Large deviations on longest runs

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Abstract

The study on the longest stretch of consecutive successes in “random” trials dates back to 1916 when the German philosopher Karl Marbe wrote a paper concerning the longest stretch of consecutive births of children of the same sex as appearing in the birth register of a Bavarian town. The result was actually used by parents to “predict” the sex of their children. The longest stretch of same-sex births during that time in 200 thousand birth registrations was actually $17 \approx \log_2(200 \times 10^3)$. During the past century, the research of longest stretch of consecutive successes (longest runs) has found applications in various areas, especially in the theory of reliability.

The aim of this thesis is to study large deviations on longest runs in the setting of Markov chains. More precisely, we establish a general large deviation principle for the longest success run in a two-state (success or failure) Markov chain. Our tool is based on a recent result regarding a general large deviation for the longest success run in Bernoulli trails. It turns out that the main ingredient in the proof is to implement several global and local estimates of the cumulative distribution function of the longest success run.

**Keywords:** Large deviation principle, Markov chain, Reliability theory, K-out-of-n system.

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Chapter 1

Introduction

1.1 Background and purpose

A consecutive-k-out-of-n: F system is a sequence of n ordered components where the system fails if and only if at least k consecutive components are failed. This system has a wide range of applications, such as the design of integrated circuitry [2], telecommunications and pipeline pumping stations [3], etc.

The longest success run \( L(n) \), namely the longest stretch of consecutive successes, has been applied in many fields, for instance: hypothesis testing, system reliability, quality control and DNA sequences (cf. [1]). Therefore considerable attention has been attracted to study the distribution and limiting behaviors of longest success runs in cases of Bernoulli trials, Markov dependent trials, and exchangeable binary trials (see for instance [5], [6], [7], [8], [9], [10], [11] and [18]), meanwhile we also refer to [12], [13], [14], [15] and [16] for the latest progress on this aspect.

Since the longest runs in Bernoulli trails have been extensively and fully studied, in this thesis we will focus on longest runs in Markov dependent trails - Markov chains. Let \( \{X_k\}_{1 \leq k \leq n} \) be the first \( n \) steps in a time-homogeneous two-state (success and failure) Markov chain with ‘1’ and ‘0’ representing ‘success’ and ‘failure’ respectively. Moreover, we suppose that the initial distribution is \( \mathbb{P}(X_1 = 0) = p_0, \mathbb{P}(X_1 = 1) = p_1 = 1 - p_0 \), and the transition matrix is:

\[
A = \begin{pmatrix}
     F & S \\
     p_{00} & p_{01} \\
     p_{10} & p_{11}
\end{pmatrix}
\]

We assume \( 0 < p_0 < 1 \) and \( 0 < p_{ij} < 1 (i, j = 0, 1) \), in order to avoid triviality. Let us define the longest success run \( L(n) \) as the longest stretch of consecutive successes in the first \( n \) step of the chain. Although there is an explicit formula to express the exact distribution of \( L(n) \) (see [11]), we can hardly derive the limiting distribution of \( L(n) \) as \( n \to \infty \) due to the complexity. Even for the IID (independent and identical distribution) case, there is no information we can get as \( n \to \infty \). Therefore, offering appropriately probability estimating of \( L(n) \) has been of great importance. In [14], a general large deviation principle
(LDP) was fully investigated for the longest success run $L(n)$ in a sequence of independent Bernoulli trials. Likewise the aim of this thesis is to generalize the results in [14] and establish a general LDP of $L(n)$ in a Markov chain defined above. To this end, let us recall several facts on $L(n)$.

We firstly recall a law of large numbers (cf. [19] for example):

$$\frac{L(n)}{\log_{1/p_{11}} n} \to 1, \quad \text{as } n \to \infty. \quad (1.1.1)$$

The above convergence is in the sense of convergence in probability, that is,

$$P\left(\left|\frac{L(n)}{\log_{1/p_{11}} n} - 1\right| < \varepsilon \right) \to 1, \quad \text{as } n \to \infty \quad (1.1.2)$$

for any $\varepsilon > 0$. It is natural that such a law of large numbers suggests to study the large deviation probabilities $P\left(\frac{L(n)}{\log_{1/p_{11}} n} \in A\right)$, where $A$ does not contain the most probable point ‘1’. Such large deviation probabilities have been comprehensively investigated in [15, Theorem 1.1].

Furthermore, the above law of large numbers also naturally leads to study LDP for the family of normalized longest success runs $\{\frac{L(n)}{\log_{1/p_{11}} n}\}$. Actually, such a LDP, in particular, when $p_{10} \leq p_{00} + p_{11}$, has been investigated in [15] based on the moment generating function of $L(n)$. In [15], a LDP for another family of normalized longest success runs $\{\frac{L(n)}{n}\}$ has been obtained as well. This LDP corresponds to the law of large numbers $\frac{L(n)}{n} \to 0$ which is directly from (1.1.1).

It is obvious that $\log_{1/p_{11}} n$ is the most accepted speed in LDP in view of (1.1.1), however, other speeds provide useful information as well, which can be seen from [15]. Based on this perspective, it makes sense to ask if a LDP can be establish for other families of normalized longest success runs in a Markov chain? A complete answer to this question in Bernoulli trials is recently given in [14]. In this thesis, our aim to give a complete answer to the same question in the setting of Markov chains. More specifically, we will present a LDP for the family of normalized longest success runs $\{\frac{L(n)}{\alpha(n)}\}$ with a general speed $\alpha(n)$, which not only includes those two LDPs derived in [15], but also gives new LDPs. Literally, all possible LDPs for normalized longest success runs in Markov chains can be recovered and unified in this thesis.

