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Abstract

We construct fully discrete stable and accurate numerical schemes for solving partial differential equations posed on non-simply connected spatial domains. The schemes are constructed using summation-by-parts operators in combination with a weak imposition of initial and boundary conditions using the simultaneous approximation term technique.

In the theoretical part, we consider the two dimensional constant coefficient advection equation posed on a rectangular spatial domain with a hole. We construct the new scheme and study well-posedness and stability. Once the theoretical development is done, the technique is extended to more complex non-simply connected geometries.

Numerical experiments corroborate the theoretical results and show the applicability of the new approach and its advantages over the standard multi-block technique. Finally, an application using the linearized Euler equations for sound propagation is presented.

Keywords: Initial boundary value problems, Stability, Well-posedness, Boundary conditions, Non-simply connected domains, Complex geometries

1. Introduction

High order Summation-by-Parts (SBP) operators, together with a weak imposition of initial and well-posed boundary conditions using the Simultaneous Approximation Term (SAT) technique, provide provably fully discrete unconditionally stable schemes for steady or time-dependent spatial domains [7, 8, 9, 15]. These schemes have so far been mostly developed for spatial domains consisting
of simply connected regions. To handle more complicated geometries, hybrid formulations utilizing finite volume and finite difference methods [1, 2, 3, 10] have been proposed. Other alternatives within the finite difference community for complex geometries include finite difference schemes using over-set mesh discretizations [6, 17, 18] (even though stability proofs are missing), multi-block techniques [4, 21, 22, 5] as well as SBP extensions to unstructured grids [19, 20].

In this article, we extend the SBP-SAT technique to handle partial differential equations posed on non-simply connected multi-dimensional geometries. Our final scheme is numerically stable, and minimizes the number of multi-block couplings with reduced accuracy.

The rest of the article proceeds as follows. In section 2, we study the two dimensional constant coefficient advection equation posed on a rectangular geometry with a hole. In section 3, the discrete problem and a new combination of SBP operators are presented. Stability of the new scheme is investigated in section 4. We extend the new approach to more complex geometries in section 5. Numerical calculations are shown in section 6, where we measure the accuracy and efficiency of our scheme and compare it to the standard SBP-SAT multi-block technique. Finally, conclusions are drawn in section 7.

2. Well-posedness

To develop the theory, we consider the constant coefficient scalar advection equation in two space dimensions

\[ u_t + \alpha u_x + \beta u_y = 0, \quad (x, y) \in \Omega, \quad t \in [0, T], \]  

where the subscripts \( t \), \( x \) and \( y \) denote partial derivatives. The computational domain \( \Omega \) is depicted in Figure 1. The spatial region \( H \) is not a part of the computational domain; it forms a hole in \( \Omega \).

The energy method (multiplying (1) with the solution and integrating over \( \Omega \)) together with the use of the Gauss-Green theorem gives

\[ ||u||_t^2 = -\int_{\partial \Omega} u^2(\alpha, \beta) \cdot n \, ds, \]

where \( \partial \Omega = \{A \cup B \cup C \cup D \cup a \cup b \cup c \cup d\} \) is the boundary of \( \Omega \). Moreover, \( n = (n_1, n_2) \) is the outward pointing normal vector from \( \Omega \) and \( ds \) is an infinitesimal element along \( \partial \Omega \). The norm is defined as \( ||u||^2 = \iint_{\Omega} u^2 \, dx \, dy \).

In order to bound the energy rate of the solution in (2), we specify

\[ u = g \text{ if } (\alpha, \beta) \cdot n < 0, \]  

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where \((\alpha, \beta) \cdot n = n_1\alpha + n_2\beta\). Assuming for example that \(\alpha, \beta > 0\), (3) leads to

\[ u_s = g_s \quad \text{and} \quad u_w = g_w \tag{4} \]

where \(s \in \{C, c\}\) and \(w \in \{D, d\}\), see Figure 1. We insert (4) into (2), integrate in time and consider an initial condition \(u = f\). The continuous energy estimate becomes

\[ ||u(T)||^2 = ||f||^2 + \beta \sum_{s \in \{C, c\}} \int_0^T \int_s g_s^2 \, dx \, dt + \alpha \sum_{w \in \{D, d\}} \int_0^T \int_w g_w^2 \, dy \, dt + BT. \tag{5} \]

In (5), \(\sum\) denotes summation and \(BT\) is the negative contribution from the outflow boundaries \(\{A, B, a, b\}\) as

\[ BT = -\beta \sum_{n \in \{A, a\}} \int_0^T \int_n u^2 \, dx \, dt - \alpha \sum_{e \in \{B, b\}} \int_0^T \int_e u^2 \, dy \, dt. \tag{6} \]

We summarize the result in

**Proposition 1.** The continuous problem (1) for \(\alpha, \beta > 0\) augmented with boundary conditions (4) is strongly well-posed and has the bound (5).
3. Summation-by-part operators

The domain $\Phi = \{\Omega \cup H\}$ is a rectangle and we discretize it using $N$ and $M$ grid points in the $x$ and $y$ directions, respectively. In time we use $L$ time levels. The boundaries of $H$ coincide with the coordinate lines after the discretization of $\Phi$. We allocate a column vector of size $LMN$ to the grid as

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_k \\ \vdots \\ v_L \end{bmatrix}, \quad [v_k] = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_N \end{bmatrix}_k, \quad [v_i]_k = \begin{bmatrix} v_1 \\ \vdots \\ v_{j} \\ \vdots \\ v_M \end{bmatrix}_{ki}$$

(7)

in which

$$\begin{align*}
    v_{ki} &\approx u(t_k, x_i, y_j) & \text{for} & \quad (x_i, y_j) \in \Omega, \\
v_{ki} : \text{not defined} & & \text{for} & \quad (x_i, y_j) \in H.
\end{align*}$$

(8)

Inside the hole $v_{ki}$ has neither any relation to the solution $u$, nor any contribution to the calculations (we will show the latter below). We have related $v$ to these grid points for the convenience of using tensor products in the formulations below.

