mass center E_{pot}

$$E = E_{\text{trans}} + E_{\text{rot}} + E_{\text{pot}} = \frac{1}{2}m\dot{\mathbf{s}}^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega} + mgs \cdot \hat{\mathbf{z}}. \quad (2.4)$$

The change of the energy in time is given by the derivative

$$\begin{aligned} \dot{E} &= \frac{d}{dt} \left(\frac{1}{2}m\dot{\mathbf{s}}^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} + mgs \cdot \hat{\mathbf{z}} \right) = \dot{\mathbf{s}} \cdot (\mathbf{F}_R + \mathbf{F}_f - mg\hat{\mathbf{z}}) + \boldsymbol{\omega} \cdot [\mathbf{a} \times (\mathbf{F}_R + \mathbf{F}_f)] + mgs\dot{\mathbf{s}} \cdot \hat{\mathbf{z}} \\ &= (\mathbf{v}_A - \boldsymbol{\omega} \times \mathbf{a}) \cdot (\mathbf{F}_R + \mathbf{F}_f) - mgs\dot{\mathbf{s}} \cdot \hat{\mathbf{z}} + \boldsymbol{\omega} \cdot [\mathbf{a} \times (\mathbf{F}_R + \mathbf{F}_f)] + mgs\dot{\mathbf{s}} \cdot \hat{\mathbf{z}} = \mathbf{v}_A \cdot \mathbf{F}_f = -\mu g_n \mathbf{v}_A^2, \end{aligned}$$

where we have used Eqs. (2.3), as well as the equality $\dot{\boldsymbol{\omega}} \cdot \mathbf{L} = \dot{\boldsymbol{\omega}} \cdot \mathbb{I}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \dot{\mathbf{L}}$, valid due to the axial symmetry of the disk, and $\mathbf{v}_A \cdot \hat{\mathbf{z}} = 0$. In this calculation, $\mathbf{v}_A \cdot \mathbf{F}_f = -\mu g_n \mathbf{v}_A^2$ is the rate of frictional loss of energy and $mgs\dot{\mathbf{s}} \cdot \hat{\mathbf{z}}$ is the rate of energy transfer from the translational part into potential energy E_{pot} . The term $\boldsymbol{\omega} \cdot [\mathbf{a} \times (\mathbf{F}_R + \mathbf{F}_f)]$ is the work performed in unit time by the torque $\mathbf{a} \times (\mathbf{F}_R + \mathbf{F}_f)$ and it transfers energy between the translational and rotational components.

For a disk in contact with the plane of support there are two natural constraints:

- pure rolling condition when the sliding velocity vanishes identically in time

$$\mathbf{v}_A(t) = \dot{\mathbf{s}}(t) + \boldsymbol{\omega}(t) \times \mathbf{a}(t) = 0, \quad (2.5)$$

i.e., the contact point is not sliding. This vector equation gives three scalar conditions. Because (2.5) is an identity in time, all derivatives of $\mathbf{v}_A$ must also vanish. From (2.5) we obtain $\dot{\mathbf{s}} = \mathbf{a} \times \boldsymbol{\omega}$ and by differentiating this expression we get

$$m\ddot{\mathbf{s}} = m(\mathbf{a} \times \dot{\boldsymbol{\omega}}) = \mathbf{F}_R + \mathbf{F}_f - mg\hat{\mathbf{z}}, \quad (2.6)$$

where we have used the first Eq. (2.3). From this equality the friction-reaction force is determined dynamically as $\mathbf{F}_R + \mathbf{F}_f = m(\mathbf{a} \times \dot{\boldsymbol{\omega}}) + mg\hat{\mathbf{z}}$ meaning that at each instant of time it is exactly such that it sustains pure rolling. Then the second and the third equation in (2.3) become a closed system of equations

$$\dot{\mathbf{L}} = \mathbf{a} \times \left[m(\mathbf{a} \times \dot{\boldsymbol{\omega}}) + mg\hat{\mathbf{z}} \right], \quad \dot{\hat{\mathbf{z}}} = \boldsymbol{\omega} \times \hat{\mathbf{z}} = \frac{1}{I_1} (\mathbf{L} \times \hat{\mathbf{z}}), \quad (2.7)$$

which in the Euler angles describes the classical problem [1, 4, 15] of a purely rolling disk having four degrees of freedom.

- the rolling and sliding condition

$$(\mathbf{s}(t) + \mathbf{a}(t)) \cdot \hat{\mathbf{z}} = 0 \quad (2.8)$$

holds all time and so all its time derivatives vanish. The first derivative $0 = (\dot{\mathbf{s}} + \dot{\mathbf{a}}) \cdot \hat{\mathbf{z}} = (\dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{a}) \cdot \hat{\mathbf{z}} = \mathbf{v}_A \cdot \hat{\mathbf{z}}$ says that the sliding velocity, indeed, stays in the horizontal plane of support. From the second derivative multiplied by mass m we obtain

$$\begin{aligned} m(\dot{\mathbf{s}} + \dot{\mathbf{a}}) \cdot \hat{\mathbf{z}} &= m[\ddot{\mathbf{s}} + (\boldsymbol{\omega} \times \dot{\mathbf{a}})] \cdot \hat{\mathbf{z}} = (\mathbf{F}_R + \mathbf{F}_f - mg\hat{\mathbf{z}} + m(\boldsymbol{\omega} \times \dot{\mathbf{a}})) \cdot \hat{\mathbf{z}} \\ &= \left[g_n(\hat{\mathbf{z}} - \mu\mathbf{v}_A) - mg\hat{\mathbf{z}} + m(\boldsymbol{\omega} \times \dot{\mathbf{a}}) \right] \cdot \hat{\mathbf{z}} = 0, \end{aligned}$$

thus the value of the reaction force is equal to

$$g_n = mg + m(\mathbf{a} \times \dot{\boldsymbol{\omega}}) \cdot \hat{\mathbf{z}}. \quad (2.9)$$

From the constraint (2.8) we can calculate the vertical component z of motion of the center of mass, $s_z = \mathbf{s} \cdot \hat{\mathbf{z}} = -\mathbf{a} \cdot \hat{\mathbf{z}}$ and $\dot{s}_z = -\dot{\mathbf{a}} \cdot \hat{\mathbf{z}} = -(\boldsymbol{\omega} \times \mathbf{a}) \cdot \hat{\mathbf{z}}$. We introduce a vector $\mathbf{r}$ that is a projection of $\mathbf{s}$ on the horizontal plane, i.e., $\mathbf{r} = \mathbf{s} - s_z\hat{\mathbf{z}}$, and then system (2.3) reduces to

$$m\ddot{\mathbf{r}} = \mathbf{F}_f, \quad \dot{\mathbf{L}} = \mathbf{a} \times (\mathbf{F}_R + \mathbf{F}_f), \quad \dot{\hat{\mathbf{z}}} = \boldsymbol{\omega} \times \hat{\mathbf{z}} = \frac{1}{I_1} (\mathbf{L} \times \hat{\mathbf{z}}). \quad (2.10)$$

The orientation of the disk with respect to $\mathbf{K}_0$ is described by the Euler angles (θ, φ, ψ) . As usual, the angle θ is the inclination of the symmetry axis $\hat{\mathbf{z}}$ with respect to $\hat{\mathbf{z}}$, see Fig. 1. We note that for $\theta = \pm\pi/2$ the disk is vertical and for $\theta = 0, \pi$ it is horizontal. Since $\hat{\mathbf{y}} = \hat{\mathbf{z}}$, the reference frame $\tilde{\mathbf{K}}$ is obtained from $\mathbf{K}$ by rotating about $\hat{\mathbf{y}}$ by an angle θ , we have

$$\begin{aligned} \hat{\mathbf{x}} &= \cos\theta\hat{\mathbf{1}} + \sin\theta\hat{\mathbf{3}}, & \hat{\mathbf{1}} &= \cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{z}}, \\ \hat{\mathbf{y}} &= \hat{\mathbf{2}}, & \hat{\mathbf{2}} &= \hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &= -\sin\theta\hat{\mathbf{1}} + \cos\theta\hat{\mathbf{3}}, & \hat{\mathbf{3}} &= \sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}}. \end{aligned} \quad (2.11)$$

We set φ as the rotation angle around the $\hat{\mathbf{z}}$ -axis and ψ is the rotation around the the symmetry axis $\hat{\mathbf{3}}$.

