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Armen Asratian and Carl Johan Casselgren

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Solution of Vizing's Problem on Interchanges for the case of Graphs with Maximum Degree 4 and Related Results

Armen S. Asratian*

Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden

Carl Johan Casselgren†

Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden

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Abstract. Let G be a Class 1 graph with maximum degree 4 and let $t \geq 5$ be an integer. We show that any proper t -edge coloring of G can be transformed to any proper 4-edge coloring of G using only transformations on 2-colored subgraphs (so-called interchanges). This settles the smallest previously unsolved case of a well-known problem of Vizing on interchanges, posed in 1965. Using our result we give an affirmative answer to a question of Mohar for two classes of graphs: we show that all proper 5-edge colorings of a Class 1 graph with maximum degree 4 are Kempe equivalent, that is, can be transformed to each other by interchanges, and that all proper 7-edge colorings of a Class 2 graph with maximum degree 5 are Kempe equivalent.

1 Introduction

We consider finite graphs without loops and multiple edges. A proper t -edge coloring of a graph G is a mapping $f : E(G) \rightarrow \{1, \dots, t\}$ such that $f(e) \neq f(e')$ for every pair of adjacent edges e and e' in G . If $e \in E(G)$ and $f(e) = k$ then we say that the edge e is *colored k under f* . The set of edges colored k under f we denote by $M(f, k)$. We denote by $\Delta(G)$ the maximum degree of vertices of a graph G , and by $d_G(v)$ the degree of a vertex v of G . The *chromatic index* $\chi'(G)$ of a graph G is the minimum number k for which there exists a proper k -edge coloring of G . For a proper t -edge coloring f of G and any two colors $a, b \in \{1, \dots, t\}$ we denote by $G_f(a, b)$ the subgraph induced by the set $M(f, a) \cup M(f, b)$. By switching colors a and b on a connected component of $G_f(a, b)$, we obtain another proper t -edge coloring of G . This operation is called an *interchange* or a *Kempe change*. In the following, if a t -edge coloring is transformed to a t' -edge coloring by a sequence of interchanges and $t' \leq t$, then

*E-mail address: armen.asratian@liu.se

†E-mail address: carl.johan.casselgren@liu.se Part of the work done while the author was a postdoc at University of Southern Denmark. Research supported by SVEFUM.

we only allow colors $1, 2, \dots, t$ to be involved in the interchanges. Two t -edge colorings f, g are *Kempe equivalent* if g can be transformed to f by a sequence of interchanges.

Interchanges play a key role in investigations on edge colorings. The proofs of many results in this area are based on transformations of one proper edge coloring of a graph G to another using interchanges. For example, Vizing's theorem on the chromatic index [9] can be reformulated, taking into consideration the proof of it, in the following way.

Theorem A. For every graph G , $\chi'(G) \leq \Delta(G) + 1$. Moreover every proper t -edge coloring of G , where $t \geq \Delta(G) + 2$, can be transformed to a proper $(\Delta(G) + 1)$ -edge coloring of G by a sequence of interchanges.

Vizing's result implies that for any graph G , $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. In the former case G is said to be *Class 1*, and in the latter G is *Class 2*. For a Class 2 graph G the result of Vizing means that any proper edge coloring of G can be transformed to a proper $\chi'(G)$ -edge coloring by using interchanges only. The following problem posed by Vizing [10, 11] (see also [8, 5]) for Class 1 graphs is still open:

Problem 1. *Is it true that any Class 1 graph G satisfies the following property: every proper t -edge coloring of G , $t \geq \Delta(G) + 1$, can be transformed to a proper $\Delta(G)$ -edge coloring of G by a sequence of interchanges? (This property we shall call the Vizing property.)*

Mohar [7] used the edge coloring algorithm suggested by Vizing [10] to prove that for a graph G with a proper t -edge coloring f , every proper $\chi'(G)$ -edge coloring can be obtained from f by a sequence of interchanges, provided that $t \geq \chi'(G) + 2$, i.e. he solved Problem 1 for the case when $t \geq \Delta(G) + 2$. This means that all proper t -edge colorings of G are Kempe equivalent if $t \geq \chi'(G) + 2$. Mohar also posed the following:

Problem 2. *Is it true that all proper $(\chi'(G) + 1)$ -edge colorings of a graph G are Kempe equivalent?*

For graphs G with $\Delta(G) \leq 3$, Problem 2 was resolved to the positive by McDonald et al. [6]. In particular they proved the following:

Theorem B [6]. Let G be a graph with $\Delta(G) \leq 3$. Then all proper $(\Delta(G) + 1)$ -edge colorings of G are Kempe equivalent.

In the general case the situation is different. It is shown in [6] that if $\chi'(G)$ is replaced by $\Delta(G)$ in Problem 2, then we get a problem which in general has a negative answer.

Note that it follows from Theorems A and B that all 5-edge colorings of a graph with maximum degree 3 are Kempe equivalent; so Theorem B combined with Theorem A settles Problem 2 for graphs with maximum degree 3. McDonald et al. [6] obtained also an affirmative answer to Problem 2 for some Class 2 graphs:

Theorem C [6]. Let G be a graph with $\Delta(G) = 4$ and $\chi'(G) = 5$. Then all proper 6-edge colorings of G are Kempe equivalent.

Our first result establishes a connection between Problem 1 and Problem 2.

Theorem 1.1. *Let $k \geq 2$ be an integer. Then all Class 1 graphs with maximum degree at most k have the Vizing property if and only if for every graph G with $\Delta(G) \leq k$ all proper $(\chi'(G) + 1)$ -edge colorings are Kempe equivalent.*

Our second result establishes a connection between the solution of Problem 2 for graphs with maximum degree $k - 1$ and Class 2 graphs with maximum degree k .

Theorem 1.2. *Let $k \geq 4$ be an integer. If for every graph H with $\Delta(H) = k - 1$ all proper $(\chi'(H) + 1)$ -edge colorings are Kempe equivalent, then for every Class 2 graph G with maximum degree k all proper $(k + 2)$ -edge colorings are Kempe equivalent.*

In the case $k = 4$, Theorem 1.2 actually implies that Theorem C follows from Theorem B.

The main result of this paper is the following theorem which gives a positive answer to the smallest previously unsolved case of Problem 1.

Theorem 1.3. *Let G be a Class 1 graph with $\Delta(G) = 4$ and t be an integer, $t \geq 5$. Then every proper t -edge coloring of G can be transformed to any proper 4-edge coloring of G by a sequence of interchanges.*

By Theorem 1.3, any two proper 5-edge colorings of a Class 1 graph G with $\Delta(G) = 4$ can be transformed by interchanges to the same proper 4-edge coloring of G . This implies the next result, which gives a positive answer to the smallest previously unsolved case of Problem 2.

Corollary 1.4. *If G is a Class 1 graph with $\Delta(G) = 4$, then all proper 5-edge colorings of G are Kempe equivalent.*

Theorem 1.2, Theorem C and Corollary 1.4 imply the following result:

Corollary 1.5. *Let G be a Class 2 graph with $\Delta(G) = 5$. Then all proper 7-edge colorings of G are Kempe equivalent.*

Note that Corollary 1.4, Theorem B, and Theorem 1.2 imply that Problem 2 has an affirmative answer for all graphs with maximum degree at most 4. Additionally, Corollary 1.5 settles the question for Class 2 graphs with maximum degree 5, so the smallest unsolved case is Class 1 graphs of maximum degree 5.

Next, we describe some Class 1 graphs G with $\Delta(G) \geq 5$ for which all proper $(\chi'(G) + 1)$ -edge colorings are Kempe equivalent. Denote by $G_{\geq 5}$ the subgraph of G induced by the set of vertices with degree at least 5.

Theorem 1.6. *For every graph G with $\Delta(G) \geq 5$ where the subgraph $G_{\geq 5}$ is acyclic, all proper $(\chi'(G) + 1)$ -edge colorings of G are Kempe equivalent.*

By Corollary 1.4, all proper 5-edge colorings of a Class 1 graph with maximum degree 4 are Kempe equivalent. Let us now briefly consider the problem of transforming a proper 4-edge coloring of such a graph G to another proper 4-edge coloring of G . The next example shows that there are such graphs with two proper 4-edge colorings f and g such that f cannot

Figure 1: A graph with the edge colorings f (to the left) and g (to the right).

be transformed to g even if, in addition to interchanges, one is allowed to use transformations on 3-edge colored subgraphs.

Consider the graph G in Figure 1 with two different proper 4-edge colorings f (to the left) and g (to the right). Clearly, g cannot be obtained from f by renaming the colors. On the other hand any proper 3-edge coloring of a subgraph $G(t_1, t_2, t_3)$ induced by the set of edges $M(f, t_1) \cup M(f, t_2) \cup M(f, t_3)$ gives the same partition of edges of $G(t_1, t_2, t_3)$ into perfect matchings, for any $1 \leq t_1 < t_2 < t_3 \leq 5$. This follows from the fact that $G(t_1, t_2, t_3)$ contains a triangle and edges of this triangle belong to disjoint, uniquely defined perfect matchings. Therefore g cannot be obtained from f even by transformations on 2-edge colored and 3-edge colored subgraphs.

For a bipartite graph G with $\Delta(G) = 4$, any two proper 4-edge colorings can always be transformed into each other by recoloring 2-edge colored and 3-edge colored subgraphs. This follows from the result of Asratian and Mirumian [2] (see also [1]) that if G is a bipartite graph with $\Delta(G) \geq 4$, then any proper $\Delta(G)$ -edge coloring of G can be transformed to any other proper $\Delta(G)$ -edge coloring of G in such a way that each intermediate coloring is a proper $\Delta(G)$ -edge coloring and differs from the previous coloring by a 2- or 3-edge colored subgraph.

2 Proofs of Theorem 1.1 and Theorem 1.2

In this section we prove Theorem 1.1 and Theorem 1.2.

First we introduce some terminology and notation. Let φ be a proper t -edge coloring of G . For a vertex $v \in V(G)$, we say that a color i *appears at v under φ* if there is an edge e incident to v with $\varphi(e) = i$, and we set

$$\varphi(v) = \{\varphi(e) : e \in E(G) \text{ and } e \text{ is incident to } v\}.$$

If φ is a proper t -edge coloring of G and $1 \leq a, b \leq t$, then a path or cycle in $G_\varphi(a, b)$ is called *(a, b) -colored under φ* . We also say that such a path or cycle is *bicolored under φ* . A φ -*fan* at a vertex $v \in V(G)$ is a sequence $(e_1, e_2, \dots, e_n)$ of edges such that

- (i) the edges $e_1, e_2, \dots, e_n$ are distinct and incident to v ,

- (ii) if $e_i = vu_i$, then $\varphi(e_i)$ does not appear at u_{i-1} , $i = 2, \dots, n$.

For such a φ -fan, we say that color $\varphi(e_i)$ is *associated with* u_{i-1} . A φ -fan $(e_1, e_2, \dots, e_k)$ at v with $e_k = vu_k$ is *saturated* if there is a color c such that $c \notin \varphi(v) \cup \varphi(u_k)$. For such a saturated φ -fan $(e_1, e_2, \dots, e_k)$ we define a *downshift* as the following operation: define a new coloring φ' from φ by setting $\varphi'(e_k) = c$, $\varphi'(e_j) = \varphi(e_{j+1})$, $j = 1, \dots, k-1$, and retaining the color of every other edge of G . Note that a downshift produces a new proper edge coloring and that it may be seen as a sequence of interchanges.