The first derivative $u_y$ at $x_i$ for all $i \in \{1, \ldots, N_l\} \cup \{N - N_r + 1, \ldots, N\}$, see Figure 2, is approximated by $D_y^M u$. In Figure 2, and also in the remainder of this article, the super-/subscripts $l$, $r$, $a$ and $b$ stand for left, right, above and below, respectively. $D_y^M$ is a so-called SBP operator of the form

$$D_y^M = (P_y^M)^{-1}Q_y^M,$$

(9)

and $u = [u_1, \ldots, u_M]^T$ is a smooth function injected in each grid point in the $y$ direction. The superscript $M$ denotes the size of the operator and the subscript $y$ denotes the direction along which the operator is acting. Moreover, $P_y^M$ is a symmetric positive definite matrix, and $Q_y^M$ is an almost skew-symmetric matrix that satisfies

$$Q_y^M + (Q_y^M)^T = E_y^M - E_y^0 = B_y^M = \text{diag}(-1,0,\ldots,0,1).$$

(10)

In (10), $E_y^0 = \text{diag}(1,0,\ldots,0)$ and $E_y^M = \text{diag}(0,\ldots,0,1)$. The first derivative in the $x$ direction, $D_x^N = (P_x^N)^{-1}Q_x^N$, at $y_j$ for all $j \in \{1, \ldots, M_b\} \cup \{M - M_a + 1, \ldots, M\}$, see Figure 2, and the first derivative in time, $D_t^L = (P_t^L)^{-1}Q_t^L$, are defined in analogous ways.
The first derivative $u_y$ at $x_i$ for all $i \in \{N_l + 1, \ldots, N - N_r\}$, is approximated by $\tilde{D}_y u$, where $\tilde{D}_y$ is

$$\tilde{D}_y = \begin{bmatrix} D_y^{M_b} & 0^{M-(M_a+M_b)} \\ 0^{M-(M_a+M_b)} & D_y^{M_a} \end{bmatrix}.$$  \hspace{1cm} (11)

In (11), $D_y^{M_b} = (P_y^{M_b})^{-1}Q_y^{M_b}$ and $D_y^{M_a} = (P_y^{M_a})^{-1}Q_y^{M_a}$ are the same type of SBP operators as in (9), but smaller in size ($M_b$ and $M_a$, respectively). The notation $0$ denotes a zero matrix and the super-script denotes its size (this notation is used throughout the rest of the paper). The zero matrix corresponds to the grid points inside the hole. The derivative in the $x$ direction at $y_j$ for all $j \in \{M_b+1, \ldots, M-M_a\}$, is constructed in the same way, as

$$\tilde{D}_x = \begin{bmatrix} D_x^{N_l} & 0^{N-(N_l+N_r)} \\ 0^{N-(N_l+N_r)} & D_x^{N_r} \end{bmatrix}.$$  \hspace{1cm} (12)

A finite difference approximation including the time discretization [23, 24], on SBP form, is constructed by extending the one-dimensional SBP operators in a tensor product fashion as

$$D_t = D_t^L \otimes \left( [I_{x}^{\Omega} \otimes I_{y}^{\Omega}] + [I_{x}^{H} \otimes I_{y}^{\Omega}] + [I_{x}^{\Omega} \otimes I_{y}^{H}] \right),$$

$$D_x = I_t \otimes \left( [D_{x}^{N} \otimes I_{y}^{\Omega}] + [D_{x}^{H} \otimes I_{y}^{\Omega}] \right),$$

$$D_y = I_t \otimes \left( [I_{x}^{\Omega} \otimes D_{y}^{M}] + [I_{x}^{H} \otimes \tilde{D}_y] \right),$$

where $I_t$ is the identity matrix in time and has the size $L$. In (13), $\otimes$ represents the Kronecker product which is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{bmatrix},$$  \hspace{1cm} (14)
for arbitrary matrices $A$ and $B$. For an $m \times n$ matrix $A$ and $k \times l$ matrix $B$, the size of $A \otimes B$ is $(mk) \times (nl)$. More details on Kronecker products and their properties can be found in [26, 27].

Moreover, $I_y^H$ and $I_x^H$ are diagonal matrices of size $M$ and $N$ respectively given by

$$
I_y^H = \begin{bmatrix} 0^M_a & I^M-(M_a+M_b) & 0^M_a \\ I^M-(M_a+M_b) & 0^M_b \\ 0^M_b \\ \end{bmatrix},
$$

$$
I_x^H = \begin{bmatrix} 0^N_l & I^N-(N_l+N_r) & 0^N_l \\ I^N-(N_l+N_r) & 0^N_r \\ 0^N_r \\ \end{bmatrix},
$$

where $I$ (with a slight abuse of notation) denotes the identity matrix and the superscripts the size of the matrices. (In other words; $I_x^H$ or $I_y^H$ have diagonal elements equal to one corresponding to the points inside the hole, respectively, and zero diagonal elements otherwise.) Moreover, $I_{x,\Omega} = I^N - I_x^H$ and $I_{y,\Omega} = I^M - I_y^H$, where $I^{M,N}$ are identity matrices of size $M$ and $N$, respectively.

A schematic of different regions of $\Omega$ influenced by the different identity matrices $I_{x,\Omega}$ and $I_{y,\Omega}$, is shown in Figure 3. Additionally, schematics of the different regions where the matrices $D_x$ and $D_y$ are active, as well as the underlying matrices involved in their construction, are shown in Figures 4 and 5.