The reference frame $\mathbf{K}$ rotates with angular velocity $\dot{\varphi}\hat{\mathbf{z}}$ with respect to the fixed frame $\mathbf{K}_0$ and $\tilde{\mathbf{K}}$ is obtained from $\mathbf{K}$ by rotation about $\hat{\mathbf{y}}$ by an angle θ . Thus, the angular velocity of the frame $\tilde{\mathbf{K}}$ with respect to $\mathbf{K}_0$ is equal to

$$\boldsymbol{\omega}_{\text{ref}} = \dot{\theta}\hat{\mathbf{z}} + \dot{\varphi}\hat{\mathbf{z}} = -\dot{\varphi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{z}} + \dot{\varphi}\cos\theta\hat{\mathbf{3}}.$$

The total angular velocity of the disk is obtained by adding to $\boldsymbol{\omega}_{\text{ref}}$ the angular velocity of rotation about the symmetry axis $\hat{\mathbf{3}}$ by angle ψ :

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{ref}} + \dot{\psi}\hat{\mathbf{3}} = -\dot{\varphi}\sin\theta\hat{\mathbf{1}} + \dot{\theta}\hat{\mathbf{z}} + (\dot{\psi} + \dot{\varphi}\cos\theta)\hat{\mathbf{3}}.$$

As usual, we denote $\omega_3 = \dot{\psi} + \dot{\varphi}\cos\theta$. The angular momentum of the disk is

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega} = -I_1\dot{\varphi}\sin\theta\hat{\mathbf{1}} + I_1\dot{\theta}\hat{\mathbf{z}} + I_3(\dot{\psi} + \dot{\varphi}\cos\theta)\hat{\mathbf{3}}.$$

The kinematic equations describing the rotation of the axes $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ are the following:

$$\begin{aligned}\dot{\hat{\mathbf{1}}} &= \boldsymbol{\omega}_{\text{ref}} \times \hat{\mathbf{1}} = \dot{\varphi}\cos\theta\hat{\mathbf{2}} - \dot{\theta}\hat{\mathbf{3}}, \\ \dot{\hat{\mathbf{2}}} &= \boldsymbol{\omega}_{\text{ref}} \times \hat{\mathbf{2}} = -\dot{\varphi}\cos\theta\hat{\mathbf{1}} - \dot{\varphi}\sin\theta\hat{\mathbf{3}}, \\ \dot{\hat{\mathbf{3}}} &= \boldsymbol{\omega}_{\text{ref}} \times \hat{\mathbf{3}} = \boldsymbol{\omega} \times \hat{\mathbf{3}} = \dot{\theta}\hat{\mathbf{1}} + \dot{\varphi}\sin\theta\hat{\mathbf{2}}.\end{aligned}$$

The velocity of the point of contact with the horizontal plane (due to (2.11)) is

$$\mathbf{v}_A = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} = v_x\cos\theta\hat{\mathbf{1}} + v_y\hat{\mathbf{2}} + v_x\sin\theta\hat{\mathbf{3}}. \quad (2.12)$$

Substitution of $\boldsymbol{\omega}$, $\mathbf{v}_A$ and $\hat{\mathbf{z}}$ expressed in terms of the Euler angles into the equations of motion (2.10) gives us a system of equations that, when solved with respect to $(\ddot{\theta}, \ddot{\varphi}, \dot{\omega}_3, \dot{v}_x, \dot{v}_y)$, takes the form

$$\ddot{\theta} = \frac{\sin\theta}{I_1} (I_1\dot{\varphi}^2\cos\theta - I_3\omega_3\dot{\varphi}) + \frac{\mu g_n v_x R \sin\theta}{I_1} - \frac{R g_n \cos\theta}{I_1}, \quad (2.13)$$

$$\ddot{\varphi} = \frac{1}{I_1 \sin\theta} (I_3\omega_3\dot{\theta} - 2I_1\dot{\theta}\dot{\varphi}\cos\theta), \quad (2.14)$$

$$\dot{\omega}_3 = -\frac{\mu R g_n v_y}{I_3}, \quad (2.15)$$

$$\begin{aligned}\dot{v}_x &= -(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta)R\cos\theta - \frac{R\omega_3\dot{\varphi}}{I_1}(I_1 - I_3\sin^2\theta) - \frac{\mu g_n v_x}{mI_1}(I_1 + mR^2\sin^2\theta) \\ &\quad + \frac{R^2 g_n \sin\theta}{I_1}\cos\theta + \dot{\varphi}v_y, \quad (2.16)\end{aligned}$$

$$\dot{v}_y = -R\dot{\varphi}\dot{\theta}\sin\theta - \frac{(I_3 + mR^2)\mu g_n v_y}{mI_3} - \dot{\varphi}v_x. \quad (2.17)$$

It effectively is a system of sixth order w.r.t. $\theta, \dot{\theta}, \dot{\varphi}, \omega_3, v_x, v_y$. Here the value of the reaction force is

$$g_n = \frac{mgI_1 - mRI_1\sin\theta\dot{\theta}^2 + mRI_1\cos^2\theta\dot{\varphi}^2\sin\theta - mRI_3\omega_3\dot{\varphi}\sin\theta\cos\theta}{I_1 + mR^2\cos^2\theta - \mu mR^2v_x\sin\theta\cos\theta}. \quad (2.18)$$

Notice that this system has an invariant manifold $\mathcal{N} = \{(\theta, \dot{\theta}, \dot{\varphi}, \omega_3, v_x, v_y) \in \mathbb{R}^6 \mid \dot{\varphi} = v_y = \omega_3 = 0\}$.

The energy of the system in the Euler angles takes the form

$$\begin{aligned}E = E_{\text{trans}} + E_{\text{rot}} + E_{\text{pot}} &= \frac{m}{2} \left[v_x^2 \cos^2\theta + (v_y - R\omega_3)^2 + (v_x \sin\theta + R\dot{\theta})^2 \right] \\ &\quad + \frac{1}{2} \left[I_3\omega_3^2 + I_1\dot{\theta}^2 + I_1\dot{\varphi}^2 \sin^2\theta \right] + mgR\sin\theta. \quad (2.19)\end{aligned}$$

3. ASYMPTOTIC SOLUTIONS

We define asymptotic solutions of Eqs. (2.13)–(2.17) as nondissipative solutions of constant energy $\dot{E} = -\mu g_n \mathbf{v}_A^2 = 0$, and so satisfying $\mathbf{v}_A = \mathbf{0}$. As a consequence, for asymptotic solutions also the friction force vanishes $\mathbf{F}_f = -\mu g_n \mathbf{v}_A = \mathbf{0}$.

Proposition 1. *Asymptotic solutions $\mathbf{v}_A = \mathbf{0}$ of the rolling and sliding disk are of the following three types:*

- *straight rolling solutions with $\theta = \pi/2$, $\dot{\varphi} = 0$ and ω_3 an arbitrary constant,*
- *spinning solutions with $\theta = \pi/2$, $\omega_3 = 0$ and $\dot{\varphi}$ an arbitrary constant,*
- *tumbling solutions with $\theta \in (0, \pi)$ and*

$$\omega_3 = 0, \quad \dot{\varphi} = \pm \sqrt{\frac{gmR}{I_1 \sin \theta}}. \quad (3.1)$$

They are also stationary solutions of Eqs. (2.13)–(2.17). The reaction force for all these solutions is $g_n(t) = mg$. For spinning and tumbling solutions the center of mass CM is fixed and for the rolling solutions it is moving along a straight line with speed $|\dot{\mathbf{s}}| = \omega_3 R$.

Proof. The asymptotic solutions are determined by the system

$$\dot{\omega}_3 = 0, \quad (3.2)$$

$$\ddot{\varphi} = \frac{1}{I_1 \sin \theta} (I_3 \omega_3 \dot{\theta} - 2I_1 \dot{\theta} \dot{\varphi} \cos \theta), \quad (3.3)$$

$$0 = \dot{v}_y = -R\dot{\varphi} \dot{\theta} \sin \theta, \quad (3.4)$$

$$0 = \dot{v}_x = -(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta)R \cos \theta - \frac{R\omega_3 \dot{\varphi}}{I_1} (I_1 - I_3 \sin^2 \theta) + \frac{R^2 g_n \sin \theta}{I_1} \cos \theta, \quad (3.5)$$

$$\ddot{\theta} = \frac{\sin \theta}{I_1} (I_1 \dot{\varphi}^2 \cos \theta - I_3 \omega_3 \dot{\varphi}) - \frac{Rg_n \cos \theta}{I_1}, \quad (3.6)$$

where

$$g_n = \frac{mgI_1 - mRI_1 \sin \theta \dot{\theta}^2 - mR \cos \theta \sin \theta (I_3 \omega_3 \dot{\varphi} - I_1 \cos \theta \dot{\varphi}^2)}{I_1 + mR^2 \cos^2 \theta}. \quad (3.7)$$

From (3.2) $\omega_3 = \text{const}$ and then one can integrate Eq. (3.3) to get

$$(I_1 \dot{\varphi} \sin^2 \theta + I_3 \omega_3 \cos \theta) = 0 \quad \Rightarrow \quad I_1 \dot{\varphi} \sin^2 \theta + I_3 \omega_3 \cos \theta = C = \text{const}. \quad (3.8)$$

Thus, all asymptotic motions have a first integral C .

Let us note that $\sin \theta = 0$ does not give asymptotic solutions because then vanishing of $\ddot{\theta}$ determined in (3.6) requires that $g_n = 0$, while in this case $g_n = mgI_1/(I_1 + mR^2) \neq 0$. Thus, we can assume that $\sin \theta \neq 0$, which enables expressing $\dot{\varphi}$ as a function of $\cos \theta$

$$\dot{\varphi} = \frac{C - I_3 \omega_3 \cos \theta}{I_1 \sin^2 \theta}. \quad (3.9)$$

Equation (3.4) is now satisfied when either $\dot{\varphi} = 0$ or $\dot{\theta} = 0$. Then Eq. (3.9) implies that $\theta = \text{const}$ for every asymptotic solution and $\dot{\varphi} = \text{const}$. Then the condition of vanishing of the linear combination $R \sin \theta (\ddot{\theta}) + (\dot{v}_x)$ (see (3.6) and (3.5)) after dividing by $-R$ gives

$$\omega_3 \dot{\varphi} = 0. \quad (3.10)$$

So, for asymptotic solutions either $\dot{\varphi} = 0$ or $\omega_3 = 0$.