1.2 Structure of the thesis

Markov chains are the basic objects of the thesis, therefore in Chapter 2 we will make a summary review on Markov chains in the setting of both discrete-time and continuous-time, together with their possible applications. In Chapter 3 the main result of the thesis will be presented, whose complete proof will be given in Chapter 4. At the end of the thesis some conclusion remarks will be given.
Chapter 2

Markov chains

Markov chains are the basic objects used in this thesis, and in this chapter we briefly recall the definitions and several properties of them. Loosely speaking, a Markov chain is a stochastic process that starts from one state and moves successively to another on a state space, and the current state can affect and only affect the outcome of next state. Furthermore, it must possess a property which is called Markov property (memorylessness): the probability distribution of next state depends only on current state and not on the sequence of events that preceded it. In the literature, Markov chains consist of two different kinds: discrete-time Markov chain and continuous-time Markov chain. While usually the time parameter is discrete, i.e. a discrete-time Markov chain, the state space may be arbitrary. However, in most situations, Markov chains employ finite or countably infinite (that is, discrete) state space, which have a more straightforward statistical analysis. What we focus on in this thesis is exactly the discrete-time, discrete-state-space Markov chain. Since the system changes randomly, it is generally impossible to predict with certainty the state of a Markov Chain at a given point in the future. However, the statistical properties can be predicted, and these statistical properties are still of a lot of importance.

2.1 Discrete-time Markov chains

The definition of a Markov chain (discrete-time, discrete-state space Markov chain) can be simply given as follows: A Markov chain is a sequence of random variables $X_1, X_2, X_3, \ldots X_i \ldots$ with the Markov property. All possible values of $X_i$ form a countable set $S$ called the state space of the chain. The changes of states of the system are called transitions. If the current state is $i$ in the chain, then it moves to the next state $j$ with a probability denoted by $p_{ij}$, the probabilities associated with state changes are called transition probabilities. It should be remarked that the transition probability depends only on the present state of the system and not on any previous states. Mathematically speaking, a Markov chain is characterized by a state space, a transition matrix describing the probabilities of particular transitions, and an initial state (or initial distribution) across the state space. Below is the formal definition of a Markov chain.

**Definition 1.** For a stochastic process $\{X_n, n \in \mathbb{N}\}$ to be a Markov chain, it is required that for any $n \in \mathbb{N}$ and any states $i_0, i_1, \ldots, i_{n+1} \in S$, the conditional
Chapter 2. Markov chains

probabilities satisfy:
\[ P[X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0] = P[X_{n+1} = i_{n+1} | X_n = i] \]

where the conditional probabilities are well defined, i.e.
\[ P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) > 0. \]

Based on the above definition, we can now define the following general transition probabilities as

**Definition 2** (transition probabilities). Normally, the conditional probabilities are designated as transition probabilities, that is:
\[ p_{ij}(n) = P\{X_{n+1} = j | X_n = i\} \quad \text{where } i, j \in S, \]

which denote the transition probability at time \( n \).

In general, the transition probability is not only related to states \( i \), or \( j \), but also associated with time \( n \). If the transition probabilities in a Markov chain are independent of \( n \), then the chain is said to be time-homogeneous, i.e., \( p_{ij}(n) = p_{ij} \). Throughout the thesis, we will be restricted to time-homogeneous Markov chains. If we put all the transition probabilities together as a matrix, then we reach the so-called transition matrix:

\[
A = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1j} & \cdots \\
p_{21} & p_{22} & \cdots & p_{2j} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

From the above definition, we can also derive the joint distribution as follows.

**Property 1.** Using the initial distribution of the chain and the transition probabilities, we can easily compute the probability distribution of the random vector \( (X_0, \ldots, X_n) \):

\[
P(X_0 = i_0, \ldots, X_n = i_n) = P(X_n = i_n | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) \\
= \cdots \\
= P(X_n = i_n | X_{n-1} = i_{n-1})P(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2}) \cdots P(X_1 = i_1 | X_0 = i_0)P(X_0 = i_0)
\]

We remark here that the transition probabilities should satisfy the following requirements.

**Property 2.** For transition probabilities, it holds that
\[ p_{ij} > 0, \quad i, j \in S, \]
\[ \sum_{j \in S} p_{ij} = 1, \quad i \in S \]
2.2 Continuous-time Markov chains

In summary, it is clear that we have three different ways to describe a Markov chain which can be seen from the following simple example.

- States transfer diagram

![Figure 2.1: a Markov chain process](image)

- Transition probabilities matrix

\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 \\
0 & 1/3 & 1/3 & 1/3 \\
0 & 0 & 1 & 0 
\end{pmatrix}
\]

- Function expression

\[p_{ij} = f(i, j)\]

where \(f(i, j)\) are given explicitly, for instance \(f(1, 1) = 1\), \(f(2, 3) = 1/3\), and so on.

2.2 Continuous-time Markov chains

The other type of Markov chains is continuous-time Markov chains. As before, loosely speaking a continuous-time Markov chain is a mathematical model which takes values in some finite state space and for which the time spent in each state takes non-negative real values and has an exponential distribution. It is a continuous-time random process and also holds Markov property, which means the future outcomes of the model depends only on the current state of the model and not on any historical outcomes. Below is a formal definition.

**Definition 3.** Suppose that \(\{X(t), t \geq 0\}\) is a continuous-time stochastic process taking values in a state space \(S = \{i_n, n \geq 0\}\). For \(\{X(t), t \geq 0\}\) to be a time-continuous Markov chain, it is required that for any \(i_1, \ldots, i_{n+1} \in S\) and any \(0 \leq t_1 < t_2 < \ldots < t_{n+1}\), it holds

\[
P\{X(t_{n+1}) = i_{n+1} | X(t_1) = i_1, X(t_2) = i_2, \ldots, X(t_n) = i_n\}
= P\{X(t_{n+1}) = i_{n+1} | X(t_n) = i_n\}.
\]
The transition probabilities can be similarly given as

**Definition 4 (transition probabilities).** For \( s,t \geq 0 \), the transition probabilities \( p_{ij}(t+s,t) \) are defined as

\[
p_{ij}(t+s,t) = \mathbb{P}(X(t+s) = j|X(t) = i).
\]

As the discrete-time case, if the transition probabilities \( p_{ij}(t+s,t) \) depend only on the difference \( (t+s) - t = s \), then the Markov chain is called time-homogeneous. Similarly, the joint distribution of a continuous-time Markov chains can be obtained as follows.