4. Stability

The fully discrete SBP-SAT approximation of (1), including (4) can be written as

$$
D_t v + \alpha D_x v + \beta D_y v = P_i^{-1} \sigma_i (v_i - f) + \sum_{s=\{C, c\}} P_s^{-1} \sigma_s (v_s - g_s) + \sum_{w=\{D, d\}} P_w^{-1} \sigma_w (v_w - g_w).
$$

In (16), $\sigma_{i,s,w}$ are penalty coefficients for the weak initial and boundary conditions. $g_{s,w}$ are zero vectors of the same size as $v$, except at the positions corresponding to the inflow boundaries where the zeros are replaced with the boundary data. Moreover $f$ is a zero vector, of the same size as $v$, except at the positions corresponding to $t = 0$ where the initial data (compatible with the boundary conditions) is injected. The subscripts $i, s$ and $w$ on the solution restrict the solution to the initial
Figure 2: A schematic of the regions where difference operators in x and y directions are defined.

Figure 3: A schematic of the geometry influenced with different combinations of $I^\Omega_x$ and $I^H_{x,y}$.

Figure 4: A schematic that shows where the matrices, $D^N_x$, $\tilde{D}_x$, $I^\Omega_y$ and $I^H_y$, are defined; the solid lines correspond to the non-zero contributions.

Figure 5: A schematic that shows where the matrices, $D^M_y$, $\tilde{D}_y$, $I^\Omega_x$ and $I^H_x$, are defined; the solid lines correspond to the non-zero contributions.
time, and the s and w boundary locations. Additionally,

\[ P_{i}^{-1} = (P_{i}^{L})^{-1} E_{0}^{L} \otimes \left( [I_{x}^{\Omega} \otimes I_{y}^{\Omega}] + [I_{x}^{H} \otimes I_{y}^{H}] + [I_{x}^{\Omega} \otimes I_{y}^{H}] \right), \]

\[ P_{C}^{-1} = I_{t} \otimes \left( [I_{x}^{\Omega} \otimes (P_{y}^{M})^{-1} E_{0}^{M}] + [I_{x}^{H} \otimes (\tilde{P}_{y})^{-1} \tilde{E}_{0b}] \right), \]

\[ P_{c}^{-1} = I_{t} \otimes I_{x}^{H} \otimes (\tilde{P}_{y})^{-1} \tilde{E}_{0a}, \]

\[ P_{D}^{-1} = I_{t} \otimes \left( [(P_{x}^{N})^{-1} E_{0}^{N} \otimes I_{y}^{\Omega}] + [(\tilde{P}_{x})^{-1} \tilde{E}_{0} \otimes I_{y}^{H}] \right), \]

\[ P_{d}^{-1} = I_{t} \otimes (\tilde{P}_{x})^{-1} \tilde{E}_{0a} \otimes I_{y}^{H}, \]

where

\[ (\tilde{P}_{y})^{-1} = \begin{bmatrix} (P_{y}^{M})^{-1} & 0 \end{bmatrix}, \quad (\tilde{P}_{x})^{-1} = \begin{bmatrix} (P_{x}^{N})^{-1} & 0 \end{bmatrix}, \]

\[ \tilde{E}_{0b} = \begin{bmatrix} E_{0}^{M} & 0 \\ 0 & E_{0}^{M} \end{bmatrix}, \quad \tilde{E}_{0a} = \begin{bmatrix} 0 & 0 \\ E_{0}^{N} & 0 \end{bmatrix}, \]

\[ \tilde{E}_{0a} = \begin{bmatrix} 0 & 0 \\ E_{0}^{N} & 0 \end{bmatrix}, \quad \tilde{E}_{0b} = \begin{bmatrix} 0 & 0 \\ E_{0}^{N} & 0 \end{bmatrix}. \]

Note that we have again slightly abused notation in (18) by applying the inverse sign on \( \tilde{P}_{y} \) and \( \tilde{P}_{x} \). In Figure 6, we graphically show the different parts of \( \delta \Omega \) (the inflow parts) that are influenced by non-zero contributions from the penalty terms in (17).

Next, we apply the discrete energy method, by multiplying (16) from the left with \( v^{T} P \), where

\[ P = P_{i}^{L} \otimes \left( [I_{x}^{\Omega} \otimes I_{y}^{\Omega}] + [\tilde{P}_{x} \otimes I_{y}^{H}] \right) \left( [I_{x}^{\Omega} \otimes P_{y}^{M}] + [I_{x}^{H} \otimes \tilde{P}_{y}] \right), \]

\[ := P_{x} \quad := P_{y}. \]
\[\tilde{P}_y = \begin{bmatrix} P_{y}^{M_b} \\ 0^{M-(M_a+M_b)} \\ P_{y}^{M_a} \end{bmatrix} \text{ and } \tilde{P}_x = \begin{bmatrix} P_{x}^{N_l} \\ 0^{N-(N_l+N_r)} \\ P_{x}^{N_r} \end{bmatrix}. \quad (20)\]

Schematics of the different regions in the geometry influenced by \(P_x\) and \(P_y\), as well as the underlying matrices involved in their construction, are shown in Figures 7 and 8. By using the properties of the Kronecker product [26, 27] one can rewrite (19) as

\[P = P^L_t \otimes \left( \tilde{I}_y \otimes \tilde{P}_y \right) + I^H_x \otimes \tilde{P}_y + \tilde{P}_x \otimes \tilde{I}_y \left( P^M_x \right). \quad (21)\]

In Figure 9, the non-zero contributions from \(P_{1,2,3}\) to the regions of \(\Omega\) are shown.

We need

**Lemma 1.** The matrix \(P\) in (21) is non-singular.

**Proof.** Consider the matrix

\[S = (P^L_t)^{-1} \otimes \left( S_1 + S_2 + S_3 \right) \quad (22)\]

in which
Figure 7: A schematic that shows where the matrices, $P_M^y$, $\tilde{P}_y$, $I_\Omega^x$ and $I_H^x$, are defined; the solid lines correspond to the non-zero contributions.

