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Shakedown in an elastic-plastic solid with a frictional crack

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Abstract

When subjected to periodic loading, elastic systems containing contact interfaces might exhibit frictional slip which ceases after some loading cycles. In such cases, it is said that the system shakes down. For elastic discrete systems presenting complete contacts, it has been proved that Melan’s theorem, originally proposed for elastic-plastic problems, offers a sufficient condition for the system to shake down, provided that the contact is of an uncoupled type. In the present paper, the application of Melan’s theorem is speculated for systems involving plasticity and friction. A finite element example of an elastic-plastic solid containing a frictional crack is discussed.

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1. Introduction

A solid or structure under oscillatory loading might develop permanent deformations which after a certain number of load cycles no longer progress, so that the structural response to load becomes purely elastic. Under such circumstances, it is said that the solid or structure elastically shakes down. The knowledge of shakedown limit is relevant if one has to assess energy dissipation conditions in a material, possibly with the aim of quantifying hysteretic damping under vibratory loading and/or damage under fatigue loading. The problem of shakedown has been deeply explored in the context of monolithic bodies whose mechanical behaviour is described by elastic-plastic material models. One of the main achievement is the development of limit analysis methods which allow the determination of shakedown conditions with simpler procedures in comparison to more laborious step-by-step incremental solutions. Shakedown limit analysis is centred on Melan’s theorem, Melan (1936), which following a static approach was originally conceived for the special case of elastic-perfectly plastic bodies with associative flow rule. The dual kinematic approach is due to Koiter (1960).

In the case of frictional contact of elastic bodies transmitting normal and tangential loads through the contact surface, microslips can occur along limited portions of the contact surface at load levels well below those needed

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to produce a gross slip. Such localized microslips play the role of the plastic strains in the counterpart problem of monolithic elastic-plastic bodies, since they are sources of energy dissipation. The formalization of the shakedown problem in frictional contact, described by the Coulomb’s law, of elastic bodies can be found in Klarbring et al. (2007) and Barber et al. (2008) for discrete and continuum systems, respectively. For discrete systems, it is shown that Melan’s theorem holds if the complete contact is uncoupled (no coupling between relative tangential displacements at the interface and the corresponding normal contact tractions), otherwise there is a limited range of the friction coefficient within which the theorem holds.

The problem of shakedown in the presence of both plasticity and frictional contact has attracted a limited attention. In the present paper, the problem of elastic-plastic bodies in contact with Coulomb friction is explored in their discrete formulation. An example of a finite element elastic-plastic solid containing a frictional cracks is presented for illustrative purposes.

2. Formulation of the problem

Consider a structure discretized by finite elements such that we can define a vector of nodal displacements \( \mathbf{u} \) and a work conjugate vector of nodal forces. The later is composed of given external forces \( \mathbf{F} \) and contact forces \( \mathbf{r} \). The components of the contact force are expressed in local coordinate systems, aligned in normal and tangential contact directions: there is a transformation matrix \( \mathbf{C} \) such that the total force is given by \( \mathbf{F} + \mathbf{C}^T \mathbf{r} \). Similarly, the contact displacements, which are relative displacements in case of a two-body problem, are collected in a vector \( \mathbf{w} \) and are related to nodal displacements by the equation

\[
\mathbf{w} = \mathbf{C} \mathbf{u}.
\]

If we evaluate strains at integration points of finite elements, these strains, collected in a vector \( \mathbf{\varepsilon} \), will be linear functions of the nodal displacements, i.e., there is a matrix \( \mathbf{B} \) such that

\[
\mathbf{\varepsilon} = \mathbf{B} \mathbf{u}.
\]

A vector of stresses \( \mathbf{\sigma} \), work conjugate to \( \mathbf{\varepsilon} \), can be defined such that equilibrium is described by

\[
\mathbf{F} + \mathbf{C}^T \mathbf{r} = \mathbf{D}^T \mathbf{\sigma},
\]

where

\[
\mathbf{D} = \mathbf{V} \mathbf{B},
\]

with \( \mathbf{V} \) being a diagonal matrix containing the volume associated to each integration point of finite elements.