In all the above definitions, we often leave out the explicit reference to a coloring φ , if the coloring is clear from the context.

Now we present a family of graphs permitting an affirmative answer to Problem 1.

Denote by G_Δ the graph induced by the vertices of maximum degree in a graph G . It is known that if G_Δ is acyclic, then G is a Class 1 graph [4]. In fact, Fournier [4] (see also [3]) described an algorithm for sequentially coloring the edges properly of such a graph using $\Delta(G)$ colors. This result also follows from Vizing's well-known adjacency lemma, see e.g. [5], p. 196. Here, we use a similar algorithm to prove the following proposition:

Proposition 2.1. *Let G be a graph where G_Δ is acyclic. Then any proper t -edge coloring of G with $t > \Delta(G)$ can be transformed into a proper $\Delta(G)$ -edge coloring by a sequence of interchanges.*

Proof. The major parts of the proof is similar to the proof of Vizing's theorem on the chromatic index. Let G be a graph with maximum degree Δ such that G_Δ is acyclic, t an integer satisfying $t > \Delta$, and suppose that f is a proper t -edge coloring of G .

Taking Theorem A into consideration, it is sufficient to prove the proposition in the case $t = \Delta + 1$.

Suppose that there is at least one edge e colored $\Delta + 1$ in G . We will show that using only interchanges we can transform f to another proper $(\Delta + 1)$ -edge coloring with fewer edges colored $\Delta + 1$.

Case A. There is an edge $e = vu_1 \in E(G_\Delta)$ with $f(e) = \Delta + 1$ and $d_{G_\Delta}(v) = 1$.

Since every neighbor $x \neq u_1$ of v has degree at most $\Delta - 1$, there is a maximal fan $F = (e_1, e_2, \dots, e_k)$ at v with $e_i = vu_i$ and with no vertex in $\{u_1, \dots, u_k\}$ being associated with color $\Delta + 1$. Suppose that $f(e_i) = c_i$, $i = 1, \dots, k$, where $c_1 = \Delta + 1$, and that color $c_{k+1} \neq \Delta + 1$ does not appear at u_k .

Since F is maximal, either

- (i) there is no edge incident to v colored c_{k+1} , or
- (ii) $c_{k+1} = f(e_{j+1})$ for some j , $1 \leq j < k-1$, that is, $c_{k+1} = c_{j+1}$ and c_{k+1} does not appear at u_j .

In the case (i) we downshift on F , and since no vertex in $\{u_1, \dots, u_k\}$ is associated with color $\Delta + 1$, we obtain a proper $(\Delta + 1)$ -edge coloring with fewer edges colored $\Delta + 1$.

Suppose now that (ii) holds. Choose a color c_0 that does not appear at v . If c_0 does not appear also at u_k , then we replace c_{k+1} with c_0 and proceed as in the case (i). Otherwise, we consider a maximal (c_0, c_{k+1}) -colored path Q with origin at u_k .

If u_{j+1} is on Q , then v is an endpoint of Q . We interchange the colors on Q and denote the obtained coloring by f_1 . The color c_{k+1} does not appear now at vertices v and u_j , so we may downshift on the saturated f_1 -fan $(e_1, e_2, \dots, e_j)$ to obtain the desired result.

If u_j is on Q , then this is an endpoint of Q . We interchange the colors on Q and obtain a coloring f_1 where the color c_0 does not appear at vertices v and u_j . Now we downshift on the saturated f_1 -fan $(e_1, \dots, e_j)$.

If $\{u_{j-1}, u_j\} \cap V(Q) = \emptyset$, then we interchange colors on Q and then downshift on $(e_1, \dots, e_k)$.

Case B. There is no edge $e = uv \in E(G_\Delta)$ of color $\Delta + 1$ with $d_{G_\Delta}(u) = 1$ or $d_{G_\Delta}(v) = 1$.

The following cases are possible:

Case B.1 $E(G_\Delta) \cap M(f, \Delta + 1) \neq \emptyset$.

Let $e = uv \in E(G_\Delta) \cap M(f, \Delta + 1)$. We will describe an algorithm for constructing a sequence of distinct vertices $a_0, a_1, \dots, a_n$ in G such that $a_i \in V(G_\Delta)$, $i = 0, \dots, n$, and a_i and a_{i+1} are adjacent in G . The algorithm will also construct a related sequence $g_1, \dots, g_{n-1}$ of proper $(\Delta + 1)$ -edge colorings of G , such that the edge $a_i a_{i+1}$ is colored $\Delta + 1$ under g_i . Additionally $|M(g_i, \Delta + 1)| \leq |M(f, \Delta + 1)|$ for each $i \in \{1, \dots, n - 1\}$.

Algorithm

Step 1: We first set $g_1 = f$ and $a_0 = u$ and $a_1 = v$.

Step i ($i \geq 2$): Suppose that we have constructed the sequence $a_0, a_1, \dots, a_{i-1}$ of distinct vertices in G , such that a_{j-1} and a_j are adjacent, $j = 1, \dots, i - 1$, and a sequence of proper $(\Delta + 1)$ -edge colorings $g_1, \dots, g_{i-1}$, satisfying $|M(g_j, \Delta + 1)| \leq |M(f, \Delta + 1)|$ and $g_j(a_{j-1}a_j) = \Delta + 1$, $j = 1, \dots, i - 1$.

If $d_{G_\Delta}(a_{i-1}) = 1$, then Stop. The coloring g_{i-1} satisfies the conditions in Case A and we can transform g_{i-1} to a proper $(\Delta + 1)$ -edge coloring with fewer edges colored $\Delta + 1$.

If $d_{G_\Delta}(a_{i-1}) > 1$, we consider a maximal g_{i-1} -fan $F = (e_1, e_2, \dots, e_k)$ at a_{i-1} with $e_1 = a_{i-1}a_{i-2}$ and $e_j = a_{i-1}u_j$, for $j = 2, \dots, k$. If there is some color $c_{k+1} \neq \Delta + 1$ missing at u_k under g_{i-1} , then by proceeding similarly as in Case A we may transform g_{i-1} to a proper $(\Delta + 1)$ -edge coloring α using only interchanges, where α satisfies $|M(\alpha, \Delta + 1)| < |M(f, \Delta + 1)|$.

So suppose that $\Delta + 1$ is the only color that does not appear at u_k under g_{i-1} . Note that this implies that $u_k \in V(G_\Delta)$. Additionally, u_k is distinct from $a_0, a_1, \dots, a_{i-1}$, because $u_k \neq a_{i-2}$ and G_Δ is acyclic.

We will now make a series of interchanges to obtain g_i from g_{i-1} . Suppose that $g_{i-1}(e_j) = c_j$, $j = 1, \dots, k$, where $c_1 = \Delta + 1$. We first interchange colors on the maximal $(\Delta + 1, c_2)$ -colored path with origin at a_{i-2} and denote the obtained coloring by f_1 . Since the first edge of this path is colored $\Delta + 1$, $|M(f_1, \Delta + 1)| \leq |M(g_{i-1}, \Delta + 1)|$. Now, for each $j = 2, \dots, k - 1$, from f_{i-1} we

construct the coloring f_i by interchanging colors on the maximal $(\Delta + 1, c_{i+1})$ -colored path with origin at u_i . All these paths begin by an edge colored $\Delta + 1$, so we do not increase the number of edges colored $\Delta + 1$ in G by making these interchanges. Note that $f_{k-1}(a_{i-1}u_k) = \Delta + 1$. We set $g_i = f_{k-1}$ and $a_i = u_k$, and go to Step $(i + 1)$.

Since G_Δ is acyclic, the algorithm above will stop after a finite number of steps, when we reach a vertex of degree 1 in G_Δ . Moreover, the algorithm produces a proper $(\Delta + 1)$ -edge coloring g of G such that $|M(g, \Delta + 1)| \leq |M(f, \Delta + 1)|$, and there is an edge $e \in E(G_\Delta) \cap M(g, \Delta + 1)$ which is incident to a vertex of degree 1 in G_Δ . Hence, this case has been reduced to Case A, and thus we may, using only interchanges, from g construct a proper $(\Delta + 1)$ -edge coloring with fewer edges colored $\Delta + 1$.

Case B.2 $E(G_\Delta) \cap M(f, \Delta + 1) = \emptyset$ but some edge colored $\Delta + 1$ is incident to a vertex of degree Δ .

Let $e_1 = u_1v$ be an edge colored $\Delta + 1$ and suppose that $d_G(v) = \Delta$. We consider a maximal fan $F = (e_1, e_2, \dots, e_k)$, at v with $e_i = vu_i$, for $i = 1, \dots, k$. If there is a color $c \neq \Delta + 1$ missing at u_k , then we proceed as in Case A and construct a proper $(\Delta + 1)$ -edge coloring α from f via interchanges, such that $|M(\alpha, \Delta + 1)| < |M(f, \Delta + 1)|$.

So suppose $\Delta + 1$ is the only color missing at u_k under f . This implies that $u_k \in V(G_\Delta)$. We may now proceed exactly as described in Step i of the algorithm above: by making a series of interchanges on maximal bicolored paths (where $\Delta + 1$ is one of the colors on every such path), we obtain a proper $(\Delta + 1)$ -edge coloring g of G such that $|M(g, \Delta + 1)| \leq |M(f, \Delta + 1)|$ and $g(vu_k) = \Delta + 1$; that is, there is an edge colored $\Delta + 1$ under g both ends of which have degree Δ . Then the edge $e = vu_k$ satisfies the conditions of Case A or Case B.1; as before, we can construct a proper $(\Delta + 1)$ -edge coloring with fewer edges colored $\Delta + 1$ than under g .

Case B.3 $E(G_\Delta) \cap M(f, \Delta + 1) = \emptyset$ and no edge colored $\Delta + 1$ is incident to a vertex of degree Δ .

Suppose that $e_1 = u_1v$ is colored $\Delta + 1$ under f , and consider a maximal fan $F = (e_1, e_2, \dots, e_k)$ at v , where $e_i = vu_i$, for $i = 1, \dots, k$. If there is a color $c \neq \Delta + 1$ missing at u_k , then we proceed as in Case A and construct a proper $(\Delta + 1)$ -edge coloring α from f via interchanges, such that $|M(\alpha, \Delta + 1)| < |M(f, \Delta + 1)|$.

So assume that $\Delta + 1$ is the only color missing at u_k . Then $u_k \in V(G_\Delta)$ and we may now proceed as in Step i of the algorithm above for obtaining a proper $(\Delta + 1)$ -edge coloring g of G from f , such that $|M(g, \Delta + 1)| \leq |M(f, \Delta + 1)|$ and there is an edge e colored $\Delta + 1$ under g such that one end of e is u_k , and thus has degree Δ . Hence, this case has been reduced to Case B.2. This completes the proof of Proposition 2.1. $\square$

Let us now prove Theorem 1.1.

Proof of Theorem 1.1. If for every graph G with $\Delta(G) \leq k$ all proper $(\chi'(G) + 1)$ -edge colorings of G are Kempe equivalent, then, using Theorem A, it is not difficult to show that every Class 1 graph H with $\Delta(H) \leq k$ has the Vizing property.

Conversely, assume that all Class 1 graphs with maximum degree at most k have the Vizing property. We will prove that for each graph G with $\Delta(G) \leq k$ all proper $(\chi'(G) + 1)$ -edge colorings of G are Kempe equivalent. The proof is by induction on k .