Figure 8: A schematic that shows where the matrices, $P_N^x$, $\tilde{P}_x$, $I_\Omega^y$ and $I_H^y$, are defined; the solid lines correspond to the non-zero contributions.

Figure 9: A schematic of the geometry and its relation to different parts of the matrix $P$.

Figure 10: A geometry with two holes.
\[ S_1 = I_X^\Omega (P_N^x)^{-1} \otimes I_Y^\Omega (P_M^y)^{-1}, \]
\[ S_2 = I_X^H (P_N^x)^{-1} \otimes I_Y^\Omega (P_y^x)^{-1}, \]
\[ S_3 = I_X^\Omega (P_x^x)^{-1} \otimes I_Y^H (P_M^y)^{-1}. \]

The properties of the Kronecker product give
\[ SP = I_t \otimes [(S_1 + S_2 + S_3)(P_1 + P_2 + P_3)]. \]

Now, we substitute \( S_{1,2,3} \) from (23) and \( P_{1,2,3} \) from (21) in (24) and find
\[ (S_1 + S_2 + S_3)(P_1 + P_2 + P_3) = S_1P_1 + S_2P_2 + S_3P_3 \]
\[ = \left( [I_X^\Omega \otimes I_Y^\Omega] + [I_X^H \otimes I_Y^\Omega] + [I_X^\Omega \otimes I_Y^H] \right) =: I, \]
by the fact that \( S_iP_j = 0 \) if \( i \neq j \). In the same way, we find that \( PS = I_t \otimes I^\Omega \). □

Applying the discrete energy method to (16) and considering zero data gives
\[ v^T P_D v + \alpha v^T P_D v + \beta v^T P_D v = v^T P_P^{-1} \sigma_i \nu_i + v^T P_P^{-1} \sigma_c \nu_c + v^T P_P^{-1} \sigma_c \nu_c + v^T P_P^{-1} \sigma_D \nu_D, \]
\[ + v^T P_P^{-1} \sigma_D \nu_D + v^T P_P^{-1} \sigma_D \nu_D. \]

Next, we evaluate the matrix products in (25), as follows
\[ P_D = Q_L^t \otimes (P_1 + P_2 + P_3), \]
\[ PD_x = P^L_t \otimes \left( [I_X^\Omega Q_N^x \otimes I_Y^\Omega P_y^x] + [I^H_X Q_N^x \otimes I_Y^\Omega P_y^x] \right), \]
\[ PD_y = P^L_t \otimes \left( [I_X^\Omega P_N^x \otimes I_Y^\Omega Q_y^x] + [I_X^\Omega P_x^x \otimes I_Y^\Omega Q_y^x] \right), \]
where
\[ \tilde{Q}_x = \begin{bmatrix} Q^N_x & 0^{N-(N_t+N_c)} \\ 0^{N-(N_t+N_c)} & Q^N_x \end{bmatrix} \quad \text{and} \quad \tilde{Q}_y = \begin{bmatrix} Q^M_y & 0^{M-(M_u+M_b)} \\ 0^{M-(M_u+M_b)} & Q^M_y \end{bmatrix}. \]
Further,

\[ PP_i^{-1} = E_0^L \otimes (P_1 + P_2 + P_3) := H_i, \]

\[ PP_C^{-1} = P^L_i \otimes \left( [I_x^O P_N^X \otimes I_y^O E_0^M] + [I_x^H P_N^X \otimes I_y^H \tilde{E}_{0_a}] \right) := H_C, \]

\[ PP_c^{-1} = P^L_i \otimes I_x^H \otimes I_y^O \tilde{E}_{0_a} := H_c, \]

\[ PP_D^{-1} = P^L_i \otimes \left( [I_x^O E_0^N \otimes I_y^O P_M^y] + [I_x^O \tilde{E}_{0_o} \otimes I_y^H P_M^y] \right) := H_D, \]

\[ PP_d^{-1} = P^L_i \otimes I_x^O \tilde{E}_{0_r} \otimes I_y^H P_M^y := H_d. \] (28)

The details of the computations in (26) and (28) are given in appendix A.

By substituting (26) and (28) into (25) and adding the transpose, we obtain

\[ v^T H_f v - v^T (1 + 2\sigma_i) H_i v = (\beta + 2\sigma_c) v^T H_c v + (\beta + 2\sigma_c) v^T H_C v \]

\[ + (\alpha + 2\sigma_d) v^T H_d v + (\alpha + 2\sigma_d) v^T H_D v \]

\[ + CT \] (29)

where \( H_f = E_1^L \otimes (P_1 + P_2 + P_3) \) and \( CT \) stands for Corner Terms. Details of the derivation of (29) are given in appendix B.

In (29), \( CT \) is an indefinite term that involves the solution on a few grid points enclosed in two small blocks around each corner. These blocks are results of using central differences in \( \xi \) and \( \eta \) directions around the corners while having different norms in the perpendicular directions, due to the hole. Hence, the skew-symmetric property of the difference operators cannot be preserved and the interior grid contribution is not removed in \( CT \). In Figure 11 we show schematically where \( CT \) is located. The size of each block depends on the order of the difference operators and is independent of the number of grid points, see Table 1. In Figure 12, a more

<table>
<thead>
<tr>
<th>SBP</th>
<th>21</th>
<th>42</th>
<th>63</th>
</tr>
</thead>
<tbody>
<tr>
<td>size of ( CT )</td>
<td>2 \times 1</td>
<td>4 \times 4</td>
<td>6 \times 6</td>
</tr>
</tbody>
</table>

Table 1: The size of each block in \( CT \) around one corner for schemes of different order of accuracy.
Figure 11: A schematic of where $CT$ is active; red and blue colors mark the area affected by the $x$ and $y$ difference operators, respectively.