If we substitute g_n given by (3.7) and the condition $\dot{\theta} = 0$ into (3.6), then we obtain the equality

$$mgR \cos \theta + \sin \theta (I_3 \omega_3 \dot{\varphi} - I_1 \cos \theta \dot{\varphi}^2) = 0, \quad (3.11)$$

which substituted into (3.7) gives $g_n = mg$ for every asymptotic solution. Since $\omega_3 \dot{\varphi} = 0$, condition (3.6) says that for every solution

$$\cos \theta [I_1 \dot{\varphi}^2 \sin \theta - gmR] = 0. \quad (3.12)$$

Thus, every asymptotic solution satisfies

$$\dot{\theta} = 0, \quad \omega_3 \dot{\varphi} = 0 \quad \text{and} \quad \cos \theta [I_1 \dot{\varphi}^2 \sin \theta - gmR] = 0.$$

This gives either $\cos \theta = 0$ with solution $\theta = \pm \pi/2$ or $I_1 \dot{\varphi}^2 \sin \theta - gmR = 0$ with solution $\dot{\varphi} = \pm \sqrt{\frac{gmR}{I_1 \sin \theta}}$ and then $\omega_3 = 0$. Solutions $\theta = \pm \pi/2$ require either $\omega_3 = 0$ with arbitrary $\dot{\varphi}$ or $\dot{\varphi} = 0$ with arbitrary ω_3 . Thus we obtain all solutions mentioned in Proposition 1.

Every asymptotic solution satisfies $\mathbf{0} = \mathbf{v}_A = \dot{\mathbf{s}} + \boldsymbol{\omega} \times R\hat{\mathbf{1}}$ and either $\dot{\mathbf{s}} = \mathbf{0}$ or for rolling solutions the velocity of the center of mass is

$$\dot{\mathbf{s}} = -\boldsymbol{\omega} \times R\hat{\mathbf{1}} = -\omega_3 \hat{\mathbf{3}} \times R\hat{\mathbf{1}} = -\omega_3 R\hat{\mathbf{2}}$$

and CM moves with constant velocity directed parallel to the plane of support since here $\dot{\varphi} = 0$, $\omega_3 \in \mathbb{R}$. For spinning and tumbling solutions $\dot{\mathbf{s}} = \mathbf{0}$, CM is fixed and $\boldsymbol{\omega} \parallel \hat{\mathbf{1}}$. For spinning solutions $\cos \theta = 0$ and $\dot{\varphi} \in \mathbb{R}$ may be arbitrary. For tumbling solutions $\cos \theta \neq 0$, $\dot{\varphi} = \pm \sqrt{\frac{gmR}{I_1 \sin \theta}}$ and $\omega_3 = \cos \theta \dot{\varphi} + \dot{\psi} = 0$.

The asymptotic solutions of Proposition 1 turn the r.h.s. of Eqs. (2.13)–(2.17), together with the equation $\dot{\theta} = 0$, into zero. So they are also stationary solutions. To show that they are all stationary solutions, we need to solve the stationary equations of (2.13)–(2.17).

From (2.15) $\dot{\omega}_3 = -\frac{\mu R g_n v_y}{I_3}$ and for $g_n \neq 0$ there is $v_y = 0$. Then from the Eq. (2.17) $0 = \dot{v}_y = -\dot{\varphi} v_x$ either $v_x = 0$ or $\dot{\varphi} = 0$.

Fig. 2. Bifurcation diagram.

The linear combination

$$0 = (\ddot{\theta})R \sin \theta + (\dot{v}_x) = -\frac{\mu}{m} g_n v_x - R \omega_3 \dot{\varphi} + \dot{\varphi} v_y$$

says that if $\dot{\varphi} = 0$, then $v_x = 0$. So stationary solutions with $g_n \neq 0$ have $v_x = 0$.

Thus, stationary solutions have $v_x = v_y = 0$, $\dot{\theta} = 0$ and they are described by Eqs. (3.2)–(3.7) with $\dot{\theta} = 0$. Obviously they have the same solutions.

If it were $g_n = 0$, then by (2.17) there is $v_x \dot{\varphi} = 0$ and either $v_x = 0$ or $\dot{\varphi} = 0$. If $\dot{\varphi} = 0$, then all stationary equations are fulfilled but the formula (2.18) gives $g_n \neq 0$ and we have a contradiction. If $v_x = 0$, then from (2.13) either $\sin \theta = 0$, this describes a disk laying on a plane, or $(I_1 \dot{\varphi}^2 \cos \theta - I_3 \omega_3 \dot{\varphi}) = 0$. By substituting $(I_1 \dot{\varphi}^2 \cos \theta - I_3 \omega_3 \dot{\varphi}) = 0$ into expression (2.18) we see that again $g_n \neq 0$ and the assumption $g_n = 0$ leads to a contradiction. $\square$

Using the values of integral C of (3.8), we graphically represent asymptotic solutions of the rolling disk in Fig. 2. The bold lines indicate (possibly) stable solutions and the dotted lines unstable ones.

4. LINEAR STABILITY OF ASYMPTOTIC SOLUTIONS

In this section we analyze stability of the asymptotic solutions found in the previous section. In order to reduce the number of parameters, we rescale the variables of system (2.13)–(2.18) in the following way:

$$(\dot{\theta}, \dot{\varphi}, \omega_3) = \frac{1}{T}(\theta', \varphi', \Omega_3), \quad (v_x, v_y) = \frac{L}{T}(V_x, V_y), \quad \tau = Tt, \quad (4.1)$$

where L and T are the respective scales of length and time. The variables Ω_3 , V_x , V_y denote now nondimensional quantities. The derivative $'$ denotes differentiation with respect to new nondimensional time τ . We choose normalization in such a way that $T = \sqrt{R/g}$ and $L = \sqrt{R/g}/\mu$. In these variables the equations of motion take the form

$$\theta'' = \frac{\cos(\theta) \left[\sin(\theta) (\theta'^2 + k\varphi'^2) - 1 \right] + \sin(\theta) \left[V_x - 2\varphi'k\Omega_3 - \theta'^2 V_x \sin(\theta) \right]}{k + \cos^2(\theta) - V_x \sin(\theta) \cos(\theta)}, \quad (4.2)$$

$$\varphi'' = \frac{2\theta'}{\sin \theta} (\Omega_3 - \varphi' \cos \theta), \quad (4.3)$$

$$\Omega_3' = \frac{V_y \left[\sin(\theta) (\theta'^2 - \phi'^2 \cos^2(\theta)) + \varphi' \Omega_3 \sin(2\theta) - 1 \right]}{2(\cos^2(\theta) + k - V_x \sin(\theta) \cos(\theta))}, \quad (4.4)$$

$$\begin{aligned} V_x' = & \left[4\alpha \sin \theta \cos \theta - 4\alpha \theta'^2 (k+1) \cos \theta - 4\alpha \phi'^2 k \sin^2 \theta \cos \theta - 2\alpha \phi' \Omega_3 ((2k+1) \cos(2\theta) \right. \\ & + 1) + 2\alpha V_x (2\theta'^2 (k+1) \sin \theta - 2\phi'^2 k \sin \theta \cos^2 \theta + (2k+1)(-1 + \phi' \Omega_3 \sin(2\theta)) \\ & \left. + \cos(2\theta)) + 4\phi' V_y (\cos^2 \theta + k) - 4\phi' V_x V_y \sin \theta \cos \theta \right] / \left[4(\cos^2 \theta + k - V_x \sin \theta \cos \theta) \right], \end{aligned} \quad (4.5)$$

$$\begin{aligned} V_y' = & \left[-8\alpha \theta' \varphi' \sin \theta (\cos^2 \theta + k) + V_x (8\alpha \theta' \varphi' \sin^2 \theta \cos \theta - 8\varphi' (\cos^2 \theta + k)) \right. \\ & + V_y (4\alpha \theta'^2 (2k+1) \sin(\theta) - 4\alpha \varphi'^2 (2k+1) \sin \theta \cos^2 \theta + 4\alpha (2k+1)(\varphi' \Omega_3 \sin(2\theta) - 1)) \\ & \left. + 8\varphi' V_x^2 \sin \theta \cos \theta \right] / \left[8(\cos^2 \theta + k - V_x \sin \theta \cos \theta) \right], \end{aligned} \quad (4.6)$$

where the parameter α is defined by

$$\alpha = \mu \sqrt{gR}.$$

In these equations we have already substituted the expression for g_n given by (2.18) and k is either 1/4 for a uniform disk or 1/2 for a hoop. After this rescaling of variables the description of three types of asymptotic solutions takes the form

- straight rolling solutions

$$\theta = \pi/2, \quad \theta' = \varphi' = V_x = V_y = 0, \quad \Omega_3 = \Omega_{30},$$

- spinning solutions

$$\theta = \pi/2, \quad \theta' = \Omega_3 = V_x = V_y = 0, \quad \varphi' = \varphi'_0,$$

- tumbling solutions

$$\theta = \theta_0, \quad \Omega_3 = V_x = V_y = 0, \quad \varphi' = \pm \frac{1}{\sqrt{k \sin \theta_0}}.$$

Recall that all these asymptotic solutions are equilibria of the vector field $\mathbf{v} = [\theta', \theta'', \Omega'_3, V'_x, V'_y]^T$ defined by system (4.2)–(4.6). Equations of the linearized vector field around an equilibrium $\mathbf{x}_0$ have the form

$$\Delta \mathbf{x}' = L \Delta \mathbf{x}, \quad (4.7)$$

with the Jacobian matrix

$$L = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x}_0), \quad (4.8)$$

where $\Delta \mathbf{x} = [\Delta \theta, \Delta \theta', \Delta \varphi', \Delta \Omega_3, \Delta V_x, \Delta V_y]^T$ denote variations of variables $\mathbf{x} = [\theta, \theta', \varphi', \Omega_3, V_x, V_y]^T$, respectively, and $\mathbf{v}$ in (4.8) is a vector field defined by the r.h.s. of Eqs. (4.2)–(4.6).