**Property 3.** Using the transition probabilities, we can easily compute the probability distribution of the random vector \((X(t_0), \ldots, X(t_n))\):

\[
\mathbb{P}(X(t_0) = i_0, \ldots, X(t_n) = i_n) = \mathbb{P}(X(t_n) = i_n|X(t_{n-1}) = i_{n-1}) \mathbb{P}(X(t_{n-1}) = i_{n-1}|X(t_{n-2}) = i_{n-2}) \cdots \mathbb{P}(X(t_1) = i_1|X(t_0) = i_0) \mathbb{P}(X(t_0) = i_0) \cdot p_{i_0i_1}(t_1 - t_0) \cdots p_{i_{n-1}i_n}(t_n - t_{n-1}).
\]

### 2.3 Applications of Markov chains

Because of the Markov property: the future depends only on current (not on the past), nowadays Markov chains find a wide range of applications including physics, chemistry, information science, economics and finance, statistics, medicine, music, game theory and sports. Here we would like to introduce one of the applications which will in the theory of reliability. Such an application also has close connections with the longest runs which are the main topic of this thesis.

In a signal station, according to the outcome of current station, the outcome of next station can be adjusted to ensuring successful signal transmissions in the most effective way. Suppose that one station takes into account information only from the closest previous station, then the signal transmission process is a Markov chain; see the following graph for such a scenario.

![Markov chain model](image-url)
The Markov chain is determined by the current state $i_t, (t = 0, 1, \cdots, n)$ and the transition matrix $A$. For this time-homogeneous two-state (success and failure) Markov chain model, the state for each signal station $X_t, (t = 0, 1, \cdots, n)$ only takes two possibilities, i.e.

$$i_t = \begin{cases} 
0, & \text{failure}, \\
1, & \text{success}.
\end{cases}$$

Then the transition matrix of $\{i_t\}$ can be written as

$$A = \begin{pmatrix} F & S \\
S & S \end{pmatrix} \begin{pmatrix} p_{00} & p_{01} \\
p_{10} & p_{11} \end{pmatrix}.$$
Chapter 3

Large deviations on longest runs

Let \( \{X_k\}_{1 \leq k \leq n} \) be the first \( n \) steps in a time-homogeneous two-state (success and failure) Markov chain with ‘1’ and ‘0’ representing ‘success’ and ‘failure’ respectively. Suppose that the initial distribution is \( \mathbb{P}(X_1 = 0) = p_0, \mathbb{P}(X_1 = 1) = p_1 = 1 - p_0 \), and the transition matrix is:

\[
A = \begin{pmatrix}
F & S \\
S & \end{pmatrix}
\begin{pmatrix}
p_{00} & p_{01} \\
p_{10} & p_{11}
\end{pmatrix}
\]

We assume \( 0 < p_0 < 1 \) and \( 0 < p_{ij} < 1 \) \((i, j = 0, 1)\), in order to avoid triviality. In this chapter, we focus on the longest success run \( L(n) \) which is the longest stretch of consecutive successes in the first \( n \) step of the chain. More precisely, we will derive a general LDP which recovers the two special LDPs recently obtained in [15]. The main result of the thesis is formulated as follows.

3.1 The main result

**Theorem 3.1.1.** We assume \( p_{10} \leq p_{00} + p_{11} \), and suppose that the speed \( \alpha(n) \) satisfies the following two conditions:

1. \( \log_1/p_{11} \, n \leq \alpha(n) \leq n \);
2. the limit \( \lim_{n \to \infty} \ln(n)/\alpha(n) =: \beta \) exists.

Then the family of normalized longest success runs \( \{L(n)/\alpha(n)\} \) satisfies the LDP with the speed \( \alpha(n) \) and a good rate function \( S(x) \) defined as

\[
S(x) = \begin{cases}
x \cdot \ln(1/p_{11}) - \beta, & x \in D, \\
+\infty, & x \notin D,
\end{cases}
\]

(3.1.1)

where the interval \( D \) is given by

\[
D = \{x \in \mathbb{R} : \beta/\ln(1/p_{11}) \leq x \leq \limsup_{n \to \infty} n/\alpha(n)\}.
\]

That is,
(i) for any open set $O \subseteq \mathbb{R}$,
\[
\liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln P \left( \frac{L(n)}{\alpha(n)} \in O \right) \geq - \inf_{x \in O} S(x); \quad (3.1.2)
\]

(ii) for any closed set $F \subseteq \mathbb{R}$,
\[
\limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln P \left( \frac{L(n)}{\alpha(n)} \in F \right) \leq - \inf_{x \in F} S(x). \quad (3.1.3)
\]

A few special cases of Theorem 3.1.1 are listed as follows, together with their connections with the LDPs obtained in [15].

- $\alpha(n) = \log_{1/p_{11}} n$. In this case the interval $D = \{x : x \geq 1\}$, $\beta = \ln(1/p_{11})$ and $S(x) = (x - 1) \ln(1/p_{11})$ for $x \geq 1$. This LDP has been proved in [15].

- $\alpha(n) = n$. The interval $D = \{x : 0 \leq x \leq 1\}$, $\beta = 0$ and $S(x) = x \ln(1/p_{11})$ for $0 \leq x \leq 1$. This LDP has also been obtained in [15].

- $\alpha(n) = n^\alpha$ with $0 < \alpha < 1$. This is a new type of LDP and the speed is between $\log_{1/p_{11}} n$ and $n$. The interval $D = \{x : x \geq 0\}$, $\beta = 0$ and $S(x) = x \ln(1/p_{11})$ for $x \geq 0$.

### 3.2 Auxiliary formulas

The proof of Theorem 3.1.1 will be based on the following simple auxiliary formulas whose proofs are given explicitly for the purpose of clarity.