A precise schematic of the $\xi$ and $\eta$ blocks involved in $CT$ is shown for $SBP21$ and $SBP42$.

In order to investigate the stability, we consider the semi-discrete version of (1) on SBP-SAT form, written as

$$v_t + Av_x = 0, \quad (30)$$

and investigate the spectrum of $A$. In (30), $A$ is given by

$$A = \alpha D_x + \beta D_y - \sum_{s=\{C,c\}} P_s^{-1} \sigma_s - \sum_{w=\{D,d\}} P_w^{-1} \sigma_w, \quad (31)$$

and $\sigma_{s,w}$ and $P_{C,c,D,d}^{-1}$ are given in (16) (discard $I_t$ in (17)). We assume $\alpha = 1$, $\beta = 1$ and use the following stability conditions for the penalty parameters,

$$1 + 2\sigma_i \leq 0, \quad \beta + 2\sigma_{c,c} \leq 0, \quad \alpha + 2\sigma_{d,d} \leq 0. \quad (32)$$

We choose $\sigma_i = -1$, $\sigma_C = \sigma_c = -\beta$, and $\sigma_D = \sigma_d = -\alpha$.

The eigenvalue distribution of $A$ with SBP operators of different orders on a grid of size $91 \times 91$ are shown in Figures 13-20. The minimum real part of the spectrum for a sequence of mesh refinements is given in Figure 21. As Figures 13-21 show, the eigenvalues of $A$ for all orders of accuracy have the correct sign with a minimum real part clearly positive. In combination with SBP-SAT in time, this implies stability.
Figure 12: A closer look at $CT$, near one of the corners, for $SBP21$ and $SBP42$ schemes.

Figure 13: The discrete spectrum, for the second order case ($SBP21$), $N=M=91$.

Figure 14: A blow-up of the spectrum near imaginary axis, $SBP21$, $N=M=91$. 

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Figure 15: The discrete spectrum, for the third order case (SBP42), $N=M=91$.

Figure 16: A blow-up of the spectrum near imaginary axis, SBP42, $N=M=91$.

Figure 17: The discrete spectrum for the fourth order case (SBP63), $N=M=91$.

Figure 18: A blow-up of the spectrum near imaginary axis, SBP63, $N=M=91$.

Figure 19: The discrete spectrum for the fifth order case (SBP84), $N=M=91$.

Figure 20: A blow-up of the spectrum near imaginary axis, SBP84, $N=M=91$. 

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5. Extension to geometries with multiple holes

One can readily extend the techniques presented in section 3 and 4, to construct SBP operators and stable schemes for geometries with multiple holes, see Figure 10. In this example, we partition $\Omega$ along the $x$ and $y$ axes, as seen in Figure 22. The partitioning along the $x$ axis is done such that in each partition, only one difference operator approximates the $y$ derivative. The partitioning along $y$ axis is analogous.

The difference operators in the $y$ and $x$ directions are defined using tensor products in the following way

$$
D_y = \sum_{i=1}^{5} (I_{\mathcal{N}_x}^{N_i} \otimes D_y^i), \quad D_x = \sum_{j=1}^{5} (D_x^j \otimes I_{\mathcal{M}_y}^{M_j}).
$$

(33)

In (33), the matrices $I_{\mathcal{N}_x}^{N_i}$ and $I_{\mathcal{M}_y}^{M_j}$ are defined to single out the different segments on the $x$ and $y$ axes over which one partition is defined. As an example, $I_{\mathcal{N}_x}^{N_i}$ is a $N \times N$ matrix ($N = \sum_{i=1}^{5} N_i$) which has elements equal to one on the main diagonal corresponding to the $N_1$ grid points shown in Figure 23. Other matrices are defined.
similarly. Moreover, in (33), we have used

\[ D_1^y = (P_y^M)^{-1} Q_y^M, \]

\[ D_2^y = \begin{bmatrix} (P_y^{M_1+M_2+M_3})^{-1} Q_y^{M_1+M_2+M_3} & 0^{M_4} \\ (P_y^{M_5})^{-1} Q_y^{M_5} & 0^{M_4} \end{bmatrix}, \]

\[ D_3^y = \begin{bmatrix} (P_y^{M_1})^{-1} Q_y^{M_1} & 0^{M_2} \\ 0^{M_2} & (P_y^{M_5})^{-1} Q_y^{M_5} \end{bmatrix}, \]

(34)

\[ D_4^y = \begin{bmatrix} (P_y^{M_1})^{-1} Q_y^{M_1} & 0^{M_2} \\ 0^{M_2} & (P_y^{M_5+M_4+M_5})^{-1} Q_y^{M_5+M_4+M_5} \end{bmatrix}, \]

\[ D_5^y = D_1^y = (P_y^M)^{-1} Q_y^M. \]
Similarly we have,

\[
D^1_x = (p_x^N)^{-1} Q_x^N,
\]

\[
D^2_x = \begin{bmatrix}
(p_x^{N_1+N_2})^{-1} Q_x^{N_1+N_2} & 0^{N_3+N_4} \\
0^{N_2+N_3} & (p_x^{N_5})^{-1} Q_x^{N_5}
\end{bmatrix},
\]

\[
D^3_x = D^1_x = (p_x^N)^{-1} Q_x^N,
\]

\[
D^4_x = \begin{bmatrix}
(p_x^{N_1})^{-1} Q_x^{N_1} & 0^{N_2+N_3} \\
0^{N_2+N_3} & (p_x^{N_4+N_5})^{-1} Q_x^{N_4+N_5}
\end{bmatrix},
\]

\[
D^5_x = D^1_x = (p_x^N)^{-1} Q_x^N.
\]

The SBP-SAT scheme is similar but more technically involved than the ones in section 3.
6. Numerical experiments