The Jacobian matrices of linearization around our asymptotic solutions have the following forms: for straight rolling solutions

$$L_{\text{straight}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{k} & 0 & -2\Omega_{30} & 0 & \frac{1}{k} & 0 \\ 0 & 2\Omega_{30} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2k} \\ -\frac{\alpha}{k} & 0 & \alpha\Omega_{30} & 0 & -\frac{(k+1)\alpha}{k} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2k\alpha+\alpha}{2k} \end{bmatrix}, \quad (4.9)$$

for spinning solutions

$$L_{\text{spin}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{k} - \varphi_0'^2 & 0 & 0 & -2\varphi_0' & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2k} \\ \left(\varphi_0'^2 - \frac{1}{k}\right)\alpha & 0 & 0 & \varphi_0'\alpha & -\frac{(k+1)\alpha}{k} & \varphi_0' \\ 0 & -\varphi_0'\alpha & 0 & 0 & -\varphi_0' & -\frac{2k\alpha+\alpha}{2k} \end{bmatrix}, \quad (4.10)$$

for tumbling solutions

$$L_{\text{tumb}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1-\kappa^2}{-\kappa^3+k\kappa+\kappa} & 0 & \pm \frac{2\sqrt{k(\kappa-\kappa^3)}}{-\kappa^2+k+1} & \mp \frac{2\sqrt{k\kappa}}{-\kappa^2+k+1} & \frac{\kappa}{-\kappa^2+k+1} & 0 \\ 0 & \mp 2\sqrt{\frac{1-\kappa^2}{k\kappa^3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2k} \\ \frac{\alpha(\kappa^2-1)}{-\kappa^2+k+1} & 0 & \pm \frac{2\alpha\kappa^{3/2}\sqrt{k-k\kappa^2}}{\kappa^2-k-1} & \frac{\alpha(\kappa^2+k(2\kappa^2-1)-1)}{\sqrt{k\kappa}(-\kappa^2+k+1)} & -\frac{(k+1)\alpha}{-\kappa^2+k+1} & \frac{1}{\sqrt{k\kappa}} \\ 0 & \mp \alpha\sqrt{\frac{\kappa}{k}} & 0 & 0 & \mp \frac{1}{\sqrt{k\kappa}} & -\frac{2k\alpha+\alpha}{2k} \end{bmatrix}, \quad (4.11)$$

where we have introduced the notation $\kappa = \sin \theta_0$ and double signs in some entries correspond to two choices of the sign of φ' for tumbling solutions.

As is well known, a sufficient condition for stability of an equilibrium point is that real parts of all eigenvalues of the Jacobian matrix are negative. For the eigenvalues of matrices (4.9), (4.10) and (4.11) we apply the Routh–Hurwitz conditions to its characteristic polynomials.

Recall that for a polynomial of degree n with real coefficients

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad (4.12)$$

with $a_0 > 0$, the Routh–Hurwitz conditions are necessary and sufficient for all roots of $f(z)$ to have negative real parts. The Routh–Hurwitz matrix related to polynomial (4.12) is

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad (4.13)$$

with entries $a_k = 0$ for $k > n$. The Routh–Hurwitz conditions for all real parts of the eigenvalues to be negative is a sequence of inequalities

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \dots \quad \Delta_n > 0 \quad (4.14)$$

for all principal minors Δ_i of the Routh–Hurwitz matrix H , where $i = 1, \dots, n$, see [10, Vol II, Chap. XV, §13].

The characteristic polynomial for the Jacobian matrix related to the linearization along rolling solutions has the form

$$\begin{aligned} p_1(\lambda) &= \frac{\lambda^2}{k} \left[\lambda + \frac{(2k+1)\alpha}{2k} \right] q_1(\lambda), \\ q_1(\lambda) &= k\lambda^3 + (k+1)\alpha\lambda^2 + (4k\Omega_{30}^2 - 1)\lambda + \alpha(2(2k+1)\Omega_{30}^2 - 1). \end{aligned} \quad (4.15)$$

We have here a double zero eigenvalue and a negative eigenvalue $-(2k+1)\alpha/(2k)$. The principal minors of the Routh–Hurwitz matrix for the third-order polynomial $q_1(\lambda)$ are

$$\Delta_1 = \alpha(k+1), \quad \Delta_2 = \alpha(2k\Omega_{30}^2 - 1), \quad \Delta_3 = \alpha^2(2k\Omega_{30}^2 - 1)(2(2k+1)\Omega_{30}^2 - 1). \quad (4.16)$$

The requirement that all these determinants be greater than zero gives $\Omega_{30}^2 > 1/(2k)$. Thus, for a uniform disk with $k = 1/4$ asymptotic rolling solutions may be linearly stable only when $\Omega_{30}^2 > 2$ and for a ring ($k = 1/2$) when $\Omega_{30}^2 > 1$. Notice that this condition is independent of the value of the parameter α .

For spinning solutions the characteristic polynomial of the Jacobian matrix has the form

$$\begin{aligned} p_2(\lambda) &= \frac{\lambda}{2k^2} q_2(\lambda), \\ q_2(\lambda) &= 2k^2\lambda^5 + k(4k+3)\alpha\lambda^4 + [\alpha^2 + k(4\varphi_0'^2 k + \alpha^2(2k+3) - 2)]\lambda^3 + \alpha(4k(\varphi_0'^2(k+1) \\ &\quad - 1) - 1)\lambda^2 + [\alpha^2(2k+1)(\varphi_0'^2(k+1) - 1) + 2\varphi_0'^2 k(\varphi_0'^2 k - 1)]\lambda + \alpha\varphi_0'^2(\varphi_0'^2 k - 1). \end{aligned} \quad (4.17)$$

We have a simple zero eigenvalue and the Routh–Hurwitz criterion applied to the fifth-order polynomial $q_2(\lambda)$ gives the conditions

$$\begin{aligned} \Delta_1 &= \alpha k(4k+3) > 0, \quad \Delta_2 = \alpha k \left(4k \left(\varphi_0'^2 k(2k+1) - 1 \right) + \alpha^2(k+1)(2k+1)(4k+3) \right) > 0, \\ \Delta_3 &= \alpha^2 k \left[\alpha^2(2k+1)(4k+3) \left(k \left(\varphi_0'^2(k+1) - 2 \right) - 1 \right) + 4k \left(k \left(\varphi_0'^4 k(2k+1) - 2\varphi_0'^2 + 4 \right) \right. \right. \\ &\quad \left. \left. + 1 \right) \right] > 0, \quad \Delta_4 = \alpha^2 k C > 0, \quad \Delta_5 = \alpha^3 \varphi_0'^2 k \left(\varphi_0'^2 k - 1 \right) C > 0, \quad \text{where} \\ C &= \left[32\varphi_0'^2 k^3 \left(\varphi_0'^2 k - 1 \right) - \alpha^4(2k+1)^2(4k+3) \left(2k \left(\varphi_0'^2(k+1) - 1 \right) - 1 \right) \right. \\ &\quad \left. - 4\alpha^2 k(2k+1) \left(k \left(2\varphi_0'^4 k(2k+1) - \varphi_0'^2(8k+9) + 4 \right) + 1 \right) \right]. \end{aligned} \quad (4.18)$$

The regions, in a plane $(\alpha, (\varphi_0')^2)$, corresponding to linearly stable asymptotic spinning solutions for a uniform disk and for a hoop are shown in Figs. 3a and 3b, respectively.

Fig. 3. Regions of linear stability for spinning solutions.