**Lemma 3.2.1.**
\[
\limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \ln (A + B) = \max \left[ \limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \ln (A), \ln (B) \right].
\]

**Proof of Lemma 3.2.1.** It holds that
\[
\ln (A + B) = \ln \{\exp[\ln(A)] + \exp[\ln(B)]\}
\]
\[
= \ln \left\{ \exp[\ln(A)] \cdot \left[ 1 + \frac{\exp[\ln(B)]}{\exp[\ln(A)]]}ight] \right\}
\]
\[
= \ln \{\exp[\ln(A)] + \ln [1 + \exp[\ln(B) - \ln(A)]]\}
\]
\[
= \ln(A) + \ln \{1 + \exp[-(\ln(A) - \ln(B)))]\}
\]
\[
= \ln(B) + \ln \{1 + \exp[-(\ln(B) - \ln(A))]]\}
\]
\[
= \max(\ln(A), \ln(B)) + \ln \{1 + \exp[-|\ln(A) - \ln(B)|]\},
\]

therefore,
\[
\limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \ln (A + B)
\]
\[
= \limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \{\max(\ln(A), \ln(B)) + \ln [1 + \exp(-|\ln(A) - \ln(B)|)]\}
\]
\[
= \max \left[ \limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \{\ln(A), \ln(B)\} \right].
\]
3.2. Auxiliary formulas

\[ + \limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \ln \{1 + \exp [-|\ln(A) - \ln(B)|]\}. \]

Trivially \(-|\ln(A) - \ln(B)| \leq 0\), which implies \(0 \leq \exp [-|\ln(A) - \ln(B)|] \leq 1\). Thus,

\[ \limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \ln \{1 + \exp [-|\ln(A) - \ln(B)|]\} \to 0, \text{ as } \alpha(n) \to 0, \]

which yields that

\[ \limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \ln (A + B) = \max \left[ \limsup_{\alpha(n) \to \infty} \frac{1}{\alpha(n)} \{\ln(A), \ln(B)\} \right]. \]

Lemma 3.2.2. For every \(a > b > 0\), it holds that

\[ \ln(a - b) \geq \ln(a) - \frac{b}{a - b}. \]

Proof of Lemma 3.2.2. From Taylor series,

\[ f(x) = f(c) + f'(c) \frac{x - c}{1!} + f''(c) \frac{(x - c)^2}{2!} + \ldots \]

We have, when \(f(x) = \ln x\),

\[ \ln x = \ln c + \frac{1}{c}(x - c) - \frac{1}{2c^2}(x - c)^2 + \ldots \]

\[ \leq \ln c + \frac{1}{c}(x - c) \]

It follows from replacing \(x, c\) with \(a, (a - b)\) respectively that

\[ \ln a \leq \ln(a - b) + \frac{b}{a - b} \]

\[ \implies \ln(a - b) \geq \ln(a) - \frac{b}{a - b}. \]

\[ \square \]
Chapter 4

Proof of Theorem 3.1.1

The proof is based on Bryc’s Inverse Varadhan Lemma (cf. [4]) and an appropriate estimate on the limit of normalized longest success runs \(\{L(n)/\alpha(n)\}\).

4.1 An important limit

First of all, we would like to introduce an important limit which will be constantly used in the following sections.

**Lemma 4.1.1.** Under the assumption \(p_{10} \leq p_{00} + p_{11}\), it holds that

\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{P}(L(n)/\alpha(n) \geq x) = -S(x),
\]

where \(x > \beta / \ln(1/p_{11})\).

**Proof of Lemma 4.1.1.** If \(x > \lim \sup_{n \to \infty} n/\alpha(n)\), then \(\mathbb{P}(L(n)/\alpha(n) \geq x) \leq \mathbb{P}(L(n) > n) = 0\), as \(n \to \infty\), then Lemma 4.1.1 trivially holds. Therefore, now we assume \(x \in D\) and \(x \neq \beta / \ln(1/p_{11})\). Let \(k = \lfloor x \alpha(n) \rfloor\), the following estimates hold (see [15, section 2.2])

\[
1 - (1 - C_{4p_{11}})^{n-k+1} \leq \mathbb{P}(L(n) \geq k) \leq 1 - (1 - C_{5p_{11}})^{n-k+1} + c(n), \quad (4.1.1)
\]

where

\[
e(n) = \frac{C_2(p_{01} + p_{10})}{p_{01}} \frac{(p_{00} - p_{10})^{n-1}}{p_{11}} + \frac{C_2(p_{01} + p_{10})}{p_{11}} \frac{(p_{00} - p_{10})^{n} - p_{11}^{k-1}(p_{00} - p_{10})^{n-k}}{p_{00} - p_{10} - p_{11}}.
\]

According to the properties of a Markov chain, we have

\[
p_{00} + p_{01} = 1, p_{10} + p_{11} = 1 \implies p_{00} - p_{10} - p_{11} = -p_{01} \implies c(n) = \frac{C_2(p_{01} + p_{10})}{p_{01}} \times
\]
\[(p_{00} - p_{10})^{n-1} - \frac{(p_{00} - p_{10})^n - p_{k1}^{k-1}(p_{00} - p_{10})^{n-k}}{p_{11}}\]

\[
\Rightarrow e(n) = \frac{C_2(p_{01} + p_{10})}{p_{01}} \times \\
\times \left[\frac{1}{p_{00} - p_{10}} - \frac{1}{p_{11}}\right](p_{00} - p_{10})^n + p_{11}^{-2}p_{k1}^k(p_{00} - p_{10})^{n-k}\),
\]

for \(C_2 = \frac{p_{01}(p_{00} - p_{10})}{(p_{01} + p_{10})}\). We use absolute value of \(e(n)\) and obtain

\[
|e(n)| = \frac{|p_{00}p_{01} - p_{10}p_{10}|}{p_{01} + p_{10}} \times \\
\times \left|\frac{1}{p_{00} - p_{10}} - \frac{1}{p_{11}}\right|[p_{00} - p_{10}]^n + p_{11}^{-2}p_{k1}^k[p_{00} - p_{10}]^{n-k}\).
\]

Obviously \(|e(n)| \leq 1\), and there must exist \(\text{const}_1, \text{const}_2 > 0\) such that

\[|e(n)| \leq \text{const}_1 \cdot |p_{00} - p_{10}|^n + \text{const}_2 p_{k1}^k |p_{00} - p_{10}|^{n-k} < 1.\]

It now follows that

\[
\limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln |e(n)|
\leq \limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln \left(\text{const}_1 \cdot |p_{00} - p_{10}|^n + \text{const}_2 p_{k1}^k |p_{00} - p_{10}|^{n-k}\right)
\leq \max\{ -\ln \frac{1}{|p_{00} - p_{10}|}, -x \ln (1/p_{11})\}.
\]