We consider the two-dimensional constant coefficient symmetrized Euler equations [25]

\[ U_t + \hat{A} U_x + \hat{B} U_y = 0, \quad (x, y) \in \Omega, \quad t \in [0, T], \] (36)

where \( U = [\hat{\rho} \sqrt{\gamma}, \ u, \ v, \ \theta/\sqrt{\gamma(\gamma - 1)}]^T \). In (36), \( \rho, \ u, \ v, \ \theta, \) and \( \gamma \) are the density, the \( x \) and \( y \) velocity components, the temperature and the ratio of specific heats, respectively. An equation of state of the form \( \gamma p = \hat{\rho} \theta + \rho \hat{\theta} \), where \( p \) is the pressure, closes the system (36). Moreover, the bar sign denotes the state around which we have linearized. The matrices in (36) are

\[
\hat{A} = \begin{bmatrix}
\hat{u} & \hat{c}/\sqrt{\gamma} & 0 & 0 \\
\hat{c}/\sqrt{\gamma} & \hat{u} & 0 & \sqrt{\gamma^{-1}} \hat{c} \\
0 & 0 & \hat{u} & 0 \\
0 & \sqrt{\gamma^{-1}} \hat{c} & 0 & \hat{u}
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
\hat{v} & 0 & \hat{c}/\sqrt{\gamma} & 0 \\
0 & \hat{v} & 0 & 0 \\
0 & \sqrt{\gamma^{-1}} \hat{c} & \hat{v} & 0 \\
0 & 0 & \sqrt{\gamma^{-1}} \hat{c} & \hat{v}
\end{bmatrix}. \] (37)

In the remainder of this paper, we use \( \gamma = 1.4, \ \hat{c} = 2 \) and \( \hat{\rho} = 1 \).

6.1. Accuracy

To verify the order of accuracy of the scheme we use the domain in Figure 1 where \( \Phi = [0, 1] \times [0, 1] \) and \( H = [1/3, 2/3] \times [1/3, 2/3] \). We prescribe the mean velocity field to be \((\bar{u}, \bar{v}) = (1, 1)\), and use the manufactured solution

\[ U_\infty = [\sin(x-t), \ \cos(x-t), \ \sin(y-t), \ \cos(y-t)]^T \] (38)

for the forcing function, initial and boundary data in (36). Moreover, characteristic boundary conditions [9] are used.

We examine the scheme for SBP operators of order \( 2s \) in the interior and \( s \) close to the boundaries in space, for \( s \in \{1, 2, 3, 4\} \). The fifth order accurate SBP operator, with sufficiently large \( L \) to minimize the time error, is used in time. The rates of convergence are shown in Figure 24. According to [12, 13, 14, 15, 16], for a scheme with first derivative SBP operators which are \( 2s \)-order accurate in the interior, and \( s \)-order accurate close to the boundaries (where diagonal norms are used), should yield \( s + 1 \) order of accuracy globally. The results in Figure 24 converge at these rates.

6.2. Comparison with the multi-block technique

In the standard SBP-SAT multi-block approach, the domain is divided into sub-domains and interface penalties are used to couple the blocks. A schematic
The error
SBP21
s=2
SBP42
s=3
SBP63
s=4
SBP84
s=5

Figure 24: Mesh refinement, the errors and the convergence rates.

Figure 25: A schematic of the multi-domains and interfaces.

of such multi-block division including the interfaces (dashed lines) is shown in Figure 25.

One drawback with the standard multi-block schemes is that close to the interfaces the accuracy of the approximation is reduced from $2s$ to $s + 1$. In order to compare this approach with our new one, we use the domain in Figure 1. To compare these schemes we consider the manufactured solution

$$U_{\infty} = [0, 0, 0, e^{-10((x-t)^2+(y-t)^2)}]T.$$  \hspace{1cm} (39)

The resulting solution is a pressure pulse that starts from $(x, y) = (0, 0)$ and travels to $(x, y) = (1, 1)$ as time passes.

For low order operators and coarse meshes the influence of the interfaces is visible on the solution, as seen in Figures 26-29 where the pressure distribution for different times are shown. We have used $SBP21$ and a coarse grid of size $19 \times 19$ in space together with a fifth order SBP operator (with sufficiently small time steps) in time. The interfaces are clearly visible in the multi-block approach while in our approach they are not. For finer grids and higher order operators, it is more instructive to consider the error plots. For a mesh of size $61 \times 61$ and $SBP42$ in space, together with a sufficiently accurate $SBP84$ in time, the errors at different times are presented in Figures 30-37.

To quantitatively compare the effects of the interfaces on accuracy, the error is integrated numerically over the spatial domain. The integrated errors for the two approaches when using $SBP21$ and $SBP42$ in space, are shown in Figure 38 and
The pressure distribution at $t = 0.400$

Figure 26: The standard multi-block.

The pressure distribution at $t = 0.400$

Figure 27: The new approach.

The pressure distribution at $t = 0.700$

Figure 28: The standard multi-block.

The pressure distribution at $t = 0.200$

Figure 29: The new approach.

The error at $t = 0.020$

Figure 30: The error, standard multi-block.

The error at $t = 0.020$

Figure 31: The error, new approach.

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The error at $t = 0.070$

Figure 32: The error, standard multi-block.

Figure 33: The error, new approach.

The error at $t = 0.190$

Figure 34: The error, standard multi-block.

Figure 35: The error, new approach.

The error at $t = 0.213$

Figure 36: The error, standard multi-block.

Figure 37: The error, new approach.
40. Additionally, we show the resulting CPU time for both methods in Figures 39 and 41. As can be seen, the gain from the new approach is two-fold; we obtain lower error levels in the solution while spending less CPU time (the gain in CPU time for the 42 case is minimal). The convergence rates are of course identical.