The proposition is evident for $k = 2$. Suppose that the proposition is true for all integers less than k , and let G be an arbitrary graph with $\Delta(G) = k$ and $\chi'(G) = \chi'$. Consider a proper χ' -edge coloring h of G . We assume that $M(h, \chi')$ is a maximal matching of G . Otherwise we can consider instead of h another proper χ' -edge coloring which can be obtained from h by a sequence of interchanges as follows: if there is an edge xy such that the color χ' is missing at x and at y , then recolor the edge xy with color χ' . We repeat this procedure until we obtain a proper χ' -edge coloring where the edges colored χ' form a maximal matching.

In order to show that any two proper $(\chi' + 1)$ -edge colorings of G are Kempe equivalent it is sufficient to show that any proper $(\chi' + 1)$ -edge coloring of G can be transformed to h by a sequence of interchanges.

Let φ be a proper $(\chi' + 1)$ -edge coloring of G . First we transform φ by a sequence of interchanges to a proper χ' -edge coloring. In the case when G is a Class 1 graph it is possible because G has the Vizing property, and in the case when G is a Class 2 graph it is possible by Theorem A. The obtained coloring we denote by φ_1 .

Then the coloring φ_1 can be transformed to a proper $(\chi' + 1)$ -edge coloring by sequentially recoloring the edges in the matching $M(h, \chi')$ with color $\chi' + 1$. Clearly, the obtained proper $(\chi' + 1)$ -coloring f satisfies the condition $M(h, \chi') = M(f, \chi' + 1)$. Furthermore, f is obtained from φ by a sequence of interchanges.

Consider the graph $G_1 = G - M(h, \chi')$. The proper $\chi'(G)$ -edge coloring h of G induces a proper $(\chi'(G) - 1)$ -edge coloring h' of G_1 and the proper $(\chi'(G) + 1)$ -edge coloring f of G induces a proper $\chi'(G)$ -edge coloring f' of G_1 . Clearly, $\chi'(G_1) = \chi'(G) - 1$. Therefore, h' is a proper $\chi'(G_1)$ -edge coloring and f' is a proper $(\chi'(G_1) + 1)$ -edge coloring of G_1 .

If $\Delta(G_1) = k - 1$ then, by the induction hypothesis, f' and h' are Kempe equivalent. Then the coloring φ of G can be transformed to the coloring h by a sequence of interchanges.

Suppose now that $\Delta(G_1) = k$. This is possible only if G is a Class 2 graph and G_1 is a Class 1 graph, that is, if $\chi'(G) = k + 1$ and $\chi'(G_1) = k$. Since $M(h, k + 1)$ is a maximal matching in G , every two vertices of degree k in G_1 are nonadjacent. By Proposition 2.1 the coloring f' can be transformed by interchanges to a proper $(k + 1)$ -edge coloring f'' such that $M(f'', k + 1) = \emptyset$. Let g be the proper $(k + 1)$ -edge coloring obtained from f'' by sequentially recoloring the edges in the set $M(h', k)$ with color $k + 1$. Clearly, g is obtained from f'' by a sequence of interchanges and $M(g, k + 1) = M(h', k)$. Then the graph $G_2 = G_1 - M(h', k)$ has the maximum degree $k - 1$. Let h_1 be the proper $(k - 1)$ -edge coloring of G_2 induced by the matchings $M(h', 1), M(h', 2), \dots, M(h', k - 1)$ and g_1 be the proper k -edge coloring of G_2 induced by the matchings $M(g, 1), M(g, 2), \dots, M(g, k)$. By the induction hypothesis, g_1 can be transformed to h_1 by a sequence of interchanges. Then the coloring φ also can be transformed to h by a sequence of interchanges.

We have thus proved that for every graph G with $\Delta(G) = k$ all proper $(\chi'(G) + 1)$ -edge colorings of G are Kempe equivalent. The proposition follows by the principle of induction. $\square$

Next, we will prove Theorem 1.2. The proof is similar to the proof of Theorem 1.1.

Proof of Theorem 1.2. Assume that for every graph H with $\Delta(H) = k - 1$ all proper

$(\chi'(H) + 1)$ -edge colorings of H are Kempe equivalent, and let G be a Class 2 graph with $\Delta(G) = k$ and $\chi'(G) = k + 1$.

Consider a proper $(k + 1)$ -edge coloring h of G . Again as in the proof of Theorem 1.1 we may assume that $M(h, k + 1)$ is a maximal matching of G . Our goal is then to show that any proper $(k + 2)$ -edge coloring of G can be transformed to h by a sequence of interchanges.

Let φ be a proper $(k + 2)$ -edge coloring of G . By Theorem A, we can transform φ by a sequence of interchanges to a proper $(k + 1)$ -edge coloring. The obtained coloring we denote by φ_1 .

Then the coloring φ_1 can be transformed to a proper $(k + 2)$ -edge coloring by sequentially recoloring the edges in the matching $M(h, k + 1)$ with color $k + 2$. Clearly, the obtained proper $(k + 2)$ -coloring f satisfies the condition $M(h, k + 1) = M(f, k + 2)$. Furthermore, f is obtained from φ by a sequence of interchanges.

Consider the graph $G_1 = G - M(h, k + 1)$. The proper $(k + 1)$ -edge coloring h of G induces a proper k -edge coloring h' of G_1 and the proper $(k + 2)$ -edge coloring f of G induces a proper $(k + 1)$ -edge coloring f' of G_1 . Clearly, $\chi'(G_1) = k$. Therefore, h' is a proper $\chi'(G_1)$ -edge coloring and f' is a proper $(\chi'(G_1) + 1)$ -edge coloring of G_1 .

If $\Delta(G_1) = k - 1$, then G_1 is Class 2, and it follows from our assumption that f' can be transformed to h' by a sequence of interchanges. Then the coloring φ of G also can be transformed to the coloring h by a sequence of interchanges.

In the case when $\Delta(G_1) = k$, the argument is the same as in the proof of Theorem 1.1 with only one difference: instead of the induction hypothesis we use the above assumption of the theorem. The details are omitted. $\square$

3 Proof of Theorem 1.3

In this section we will prove the main result of our paper. First we prove the result for 4-regular graphs (see Theorem 3.4 below) and then, using it, for any graph with maximum degree 4.

The following three lemmas are fundamental for our proofs of Theorems 1.3 and 3.4.

Lemma 3.1. *Let G be a 4-regular graph, f be a proper 5-edge coloring of G and assume that P is a $(1, 2)$ -colored path under f in G .*

(a) *Suppose that P has length 4 and the internal vertices along P are v_1, v_2, v_3 . Assume further that at least one of the following conditions does not hold:*

- (i) $|f(v_i) \cap f(v_j) \setminus \{1, 2\}| = 1$ for $i, j \in \{1, 2, 3\}$ satisfying $i \neq j$,
- (ii) *if x_1 and x_2 are the vertices distinct from v_1 and v_3 that are adjacent to v_2 , then colors 3, 4, 5 appear at x_1 and x_2 under f .*

Then there is a proper 5-edge coloring g of G which can be obtained from f by a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$, such that one of colors 3, 4, 5 is missing either in the set $g(v_1) \cup g(v_2)$ or in $g(v_2) \cup g(v_3)$.

- (b) If P has length 5 and v_1, v_2, v_3, v_4 are the internal vertices along P , then there is a proper 5-edge coloring g of G which can be obtained from f by a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$, such that one of colors 3, 4, 5 is missing in the set $g(v_i) \cup g(v_{i+1})$, for some $i \in \{1, 2, 3\}$.

Proof. We first prove (a). By the definition of the path P , $\{1, 2\} \subseteq f(v_i)$, for $i = 1, 2, 3$. We show that if the conclusion is false, then (i) and (ii) hold. So suppose that the conclusion of the lemma is false. Then $\{3, 4, 5\} \subseteq f(v_1) \cup f(v_2)$ and $\{3, 4, 5\} \subseteq f(v_2) \cup f(v_3)$, since otherwise the conclusion follows immediately by setting $g = f$. These two properties together with the conditions $\Delta(G) = 4$ and $\{1, 2\} \subseteq f(v_i)$, for $i = 1, 2, 3$, imply that $|f(v_1) \cap f(v_2) \setminus \{1, 2\}| \leq 1$ and $|f(v_2) \cap f(v_3) \setminus \{1, 2\}| \leq 1$. Since G is 4-regular, we thus have $|f(v_1) \cap f(v_2) \setminus \{1, 2\}| = 1$ and $|f(v_2) \cap f(v_3) \setminus \{1, 2\}| = 1$.

We now prove that $|f(v_1) \cap f(v_3) \setminus \{1, 2\}| = 1$. Suppose that this is not true, that is, $|f(v_1) \cap f(v_3) \setminus \{1, 2\}| = 2$. Without loss of generality we may assume that $f(v_1) = f(v_3) = \{1, 2, 3, 5\}$. Then $f(v_2) = \{1, 2, 4, 5\}$ because $\{3, 4, 5\} \subseteq f(v_2) \cup f(v_3)$. Consider a maximal path Q that is $(3, 4)$ -colored under f and has origin at v_2 . If v_1 is not an endpoint of this path, then color 4 neither appears at v_1 nor at v_2 after an interchange on Q ; if v_1 is an endpoint of Q , then color 4 neither appears at v_2 nor at v_3 after an interchange on Q . In both cases the conclusion of the lemma holds; a contradiction. Hence, $|f(v_1) \cap f(v_3) \setminus \{1, 2\}| = 1$.

Suppose now that (i) holds but not (ii). Without loss of generality we can assume that

$$f(v_1) = \{1, 2, 3, 5\}, f(v_2) = \{1, 2, 4, 5\}, f(v_3) = \{1, 2, 3, 4\},$$

and that $f(x_1 v_2) = 5$. If color 3 does not appear at x_1 , then we may recolor the edge $v_2 x_1$ with color 3, obtaining a coloring where 5 does not appear at v_2 or v_3 ; a contradiction. Similarly, color 3 must appear at x_2 . Hence, $\{3, 5\} \subseteq f(x_1)$ and $\{3, 4\} \subseteq f(x_2)$.

Let us now prove that color 4 appears at x_1 . Consider a maximal $(3, 4)$ -colored path Q with origin v_1 . If v_2 is not an endpoint of this path, then by interchanging colors on Q we get a coloring where 3 does not appear at v_1 or v_2 . So suppose that v_2 is an endpoint of Q , interchange colors on Q and denote the obtained coloring by f_1 . If 4 does not appear at x_1 under f_1 , then we may recolor $v_2 x_1$ with color 4 to obtain a coloring where 5 does not appear at v_2 or v_3 . Hence, $4 \in f(x_1)$. That color 5 must appear at x_2 under f can be proved similarly by considering a maximal $(3, 5)$ -colored path with origin at v_3 . Thus $\{3, 4, 5\} \subseteq f(x_i)$, $i = 1, 2$.