Since \(p_{10} \leq p_{00} + p_{11}\),

\[
\Rightarrow |p_{00} - p_{10}| \leq p_{11} \leq 0 \\
\Rightarrow \frac{1}{|p_{00} - p_{10}|} \geq \frac{1}{p_{11}} \geq 1 \\
\Rightarrow \ln \frac{1}{|p_{00} - p_{10}|} \geq \ln \frac{1}{p_{11}} \\
\Rightarrow \ln \frac{1}{|p_{00} - p_{10}|} \geq x \ln \frac{1}{p_{11}}, \text{ where } 0 < x < 1 \\
\Rightarrow -\ln \frac{1}{|p_{00} - p_{10}|} \leq -x \ln \frac{1}{p_{11}}
\]

Finally, we obtain

\[
\limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln |e(n)| \leq -x \ln (1/p_{11}). \tag{4.1.2}
\]

On the other hand, by using \(1 - a^x = -a^\theta \cdot \ln(a) \cdot x \text{ with } \theta \in [0, x]\), we obtain

\[
1 - (1 - C_5 p_{11}^k)^{n-k+1} = 1 - \left(1 - C_5 p_{11}^k\right)^{1/(C_5 p_{11}^k)} C_5 p_{11}^k(n-k+1)
\]
Similarly, we can prove that

\[
\therefore \text{It now follows that}
\]

\[
\theta_n \in [0, C_5 p_{11}^k (n - k + 1)]. \text{ Notice that}
\]

\[
p_{11}^k (n - k + 1) \leq p_{11}^{\alpha(n)-1} \cdot \frac{1}{p_{11}^{\alpha(n)-\log_5(p_{11})}} = \frac{1}{p_{11}^{\alpha(n)(x-\log_5(p_{11})-1)}}.
\]

With the assumption \( x > \beta/\ln(1/p_{11}) \), we can have

\[
x > \beta/\ln(1/p_{11}) \implies x > \lim_{n \to \infty} \frac{\log(n)/\alpha(n)}{\ln(1/p_{11})} \implies x > \frac{\log_5(p_{11})}{\alpha(n)}.
\]

Namely, \( x - \frac{\log_5(p_{11})}{\alpha(n)} > 0 \), for large enough \( n \). This implies that

\[
\lim_{n \to \infty} p_{11}^k (n - k + 1) = 0.
\]

Then we can derive

\[
1 - (1 - C_5 p_{11}^k)^{n-k+1} = - (1 - C_5 p_{11}^k)^{1/(C_5 p_{11}^k)} \cdot \ln \left( \frac{1 - C_5 p_{11}^k}{C_5 p_{11}^k} \right) \cdot C_5 p_{11}^k (n - k + 1)
\]

Similarly, we can prove that

\[
\mathbb{P} \left( L(n) \geq k \right) \geq 1 - (1 - C_4 p_{11}^k)^{n-k+1} = \text{const} \cdot p_{11}^k (n - k + 1).
\]

It now follows that

\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \{ \text{const} \cdot p_{11}^k (n - k + 1) \}
\]

\[
\leq \lim_{n \to \infty} \frac{1}{\alpha(n)} \mathbb{P} \left( \frac{L(n)}{\alpha(n)} \geq x \right) \leq \lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \{ \text{const} \cdot p_{11}^k (n - k + 1) + e(n) \};
\]

\[
x \ln p_{11} + \beta
\]

\[
\leq \lim_{n \to \infty} \frac{1}{\alpha(n)} \mathbb{P} \left( \frac{L(n)}{\alpha(n)} \geq x \right) \leq \max \{ (x \ln p_{11} + \beta), x \ln p_{11} \};
\]

\[
x \ln p_{11} + \beta
\]

\[
\leq \lim_{n \to \infty} \frac{1}{\alpha(n)} \mathbb{P} \left( \frac{L(n)}{\alpha(n)} \geq x \right) \leq x \ln p_{11} + \beta.
\]

Now, we can easily take the limit and obtain

\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \mathbb{P} \left( \frac{L(n)}{\alpha(n)} \geq x \right) = x \ln p_{11} + \beta = -S(x),
\]

which prove the lemma. Likewise, we can also prove

\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{P} \left( \frac{L(n)}{\alpha(n)} > x \right) = -S(x)
\]
for $x > \beta/\ln(1/p_{11})$. To see this, we rewrite the equation (4.1.1) using $k + 1$ instead of $k$ when $L(n)/\alpha(n) > x$. It follows now that
\[
1 - (1 - C_4 p_{11}^{k+1})^{n-k+2} \leq P \left( L(n)/\alpha(n) > x \right) = P \left( L(n) > k + 1 \right) \\
\leq 1 - (1 - C_5 p_{11}^k + 1)^{n-k+2} + e(n),
\]
where
\[
e(n) = \frac{C_2(p_{01} + p_{10})}{p_{01}} (p_{00} - p_{10})^{n-1} + \frac{C_2(p_{01} + p_{10})}{p_{11}} \frac{(p_{00} - p_{10})^n - p_{11}^k (p_{00} - p_{10})^{n-k-1}}{p_{00} - p_{10} - p_{11}}.
\]
Then the conclusion trivially holds after taking $n \to \infty$. 

4.2 Bryc’s inverse Varadhan lemma

Lemma 4.2.1 (Bryc’s Inverse Varadhan Lemma). If the family of probability measures of the normalized longest success runs $\{L(n)/\alpha(n)\}$ is exponentially tight (whose definition can be found in Lemma 4.3.1 below), and assume the following limit exists
\[
\Lambda(\phi) := \lim_{n \to \infty} \frac{1}{\alpha(n)} \ln E \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right]
\]
for all continuous and bounded functions $\phi(x)$. Then the family $\{L(n)/\alpha(n)\}$ satisfies a LDP with a good rate function $S(x)$ defined as
\[
S(x) = \sup_{x \in \mathbb{R}} \left[ \phi(x) - \Lambda(\phi) \right].
\]
Furthermore we can derive
\[
\Lambda(\phi) = \sup_{x \in \mathbb{R}} \left[ \phi(x) - S(x) \right].
\]