A characteristic polynomial of the Jacobian matrix for tumbling solutions is the same for both choices of signs in the matrix L_{tumb} entries and it also has a factor λ

$$\begin{aligned} p_3(\lambda) &= \frac{\lambda q_3(\lambda)}{2\kappa^2 k^2 (-\kappa^2 + k + 1)^2}, \\ q_3(\lambda) &= 2\kappa^2 k^2 (-\kappa^2 + k + 1)^2 \lambda^5 + \alpha \kappa^2 k (-\kappa^2 + k + 1) (4k^2 - \kappa^2(2k+1) + 5k + 1) \lambda^4 \\ &\quad + \kappa k (-\kappa^2 + k + 1) (\alpha^2 \kappa(k+1)(2k+1) + 2\kappa^2(5k-1) - 8k + 2) \lambda^3 + \alpha \kappa \left(-2\kappa^2 \right. \\ &\quad \left. + \kappa^4(1 - k(4k+7)) + \kappa^2 k(k(20k+31) + 10) - k(4k(5k+6) + 3) + 1 \right) \lambda^2 \\ &\quad + \left[10k (\kappa^2(k+2) - k - 1) + \alpha^2 \kappa^5 (6k^2 + k - 1) + \alpha^2 \kappa^3(2k+1)(k(5k+3) + 1) \right. \\ &\quad \left. - 5\alpha^2 \kappa k(k+1)(2k+1) - 10\kappa^4 k \right] - \alpha (\kappa^2 - 1) (\kappa^2(16k+5) - 5(k+1)). \end{aligned} \quad (4.19)$$

For tumbling solutions the Routh–Hurwitz conditions applied to the fifth-order polynomial $q_3(\lambda)$ give the following inequalities:

$$\begin{aligned}
 \Delta_1 &= \alpha \kappa^2 k (-\kappa^2 + k + 1) (4k^2 - \kappa^2(2k + 1) + 5k + 1) > 0, \\
 \Delta_2 &= \alpha \kappa^3 k^2 (-\kappa^2 + k + 1)^2 \left(\alpha^2 \kappa (k + 1)^2 (8k^2 + 6k + 1) - \kappa^3 (k + 1)(\alpha + 2\alpha k)^2 \right. \\
 &\quad \left. + 4\kappa^4 (2 - 3k)k - 4\kappa^2 k(k + 4) + 8k(k + 1)^2 \right) > 0, \quad \Delta_3 = \alpha^2 \kappa^4 (-k^2) (-\kappa^2 + k + 1)^2 \\
 &\quad \times \left[\alpha^2 \kappa^5 (k + 1)(2k + 1) (k (k (24k^2 + 58k + 7) - 21) - 6) - \alpha^2 \kappa (k + 1)^4 (8k^2 + 6k + 1) \right. \\
 &\quad \left. + 4\kappa^6 k (k (k (60k^2 + 51k - 55) - 52) + 8) - 4\kappa^4 k (k (k (80k^2 + 6k - 119) - 36) + 12) \right. \\
 &\quad \left. + \alpha^2 \kappa^9 (2k + 1)^3 (3k - 1) - \alpha^2 \kappa^3 (k + 1)^2 (2k + 1) (k (2k (8k - 3) - 17) - 4) \right. \\
 &\quad \left. - \kappa^7 (k (2k (9k + 11) - 3) - 4) (\alpha + 2\alpha k)^2 - 4\kappa^8 k (k (k (12k + 7) - 20) + 2) \right. \\
 &\quad \left. + 4\kappa^2 k (k + 1) (k (k (16k - 41) - 4) + 8) - 8k(k + 1)^4 \right] > 0, \quad \Delta_4 = -\alpha^2 k^2 \kappa^6 D > 0, \\
 \Delta_5 &= \alpha^3 (\kappa - 1) \kappa^6 (\kappa + 1) k^2 (\kappa^2 (16k + 5) - 5(k + 1)) D > 0,
 \end{aligned} \tag{4.20}$$

where

$$\begin{aligned}
 D &= (-\kappa^2 + k + 1)^2 (-6k^2 + \kappa^4 (3k - 1) + \kappa^2 (k(5k + 4) + 2) - 7k - 1) \\
 &\quad \left(\kappa^4 (32k^3 (k(15k + 19) - 39) + \alpha^4 (k + 1)(2k + 1)^2 (k(4k + 3)(k(6k + 13) - 2) - 3)) \right. \\
 &\quad \left. + 32k^3 (k + 1)(2k - 13) - 4\alpha^2 \kappa^7 k(2k + 1) (k (12k^2 + k - 23) + 2) - 4\alpha^2 \kappa k(k + 1)(2k + 1) \right. \\
 &\quad \times (k (4k^2 + 66k + 15) - 2) + \kappa^6 (32(13 - 15k)k^3 - 3\alpha^4 (2k + 1)^3 (6k^3 + 8k^2 - 1)) \\
 &\quad \left. + \kappa^2 (32k^3 ((7 - 17k)k + 39) - \alpha^4 (k + 1)^2 (2k + 1)^2 (4k + 1) (6k^2 + 2k - 1)) + 4\alpha^2 \kappa^5 k \right. \\
 &\quad \times (2k + 1) (k(3k(k(20k + 7) - 31) - 59) + 6) + 4\alpha^2 \kappa^3 k (k(k(k(2(39 - 56k)k + 417) + 273) \\
 &\quad \left. + 37) - 6) + \kappa^8 (3k - 1)(\alpha + 2\alpha k)^4 \right).
 \end{aligned}$$

Fig. 4. Regions of linear stability for tumbling solutions ($\kappa = \sin \theta_0$).

The regions, in a plane $(\alpha, \kappa = \sin \theta_0)$, corresponding to (possibly) linearly stable asymptotic tumbling solutions for a uniform disk and for a hoop are shown in Figs. 4a and 4b, respectively. Since these regions are qualitatively different, we considered an annulus with external radius R and internal radius r . Its principal moments of inertia are equal to

$$I_1 = I_2 = \frac{m}{4}(R^2 + r^2) = kmR^2, \quad I_3 = 2kmR^2, \quad k = \frac{1}{4} \left[1 + \left(\frac{r}{R} \right)^2 \right],$$

where the internal radius $r \in [0, R]$, which gives $k \in [\frac{1}{4}, \frac{1}{2}]$. The limiting cases $r = 0$ correspond to a solid disk with minimal value $k = 1/4$ and $r = R$ corresponds to a hoop with maximal value of $k = 1/2$. We notice that already very small perturbations of the asymptotic case of the maximal value of the principal moments of inertia change qualitatively the region of linear stability, see Fig. 5a, where $k = \frac{1}{2} - \frac{1}{1000}$ corresponding to $r/R = 0.997998$ was chosen. In general the region of linear stability as a function of parameters (α, κ, k) with $k \in [\frac{1}{4}, \frac{1}{2}]$ is given in Fig. 5b. On the upper surface $k = \frac{1}{2}$ we recognize the region 4b for a hoop and on the bottom surface $k = \frac{1}{4}$ we observe the region 4a for a solid disk.

Fig. 5. Regions of linear stability for tumbling solutions for a) a very thin annulus with $k = \frac{1}{2} - \frac{1}{1000}$ and b) general case with $k \in [\frac{1}{4}, \frac{1}{2}]$. Here $\kappa = \sin \theta_0$.

Taking into account that

$$\dot{\varphi} = \sqrt{\frac{g}{R}} \varphi', \quad \omega_3 = \sqrt{\frac{g}{R}} \Omega_3,$$

the necessary conditions of linear stability for rolling and spinning solutions are

$$\omega_3^2 > \frac{1}{2k} \frac{g}{R}, \quad \dot{\varphi}^2 > \frac{1}{k} \frac{g}{R}, \quad (4.21)$$

respectively, and, moreover, for spinning solutions α must belong to the region given in Fig. 3.

It should be made clear that the linear analysis has determined only possible regions of stability for all three types of asymptotic solutions. The criterion is not conclusive because the characteristic polynomial for each Jacobian matrix (4.9), (4.10) or (4.11) admits one or two zero eigenvalues. However, it determines regions of possible stability which we can relate to the behavior of solutions studied numerically in the next section.

5. NUMERICAL SIMULATIONS

The knowledge of asymptotic solutions and of their possible stability regions provides a framework for understanding and interpreting numerical simulations. In this section we simulate the dynamics of the sliding and rolling disk with parameters $m = 0.02$ kg, $R = 0.02$ m, $\mu = 0.3$, $I_3 = \frac{1}{2}mR^2$, $I_1 = \frac{1}{4}mR^2$, and $g = 9.82$ m/s². They correspond to the value $\alpha = 3\sqrt{491}/500 \approx 0.133$ m/s. Taking into account that $\sqrt{\frac{g}{R}} = \sqrt{491}$ s⁻¹ and inequalities (4.21), the necessary conditions for linear stability of asymptotic solutions for our disk are the following:

- for rolling solutions: $|\omega_{30}| > \omega_{3\text{tresh}} = \sqrt{2}\sqrt{491} \approx 31.337 \text{ rad/s}$,
- for spinning solutions: $|\varphi_0| > \varphi_{0\text{tresh}} = 2\sqrt{491} \approx 44.31704 \text{ rad/s}$,
- for tumbling solutions: if $\alpha \leq 1$ the only condition is $\kappa = \sin \theta_0 > 0$, for $\alpha > 1$ the threshold curve is given by a solution of

$$3\alpha^4 \kappa^2 (9(\kappa^2 - 2)\kappa^2 + 5) + \alpha^2 \kappa (8\kappa^4 - 30\kappa^2 + 34) + 8\kappa^2 (\kappa^2 - 3) + 16 = 0,$$

with $\kappa > 0$ for $\alpha > 0$, see the dashed curve in Fig. 4a.