We now prove statement (b) of Lemma 3.1. By part (a), we may assume that condition (i) holds for the vertices v_1, v_2, v_3 , since otherwise the required result follows. In fact, we will as before assume (without loss of generality) that

$$f(v_1) = \{1, 2, 3, 5\}, f(v_2) = \{1, 2, 4, 5\}, f(v_3) = \{1, 2, 3, 4\}.$$

If $f(v_4) = \{1, 2, 3, 4\}$, then the result follows by setting $g = f$. If $f(v_4) = \{1, 2, 4, 5\}$, then the result follows by applying part (a) to the subpath of P with internal vertices v_2, v_3, v_4 . So we may assume that $f(v_4) = \{1, 2, 3, 5\}$. Consider a maximal $(3, 4)$ -colored path Q with origin at v_2 . Let f_1 be a proper 5-edge coloring obtained from f by interchanging colors on Q . If v_1 is not an endpoint of Q , then the result follows by setting $g = f_1$ because $4 \notin f_1(v_1) \cup f_1(v_2)$. Suppose now that v_1 is an endpoint of Q . Then $|f_1(v_2) \cap f_1(v_4) \setminus \{1, 2\}| = 2$. The result now follows by applying part (a) to the subpath of P with internal vertices v_2, v_3, v_4 . $\square$

Lemma 3.2. *Let G be a 4-regular graph with $\chi'(G) = 4$, f be a proper 5-edge coloring of G , h a proper 4-edge coloring of G , and let $P = u_1v_1v_2v_3v_4$ be a $(1, 2)$ -colored path under f in G such that $f(u_1v_1) = 2$ and $h(u_1v_1) = 1$. Assume further that*

$$f(v_1) = \{1, 2, c_1, c_3\}, f(v_2) = \{1, 2, c_2, c_3\}, f(v_3) = \{1, 2, c_1, c_2\},$$

where (c_1, c_2, c_3) is a permutation of the set $\{3, 4, 5\}$. Let x_1 and x_2 be the vertices distinct from v_1 and v_3 that are adjacent to v_2 , where $f(v_2x_1) = c_3, f(v_2x_2) = c_2$. If $u_1 \neq x_2$, then one of the following holds:

- (I) $f(x_1) = \{2, 3, 4, 5\}$ and $f(x_2) = \{1, 3, 4, 5\}$,
- (II) *there is a proper 5-edge coloring g of G which can be obtained from f by a sequence of interchanges, such that $M(h, 1) \cap M(g, 1) = M(h, 1) \cap M(f, 1)$, the color 1 is missing in $g(v_1)$ and, moreover, if $1 \notin f(u_1)$, then $1 \notin g(u_1)$,*
- (III) *there is a proper 5-edge coloring g of G which can be obtained from f by a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$, such that one of colors 3, 4, 5 is missing either in the set $g(v_1) \cup g(v_2)$ or in $g(v_2) \cup g(v_3)$.*

Proof. Suppose that (III) does not hold. Then, by Lemma 3.1, colors 3, 4, 5 appear at both x_1 and x_2 under f .

First we show that $f(x_2) = \{1, 3, 4, 5\}$ or (II) holds. Suppose that $1 \notin f(x_2)$. Then $v_1v_2x_2$ is a maximal $(1, c_2)$ -colored path under f . Since, $u \neq x_2$ we may interchange colors on this path to obtain a proper 5-edge coloring g of G for which (II) holds. Hence we may assume that $1 \in f(x_2)$, that is, $f(x_2) = \{1, 3, 4, 5\}$.

Now we show that $f(x_1) = \{2, 3, 4, 5\}$ or (II) holds. Suppose that $2 \notin f(x_1)$. Then the sequence $(v_2v_1, v_2x_2, v_2v_3, v_2x_1)$ is an f -fan at v_2 and color 2 does not appear at x_1 and x_2 . Note also that c_1 does not appear at v_2 . Consider a maximal path Q that is $(c_1, 2)$ -colored under f and has origin at x_1 .

If $\{v_3, x_2\} \cap V(Q) = \emptyset$, then we interchange colors on Q and obtain a proper coloring that we denote by f_1 . The sequence $(v_2v_1, v_2x_2, v_2v_3, v_2x_1)$ is a saturated f_1 -fan at v_2 and we may downshift to obtain a proper 5-edge coloring g under which color 1 does not appear at v_1 . Moreover, if $1 \notin f(u_1)$, then $1 \notin g(u_1)$; that is, (II) holds.

Suppose now that $v_3 \in V(Q)$. Then $x_2 \notin V(Q)$. Consider a maximal path Q' that is $(c_1, 2)$ -colored under f and has origin x_2 . Clearly, Q and Q' are disjoint and by interchanging colors on Q' we get a coloring f_1 where (v_2v_1, v_2x_2) is a saturated f_1 -fan at v_2 . By downshifting on this fan we obtain a coloring g for which (II) holds.

Suppose now that $x_2 \in V(Q)$. Then $v_3 \notin V(Q)$ and by interchanging colors on Q we get a coloring f_1 where (v_2v_1, v_2x_2) is a saturated f_1 -fan. Again, downshifting on this fan yields that (II) holds. $\square$

Let G be a graph with $\chi'(G) = \Delta(G) = 4$ and h a proper 4-edge coloring of G . For an arbitrary proper 5-edge coloring φ of G we say that an edge e is *h -correct under φ* (or just *correct under φ*) if $\varphi(e) = h(e)$. By the *distance* between two edges e and e' on a path P , we mean the number of edges between e and e' on P .

Lemma 3.3. *Let G be a graph with $\chi'(G) = \Delta(G) = 4$, h be a proper 4-edge coloring of G and f a proper 5-edge coloring of G . Furthermore, let P be a maximal path or cycle that is $(1, 2)$ -colored under f and which contains an edge xy such that $f(xy) = 2$ and $h(xy) = 1$. If there is no h -correct edge of color 1 under f at distance at most 2 from xy on P , then there is a proper 5-edge coloring f' of G such that f' can be obtained from f by a sequence of interchanges and $|M(f', 1) \cap M(h, 1)| > |M(f, 1) \cap M(h, 1)|$.*

Proof. Suppose first that P contains no h -correct edge colored 1. Then we simply make an interchange on P to obtain the required coloring f' .

Suppose now that P contains some h -correct edge colored 1. This implies that the length of P is at least 6.

Case 1. P is a path and the distance between xy and the first or the last edge of P is at most 3.

Let $P = a_l a_{l-1} \dots a_1 xy b_1 \dots b_k$, where $l \leq 4$ or $k \leq 4$. Without loss of generality we assume that $l \leq 4$. The conditions imply that the subpath $a_l a_{l-1} \dots x$ contains no h -correct edge colored 1 under f , so we must have $k \geq 5$. Consider the path $y b_1 b_2 b_3 b_4 b_5$. By Lemma 3.1 (b), there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ yielding a proper 5-edge coloring g of G from f , such that for some color $c \in \{3, 4, 5\}$ and some $i \in \{1, 2, 3\}$, we have $c \notin g(b_i) \cup g(b_{i+1})$. Note that $b_i b_{i+1}$ is not both h -correct and colored 1. Recolor the edge $b_i b_{i+1}$ with the color c and denote the obtained coloring by g_1 . Then the path $Q = a_l a_{l-1} \dots a_1 xy b_1 \dots b_i$ is a maximal $(1, 2)$ -colored path under g_1 and it contains no correct edge colored 1, so the desired result follows by making an interchange on Q .

Case 2. P is a path and the distance between xy and each of the first and last edges of P is at least 4.

Let $P = a_l a_{l-1} \dots a_1 xy b_1 \dots b_k$, where $l, k \geq 5$. Consider the path $y b_1 b_2 b_3 b_4 b_5$. By Lemma 3.1 (b), there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ yielding a proper 5-edge coloring g of G from f , such that for some color $c \in \{3, 4, 5\}$ and some $i \in \{1, 2, 3\}$, we have $c \notin g(b_i) \cup g(b_{i+1})$. Recolor the edge $b_i b_{i+1}$ with the color c and denote the obtained coloring by g_1 . Then the maximal path Q that is $(1, 2)$ -colored under g_1 and contains xy , satisfies the condition of Case 1. Therefore, the coloring g_1 can be transformed to a required coloring f' by a sequence of interchanges. Thus, the lemma is true in Case 2.

Case 3. P is a cycle:

Suppose that $P = a_k a_{k-1} \dots a_1 xy b_1 \dots b_k a_k$. If P has length at most 8, then it contains no h -correct edge colored 1, which contradicts our assumption; so assume that P has length at least 10. If P has length 10, then there is at most one h -correct edge colored 1 on P , namely $b_4 a_4$. Suppose that $b_4 a_4$ is h -correct. By applying Lemma 3.1 (b) to the path $y b_1 b_2 b_3 b_4 a_4$, and proceeding as above we may reduce this situation to Case 1.

If, on the other hand, P has length at least 12, then there is a path $y b_1 b_2 b_3 b_4 b_5$ lying on P , and we may proceed exactly as in Case 2 for reducing this case to Case 1. $\square$

Theorem 3.4. *Let G be a 4-regular graph with $\chi'(G) = 4$. Then every proper 5-edge coloring of G can be transformed to any proper 4-edge coloring of G by a sequence of interchanges.*

Proof. Let h be a proper 4-edge coloring and f be a proper 5-edge coloring of the graph G . It suffices to prove that the coloring f can, by a sequence of interchanges, be transformed to a proper 5-edge coloring g such that $M(g, 1) = M(h, 1)$. The result then follows by applying Theorem B to the graph $G' = G - M(g, 1)$ with maximum degree 3 and two proper edge colorings of G' induced by the colorings g and h , respectively.

Suppose that there is some edge $e = u_1v_1$ of G such that $h(e) = 1 \neq f(e)$. Without loss of generality we assume that $f(e) = 2$. We will prove that by a sequence of interchanges we can transform f into a proper 5-edge coloring f' which satisfies

$$|M(f', 1) \cap M(h, 1)| > |M(f, 1) \cap M(h, 1)|. \quad (3.1)$$

This suffices for proving the theorem.

The remaining part of the proof proceeds by rather extensive case analysis, and we now give a brief outline. To begin with, we distinguish between the cases that e is adjacent to at most one edge colored 1 under f (Case A below) and the case when e is adjacent to two edges colored 1 under f (Case B below). In Case A the proof consists of several different subcases and the required coloring f' is constructed in different ways depending on the structure of the component of $G_f(1, 2)$ containing e .

Case B also breaks into several different subcases, and here the analysis is considerably more involved. Several reductions are used, but we still have to consider quite a number of different cases depending on the structure of the component of $G_f(1, 2)$ containing e . The general technique employed is to reduce different subcases to previously considered ones. Let us now give the details of the case analysis.

Case A. e is adjacent to at most one edge colored 1 under f :

Without loss of generality we assume that color 1 appears at v_1 under f (if e is not adjacent to any edge colored 1, then we just recolor e with 1 and are done). Consider a maximal path $P = u_1v_1v_2 \dots v_k$ in G that is $(1, 2)$ -colored and has origin at u_1 . Note that since $h(u_1v_1) = 1$, v_1v_2 is not correct under f . If $k \leq 3$, then, by Lemma 3.3, we can transform f by interchanges to a required coloring f' satisfying (3.1). Suppose now that $k \geq 4$. If v_3v_4 is not correct, then again by Lemma 3.3, we can transform f by interchanges to a required coloring f' satisfying (3.1). So in the following we will assume that v_3v_4 is correct.

Case A.1. One of the conditions (i), (ii) in Lemma 3.1 does not hold.

Then there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ which transform f to a proper 5-edge coloring g such that $c \notin g(v_i) \cup g(v_{i+1})$ for some color $c \in \{3, 4, 5\}$ and some $i \in \{1, 2\}$. Now we recolor v_iv_{i+1} with c and denote by g_1 the proper edge coloring obtained from this operation. Then the maximal $(1, 2)$ -colored path (under g_1) with origin at u_1 has length at most two and, by Lemma 3.3, g_1 can by interchanges be transformed into a proper 5-edge coloring f' such that $|M(f', 1) \cap M(h, 1)| > |M(g_1, 1) \cap M(h, 1)|$. This and $M(g_1, 1) \cap M(h, 1) = M(f, 1) \cap M(h, 1)$ imply that $|M(f', 1) \cap M(h, 1)| > |M(f, 1) \cap M(h, 1)|$.