To pass from a continuous and bounded function $\phi$ to a probability, a rough and intuitive idea can be found in [17, Theorem 3.2]. Namely, the limit
\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln E \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] = \sup_{x \in \mathbb{R}} \left[ \phi(x) - S(x) \right]
\]
gives
\[
E \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] \approx \sum_{x \in \mathbb{R}} \exp \left[ \alpha(n)(\phi(x) - S(x)) \right]
\]
for large enough $n$. So it follows
\[
\sum_{x \in \mathbb{R}} \exp[\alpha(n)\phi(x)] \cdot P \left( \frac{L(n)}{\alpha(n)} = x \right) \approx \sum_{x \in \mathbb{R}} \exp \left[ \alpha(n)(\phi(x) - S(x)) \right]
\]\[
\approx \sum_{x \in \mathbb{R}} P \left( \frac{L(n)}{\alpha(n)} = x \right) \approx \sum_{x \in \mathbb{R}} \exp \left( - \alpha(n)S(x) \right),
\]
It now trivially follows that
\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln P \left( \frac{L(n)}{\alpha(n)} = x \right) \approx - \inf_{x \in \mathbb{R}} S(x).
\]
4.3 End of proof of Theorem 3.1.1

According to Bryc's Inverse Varadhan Lemma, we now need to prove the exponentially tightness property for the family of probability measures of the normalized longest success runs \( \{L(n)/\alpha(n)\} \).

**Lemma 4.3.1** (exponential tightness). Under the assumption \( p_{10} \leq p_{00} + p_{11} \), the family of probability measures \( \{\mu_{L(n)/\alpha(n)}\}_{n \geq 1} \) is exponentially tight, that is, for any constant \( 0 < a < \infty \), there is a compact set \( K_a \) such that

\[
\limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{P}(L(n)/\alpha(n) \in K_a^c) \leq -a
\]

where \( K_a^c \) denoting the complement of \( K_a \).

**Proof of Lemma 4.3.1.** For any given constant \( 0 < a < \infty \), we define a compact set as

\[
K_a = [0, \max \{(1 + \beta)/\ln(1/p_{11}), (\beta + |a|)/\ln(1/p_{11})\}].
\]

It follows from Lemma 4.1.1 that

\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{P}(L(n)/\alpha(n) \in K_a^c) = \lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{P} \left( \sum_{i=1}^{n} 1_{X_i \neq Y} \geq \max \left\{ \frac{1 + \beta}{\ln(1/p_{11})}, \frac{\beta + |a|}{\ln(1/p_{11})} \right\} \right) + 0
\]

\[
= -S\left( \max \left\{ \frac{1 + \beta}{\ln(1/p_{11})}, \frac{\beta + |a|}{\ln(1/p_{11})} \right\} \right)
\]

\[
\leq -\left[ \max \left\{ \frac{1 + \beta}{\ln(1/p_{11})}, \frac{\beta + |a|}{\ln(1/p_{11})} \right\} \cdot \ln(1/p_{11}) - \beta \right]
\]

\[
\leq -|a| \leq -a.
\]

\( \square \)

On the other hand, the proof of Theorem 3.1.1 is the limit in Lemma 4.2.1 as formulated below.

**Lemma 4.3.2** (Laplace transform). Under the assumption \( p_{10} \leq p_{00} + p_{11} \), it holds that for every continuous and bounded function \( \phi(x) \),

\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{E} \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] = \sup_{x \in \mathbb{R}} [\phi(x) - S(x)].
\]

**Proof of Lemma 4.3.2.** To achieve the limit, we prove the lower bound and the upper bound respectively.

**Proof of the lower bound**

It suffices to to prove that for every \( x \in \mathbb{R} \),

\[
\liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{E} \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] \geq [\phi(x) - S(x)]. \quad (4.3.1)
\]
If \( x \notin D \), then \( S(x) = +\infty \), and the inequality (4.3.1) holds trivially. When \( x \in D \), it is clear that for a small \( \delta > 0 \), there is an \( \varepsilon > 0 \) such that

\[
\inf_{y \in O_x} \phi(y) \geq \phi(x) - \delta, \quad \text{where } O_x := (x - \varepsilon, x + \varepsilon).
\]

We consider the following estimates

\[
\liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln E \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] \\
\geq \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln E \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \cdot 1_{\{L(n)/\alpha(n) \in O_x\}} \right] \\
\geq \phi(x) - \delta + \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right) - P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right) \right].
\]

(i) For \( x = \beta / \ln(1/p_{11}) = \ln(n) / (\alpha(n) \cdot \ln(1/p_{11})) = \log_{(1/p_{11})} n/\alpha(n) \), it is then obviously that \( S(x) = 0 \), and based on equation (1.1.2),

\[
P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right) = P \left( \frac{L(n)}{\alpha(n)} > \frac{\log_{(1/p_{11})} n}{\alpha(n)} - \varepsilon \right) \\
= P \left( \frac{L(n)}{\log_{(1/p_{11})} n} > 1 - \frac{\varepsilon \cdot \alpha(n)}{\log_{(1/p_{11})} n} \right) \\
\geq P \left( \frac{L(n)}{\log_{(1/p_{11})} n} = 1 \right) \\
\approx 1.
\]

Likewise,

\[
P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right) = P \left( \frac{L(n)}{\alpha(n)} \geq \frac{\log_{(1/p_{11})} n}{\alpha(n)} + \varepsilon \right) \\
= P \left( \frac{L(n)}{\log_{(1/p_{11})} n} \geq 1 + \frac{\varepsilon \cdot \alpha(n)}{\log_{(1/p_{11})} n} \right) \\
= 1 - P \left( \frac{L(n)}{\log_{(1/p_{11})} n} < 1 + \frac{\varepsilon \cdot \alpha(n)}{\log_{(1/p_{11})} n} \right) \\
\leq 1 - P \left( \frac{L(n)}{\log_{(1/p_{11})} n} = 1 \right) \\
\approx 0.
\]

Therefore,

\[
\liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right) - P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right) \right] \\
= \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln (1 - 0)
\]
=0.