To learn about the behavior of solutions, we take reference initial conditions in a neighborhood of rolling or spinning asymptotic solutions and then we study what happens under variation of certain initial variables. In order to prevent the possibility of bouncing in our simulations, we bound the maximal values of the rolling and spinning angular velocities $\omega_3(0) \leq 130 \text{ rad/s}$, $\dot{\varphi}(0) \leq 80 \text{ rad/s}$ so that the value $g_n(t) > 0$ of the reaction force is positive all time and goes in oscillatory way to the weight mg of the disk (see Fig. 8b).

As a first reference initial condition we choose the values of variables from a neighborhood of an asymptotic rolling solution

$$IC_{\text{roll}}(0) = \left\{ \theta(0) = \frac{\pi}{2} - 0.1 \text{ rad}, \dot{\theta}(0) = \dot{\varphi}(0) = 0, \omega_3(0) = 90 \text{ rad/s}, v_x(0) = v_y(0) = 0 \text{ m/s} \right\}, \quad (5.1)$$

where the disk is tilted from the vertical by 0.1 rad and the only nonzero variable is the rolling angular velocity ω_3 . This choice of initial condition is motivated by the following observations:

- all asymptotic solutions have $\dot{\theta}(\infty) = v_x(\infty) = v_y(\infty) = 0$,
- usually a sliding and rolling disk is launched close to $\theta \approx \pi/2 \approx 1.57 \text{ rad}$,
- time evolution of $IC_{\text{roll}}(0)$ having energy about 8 times larger than the threshold energy $E_{\text{tresh}} = \frac{1}{2}I_1\dot{\varphi}_{\text{tresh}}^2$ with $\dot{\varphi}_{\text{tresh}} = \sqrt{\frac{mgR}{I_1}}$ illustrates how energy is dissipated by the disk approaching an asymptotic solution.

Figures 6 and 7 show that the time evolution consists of two distinct stages: dissipative rolling and sliding up to the critical time $T_{\text{cr}} = 15 \text{ s}$ when ω_3 decreases from 90 to 0 rad/s. It ends up with a drastic jump of the inclination angle θ from the initial value $\theta(0) = \pi/2 - 0.1 \text{ rad}$ to the neighborhood of the asymptotic tumbling angle $\theta(\infty) \approx 0.61 \text{ rad}$, see Fig. 6a. Also, the angular velocity $\dot{\varphi}$ jumps from $\dot{\varphi}(0) = 0 \text{ rad/s}$ to the neighborhood of the asymptotic value

$$\dot{\varphi}(\infty) = -\sqrt{\frac{gmR}{I_1 \sin \theta(\infty)}} \approx -58.55 \text{ rad/s},$$

see Fig. 6b and formula (3.1). An oscillatory approach to the asymptotic solution takes place after $T_{\text{cr}} = 15 \text{ s}$, while the amplitude of oscillations attenuates to zero. Figures 6 and 7 indicate that after 25 s the disk gets close to the asymptotic tumbling state

$$S_{\text{tumb}} = \left\{ \theta(\infty) \approx 0.61 \text{ rad}, \dot{\varphi}(\infty) = -58.55 \text{ rad/s}, \dot{\theta}(\infty) = \omega_3(\infty) = 0 \text{ rad/s}, v_x(\infty) = v_y(\infty) = 0 \text{ m/s} \right\}. \quad (5.2)$$

The initial value of the angular velocity $\omega_3(0) = 90 \text{ rad/s}$ decreases almost monotonically to the asymptotic value $\omega_3(\infty) = 0 \text{ rad/s}$ characterizing all tumbling asymptotic solutions. It is accomplished at $t = T_{\text{cr}} = 15 \text{ s}$, see Fig. 7a. Notice that for $t < T_{\text{cr}}$ the angular velocity $\dot{\varphi}$ remains almost constant, while $\dot{\psi}$ is decreasing from the initial value $\dot{\psi}(0) = 90 \text{ rad/s}$ to about $\dot{\psi} = 20 \text{ rad/s}$. For $t > T_{\text{cr}}$ the oscillations of $\dot{\psi}$ and $\dot{\varphi}$ match each other to keep $\omega_3(t) = \dot{\psi}(t) + \dot{\varphi} \cos \theta(t) = 0$ during the final stabilization phase. Asymptotically $\omega_3(\infty) = 0$, thus $\dot{\psi}(\infty) = -\cos \theta(\infty)\dot{\varphi}(\infty) = 48.12 \text{ rad/s}$.

Fig. 6. Time evolutions of θ , $\dot{\phi}$ and $\dot{\psi}$. The plots were obtained from numerical integration of equations of motion with $IC_{\text{roll}}(0)$.

Components of velocity that start with initial values $v_x(0) = v_y(0) = 0$ m/s acquire the positive maximal value $v_x(t) = v_y(t) = 0.4$ m/s for times $t < T_{\text{cr}}$. At critical time T_{cr} they jump to a neighborhood of its asymptotic values $v_x(\infty) = v_y(\infty) = 0$ m/s and oscillate with amplitudes going to zero. The transversal velocity v_x for most of the time interval $[0, T_{\text{cr}}]$ takes the value $v_x(t) \approx 0.2$ m/s, while $v_y(t) \approx 0.02$ m/s is an order of magnitude smaller. Thus, it is the friction in the direction transversal to the disk that is responsible for prevailing loss of energy.

Fig. 7. Time evolutions of ω_3 , v_x and v_y . The plots were obtained from numerical integration of equations of motion with $IC_{\text{roll}}(0)$.

In Fig. 8a the total energy decreases monotonically in the interval $[0, T_{\text{cr}}]$. After T_{cr} all energies remain close to its asymptotic values: $E(\infty) = 0.00346$ J for the total energy, $E_{\text{trans}}(\infty) = 0$ for the energy of translational motion, $E_{\text{pot}}(\infty) = 0.002306$ J for potential energy, and $E_{\text{rot}}(\infty) = 0.001153$ J for the energy of rotational motion. But due to the oscillatory approach to the asymptotic state we see in Fig. 8a an oscillatory exchange of energy between the rotational and potential modes as the center of mass is wobbling up and down.

(a) evolution of total energy $E(t)$ and its components (b) evolution of the value of the normal force $g_n \hat{z}$

Fig. 8. Time evolution of a) total energy $E(t)$ and its components $E_{\text{trans}}(t)$, $E_{\text{rot}}(t)$, $E_{\text{pot}}(t)$ and b) the normal force $g_n \hat{z}$ which is positive at all times. Left y axis and bottom x axis concern $IC_{\text{spin}}(0)$ and $IC_{\text{mixed}}(0)$. The right y axis and the top x axis concern $IC_{\text{roll}}(0)$.

Figures 6, 7 and 8a show that when the disk starts its time evolution with initial conditions $IC_{\text{roll}}(0)$, then at the beginning up to T_{cr} its evolution takes place in a neighborhood of an asymptotic rolling solution $\theta(\infty) = \pi/2$. The inclination angle stays close to $\pi/2$, the angular velocity is close to $\dot{\varphi}(t) = 0$ for $t \in [0, T_{\text{cr}})$ and the value of ω_3 decreases, but the solution has still a rolling character. It is a stage when it dissipates most of excess energy, which initially has been about 15 times higher than the energy of the asymptotic solution S_{tumb} given by (5.2).

At $t = T_{\text{cr}}$ the solution switches to a different regime of dynamics close to an asymptotic tumbling solution and during final oscillations its energy approaches the energy of S_{tumb} . Similar behavior can be observed for many other values of the initial velocity $\omega_3(0)$, see Fig. 9. For some values of $\omega_3(0)$ the critical time is large and $\theta(t)$ may strongly oscillate.

Fig. 9. Curves of $\theta(t)$ escaping from a neighborhood of the asymptotic rolling solution $\theta(\infty) = \pi/2$ to certain asymptotic tumbling solutions for chosen initial $\omega_3(0)$, with remaining initial conditions as in $IC_{\text{roll}}(0)$. The threshold value is $\omega_{3\text{tresh}} = \sqrt{2}\sqrt{491} \approx 31.337$ rad/s.

Interesting is the dependence of the asymptotic inclination angle $\theta(\infty)$ on the initial value $\omega_3(0)$ when keeping remaining initial conditions as in $IC_{\text{roll}}(0)$. The graph presented in Fig. 10 is quite complicated:

- for $0 < \omega_3(0) < 19$ rad/s the asymptotic inclination angle is growing up to $\theta(\infty) \rightarrow 0.67$ rad. After reaching a maximum, in the interval $19 \leq \omega_3(0) < 23$ rad/s the graph is decreasing to $\theta(\infty) = 0.47$ rad,
- for $23 \leq \omega_3(0) < 45$ rad/s the asymptotic angle $\theta(\infty)$ belongs to almost the whole range $\theta(\infty) \in [0, \pi]$. The dependence $\theta(\infty) (\omega_3(0))$ shows great sensitivity to small changes of $\omega_3(0)$, as in Fig. 10 and in the magnification in Fig. 11(a).