Case A.2. The conditions (i), (ii) in Lemma 3.1 hold.

This means that $|f(v_i) \cap f(v_j) \setminus \{1, 2\}| = 1$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$; and the colors 3, 4, 5 appear at x_1 and x_2 , where x_1, x_2 denote the vertices distinct from v_1 and v_3 that are

adjacent to v_2 . Then

$$f(v_1) = \{1, 2, c_1, c_3\}, f(v_2) = \{1, 2, c_2, c_3\}, f(v_3) = \{1, 2, c_1, c_2\}$$

for some permutation (c_1, c_2, c_3) of the set $\{3, 4, 5\}$. Clearly, $\{f(v_2x_1), f(v_2x_2)\} = \{c_2, c_3\}$. Let $f(v_2x_1) = c_3$ and $f(v_2x_2) = c_2$. Without loss of generality we may assume that

$$f(v_1) = \{1, 2, 3, 5\}, f(v_2) = \{1, 2, 4, 5\}, f(v_3) = \{1, 2, 3, 4\}.$$

Then $f(v_2x_1) = 5$ and $f(v_2x_2) = 4$.

Case A.2.1. The vertex v_4 is an endpoint of P and $x_2 \neq u_1$.

First suppose that either $f(x_1) \neq \{2, 3, 4, 5\}$ or $f(x_2) \neq \{1, 3, 4, 5\}$. Then, by Lemma 3.2, either condition (II) or condition (III) of Lemma 3.2 holds.

If condition (III) holds, then arguing similarly as in Case A.1 we can by interchanges transform f into a required coloring f' satisfying (3.1).

If condition (II) holds, then we can by interchanges transform f into a proper 5-edge coloring g such that $M(g, 1) \cap M(h, 1) = M(f, 1) \cap M(h, 1)$. Moreover, since color 1 does not appear at u_1 under f , we will have $1 \notin g(v_1), 1 \notin g(u_1)$. We color the edge u_1v_1 with color 1 and obtain a required coloring f' satisfying (3.1).

Now suppose that $f(x_1) = \{2, 3, 4, 5\}$ and $f(x_2) = \{1, 3, 4, 5\}$.

We first consider the case when $u_1 \neq x_1$. We have that $f(v_2x_1) = 5$. Clearly, $x_1 \neq v_4$ because $1 \in f(v_4)$ and $f(x_1) = \{2, 3, 4, 5\}$. We make an interchange on P and denote the obtained coloring by f_1 . Then u_1v_1 is correct under f_1 , but v_3v_4 is not. Moreover, $x_1v_2v_3$ is a maximal $(5, 1)$ -colored path under f_1 , and by interchanging colors on this path we get a new coloring which we denote by f_2 . Since $x_1 \neq v_4$, we may now recolor v_3v_4 with color 1 to obtain a required coloring f' .

Suppose now that $u_1 = x_1$. We have that $f(x_2) = \{1, 3, 4, 5\}$ and $d_G(v_2) = 4$. Then v_2 is incident to an edge colored 1 under h . Furthermore,

$$h(x_1v_1) = h(v_3v_4) = 1,$$

and thus we must have $h(v_2x_2) = 1$. Let x_3 be the vertex adjacent to x_2 , such that $f(x_2x_3) = 1$. Since $h(v_2x_2) = 1$, x_2x_3 is not correct under f , which implies that $v_4 \notin \{x_2, x_3\}$. We now interchange colors on P and denote the obtained coloring by f_1 . Then x_1v_1 is correct under f_1 , but v_3v_4 is not. Note that $v_1v_2x_2$ is a maximal $(4, 2)$ -colored path under f_1 . We make an interchange on this path and denote the obtained coloring by f_2 . Let $P' = v_4v_3v_2x_2x_3 \dots$ be the maximal $(1, 2)$ -colored path under f_2 with origin at v_4 . Since, x_2x_3 is not correct under f , it is not correct under f_2 too, and the desired result now follows by applying Lemma 3.3 to the coloring f_2 , the path P' and the edge v_4v_3 instead of f, P and xy .

Case A.2.2. The vertex v_4 is an endpoint of P and $x_2 = u_1$.

Recall that colors 3, 4, 5 appear at x_1 under f . If 1 appears at x_1 under f , then the edge colored 1 incident to x_1 cannot be correct under f , because we must have $h(v_2x_1) = 1$. This implies that $v_4 \neq x_1$. Furthermore, $x_1v_2v_3$ is a maximal $(5, 2)$ -colored path under f and by making an interchange on this path, we obtain a coloring that we denote by f_1 . Now, note that $P' = x_2v_1v_2x_1 \dots$ is a maximal $(1, 2)$ -colored path under f_1 , and since the edge incident

to x_1 colored 1 is not correct, it now follows from Lemma 3.3 that there is a proper 5-edge coloring f' of G that can be obtained from f_1 via a sequence of interchanges, and such that $|M(f', 1) \cap M(h, 1)| > |M(f, 1) \cap M(h, 1)|$. If instead the color 2 appears at x_1 under f , then the proof proceeds exactly as in Case A.2.1 when $u_1 \neq x_1$.

Case A.2.3. The vertex v_4 is an internal vertex of P .

It follows from Lemma 3.1 (b) that there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ which transforms f to a proper 5-edge coloring g such that $c \notin g(v_i) \cup g(v_{i+1})$ for some color $c \in \{3, 4, 5\}$ and some $i \in \{1, 2, 3\}$.

If $i \leq 2$, then arguing similarly as in Case A.1, we can obtain a required coloring f' satisfying (3.1). Now we suppose that such a color c exists only for $i = 3$. This means that $c \notin g(v_3) \cup g(v_4)$, $\{3, 4, 5\} \subseteq g(v_1) \cup g(v_2)$ and $\{3, 4, 5\} \subseteq g(v_1) \cup g(v_2)$.

If the coloring g does not satisfy one of the conditions (i) and (ii) in Lemma 3.1 (with g instead of f), then in a similar way as in Case A.1 we can obtain a required coloring f' .

Suppose now that the conditions (i), (ii) of Lemma 3.1 for the coloring g (instead of f) hold. Then

$$g(v_1) = \{1, 2, b_1, b_3\}, g(v_2) = \{1, 2, b_2, b_3\}, g(v_3) = \{1, 2, b_1, b_2\}$$

for some permutation (b_1, b_2, b_3) of the set $\{3, 4, 5\}$. Clearly, $\{f(v_2x_1), f(v_2x_2)\} = \{b_2, b_3\}$.

Without loss of generality we can assume that

$$g(v_1) = \{1, 2, 3, 5\}, g(v_2) = \{1, 2, 4, 5\}, g(v_3) = \{1, 2, 3, 4\},$$

and that $g(x_1v_2) = 5$, $g(x_2v_2) = 4$ (possibly renaming the vertices x_1 and x_2 if this does not hold). Since there is a color $c \notin g(v_3) \cup g(v_4)$ and $g(v_3) = \{1, 2, 3, 4\}$, we have that $c = 5$ and $g(v_4) = \{1, 2, 3, 4\}$. If $u_1 \neq x_2$, then similarly as in Case A.2.1, it follows from Lemma 3.2 that either $g(x_1) = \{2, 3, 4, 5\}$ and $g(x_2) = \{1, 3, 4, 5\}$, or we can obtain a required coloring f' satisfying (3.1). Thus if $u_1 \neq x_2$, then it suffices to consider the case when $g(x_1) = \{2, 3, 4, 5\}$ and $g(x_2) = \{1, 3, 4, 5\}$.

We first consider the case when $u_1 \notin \{x_1, x_2\}$. From g we construct the coloring g_1 by recoloring v_3v_4 with color 5. Then $g_1(x_1) = \{2, 3, 4, 5\}$ and $g_1(x_2) = \{1, 3, 4, 5\}$. Now we make an interchange on the path $u_1v_1v_2v_3$ and denote the constructed coloring by g_2 . Note that $v_4v_3v_2x_1$ is a maximal $(1, 5)$ -colored path under g_2 . By making an interchange on this path we obtain a required coloring f' satisfying (3.1).

Now assume that $u_1 = x_1$. From g we construct the coloring g_1 by recoloring v_3v_4 with color 5. Then $g_1(x_1) = \{2, 3, 4, 5\}$ and $g_1(x_2) = \{1, 3, 4, 5\}$. By making an interchange on the path $x_1v_1v_2v_3$ we obtain a coloring g_2 . Note that the path $v_4v_3v_2x_1v_1$ is $(5, 1)$ -colored under g_2 and that the edge v_3v_4 is adjacent to only one edge colored 1. Moreover, v_2 and v_4 are not adjacent, because $g(x_1) \neq g(v_4)$ and $g(x_2) \neq g(v_4)$. Consider the path $P' = u'_1v'_1v'_2v'_3v'_4$, where $u'_1 = v_4, v'_1 = v_3, v'_2 = v_2, v'_3 = u_1$ and $v'_4 = v_1$. If we exchange the colors 5 and 2 in the coloring g_2 and consider the obtained coloring instead of g , the path $P' = u'_1v'_1v'_2v'_3v'_4$ instead of P and the vertices $x'_1 = v_1, x'_2 = x_2$ instead of x_1 and x_2 , then we obtain a situation which is similar to the situation considered in the previous paragraph, because here $u'_1 \notin \{x'_1, x'_2\}$.

Now assume that $u_1 = x_2$. Recall that colors 3, 4, 5 appear at both x_1 and x_2 under f . As before, we construct the coloring g_1 from g by recoloring v_3v_4 with color 5. Thereafter, we make an interchange on the path $x_2v_1v_2v_3$ and denote the obtained coloring by g_2 .

If x_1 is incident to an edge colored 2 under g_2 , then $v_4v_3v_2x_1$ is a maximal $(1, 5)$ -colored path under g_2 . By making an interchange on this path we obtain a required coloring f' satisfying (3.1).

Suppose now that x_1 is incident to an edge colored 1 under g_2 . Consider the path $v_4v_3v_2x_1$ that is $(5, 1)$ -colored path under g_2 with origin at v_4 . Clearly, the edge colored 1 incident to x_1 cannot be correct under g_2 (because we must have $h(x_1v_2) = 1$). Put $x' = v_4$, $y' = v_3$ and let $P' = v_4v_3v_2x_1 \dots$ be the maximal $(5, 1)$ -colored path with origin at $x' = v_4$. Now we obtain the required result by applying Lemma 3.3 to the coloring g_2 , the path P' and the edge $x'y'$ instead of f, P and xy .

Case B. e is adjacent to two edges colored 1 under f .

We have that $f(e) = 2$. Let u_1u_2 and v_1v_2 be the edges adjacent to e and colored 1 under f . Since $h(e) = 1$, none of the edges u_1u_2 and v_1v_2 are h -correct. Let P be the component in $G_f(1, 2)$ that contains e . It can be a path or cycle.

If there is no correct edge colored 1 under f at distance 2 from e along P , then the theorem is true by Lemma 3.3.

Suppose now that there is a correct edge at distance 2 from e on P . This implies that if P is a cycle then it has the length at least 6.

Case B.1. P is a path where the distance between e and the first or the last edge of P is at most 1.