Furthermore, we obtain

\[ \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{E} \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] \geq \phi(x) - \delta \rightarrow [\phi(x) - S(x)] \quad \text{(with } \delta \rightarrow 0). \]

(ii) For \( x > \beta / \ln(1/p) \), we can then apply Lemma 4.1.1 and the inequality in Lemma 3.2.2, and derive

\[ \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ \frac{P(L(n)}{\alpha(n)} \geq x + \varepsilon \right] \geq \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ \frac{P(L(n)}{\alpha(n)} > x - \varepsilon \right] \]

where the fact

\[ \frac{P(L(n)}{\alpha(n)} \geq x + \varepsilon \right] \geq \limsup_{n \to \infty} \frac{1}{\alpha(n)} e^{(2\varepsilon \ln(1/p) + \varepsilon_1 - \varepsilon_2)\alpha(n) - 1}; \]

comes from

\[ \frac{P(L(n)}{\alpha(n)} \geq x + \varepsilon \right] \geq \limsup_{n \to \infty} \frac{1}{\alpha(n)} \frac{P(L(n)}{\alpha(n)} \geq x + \varepsilon \right] \cdot \frac{1}{P(L(n)}{\alpha(n)} \geq x + \varepsilon \right] - 1. \]

More precisely, according to Lemma 4.1.1, it trivially holds that

\[ \lim_{n \to \infty} \ln P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right) = \alpha(n) \cdot [-S(x - \varepsilon)] \quad \text{and} \quad \lim_{n \to \infty} \ln P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right) = \alpha(n) \cdot [-S(x + \varepsilon)]; \]

\[ \exists \varepsilon_1 \text{ makes } \lim_{n \to \infty} \ln P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right) \geq \alpha(n) \cdot [-S(x - \varepsilon)] \cdot \ln(1/p_1) - \varepsilon_1 + \beta; \]

\[ \exists \varepsilon_2 \text{ makes } \lim_{n \to \infty} \ln P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right) \geq \alpha(n) \cdot [-S(x + \varepsilon)] \cdot \ln(1/p_1) - \varepsilon_1 + \beta. \]
\[ \leq \alpha(n) \cdot \left[ -(x + \varepsilon) \cdot \ln(1/p_{11}) + \varepsilon_2 + \beta \right]. \]

Henceforth,
\[ \lim_{n \to \infty} P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right) \geq e^{\alpha(n) \cdot \left[ -(x - \varepsilon) \cdot \ln(1/p_{11}) - \varepsilon_1 + \beta \right],} \]
\[ \lim_{n \to \infty} P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right) \leq e^{\alpha(n) \cdot \left[ -(x + \varepsilon) \cdot \ln(1/p_{11}) + \varepsilon_2 + \beta \right].} \]

Finally we arrive
\[ \lim_{n \to \infty} \frac{P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right)}{P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right)} = e^{\alpha(n) \cdot \left[ -(x - \varepsilon) \cdot \ln(1/p_{11}) - \varepsilon_1 + \beta \right]} \]
\[ = e^{\varepsilon \cdot \ln(1/p_{11}) - \varepsilon_1 - \varepsilon_2} \alpha(n). \]

Then the estimate (4.3.2) follows.

Now for small enough \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) such that \( 2\varepsilon \ln(1/p_{11}) - \varepsilon_1 - \varepsilon_2 > 0 \), it follows
\[ \limsup_{n \to \infty} \frac{1}{\alpha(n)} \frac{1}{e^{2\varepsilon \ln(1/p_{11}) - \varepsilon_1 - \varepsilon_2} \alpha(n)} \to 0. \]

Therefore we obtain
\[ \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ \frac{P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right)}{P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right)} \right] \geq \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ \frac{P \left( \frac{L(n)}{\alpha(n)} > x - \varepsilon \right)}{P \left( \frac{L(n)}{\alpha(n)} \geq x + \varepsilon \right)} \right] = -S(x - \varepsilon) \to -S(x) \text{ (with } \varepsilon \to 0). \]

Furthermore, the lower bound follows that
\[ \liminf_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{E} \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] \geq \phi(x) - \delta - S(x) \to [\phi(x) - S(x)] \text{ (with } \delta \to 0). \]

**Proof of the upper bound**

Now, we focus on proving the upper bound in Lemma 4.3.2, that is,
\[ \limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{E} \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] \leq \sup_{x \in \mathbb{R}} [\phi(x) - S(x)]. \quad (4.3.3) \]

For a constant \( \alpha > 0 \) and a fixed small \( \delta > 0 \), let us consider an interval
\[ I_\varepsilon := \left[ \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2}, \max \left\{ \frac{\alpha + \beta}{\ln(1/p_{11})}, \frac{\beta + 1}{\ln(1/p_{11})} \right\} \right], \]
and its partition
\[ x_0 = \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2}. \]
\[ x_1 = \frac{\beta}{\ln(1/p_{11})} + \frac{\varepsilon}{2}, \]
\[ \vdots \]
\[ x_{N(\varepsilon)} = \max \left\{ \frac{a + \beta}{\ln(1/p_{11})}, \frac{\beta + 1}{\ln(1/p_{11})} \right\}, \]

for some \( N(\varepsilon) \) depending on \( \varepsilon \), where the small constant \( \varepsilon \) is chosen such that
\[ \phi(x_i) - \delta \leq \max_{y \in [x_i, x_{i+1}]} \phi(y) \leq \phi(x_i) + \delta, \quad i = 0, 1, \ldots, N - 1. \]

It now follows that,
\[
\mathbb{E} \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \right] \\
\leq e^{\|\phi\| \cdot \alpha(n)} \cdot P \left( \frac{L(n)}{\alpha(n)} \in I_\varepsilon^c \right) \\
+ \sum_{i=0}^{N-1} \mathbb{E} \left[ \exp \left\{ \phi \left( \frac{L(n)}{\alpha(n)} \right) \cdot \alpha(n) \right\} \cdot I_{\{L(n)/\alpha(n) \in [x_i, x_{i+1}]\}} \right] \\
\leq e^{\|\phi\| \cdot \alpha(n)} \cdot P \left( \frac{L(n)}{\alpha(n)} \in I_\varepsilon^c \right) + \sum_{i=0}^{N-1} e^{(\phi(x_i) + \delta) \alpha(n)} \cdot P \left( \frac{L(n)}{\alpha(n)} \geq x_i \right).
\]