Time evolutions of $\theta(t)$ with very close initial values of $\omega_3(0) = 40.07731423$ rad/s and $\omega_3(0) = 40.077314235$ rad/s that lie on opposite sides of one jump of the function $\theta(\infty) = \theta(\infty) (\omega_3(0))$ are presented in Fig. 11b. A very small change from $\omega_3(0) = 40.07731423$ rad/s to $\omega_3(0) = 40.077314235$ rad/s causes the inclination angles to evolve for about 3 seconds in the same way, but later both curves separate and reach different asymptotic values: $\theta(\infty) = 2.65$ rad and $\theta(\infty) = 0.32$ rad, respectively.

This sensitivity shows that despite the fact that the disk is initially tilted to one side $\theta(0) = \pi/2 - 0.1$ rad, a small change of initial $\omega_3(0)$ may cause tipping of the sliding rolling disk onto the other side. This raises a question whether this behavior is not an artefact of a sharp edge of the disk.

- surprisingly starting from $\omega_3(0) = 45$ rad/s, the asymptotic angle is almost constant $\theta(\infty) = 0.62$ rad for the whole tested range $45 < \omega_3(0) < 130$ rad/s. The time of rolling, before the catastrophic fall occurs, is increasing with $\omega_3(0)$. This is understandable since the larger

Fig. 10. Asymptotic angle $\theta(\infty)$ as a function of initial condition $\omega_3(0)$ with the remaining initial conditions as in $IC_{\text{roll}}(0)$.

initial kinetic energy of a rolling and sliding disk has to be dissipated and it takes longer time as the average and maximal value of the transversal velocity $v_x(t)$ varies little for $\omega_3(0)$ in the tested interval $45 < \omega_3(0) < 130$ rad/s.

Regions of large sensitivity of the asymptotic inclination angle $\theta(\infty)$ to small changes of initial conditions are also visible in Fig. 12, where we keep $\omega_3(0) = 90$ rad/s but vary $\theta(0)$ and $\dot{\varphi}(0)$. In all these figures asymptotic values of $\theta(\infty)$ are obtained by numerical integration until $\omega_3(t)$ reaches the asymptotic value $\omega_3(\infty) = 0$, with accuracy 10^{-6} in two successive steps.

In Fig. 13 we present the time evolution of an initially vertical disk with $\theta(0) = \frac{\pi}{2}$ rad, $\dot{\theta}(0) = 10$ rad/s for two different values of the rolling velocity $\omega_3(0) = 90 > \omega_{3\text{tresh}} = \sqrt{2}\sqrt{491} \approx 31.337$ rad/s $> \omega_3(0) = 20$; one above and the other below the stability threshold value $\omega_{3\text{tresh}}$. The solution with $\omega_3(0) = 90$ rad/s remains vertical, see in Fig. 13a within a simulation time of 20 s in the picture, while the other one with $\omega_3(0) = 20$ rad/s falls to the tumbling solution with $\omega_3(\infty) = 0$ rad/s and $\theta(\infty) = 0.371$ rad/s, which is related to $\dot{\varphi}(\infty)$ by relation (3.1). In Fig. 13b we see that $\dot{\varphi}(t) \approx 0$ for the solution $\omega_3(0) = 90$ and falls to the value $\dot{\varphi}(\infty) = -73.516$ rad/s for the solution $\omega_3(0) = 20$. It is also visible in Fig. 13b that $\omega_3(t) = 90$ rad/s remains constant for the first solution. Also, sufficiently small perturbations of $v_y(0) \neq 0$ do not cause escaping of trajectories from the neighborhoods of rolling solutions in agreement with the necessary linear stability condition for rolling asymptotic solutions.

We do not plot graphs for initial conditions in a neighborhood of certain tumbling solutions because they qualitatively behave similarly as solutions with the initial condition $IC_{\text{roll}}(0)$ after $t > T_{\text{cr}}$. For our disk with $\alpha \approx 0.133 \leq 1$ the conditions for linear stability of tumbling solutions are satisfied for all values of the asymptotic inclination angle $\theta(\infty)$ and numerical experiments show that when we choose some $\theta(0)$ and calculate $\dot{\varphi}$ according to (3.1) and disturb this value, then these solutions immediately converge to a stable asymptotic tumbling solution. However, for a disk with $\alpha > 1$ for sufficiently large $\theta(0)$ tumbling solutions are unstable and escape to other tumbling solutions contained in the linear stability region presented in Fig. 4a. This is visible in Fig. 14 presenting simulations for a disk with $\mu = 4.513$ that corresponds to $\alpha = 2$. A threshold value dividing stable and unstable regions is $\theta_{\text{tresh}} = 0.8553$ rad. Calculations in Fig. 14 are made for the following initial conditions: $(\theta(0), \dot{\varphi}(0)) = (0.5, 63.9044)$, $(\pi/5, 57.7045)$, $(0.8553, 50.911)$,

(a) sensitivity interval

(b) time evolutions of $\theta(t)$

Fig. 11. Magnification of the interval of great sensitivity to small changes of $\omega_3(0)$ and time evolution of $\theta(t)$ with two very close initial $\omega_3(0) = 40.07731423$ rad/s (upper curve) and $\omega_3(0) = 40.077314235$ rad/s (lower curve).

$(1, 48.216)$, $(1.20, 45.8043)$, $(1.45, 44.3794)$, $(1.57, 44.217)$, $(1.75, 44.5762)$. They show that tumbling solutions with $\theta(0) \leq \theta_{\text{tresh}}$ keep the angle constant all time.

A second natural class of initial conditions to study are spinning solutions with $\theta(0) = \pi/2 - 0.1$ being close to the vertical spinning solution and with exchanged initial angular velocities so that $\dot{\varphi}(0) = 80$ rad/s and $\omega_3(0) = 0$ rad/s. As a spinning reference solution we take

$$IC_{\text{spin}}(0) = \left\{ \theta(0) = \pi/2 - 0.1 \text{ rad}, \dot{\varphi}(0) = 80 \text{ rad/s}, \dot{\theta}(0) = \omega_3(0) = 0 \text{ rad/s}, v_x(0) = v_y(0) = 0 \text{ m/s} \right\}. \quad (5.3)$$

(a) dependence of $\theta(\infty)$ on $\theta(0)$ (b) dependence of $\theta(\infty)$ on $\dot{\varphi}(0)$

Fig. 12. Asymptotic angle $\theta(\infty)$ as a function of initial conditions $\theta(0)$ and $\dot{\varphi}(0)$ with remaining initial conditions as in $IC_{\text{roll}}(0)$.

This solution with $\theta(0) = \pi/2 - 0.1 = 1.47$ rises to a vertically spinning solution with $\theta(\infty) = \pi/2$ as shown by the θ curve in Fig. 15. The angular velocity $\dot{\varphi}$ remains almost constant very close to $\dot{\varphi}(0) = 80$ rad/s as illustrated by the upper $\dot{\varphi}$ curve plotted in Fig. 15 with respect to the right y axis. The other variables $\dot{\psi}(t)$, $\omega_3(t)$, $v_x(t)$, $v_y(t)$ take very small values that quickly go to zero.

This behavior is a feature of all solutions with $\theta(0) = \pi/2 - 0.1$ rad and $\dot{\varphi}(0) > \dot{\varphi}_{0\text{resh}} = \sqrt{mgR/I_1} = 2\sqrt{491} \approx 44.317$ rad/s. For $\dot{\varphi}(0) < \dot{\varphi}_{0\text{resh}} = 2\sqrt{491}$ rad/s solutions go to $\theta(\infty)$

(a) evolution of $\theta(t)$ (b) evolution of $\dot{\varphi}(t)$ and $\omega_3(t)$

Fig. 13. Time evolution of the inclination angle $\theta(t)$ and of angular velocities $\dot{\varphi}(t)$ and $\omega_3(t)$ for initial conditions $\theta(0) = \frac{\pi}{2}$ rad, $\dot{\theta}(0) = 10$ rad/s and two different values of $\omega_3(0) = 90$ and 20 rad/s with zero remaining initial values. In figure (b) the left y axis provides a scale for $\dot{\varphi}$ and the right y axis provides a scale for ω_3 .

and $\dot{\varphi}(\infty)$, which are (again) related by

$$\dot{\varphi}(\infty) = \sqrt{\frac{mgR}{I_1 \sin \theta(\infty)}}.$$

Figure 16a presents the time evolution for several initial values of $\dot{\varphi}(0) = 80, 45, 2\sqrt{491}, 40$ and 30 rad/s, while the remaining initial conditions are the same as in $IC_{\text{spin}}(0)$. As expected from

Fig. 14. Time evolutions of tumbling solutions for a disk with $\alpha = 2$.