Let $P = u_lu_{l-1} \dots u_1v_1v_2 \dots v_k$, where $2 \leq l \leq 3$ or $2 \leq k \leq 3$. Without loss of generality we may assume that $2 \leq l \leq 3$. Since there is a correct edge at distance 2 from e on P , we will have that $k \geq 4$ and the edge v_3v_4 is correct. If one of the conditions (i) and (ii) of Lemma 3.1 does not hold, then arguing similarly as in Case A.1 we can by a sequence of interchanges transform f to a proper coloring f' satisfying (3.1).

Assume now that the conditions (i), (ii) of Lemma 3.1 hold. Then we may assume without loss of generality that

$$f(v_1) = \{1, 2, 3, 5\}, f(v_2) = \{1, 2, 4, 5\}, f(v_3) = \{1, 2, 3, 4\}, f(v_2x_1) = 5, f(v_2x_2) = 4,$$

and the colors 3, 4, 5 appear at x_1 and x_2 , where x_1, x_2 are the vertices distinct from v_1 and v_3 that are adjacent to v_2 . Since $l \geq 2$, we have that $\{1, 2, \} \subseteq f(u_1)$. Therefore $u_1 \notin \{x_1, x_2\}$.

Suppose that $f(x_1) \neq \{2, 3, 4, 5\}$. Then one of the conditions (II) and (III) of Lemma 3.2 holds. If the condition (II) holds, then we obtain a proper 5-edge coloring g such that the color 1 is missing at v_1 . This situation for g is similar to the situation considered in Case A for f . Therefore we can transform g by a sequence of interchanges to a required coloring f' satisfying (3.1).

If the condition (III) holds, then there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ which transform f to a proper 5-edge coloring g such that $c \notin g(v_i) \cup g(v_{i+1})$ for some color $c \in \{3, 4, 5\}$ and some $i \in \{1, 2\}$. Now we recolor v_iv_{i+1} with c and denote by g_1 the proper edge coloring obtained from this operation. Then the path $P' = u_lu_{l-1} \dots u_1v_1v_2 \dots v_i$ is the maximal $(1, 2)$ -colored (under g_1) path with origin at u_l , such that $g_1(u_1v_1) = 2, h(u_1v_1) = 1$ and there is no h -correct edge of color 1 on P' . This situation for g_1 is similar to the situation considered in Lemma 3.3 for the coloring f , the

path P and the edge xy . Therefore we can transform g_1 by a sequence of interchanges to a required coloring f' satisfying (3.1).

We now consider the case when $f(x_1) = \{2, 3, 4, 5\}$. If v_4 is an endpoint of P then we proceed as follows: make an interchange on P and denote the obtained coloring by f_1 . Note that $|M(f_1, 1) \cap M(h, 1)| = |M(f, 1) \cap M(h, 1)|$, since u_1v_1 is correct under f_1 , but v_3v_4 is not. However, v_3v_4 is adjacent to only one edge colored 1 under f_1 . Thus we have a situation that is similar to the situation considered in Case A with the coloring f_1 and the edge v_3v_4 instead of f and e . Hence we can by interchanges transform f_1 , and therefore f too, to a required coloring f' satisfying (3.1).

If v_4 is not an endpoint of P , then by Lemma 3.1 (b) there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ yielding a coloring g , such that $c \notin g(v_j) \cup g(v_{j+1})$, for some $j \in \{1, 2, 3\}$ and some color $c \in \{3, 4, 5\}$. Recolor v_jv_{j+1} with c and denote the obtained coloring by g_1 . If $j \leq 2$, the desired result follows by applying Lemma 3.3 to the coloring g_1 instead of f . Suppose now that $j = 3$. We make an interchange on the maximal $(1, 2)$ -colored path containing u_1v_1 and denote the obtained coloring by g_2 . Then u_1v_1 is correct under g_2 but v_3v_4 is not. Furthermore $|M(g_2, 1) \cap M(h, 1)| \geq |M(f, 1) \cap M(h, 1)|$ and v_3v_4 is adjacent to only one edge colored 1 under g_2 . Thus we have a situation that is similar to the situation considered in Case A with the coloring g_2 and the edge v_3v_4 instead of f and e . Therefore we can by using interchanges construct a required coloring f' satisfying (3.1).

Case B.2. P is a path with $P = u_l u_{l-1} \dots u_2 u_1 v_1 v_2 \dots v_k$ where $l \geq 4$ and $k \geq 4$, or P is a cycle of length at least 6 with $P = u_l u_{l-1} \dots u_1 v_1 v_2 \dots v_l$, where $l \geq 3$.

Case B.2.1. At least one of the conditions (i), (ii) of Lemma 3.1 does not hold.

Then, by Lemma 3.1 (a), there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ which transforms f to a proper 5-edge coloring g , such that $c \notin g(v_j) \cup g(v_{j+1})$ for some color $c \in \{3, 4, 5\}$ and some $j \in \{1, 2\}$. We recolor v_jv_{j+1} with color c . If $j = 1$, we obtain a situation similar to the situation considered in Case A. If $j = 2$, we obtain a situation similar to the situation considered in Case B.1. Hence in both cases we can transform f to a proper coloring f' satisfying (3.1) by a sequence of interchanges.

Case B.2.2. At least one of the following conditions does not hold:

$$|f(u_i) \cap f(u_j) \setminus \{1, 2\}| = 1 \text{ for } i, j \in \{1, 2, 3\} \text{ satisfying } i \neq j, \quad (3.2)$$

$$\begin{aligned} &\text{if } y_1 \text{ and } y_2 \text{ are the vertices distinct from } u_1 \text{ and } u_3 \text{ that are} \\ &\text{adjacent to } u_2, \text{ then colors } 3, 4, 5 \text{ appear at } y_1 \text{ and } y_2. \end{aligned} \quad (3.3)$$

Consider the $(1, 2)$ -colored path $v_1 u_1 u_2 u_3 u_4$. Arguing similarly as in the Case B.2.1, we can construct a required coloring f' satisfying (3.1) by a sequence of interchanges.

Case B.2.3. The conditions (i), (ii) of Lemma 3.1 hold for the vertices v_1, v_2, v_3 and the conditions (3.2), (3.3) hold for the vertices u_1, u_2, u_3 .

Suppose that it is impossible to transform the coloring f to a required coloring f' satisfying (3.1) by using only interchanges.

We will show that then f can be transformed via interchanges on bicolored paths with colors from $\{3, 4, 5\}$ to a proper 5-edge coloring f_1 such that

$$f_1(v_1) = f_1(u_1), \quad f_1(v_2) = f_1(u_2), \quad f_1(v_3) = f_1(u_3). \quad (3.4)$$

Without loss of generality we may assume that

$$f(v_1) = \{1, 2, 3, 5\}, f(v_2) = \{1, 2, 4, 5\}, f(v_3) = \{1, 2, 3, 4\}.$$

Suppose first that $f(u_1) \neq f(v_1)$, e.g. that $f(u_1) = \{1, 2, 4, 5\}$. (The case when $f(u_1) = \{1, 2, 3, 4\}$ is similar.) Then, by condition (3.2), $f(u_1) \neq f(u_2)$. This means that either $f(u_2) = \{1, 2, 3, 5\}$ or $f(u_2) = \{1, 2, 3, 4\}$.

If $f(u_2) = \{1, 2, 3, 5\}$, then, by condition (3.2), $f(u_3) = \{1, 2, 3, 4\}$. Consider a maximal $(3, 4)$ -colored path Q with origin at u_1 . We show that u_2 is an endpoint of Q . Assuming that u_2 is not an endpoint of Q and interchanging the colors on Q we obtain a coloring f_2 such that $f_2(u_1) = f_2(u_2) = \{1, 2, 3, 5\}$. Then we may recolor u_1u_2 with color 4 and obtain a situation similar to the situation considered in Case A, which is impossible because in that case we can transform f by interchanges to a required coloring f' satisfying (3.1). Thus, u_2 is an endpoint of Q . Then by interchanging colors on Q we obtain the desired coloring f_1 satisfying (3.4).

Now suppose that $f(u_2) = \{1, 2, 3, 4\}$. Then, by condition (3.2), $f(u_3) = \{1, 2, 3, 5\}$. Let Q be a maximal $(4, 3)$ -colored path with origin at u_1 . Arguing similarly as in the preceding paragraph we obtain that u_3 is an endpoint of Q . By interchanging colors on Q we get a coloring f^* . Next, we consider a maximal path Q' that is $(3, 5)$ -colored under f^* and has origin at u_2 . Arguing similarly as in the preceding paragraph we obtain that u_3 is an endpoint of Q' . Now by interchanging colors on Q' we get the desired coloring f_1 satisfying (3.4).

Suppose now that $f(u_1) = f(v_1) = \{1, 2, 3, 5\}$.

If $f(u_2) = f(v_2) = \{1, 2, 4, 5\}$ then, by (3.2), $f(u_3) = \{1, 2, 3, 4\} = f(v_3)$, that is, the coloring f itself can be considered as the desired coloring f_1 satisfying (3.4).

If $f(u_2) \neq f(v_2)$ then the conditions (3.2) and (3.3) imply that $f(u_2) = \{1, 2, 3, 4\}$ and $f(u_3) = \{1, 2, 4, 5\}$. Let Q_1 be a maximal $(3, 5)$ -colored path with origin at u_2 . Then u_3 is an endpoint of Q_1 . (Otherwise by interchanging the colors on Q_1 we may obtain a coloring f_2 such that $f_2(u_3) = f_2(u_2) = \{1, 2, 4, 5\}$. Then we may recolor u_2u_3 with color 3 and obtain a situation similar to the situation considered in Case B.1, which is impossible because in that case we can transform f by interchanges to a required coloring f' satisfying (3.1).) Now by interchanging colors on Q_1 we obtain the desired coloring f_1 satisfying (3.4).

Thus we may, without loss of generality, further assume that f can by interchanges on bicolored paths with colors from $\{3, 4, 5\}$ be transformed to a coloring f_1 such that

$$f_1(v_1) = f_1(u_1) = \{1, 2, 3, 5\}, f_1(v_2) = f_1(u_2) = \{1, 2, 4, 5\}, f_1(v_3) = f_1(u_3) = \{1, 2, 3, 4\}. \quad (3.5)$$

Furthermore we can claim that it is impossible to transform f_1 to a proper 5-edge coloring f' satisfying (3.1) by using only interchanges, because f_1 is obtained from f by a sequence

of interchanges and, by our assumption, it is impossible to transform f to a proper 5-edge coloring f' satisfying (3.1) by using only interchanges.

Denote by x_1, x_2 the vertices distinct from v_1 and v_3 that are adjacent to v_2 , where $f(v_2x_1) = 5$. We now prove that $f_1(x_1) = \{2, 3, 4, 5\}$ and $f_1(x_2) = \{1, 3, 4, 5\}$. Suppose that one of these two conditions does not hold. Then, by Lemma 3.2, one of conditions (II) and (III) of Lemma 3.2 holds. If the condition (III) holds, we may transform f_1 by interchanges on paths with colors from $\{3, 4, 5\}$ to a proper coloring g , such that $c \notin g(v_i) \cup g(v_{i+1})$ for some color $c \in \{3, 4, 5\}$ and some $i \in \{1, 2\}$. Then by recoloring $v_i v_{i+1}$ with c we obtain a situation that is similar to the situation considered in Case A or Case B.1. If the condition (II) holds, then, by Lemma 3.2, we may transform f_1 by interchanges to another proper coloring and again obtain a situation which is similar to the situation considered in Case A.