Now Lemma 4.3.1 implies
\[
\limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ P \left( \frac{L(n)}{\alpha(n)} \in I_\varepsilon^c \right) \right] \\
= \limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln \left[ P \left( \frac{L(n)}{\alpha(n)} < \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2} \right) + P \left( \frac{L(n)}{\alpha(n)} > \max \left\{ \frac{a + \beta}{\ln(1/p_{11})}, \frac{\beta + 1}{\ln(1/p_{11})} \right\} \right) \right] \\
= \max \left\{ \limsup_{n \to \infty} \frac{1}{\alpha(n)} \ln P \left( \frac{L(n)}{\alpha(n)} < \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2} \right), -a \right\} \\
= \max \{-\infty, -a\} \\
= -a.
\]

We remark here that negative infinity limit above is based on the fact that
\[ P \left( \frac{L(n)}{\alpha(n)} < \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2} \right) = 0 \]
for \( \beta = 0 \), and
\[
P \left( \frac{L(n)}{\alpha(n)} < \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2} \right) \leq P \left( \frac{L(n)}{\log_1 p_{11} n} < 1 - \frac{\varepsilon}{4} \left( \frac{\beta}{\ln(1/p_{11}) + 1} \right)^{-1} \right)
\]
for \( \beta > 0 \). The results in [15] show that for each \( 0 < x < 1 \), it holds that
\[
\lim_{n \to \infty} \frac{1}{\log_1 p_{11} n} \ln \left[ - \ln P \left( \frac{L(n)}{\log_1 p_{11} n} \leq 1 - x \right) \right] = x \cdot \ln(1/p_{11}),
\]
Now the finite sum can be estimated as, based on Lemma 4.1.1,

\[
\lim_{n \to \infty} \ln \left[ - \ln P \left( \frac{L(n)}{\log_{1/p_{11}} n} \leq 1 - x \right) \right] = \ln (1/p_{11})^{x \log_{1/p_{11}} n}
\]

\[
\lim_{n \to \infty} \ln P \left( \frac{L(n)}{\log_{1/p_{11}} n} \leq 1 - x \right) = p_{11} x \log_{p_{11}} n
\]

\[
\lim_{n \to \infty} P \left( \frac{L(n)}{\log_{1/p_{11}} n} \leq 1 - x \right) = e^{-p_{11} x \log_{1/p_{11}} n} = e^{-n x}.
\]

Thus, with trivially 0 < \(\frac{\beta}{\ln(1/p_{11})} + 1\) \(-1 < 1,

\[
\lim_{n \to \infty} \sup \frac{1}{\alpha(n)} \ln P \left( \frac{L(n)}{\alpha(n)} < \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2} \right) \leq \lim_{n \to \infty} \sup \frac{1}{\alpha(n)} \ln e^{-n x} = \lim_{n \to \infty} \frac{-n x}{\alpha(n)} \to -\infty.
\]

This also implies \(P \left( \frac{L(n)}{\alpha(n)} < \frac{\beta}{\ln(1/p_{11})} - \frac{\varepsilon}{2} \right) \to 0\), which yields

\[
P \left( \frac{L(n)}{\alpha(n)} \geq x_{0} \right) = 1 - P \left( \frac{L(n)}{\alpha(n)} < x_{0} \right) \to 1
\]

\[
\lim_{n \to \infty} \sup \frac{1}{\alpha(n)} \ln P \left( \frac{L(n)}{\alpha(n)} \geq x_{0} \right) = 0 = S \left( \frac{\beta}{\ln(1/p_{11})} \right).
\]

Now the finite sum can be estimated as, based on Lemma 4.1.1,

\[
\lim_{n \to \infty} \sup \frac{1}{\alpha(n)} \ln \left[ \sum_{i=0}^{N-1} e^{(\phi(x_i) + \delta) \alpha(n)} \cdot P \left( \frac{L(n)}{\alpha(n)} \geq x_i \right) \right]
\]

\[
\leq \max \left\{ \phi(x_0) + \delta + \lim_{n \to \infty} \sup \frac{1}{\alpha(n)} \ln P \left( \frac{L(n)}{\alpha(n)} \geq x_0 \right), \phi(x_i) + \delta - S(x_i), i = 1, \ldots, N-1 \right\}
\]

\[
= \max \left\{ \phi(x_0) + \delta - S \left( \frac{\beta}{\ln(1/p_{11})} \right), \phi(x_i) + \delta - S(x_i), i = 1, \ldots, N-1 \right\}
\]

\[
\leq \sup_{x \in \mathbb{R}} \left[ \phi(x) - S(x) \right] + 2\delta \to \sup_{x \in \mathbb{R}} \left[ \phi(x) - S(x) \right], \text{ as } \delta \to 0.
\]

Now the proof follows

\[
\lim_{n \to \infty} \frac{1}{\alpha(n)} \ln \mathbb{E} \left[ \exp \left( \frac{L(n)}{\alpha(n)} \cdot \alpha(n) \right) \right]
\]

\[
\leq \max \left\{ -a \cdot \sup_{x \in \mathbb{R}} \left[ \phi(x) - S(x) \right] \right\}
\]

\[
= \sup_{x \in \mathbb{R}} \left[ \phi(x) - S(x) \right], \text{ as } a \to \infty.
\]
4.3. End of proof of Theorem 3.1.1

The upper bound is thus proved. Finally, Theorem 3.1.1 gets proved directly based on the above Bryc’s Inverse Varadhan Lemma.
Chapter 5

Conclusions

In this thesis we first made a summary review on discrete-time and continuous-time Markov chains and some of their applications (see Chapter 2). Then we considered a time-homogeneous two-state Markov chain, and focused on the main object of the thesis: the longest success run $L(n)$, and formulated a LDP with a general speed which can recover two special LDPs recently derived in [15] (see Chapter 3). The proof of our main result (the above mentioned LDP with a general speed) was given in Chapter 4 based on Bryc’s Inverse Varadhan Lemma.

We want to stress that an assumption was imposed throughout the thesis $p_{10} \leq p_{00} + p_{11}$ in order to overcome several technical difficulties. In terms of the structure of the Markov chain, this assumption simply means that the probability that the chain moves from ‘1’ to ‘0’ should be less than or equal to the probability that the chain stays still. In this sense, this assumption seems to be completely unnecessary. This would be one of my further research topics.
Bibliography


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