Fig. 15. Time evolutions of $\theta(t)$, $\dot{\varphi}(t)$ and $\dot{\psi}(t)$ from numerical integration of equations of motion with $IC_{\text{spin}}(0)$. The left y axis provides a scale for θ and the right y axis provides a scale for angular velocities $\dot{\varphi}$ and $\dot{\psi}$.

the stability analysis, the first two solutions approach the asymptotic spinning solution and the remaining three with $\dot{\varphi}(0) \leq \dot{\varphi}_{0\text{resh}} = 2\sqrt{491}$ rad/s approach certain tumbling solutions with decreasing asymptotic inclination angle $\theta(\infty)$ as $\dot{\varphi}(0)$ is decreasing. The dependence of $\theta(\infty)$ as a function of the initial angular momentum in $IC_{\text{spin}}(0)$ for $\dot{\varphi}(0) \in [-80, 80]$ rad/s is shown in Fig. 16b. Regions of asymptotic spinning solutions and of tumbling solutions separated by the threshold value $|\dot{\varphi}_{0\text{resh}}| = 2\sqrt{491}$ rad/s are clearly visible in the graph.

(a) graphs of $\theta(t)$ for chosen values of $\dot{\varphi}(0)$ (b) dependence of $\theta(\infty)$ on $\dot{\varphi}(0)$

Fig. 16. Time evolution of $\theta(t)$ for a sequence of values $\dot{\varphi}(0) = 80, 45, 2\sqrt{491}, 40$ and 30 rad/s and dependence of the asymptotic $\theta(\infty)$ on $\dot{\varphi}(0) \in [-80, 80]$ rad/s, with remaining initial conditions as in $IC_{\text{spin}}(0)$.

Perturbations of the initial value $\dot{\theta}(0) = 0$, $v_x(0) = 0$ and $v_y(0) = 0$ give similar behavior of dynamical variables as with the initial conditions $IC_{\text{spin}}(0)$, but the time of approaching an asymptotic solution is longer since there is more energy to be dissipated.

What about solutions with both $\dot{\varphi}(0) \neq 0$ and $\omega_3(0) \neq 0$? Here the most interesting question is which type of stable asymptotic solution is approached depending on the values of $\dot{\varphi}(0)$ and $\omega_3(0)$. It seems that there is no any clear answer. As an example we give an interesting solution that

behaves in a nonoscillatory way

$$IC_{\text{mixed}}(0) = \left\{ \theta(0) = \pi/2 - 0.35 \text{ rad}, \dot{\varphi}(0) = 50 \text{ rad/s}, \dot{\theta}(0) = \omega_3(0) = 0 \text{ rad/s}, v_x(0) = v_y(0) = 0 \text{ m/s} \right\}, \quad (5.4)$$

which is going to $\theta(\infty) = 1.5 \text{ rad}$, see Figs. 17 and 18.

Fig. 17. Time evolutions of $\theta(t)$, $\dot{\varphi}(t)$ and $\dot{\psi}(t)$ from numerical integration of equations of motion with $IC_{\text{mixed}}(0)$. The left vertical y axis provides a scale for angle θ and the right vertical y axis provides a scale for angular velocities $\dot{\varphi}$ and $\dot{\psi}$.

Fig. 18. Time evolutions of $\omega_3(t)$, $v_x(t)$ and $v_y(t)$ from numerical integration of equations of motion with initial conditions $IC_{\text{mixed}}(0)$. The left vertical y axis provides scale for angular velocity ω_3 and the right vertical y axis provides scale for velocities v_x and v_y .

Equations of motion were integrated numerically using a double precision version of the Bulirsch–Stoer extrapolation method with adaptive step-size control [23]. The local relative precision of integration was fixed to 10^{-12} for most of the tests. It appeared that for some values of the parameters and initial conditions the system is stiff. However, comparison of results generated by the Bulirsch–Stoer integrator with numerical methods dedicated to integration of stiff problems shows differences within precision limits.

An analogous linear stability analysis, as in Section 4, made for the straight rolling and spinning motions of a purely rolling disk (see condition (2.5)) gives the well-known classical values of stability [25, §§242–244], see also [19, 22]. The results are summarized in Table 1 to see that the stability thresholds for the model with friction are higher than those for pure rolling. This is probably due to difference in the reaction force in both cases.

Table 1. Comparison of necessary stability conditions.

Solution	rolling & sliding	pure rolling
spinning	$\dot{\varphi}^2 > \frac{1}{k} \frac{g}{R}$	$\dot{\varphi}^2 > \frac{1}{k+1} \frac{g}{R}$
rolling	$\omega_3^2 > \frac{1}{2k} \frac{g}{R}$	$\omega_3^2 > \frac{1}{2(2k+1)} \frac{g}{R}$

Finally, let us check how the friction influences the stability of asymptotic solutions in our model. Notice that all results in Section 4 have been obtained under the assumption $\mu \neq 0$ used in our rescaling, see Eq. (4.1) with $T = \sqrt{R/g}$ and $L = \sqrt{R/g}/\mu$. The case $\mu = 0$ corresponds to the motion of a disk on an absolutely smooth plane. Repeating linear stability analysis, as in Section 4 for Eqs. (2.13)–(2.17) with $\mu = 0$, we have obtained the following results:

1. The spinning solution is linearly stable if $\dot{\varphi}^2 > \frac{1}{k} \frac{g}{R}$.
2. The rolling solution is linearly unstable. The matrix of the linearized system has the multiple eigenvalue $\lambda = 0$ with a two dimensional Jordan block.
3. The tumbling solution is linearly stable for all $\theta_0 \in (0, \pi)$.

We notice that threshold for spinning solution agrees with the value for our model and results in [22]. For rolling solutions the condition that real parts of nonvanishing eigenvalues of linearization matrix are negative gives inequality $\omega_3^2 > \frac{1}{4k} \frac{g}{R}$ that agrees with the condition in [22]. But perturbations in the variables $(\theta, \dot{\theta})$ produce a two-dimensional Jordan block corresponding to the multiple eigenvalue $\lambda = 0$. The presence of a Jordan block of dimension greater than one for an eigenvalue with a vanishing real part implies linear instability, see, e.g., Theorem 1 on page 86 in [7]. Nonlinear stability analysis for the motion of a disk on an absolutely smooth plane is given in [18, Chap. 2].

In the simulations we have tested angular momentum velocities in the range of $0 < \omega_3(0) < 130$ rad/s or $0 < \dot{\varphi} < 80$ rad/s to keep energy sufficiently low for having $g_n(t) > 0$ all time and for avoiding the possibility of bouncing solutions. In Fig. 8b we show the evolution of $g_n(t)$ for our three basic reference initial conditions. Note that two y axes are used: the left one for solutions with initial conditions $IC_{\text{spin}}(0)$ and $IC_{\text{mixed}}(0)$ and the right one for solutions with $IC_{\text{roll}}(0)$. For all these solutions the asymptotic value of $g_n(t)$ is the same and equal to mg . For this reason we observe no loss of contact of the disk with the supporting plane such as may occur in the case of the Euler disk [3, 11].

In order to keep the results of numerical simulations understandable, we have chosen reference initial conditions $IC_{\text{roll}}(0)$, $IC_{\text{spin}}(0)$ close to linearly stable vertical rolling and spinning asymptotic solutions by tilting the initial inclination angle $\theta(0)$ by 0.1 rad and having only either $\omega_3(0) = 90$ rad/s $\neq 0$ or $\dot{\varphi}(0) = 80$ rad/s $\neq 0$ different from zero. The purpose has been to see what happens

to other variables under the action of the friction force. As expected, the solutions acquire nonzero sliding velocity with the transversal component $v_x(t)$ being an order of magnitude larger than the longitudinal $v_y(t)$ speed. The reference solution $IC_{\text{roll}}(0)$ tends asymptotically to a tumbling solution with the inclination angle $\theta(\infty) = 0.62$ rad being (surprisingly) almost constant over the whole range $45 \text{ rad/s} < \omega_3(0) < 130 \text{ rad/s}$. Quite unexpectedly the value of $\theta(\infty)$ shows great sensitivity to the value of the initial angular velocity $\omega_3(0)$ in the range $25 \text{ rad/s} < \omega_3(0) < 45 \text{ rad/s}$ and is given by a regular, increasing continuous function of $\omega_3(0)$ in the range $0 < \omega_3(0) < 25 \text{ rad/s}$. It would be impossible to deduce this picture using analytical methods. The asymptotic rolling solutions with $\omega_3(0) > \omega_{\text{tresh}} = \sqrt{2}\sqrt{491} \text{ rad/s}$ behave stable in numerical simulations only for small perturbations. Otherwise they fall into tumbling solutions.

Greater stability regions have the spinning solutions as the chosen reference spinning solution $IC_{\text{spin}}(0)$ starting with the same initial inclination angle $\theta(0) = \pi/2 - 0.1 = 1.47$ rad as for $IC_{\text{roll}}(0)$ goes asymptotically to a stable spinning solution with almost the same $\dot{\varphi}(\infty) \approx 80 \text{ rad/s}$. The curve 16(b) for an asymptotic inclination angle $\theta(\infty)$ agrees with the threshold value $\dot{\varphi}(\infty) = 2\sqrt{491} \text{ rad/s}$ for the stability of spinning solutions, but the shape of the dependence of $\theta(\infty)$ on $\dot{\varphi}(0)$ cannot be found analytically. The tumbling solutions appear to be stable in simulations for all values of $\theta(\infty) \in]0, \pi/2]$ and small changes of initial conditions. This is in agreement with the linear stability analysis. The tumbling solutions seem to play a central role as they are attractors for solutions starting far away from the stable rolling and spinning solutions and for solutions with low initial energy starting even close to rolling or spinning solutions.

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