Thus if one of conditions $f_1(x_1) = \{2, 3, 4, 5\}$ and $f_1(x_2) = \{1, 3, 4, 5\}$ does not hold, then we obtain situations which are similar to situations considered in Case A or Case B.1, which is impossible because in those cases we can transform f_1 by interchanges to a required coloring f' satisfying (3.1). Therefore $f_1(x_1) = \{2, 3, 4, 5\}$ and $f_1(x_2) = \{1, 3, 4, 5\}$.

Denote by y_1 and y_2 the vertices adjacent to u_2 that are distinct from u_1 and u_3 , and where $f_1(u_2y_1) = 5$. Clearly, $x_1 \neq y_1$. Arguing similarly as in the preceding paragraph we may deduce that $f_1(y_1) = \{2, 3, 4, 5\}$ and $f_1(y_2) = \{1, 3, 4, 5\}$.

Hence, without loss of generality we will in the following assume that

$$f_1(x_1) = f_1(y_1) = \{2, 3, 4, 5\} \text{ and } f_1(x_2) = f_1(y_2) = \{1, 3, 4, 5\}. \quad (3.6)$$

In particular, this implies that x_1 and y_1 are not internal vertices of P .

We will now show that P is not a cycle of length 6 or a path of length 7.

Suppose that P is a cycle of length 6, that is, $P = u_3u_2u_1v_1v_2v_3u_3$. Then the edge v_3u_3 is h -correct, because otherwise by interchanging colors on P we can obtain a proper coloring f' satisfying (3.1), which is impossible. Using property (3.5) of f_1 , we can obtain a proper coloring f' satisfying (3.1) by first recoloring v_3u_3 with color 5, then interchanging colors on the path $v_3v_2v_1u_1u_2u_3$, and thereafter making an interchange on the maximal $(1, 5)$ -colored path $y_1u_2u_3v_3v_2x_1$. Since this is impossible, P cannot be a cycle of length 6.

Suppose now that P is a path of length 7. Then $P = u_4u_3u_2u_1v_1v_2v_3v_4$. Note that $\{v_4, u_4\} \cap \{x_1, y_1\} = \emptyset$, because $2 \notin f_1(v_4)$, $2 \notin f_1(u_4)$ and $2 \in f_1(x_1)$, $2 \in f_1(y_1)$. Make an interchange on P and denote the obtained coloring by f_2 . Then $u_3u_2y_1$ and $v_3v_2x_1$ are two disjoint maximal $(1, 5)$ -colored paths under f_2 in G . By interchanging colors on these paths and then recoloring u_4u_3 and v_4v_3 with color 1 we obtain a proper coloring f' satisfying (3.1). Since this is impossible, P cannot be a path of length 7.

Thus P has the length at least 8. We will show now that the edges v_3v_4 and u_3u_4 are h -correct under f_1 .

Since P has length at least 8, at most one of u_4 and v_4 are endpoints of P . Suppose first that u_4 is an endpoint of P . If u_3u_4 is not correct, then we may proceed exactly as in Case B.1 to obtain a coloring f' satisfying (3.1), which contradicts our assumption that this is impossible. Hence, we may assume that u_3u_4 is correct.

If v_3v_4 is not correct, then by applying Lemma 3.1 (b) to the path $u_1v_1v_2v_3v_4v_5$ we get that there is a sequence of interchanges on bicolored paths with colors from $\{3, 4, 5\}$ yielding

a coloring g , such that $c \notin g(v_j) \cup g(v_{j+1})$ for some $j \in \{1, 2, 3\}$ and some color $c \in \{3, 4, 5\}$. By recoloring $v_j v_{j+1}$ with c , we obtain a situation which is similar to one of the situation considered in Case B.1 and Case A. This is impossible because in those cases we can obtain, by interchanges, a required coloring f' satisfying (3.1).

Suppose now that none of u_4 and v_4 is an endpoint of P . If $u_3 u_4$ (or $v_3 v_4$) is not correct, then by applying Lemma 3.1 (b) to the path $v_1 u_1 u_2 u_3 u_4 u_5$ (or $u_1 v_1 v_2 v_3 v_4 v_5$, respectively) and proceeding as in the preceding paragraph we obtain a situation which is similar to one of the situations considered in Case B.1 and Case A. This is impossible because in those cases we can obtain, by interchanges, a required coloring f' satisfying (3.1).

Thus we have that

$$\text{the edges } u_3 u_4 \text{ and } v_3 v_4 \text{ are correct under } f_1. \quad (3.7)$$

We now prove the following claim that holds for the coloring f_1 in Case B.2.3 when P has the length at least 8.

Claim 1. (a) *If Q is a maximal $(3, 5)$ -colored path under f_1 with origin at u_3 (at v_3), then u_2 (respectively, v_2) is an endpoint of Q .*

(b) *If Q' is a maximal $(4, 5)$ -colored path under f_1 with origin at u_3 (at v_3), then*

- (i) *u_1 (respectively, v_1) is an endpoint of Q' ,*
- (ii) *Q' passes through u_2 (through v_2), and*
- (iii) *Q' does not pass through v_2 (through u_2).*

Proof. We first prove (a). Let Q be a maximal $(3, 5)$ -colored path with origin at u_3 . Suppose that u_2 is not an endpoint of Q . Then, by making an interchange on Q , we obtain a proper coloring f^* satisfying $f^*(u_3) = f^*(u_2)$. Thus we obtain a situation similar to the situation considered in Case B.2.2, which is impossible because in that case we can transform f by interchanges to a required coloring f' satisfying (3.1). Hence u_2 is an endpoint of Q .

We now prove (b). Let Q' be a maximal path that is $(4, 5)$ -colored under f_1 and has origin at u_3 . By interchanging colors on this path we obtain a coloring f_2 .

Then u_1 is an endpoint of Q' , because otherwise the coloring f_2 will satisfy $f_2(u_3) = f_2(u_1)$, which is impossible by the same reason as in the proof of the part (a) of this claim.

Suppose that $u_2 \notin V(Q')$. Then by applying Lemma 3.2 to the path $v_1 u_1 u_2 u_3 u_4$, we deduce that one of conditions (II) and (III) of Lemma 3.2 holds. If the condition (III) holds, we may transform f_2 by interchanges on paths with colors from $\{3, 4, 5\}$ to a proper coloring g , such that $c \notin g(u_i) \cup g(u_{i+1})$ for some color $c \in \{3, 4, 5\}$, and some $i \in \{1, 2\}$. Then by recoloring $u_i u_{i+1}$ with c we obtain a situation that is similar to the situation considered in Case A or Case B.1. If the condition (II) holds, then, by Lemma 3.2, we may transform f_2 by interchanges to another proper coloring and again obtain a situation which is similar to the situation considered in Case A.

Thus if $u_2 \notin V(Q')$, then we obtain a situation which is similar to one of the situations considered in Case A and Case B.1, which is impossible because in those cases we can transform f by interchanges to a required coloring f' satisfying (3.1). Therefore $u_2 \in V(Q')$.

Finally, $v_2 \notin V(Q')$, because if $v_2 \in V(Q')$, then $v_1v_2x_1$ is a maximal $(1, 4)$ -colored path under f_2 , and by making an interchange on this path we obtain a coloring where only one edge adjacent to u_1v_1 is colored 1, thus reducing this situation to the situation considered in Case A. But this is impossible because in that case we can transform f by interchanges to a required coloring f' satisfying (3.1). $\square$

We continue the proof of the theorem. We have that the proper 5-edge coloring f_1 satisfies the conditions (3.5), (3.6), (3.7) and P is either a cycle of length at least 8, or a path with $P = u_lu_{l-1} \dots u_2u_1v_1v_2 \dots v_k$ where $l \geq 4$, $k \geq 4$ and $k + l \geq 9$. We assumed that it is impossible, using only interchanges, to transform f (and therefore f_1 too) to a proper 5-edge coloring f' satisfying (3.1). Now we will show that for all possible P , despite our assumption, we can transform f_1 by interchanges either directly to a proper coloring f' satisfying (3.1), or to another proper coloring φ where we have a situation which is similar to one of situations considered in Case A and Case B.1. This will lead to a contradiction, because in those cases we can transform φ , and therefore f_1 too, by a sequence of interchanges to a proper coloring f' satisfying (3.1).

Case B.2.3.1. P is a path and only one of the vertices u_4 and v_4 is an endpoint of P .

Without loss of generality we assume that u_4 is an endpoint of P . If $f_1(v_4) = \{1, 2, 3, 4\}$, then we recolor v_3v_4 with color 5, and thereafter make an interchange on $u_4u_3u_2u_1v_1v_2v_3$ and denote the obtained coloring by f_2 . Then $u_3u_2y_1$ is a maximal $(1, 5)$ -colored path under f_2 . By interchanging colors on this path and then recoloring u_4u_3 with color 1 we obtain a coloring where u_4u_3 and u_1v_1 are correct but not v_3v_4 . However, v_3v_4 is incident to only one edge colored 1. We have thus obtained a situation which is similar to one of the situations considered in Case A.

Suppose now that $f_1(v_4) = \{1, 2, 3, 5\}$. Then we consider a maximal path Q that is $(4, 5)$ -colored under f_1 and has origin at v_3 . It follows from Claim 1 that v_1 is an endpoint of Q and $u_2 \notin V(Q)$. By interchanging colors on Q we obtain a proper coloring that we denote by f_2 . Then we recolor v_3v_4 with 4, make an interchange on the maximal $(1, 2)$ -colored path $u_4u_3u_2u_1v_1v_2v_3$ and denote the obtained coloring by f_3 . Then $u_3u_2y_1$ is a maximal $(1, 5)$ -colored path under f_3 in G . By interchanging colors on this path and then recoloring u_3u_4 with 1, we obtain a coloring where u_4u_3 and u_1v_1 are correct but not v_3v_4 . As before, v_3v_4 is incident to only one edge colored 1. Hence, we obtain a situation which is similar to one of the situations considered in Case A.

Suppose now that $f_1(v_4) = \{1, 2, 4, 5\}$. Then we consider a maximal $(3, 5)$ -colored path Q with origin at v_3 . By Claim 1, v_2 is an endpoint of Q . By interchanging colors on Q , we obtain a coloring that we denote by f_2 . We now recolor v_3v_4 with 3 and then interchange colors on the maximal $(1, 2)$ -colored path $u_4u_3u_2u_1v_1v_2v_3$ and denote the obtained coloring by f_3 . Then $u_3u_2y_1$ is a maximal $(1, 5)$ -colored path under f_3 . By interchanging colors on this path and then recoloring u_3u_4 with color 1 we obtain a situation similar to the situation considered in Case A.

Case B.2.3.2. P is a cycle of length at least 8 or a path where none of the vertices u_4 and v_4 is an endpoint of P .

Since $\{1, 2\} \subset f_1(v_4)$ and $\{1, 2\} \subset f_1(u_4)$, the set $f_1(v_4)$ as well as the set $f_1(u_4)$ is one of the sets $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$ and $\{1, 2, 4, 5\}$. Furthermore, by symmetry in (3.5), we have to consider the following six cases only:

- $f_1(v_4) = f_1(u_4) = \{1, 2, 3, 4\}$,
- $f_1(v_4) = f_1(u_4) = \{1, 2, 3, 5\}$,
- $f_1(v_4) = f_1(u_4) = \{1, 2, 4, 5\}$,
- $f_1(u_4) = \{1, 2, 3, 4\}$ and $f_1(v_4) = \{1, 2, 3, 5\}$,
- $f_1(u_4) = \{1, 2, 3, 4\}$ and $f_1(v_4) = \{1, 2, 4, 5\}$,
- $f_1(u_4) = \{1, 2, 3, 5\}$ and $f_1(v_4) = \{1, 2, 4, 5\}$.