Moreover, we assume an elastic-plastic material behavior such that the strain is additively decomposed into elastic and plastic parts:

\[
\mathbf{\varepsilon} = \mathbf{e} + \mathbf{p},
\]

\[
\mathbf{\sigma} = \mathbf{E} \mathbf{e},
\]

where \( \mathbf{E} \) is the elasticity matrix. In order to define the yield law we note that strain and stress vectors can be decomposed into subvectors, indicated by an index \( \mathbf{k} \) and related to individual integration point. For each such point, an elastic state defined by convex yield functions \( f_k(\mathbf{\sigma}_k) \), and the time derivative of the plastic strain \( \dot{\mathbf{p}}_k \) is governed by

\[
f_k(\mathbf{\sigma}_k) \leq 0, \quad (\mathbf{\sigma}_k - \mathbf{\sigma}_k^*)^T \dot{\mathbf{p}}_k \geq 0 \quad \text{for all } \mathbf{\sigma}_k^* \text{ such that } f_k(\mathbf{\sigma}_k^*) \leq 0.
\]

To state the conditions for frictional contact we define a unit normal vector \( \mathbf{n}_i \) for each obstacle, i.e. for each contact node, pointing from the obstacle towards the body (or, e.g., from body B to body A in case of two-body contact). Displacements and contact forces are decomposed into tangential and normal vectors at each contact node \( i \):

\[
\mathbf{w}_i = \mathbf{w}_i^t + \mathbf{w}_i^n \mathbf{n}_i, \quad \mathbf{w}_i^t \cdot \mathbf{n}_i = 0,
\]

\[
\mathbf{r}_i = \mathbf{r}_i^t + \mathbf{r}_i^n \mathbf{n}_i, \quad \mathbf{r}_i^t \cdot \mathbf{n}_i = 0,
\]
where a central dot indicates the scalar product of vectors. Clearly, \( r_{in} = r_i \cdot n_i \) and \( u_{in} = w_i \cdot n_i \). We intend to treat only problems with a known contact surface (complete contact) and therefore require that

\[
    w_{in} = 0, \quad r_{in} \geq 0. 
\]

Coulomb’s law of friction reads:

\[
    |r_{it}| \leq \mu_i r_{in} 
\]

\[
    0 < |r_{it}| = \mu_i r_{in} \implies \ddot{w}_{it} = -\lambda_i r_{it}, \quad \lambda_i \geq 0, 
\]

\[
    |r_{it}| < \mu_i r_{in} \implies \ddot{w}_{it} = 0. 
\]

where \( \mu_i \) is the coefficient of friction and a superposed dot denotes time derivative. This law may be said to have two ingredients. Firstly, there is a condition which states that the contact forces should belong to a set of admissible such forces, the so-called Coulomb’s friction cone. Secondly, there is a condition which specifies when and how sliding takes place: sliding is opposite to the friction force. Note that \( r_{in} \geq 0 \) is obviously included in (9).

2.1. Residual state

The residual state is defined by an elastic unloading \((F=0)\), keeping \( p \) and \( w \) fixed (‘welding’). The elastic strain in this state is denoted \( e_R \) and the total strain is \( e_R \), while the displacement is denoted \( u_R \). The residual stresses and contact forces are \( \sigma_R \) and \( r_R \). The governing equations become:

\[
    e_R = B u_R, \quad (12) 
\]

\[
    e_R = e_R + p, \quad (13) 
\]

\[
    C^T r_R = D^T \sigma_R, \quad (14) 
\]

\[
    \sigma_R = E e_R, \quad (15) 
\]

\[
    w = C u_R. \quad (16) 
\]

The residual state, defined by (12) to (16), is connected to the ‘real’ state by an elastic process achieved by reintroducing the force. Denoting this elastic response by index \( E \) it holds that:

\[
    e_E = B u_E, \quad (17) 
\]

\[
    e = e_R + e_E + p, \quad (18) 
\]

\[
    F + C^T r_E = D^T \sigma_E, \quad (19) 
\]

\[
    \sigma_E = E e_E, \quad (20) 
\]

\[
    0 = C u_E, \quad (21) 
\]

\[
    u = u_R + u_E, \quad (22) 
\]

\[
    r = r_R + r_E. \quad (23) 
\]

Note that \( e = e_R + e_E \) so by (6), (15) and (20) we have

\[
    \sigma = \sigma_R + \sigma_E. \quad (24) 
\]
2.2. Shakedown

An elastic-plastic frictional system is said to have reached a state of shakedown if for all future times $\dot{p} = 0$ and $\dot{w} = 0$, i.e., no plastic yielding and no frictional sliding occurs. An obvious necessary condition for this to happen is that there exists a residual state ($\delta r, \delta F$) such that, for all times, $\delta \sigma = \delta R + \sigma_F$ satisfies $f_i(\delta r) \leq 0$ and $\dot{r} = \dot{r}_R + \dot{r}_E$ satisfies $|\dot{r}_R| \leq \mu |\dot{r}_F|$. A proof of sufficient conditions for shakedown to occur can be found in Klarbring and Barber (2012), where the theorem uses the notion of no elastic coupling between normal and tangential contact directions. More in details: if for a contact displacement $w$, such that $w_{in} = 0$ for all $i$, it holds that $r = kw$ is such that $r_{in} = 0$, then we say that there is no elastic normal-tangential coupling ($k$ is the contact stiffness matrix defined by $k^{-1} = CK^{-1}C^T$, where $K = B^T E D$ is the standard stiffness matrix).