6.3. An application

As a final application, we consider a more complex geometry where no-penetration boundary conditions are imposed on the inner solid walls. At the outer boundaries, characteristic boundary conditions [9] with data from the manufactured solution

\[ U_\infty = [0, 0, 0, e^{-20((x-t)^2+(y-t)^2)}]^T \]  \hspace{1cm} (40)

are imposed.
A mesh of size $51 \times 51$ grid points in space and 201 nodes in time is constructed. Third and fifth order accurate SBP operators in space and time, respectively, are used. The pressure distribution and the velocity field at different times are shown in Figures 42-49. The resulting flow is tangential to the solid inner boundaries and the pressure pulse moves out of the domain as time passes.

7. Conclusions

We have constructed a new combination of summation-by-parts operators that is readily applicable to a variety of partial differential equations posed on non-simply connected spatial domains. To develop the theory, we considered a two dimensional constant coefficient advection equation posed on a non-simply con-
The pressure distribution at $t = 0.225 \times 10^{-3}$

Figure 46: The pressure distribution.

The pressure distribution at $t = 0.315 \times 10^{-3}$

Figure 48: The pressure distribution.

The velocity field at $t = 0.225$

Figure 47: The velocity field.

The velocity field at $t = 0.315$

Figure 49: The velocity field.
nected spatial domain and constructed an accurate and efficient scheme. Furthermore, we extended the new approach to a more complex non-simply connected geometries. Although, no proof of energy stability was obtained, correctly located spectra in combination with SBP in time indicates that the scheme is stable.

In the numerical experiments, we applied the new formulations to the linearized Euler equations. We showed that the new formulation is design order accurate by using the method of manufactured solutions. Additionally, we compared the error levels and CPU time of the new approach with the standard multi-block technique. We conclude that the new method is more accurate and efficient compared with the standard multi-block technique. An application on a more complex geometry was also presented.
Appendix A.

In (26) we computed the matrices as

\[
PD_t = \left[ P_t^L \otimes (P_1 + P_2 + P_3) \right] \left[ (P_t^L)^{-1} Q_t^L \otimes \left( [I_x^{\Omega} \otimes I_y^{\Omega}] + [I_x^H \otimes I_y^{\Omega}] + [I_x^{\Omega} \otimes I_y^H] \right) \right]
\]

\[
= Q_t^L \otimes \left( P_1 [I_x^{\Omega} \otimes I_y^{\Omega}] + P_2 [I_x^H \otimes I_y^{\Omega}] + P_3 [I_x^{\Omega} \otimes I_y^H] \right) = Q_t^L \otimes \left( P_1 + P_2 + P_3 \right),
\]

(41)

\[
PD_\chi = \left[ P_t^L \otimes (P_1 + P_2 + P_3) \right] \left[ I_t \otimes \left( [(P_x^N)^{-1} Q_x^{N} \otimes I_y^{\Omega}] + [(P_x^{\chi})^{-1} \tilde{Q}_x \otimes I_y^H] \right) \right]
\]

\[
= P_t^L \otimes \left( (P_1 + P_2) [(P_x^N)^{-1} Q_x^{N} \otimes I_y^{\Omega}] + P_3 [(P_x^{\chi})^{-1} \tilde{Q}_x \otimes I_y^H] \right)
\]

\[
= P_t^L \otimes \left( [I_x^{\Omega} Q_x^{N} \otimes I_y^{\Omega} P_y^M] + [I_x^H Q_x^{N} \otimes I_y^{\Omega} \tilde{P}_y] + [I_x^{\Omega} \tilde{Q}_x \otimes I_y^H P_y^M] \right),
\]

(42)

and

\[
PD_y = \left[ P_t^L \otimes (P_1 + P_2 + P_3) \right] \left[ I_t \otimes \left( [I_x^{\Omega} \otimes (P_y^M)^{-1} Q_y^M] + [I_x^H \otimes (\tilde{P}_y)^{-1} \tilde{Q}_y] \right) \right]
\]

\[
= P_t^L \otimes \left( (P_1 + P_3) [I_x^{\Omega} \otimes (P_y^M)^{-1} Q_y^M] + P_2 [I_x^H \otimes (\tilde{P}_y)^{-1} \tilde{Q}_y] \right)
\]

\[
= P_t^L \otimes \left( [I_x^{\Omega} P_x^{N} \otimes I_y^{\Omega} Q_y^M] + [I_x^{\Omega} \tilde{P}_x \otimes I_y^H Q_y^M] + [I_x^H P_x^{N} \otimes I_y^{\Omega} \tilde{Q}_y] \right).
\]

(43)

In (28) we used

\[
PP_t^{-1} = \left[ P_t^L \otimes (P_1 + P_2 + P_3) \right] \left[ (P_t^L)^{-1} E_0^L \otimes \left( [I_x^{\Omega} \otimes I_y^{\Omega}] + [I_x^H \otimes I_y^{\Omega}] + [I_x^{\Omega} \otimes I_y^H] \right) \right]
\]

\[
= E_0^L \otimes \left( P_1 [I_x^{\Omega} \otimes I_y^{\Omega}] + P_2 [I_x^H \otimes I_y^{\Omega}] + P_3 [I_x^{\Omega} \otimes I_y^H] \right) = E_0^L \otimes \left( P_1 + P_2 + P_3 \right),
\]