$f_1(v_4) = f_1(u_4) = \{1, 2, 3, 4\}$: Recolor v_3v_4 and u_3u_4 with 5 and then make an interchange on the maximal $(1, 2)$ -colored path containing u_1v_1 . Then $u_4u_3u_2y_1$ and $v_4v_3v_2x_1$ are maximal disjoint $(1, 5)$ -colored paths. By interchanging colors on these paths we obtain a proper coloring f' satisfying (3.1).

$f_1(u_4) = f_1(v_4) = \{1, 2, 3, 5\}$: Let Q_1 and Q_2 be maximal $(4, 5)$ -colored paths with origin at u_3 and v_3 , respectively. Then by Claim 1, u_1 is an endpoint of Q_1 , v_1 is an endpoint of Q_2 , $u_2 \notin V(Q_2)$ and $v_2 \notin V(Q_1)$. Note also that this implies that Q_1 and Q_2 are disjoint. After interchanging colors on Q_1 and Q_2 we may recolor u_3u_4 and v_3v_4 with 4. Next, interchange colors on the maximal $(1, 2)$ -colored path $u_3u_2u_1v_1v_2v_3$. By Claim 1, $u_2 \in V(Q_1)$ and $v_2 \in V(Q_2)$, which implies that both $u_4u_3u_2y_1$ and $v_4v_3v_2x_1$ are maximal $(1, 4)$ -colored paths, and by interchanging colors on these paths we obtain a proper coloring f' satisfying (3.1).

$f_1(u_4) = f_1(v_4) = \{1, 2, 4, 5\}$: We proceed similarly as in the preceding paragraph, but instead consider $(3, 5)$ -colored paths Q_1 and Q_2 with origin at u_3 and v_3 , respectively. By Claim 1, Q_1 and Q_2 have endpoints u_2 and v_2 , respectively. Note also that this implies that Q_1 and Q_2 are disjoint. Interchange colors on Q_1 and Q_2 , and then recolor u_3u_4 and v_3v_4 with 3. Next, we make an interchange on the maximal $(1, 2)$ -colored path containing u_1v_1 and denote the obtained coloring by f_2 . The paths $u_4u_3u_2y_1$ and $v_4v_3v_2x_1$ are maximal disjoint $(1, 3)$ -colored paths under f_2 . Interchange colors on these paths to obtain a proper coloring f' satisfying (3.1).

$f_1(u_4) = \{1, 2, 3, 4\}$ and $f_1(v_4) = \{1, 2, 3, 5\}$: Then we consider a maximal $(4, 5)$ -colored path Q under f_1 with origin at v_3 . By Claim 1, v_1 is an endpoint of Q , $v_2 \in V(Q)$ and $u_2 \notin V(Q)$. First by interchanging colors on Q , and thereafter recoloring u_3u_4 with 5 and v_3v_4 with 4 we obtain a coloring f_2 . Next, we interchange colors on the maximal $(1, 2)$ -colored path $u_3u_2u_1v_1v_2v_3$ and denote the obtained coloring by f_3 . Then $u_4u_3u_2y_1$ is a maximal $(5, 1)$ -colored path under f_3 and by interchanging colors on this path we obtain a proper coloring f_4 under which u_4u_3 and u_1v_1 are correct, but not v_3v_4 . However, v_3v_4 is adjacent to only

one edge colored 1, so we have obtained a situation which is similar to the one considered in Case A.

$f_1(u_4) = \{1, 2, 3, 4\}$ and $f_1(v_4) = \{1, 2, 4, 5\}$: Then we consider a maximal path Q that is $(3, 5)$ -colored under f_1 and has origin at v_3 . By Claim 1, v_2 is an endpoint of Q . We now interchange colors on Q , then recolor u_3u_4 and v_3v_4 with 5 and 3, respectively, and denote the obtained coloring by f_2 . Thereafter make an interchange on the maximal $(1, 2)$ -colored path $u_3u_2u_1v_1v_2v_3$. Then $u_4u_3u_2v_1$ is a maximal $(5, 1)$ -colored path under the new coloring f_3 . By interchanging colors on this path we obtain a proper coloring under which u_4u_3 and u_1v_1 are correct, but not v_3v_4 . However, v_3v_4 is adjacent to only one edge colored 1, so we have obtained a situation which is similar to the one considered in Case A.

$f_1(u_4) = \{1, 2, 3, 5\}$ and $f_1(v_4) = \{1, 2, 4, 5\}$: Let Q_1 be a maximal $(4, 5)$ -colored path with origin at u_3 . Then, by Claim 1, u_1 is an endpoint of this path, and $u_2 \in V(Q_1)$ and $v_2 \notin V(Q_1)$. Interchange colors on Q_1 and denote the obtained coloring by f_2 . Next, we consider a maximal $(3, 5)$ -colored path Q_2 with origin at v_3 . Similarly as in proof of Claim 1, it can be shown that v_2 is an endpoint of Q_2 . Interchange colors on Q_2 and denote the obtained coloring by f_3 . We recolor v_3v_4 and u_3u_4 with colors 3 and 4, respectively, and thereafter interchange colors on the maximal $(1, 2)$ -colored path $u_3u_2u_1v_1v_2v_3$. Denote this new coloring by f_4 . Then $u_4v_3v_2x_1$ is a maximal $(1, 3)$ -colored path under f_4 . By interchanging the colors on this path we obtain a situation which is similar to the situation considered in Case A.

We showed that for all possible P we can transform f_1 by interchanges either directly to a proper coloring f' satisfying (3.1), or to another proper coloring φ where we have a situation which is similar to one of situations considered in Case A and Case B.1. In both cases we can reach a proper coloring f' satisfying (3.1) by a sequence of interchanges. Since f_1 was obtained from f also by a sequence of interchanges, we can obtain f' from f by interchanges too. This lead to a contradiction with our assumption that it is impossible, using only interchanges, to transform f to a proper 5-edge coloring f' satisfying (3.1). $\square$

Proof of Theorem 1.3. Let G be a graph with $\chi'(G) = \Delta(G) = 4$. Furthermore, let h be a proper 4-edge coloring of G and f be a proper t -edge coloring of G , where $t \geq 5$. We will show that f can be transformed to h by a sequence of interchanges.

If $t > 5$, then by Vizing's algorithm f can be transformed into a proper 5-edge coloring using only interchanges (see e.g. [12]); the original problem now is reduced to the problem of transforming the obtained proper 5-edge coloring to h . Thus we may assume that $t = 5$.

Let m denote the minimum degree of G , $1 \leq m \leq 4$. We define graphs $G_0, G_1, \dots, G_{4-m}$ and for each G_i two proper colorings f_i and h_i in the following way:

1) $G_0 = G$ and $f_0 = f$, $h_0 = h$.

2) Suppose that G_i, f_i and h_i are already defined for some $i \geq 0$. If $i = 4 - m$, then stop. Otherwise take two disjoint copies G'_i and G''_i of G_i , with $V(G'_i) = \{x' : x \in V(G_i)\}$ and $V(G''_i) = \{x'' : x \in V(G_i)\}$. Then we define G_{i+1} as the graph obtained from G'_i and G''_i by joining x' and x'' with an edge, for each vertex $x \in V(G_i)$ with degree less than 4. The coloring f_i of G_i induces a proper 5-edge coloring f_{i+1} of G_{i+1} in the following way: we color

the copies G'_i and G''_i in the same way as G_i , and then color the edge joining x' to x'' with a color from $\{1, 2, 3, 4\}$ which is not used to color an edge incident with x in G_i . Similarly, the coloring h_i of G_i induces a proper 4-edge coloring h_{i+1} on G_{i+1} .

Clearly, the graph G_{4-m} is a 4-regular graph. Furthermore, h_i is a proper 4-edge coloring and f_i is a proper 5-edge coloring of G_i , for $i = 0, 1, \dots, 4 - m$. Then, by Theorem 3.4, the proper 5-edge coloring f_{4-m} of the graph G_{4-m} can be transformed into the proper 4-edge coloring h_{4-m} of G_{4-m} by a sequence of interchanges. It is clear that these interchanges also define a sequence of proper colorings of G , beginning with f and ending with h , such that each intermediate coloring in this sequence either is the same as the previous coloring or obtained from the previous coloring by either one interchange or a sequence of interchanges.

This completes the proof of Theorem 1.3. $\square$

4 Proof of Theorem 1.6

In this section we prove Theorem 1.6.

Proof of Theorem 1.6. Every graph G with $\Delta(G) \geq 5$ where $G_{\geq 5}$ is acyclic is a Class 1 graph because the graph G_Δ is acyclic.

Let $\mathcal{A}$ denote the set of graphs G with $\Delta(G) \geq 5$ where $G_{\geq 5}$ is acyclic and all Class 1 graphs with maximum degree 4. We will show that for every $G \in \mathcal{A}$ all proper $(\Delta(G) + 1)$ -edge colorings are Kempe equivalent. The proof is by induction on the maximum degree $\Delta(G)$.

By Corollary 1.4, the proposition is true for all graphs in $\mathcal{A}$ with maximum degree 4. Suppose that the proposition is true for all graphs in $\mathcal{A}$ with maximum degree less than k , $k \geq 5$, and let $G \in \mathcal{A}$ be a graph with $\Delta(G) = k$. Then $\chi'(G) = k$. Consider a proper k -edge coloring h of G . As in the proof of Theorem 1.1 we can assume that $M(h, k)$ is a maximal matching of G .

In order to show that any two proper $(k + 1)$ -edge colorings of G are Kempe equivalent it is sufficient to show that any proper $(k + 1)$ -edge coloring of G can be transformed to h by a sequence of interchanges. Let φ be an arbitrary proper $(k + 1)$ -edge coloring of G . By Proposition 2.1, φ can be transformed by a sequence of interchanges to a proper k -edge coloring φ_1 . Then the coloring φ_1 can be transformed to a proper $(k + 1)$ -edge coloring by sequentially recoloring the edges in the matching $M(h, k)$ with color $k + 1$. Clearly, the obtained proper $(k + 1)$ -edge coloring f satisfies the condition $M(h, k) = M(f, k + 1)$. Furthermore, f is obtained from φ by a sequence of interchanges.

Consider the graph $G_1 = G - M(h, k)$. The proper k -edge coloring h of G induces a proper $(k - 1)$ -edge coloring h' of G_1 and the proper $(k + 1)$ -edge coloring f of G induces a proper k -edge coloring f' of G_1 . Clearly, $\chi'(G_1) = k - 1$. Therefore, h' is a proper $\chi'(G_1)$ -edge coloring and f' is a proper $(\chi'(G_1) + 1)$ -edge coloring of G_1 .

Moreover, $\Delta(G_1) = k - 1$, because G is a Class 1 graph and $k = \chi'(G)$. Furthermore, $G_1 \in \mathcal{A}$. Then, by the induction hypothesis, f' and h' are Kempe equivalent. This implies that the coloring f of G can be transformed to the coloring h by a sequence of interchanges.

Thus for any graph $G \in \mathcal{A}$ with $\Delta(G) = k$ all proper $(k + 1)$ -edge colorings are Kempe equivalent. The proposition follows by the principle of induction. $\square$

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