(44)
\[ PP_-^{1} = \left[ P_L^t \otimes \left( P_1 + P_2 + P_3 \right) \right] \left[ I_t \otimes \left( [I_x^\Omega \otimes (P_y^M)^{-1}E_0^M] + [I_x^H \otimes (\tilde{P}_x)^{-1}\tilde{E}_0] \right) \right] \]

\[ = P_L^t \otimes \left( (P_1 + P_3)\left[ I_x^\Omega \otimes (P_y^M)^{-1}E_0^M \right] + P_2\left[ I_x^H \otimes (\tilde{P}_x)^{-1}\tilde{E}_0] \right) \right) = \]

\[ = P_L^t \otimes \left( [I_x^\Omega P_N^N \otimes I_y^M] + [I_x^H P_N^N \otimes I_y^M \tilde{E}_0] \right), \quad (45) \]

\[ PP_-^{1} = \left[ P_L^t \otimes \left( P_1 + P_2 + P_3 \right) \right] \left[ I_t \otimes I_x^H \otimes (\tilde{P}_x)^{-1}\tilde{E}_0] \right] \]

\[ = P_L^t \otimes P_2\left[ I_x^H \otimes (\tilde{P}_x)^{-1}\tilde{E}_0] \right] = P_L^t \otimes [I_x^H P_N^N \otimes I_y^M \tilde{E}_0], \quad (46) \]

\[ PP_-^{1} = \left[ P_L^t \otimes \left( P_1 + P_2 + P_3 \right) \right] \left[ I_t \otimes \left( [(P_N^N)^{-1}E_0^N \otimes I_y^\Omega] + [(\tilde{P}_x)^{-1}\tilde{E}_0] \otimes I_t^H] \right) \]

\[ = P_L^t \otimes \left( (P_1 + P_2)\left[ (P_N^N)^{-1}E_0^N \otimes I_y^\Omega \right] + P_3\left[ (\tilde{P}_x)^{-1}\tilde{E}_0] \otimes I_t^H] \right) \]

\[ = P_L^t \otimes \left( [I_x^\Omega E_0^N \otimes I_y^\Omega P_N^M] + [I_x^\Omega \tilde{E}_0] \otimes I_t^H P_N^M] \right), \quad (47) \]

and finally

\[ PP_-^{1} = \left[ P_L^t \otimes \left( P_1 + P_2 + P_3 \right) \right] \left[ I_t \otimes \left( (\tilde{P}_x)^{-1}\tilde{E}_0] \otimes I_t^H \right) \right] \]

\[ = P_L^t \otimes P_3\left[ (\tilde{P}_x)^{-1}\tilde{E}_0] \otimes I_t^H \right] = P_L^t \otimes I_x^\Omega \tilde{E}_0] \otimes I_t^H P_N^M]. \quad (48) \]
Appendix B.

By substituting (26) and (28) into (25) one obtains

\[ v^T \left( Q_t^L \otimes (P_1+P_2+P_3) \right) v + \alpha v^T \left[ P_t^L \otimes \left( I_x^O Q_x^N \otimes I_y^O P_y^M + I_x^H Q_x^N \otimes I_y^H P_y^M + I_x^O \tilde{Q}_x \otimes I_y^H P_y^M \right) \right] v + \beta v^T \left[ P_t^L \otimes \left( I_x^O P_x^N \otimes I_y^O Q_y^M + I_x^H P_x^N \otimes I_y^H \tilde{Q}_y \otimes I_y^H P_y^M \right) \right] v = \]

\[ \sigma_i v^T \left( E_0^L \otimes P^{\Omega} \right) v_i + \sigma_x v^T \left( P_t^L \otimes I_x^H P_x^N \otimes I_y^H x \right) v_c + \]

\[ \sigma_c v^T \left[ P_t^L \otimes \left( I_x^N P_x^N \otimes I_y^N E_y^0 + P_y^N I_x^H \tilde{Q}_y + P_y^0 \tilde{E}_y \right) \right] v_c + \]

\[ \sigma_d v^T \left[ P_t^L \otimes \left( I_x^N \tilde{E}_0 \otimes I_y^H P_y^M \right) \right] v_d + \]

\[ \sigma_D v^T \left[ P_t^L \otimes \left( I_x^N E_y^0 \otimes I_y^H P_y^M + I_x^H E_y^0 \otimes I_y^H \tilde{Q}_y + I_x^0 \tilde{E}_y \right) \right] v_D. \]

Next we add the transpose of (49) to itself. The result is

\[ v^T \left( B_t^L \otimes (P_1+P_2+P_3) \right) v + \alpha v^T \left[ P_t^L \otimes \left( I_x^O B_x^N \otimes I_y^O P_y^M + I_x^H B_x^N \otimes I_y^H \tilde{Q}_y + I_x^O \tilde{B}_x \otimes I_y^H P_y^M \right) \right] v + \]

\[ \beta v^T \left[ P_t^L \otimes \left( I_x^O P_x^N \otimes I_y^O B_x^M + I_x^H P_x^N \otimes I_y^H \tilde{B}_x + I_x^O \tilde{P}_x \otimes I_y^H B_x^M \right) \right] v = \]

\[ 2\sigma_i v^T \left( E_0^L \otimes P^{\Omega} \right) v_i + 2\sigma_x v^T \left( P_t^L \otimes I_x^H P_x^N \otimes I_y^H x \right) v_c + \]

\[ 2\sigma_c v^T \left[ P_t^L \otimes \left( I_x^N P_x^N \otimes I_y^N E_y^0 + P_y^N I_x^H \tilde{Q}_y + P_y^0 \tilde{E}_y \right) \right] v_c + \]

\[ 2\sigma_d v^T \left[ P_t^L \otimes I_x^0 \tilde{E}_0 \otimes I_y^H P_y^M \right] v_d + \]

\[ 2\sigma_D v^T \left[ P_t^L \otimes \left( I_x^N E_y^0 \otimes I_y^H P_y^M + I_x^H E_y^0 \otimes I_y^H \tilde{Q}_y + I_x^0 \tilde{E}_y \right) \right] v_D + CT, \]

where \( \tilde{B}_{x,y} = \tilde{Q}_{x,y} + \tilde{Q}_{x,y}^T. \) If we only consider the \( s, w \) boundaries, (29) is obtained.
References