3. Optimization procedure

We consider a system of quasi-statically time-varying nodal forces applied to the (finite element) discrete model. The forces are sum of a constant term and cyclically time-varying term, i.e. $F(t) = F_0 + \beta F(t)$. If we choose to describe the size of the load domain in terms of the load parameter $\beta$, the direct way of thinking of the shakedown limit is by increasing $\beta$ until the conditions stated in Section 2.2 are no more respected. By a mathematical approach we have to construct a residual field, expressed in terms of certain parameters, and alter these parameters so that $\beta$ is maximized without violating some constraints. This procedure goes by the name of optimization, the quantity we need to maximize is the objective function and the parameters for which we have the maximum value of $\beta$ constitute the optimum parameters.

The optimization problem under consideration is a non linear one due to the non linear nature of the convex yield function $f_i(\delta r)$. In the following, our attention is restricted to the quadratic yield function of Mises. In order to reduce the computational burden, the optimization problem is tackled by solving separately the maximization problem of Coulomb contact from that of Mises plasticity. The optimum vector of each maximization problem is then plugged into the other one so as to generate an initial residual state. An iterative procedure is then set up to convergence of the optimum parameters.

In details, we set $\beta$ and $w$ as the optimum vector for the Coulomb contact. The external loads according to the expression selected above generate elastic (‘welded’) reactions at the contact nodes which can be expressed as

$$r_E = r_{E,0} + \beta \tilde{r}_E(t).$$

Then, by considering expressions from (12) to (16), the residual reactions at contact nodes can written as

$$r_R = k(w - CK^{-1}D^T E p),$$

Hence, by adding the elastic term to the residual one, the total reaction vector at the contact nodes becomes

$$r = (r_{E,0} - \kappa CK^{-1}D^T E p) + \beta \tilde{r}_E(t) + kw.$$  

where the constant term (in round brackets), the time-varying term and the term function of the frictional sliding $w$ can be identified. The term $-\kappa CK^{-1}D^T E p$ represents the coupling term with plasticity.

We can get a rid of time by maximizing the projection of time-varying reaction vector $\tilde{r}_E(t)$ on the normals to the Coulomb’s cone, namely

$$M_i^w = \max_i [N_i^w \tilde{r}_E(t)]$$

where $N_i^w$ is the unit vector normal to the Coulomb cone for backward slip ($\alpha = 1$) and forward slip ($\alpha = 2$). The scalar quantities $M_i^w$ can be arranged in the vector $M$ pertaining the contact nodes. Also, the unit vector $N_i^w$ is assembled in the block diagonal matrix $N$.

Now, we can project on $N$ the vector of (27), so as to obtain the following linear optimization statement

$$\beta_S = \max_{\beta,w} [\beta \mid \beta M + N^T \kappa w \leq -N^T (r_{E,0} - \kappa CK^{-1}D^T E p); \beta \geq 0, w_{in} = 0]$$

(29)
At this point, the residual displacement vector $u_{R}$ associated with $w$ can be obtained from equilibrium (14) and compatibility (16).

Similarly to the reactions at contact nodes, the stress components at integration points can be expressed as

$$\sigma = (\sigma_{E,0} + E Bu_{R}) + \beta \hat{\sigma}_{E}(t) - Ep.$$  \hspace{1cm} (30)

where the constant term (in round brackets), the time-varying term and the term function of the plastic strain $p$ can be identified. The term $E Bu_{R}$ represents the coupling term with friction.

Finally we have the following non linear convex optimization statement

$$\beta_{S} = \max_{\beta, p} \{ \beta \mid \max_{k} f_{k}(\sigma_{k}) \leq 0; \beta \geq 0 \}$$  \hspace{1cm} (31)

where $\sigma_{k}$ is the subvector of the vector (30) related to the integration point $k$.

Note that the maximization with respect to time appearing in (28) and (31) is executed for a discrete number of time instants, corresponding to the vertex of the convex domain enveloping the load path.

4. Illustrative example

We study a plate containing a central frictional crack with a constant coefficient of friction $\mu$ (Fig. 1). The dimensions are: plate width $2b$ and height $2h$, crack length $2a$ and $\delta = 0$. Given the geometrical symmetry of the problem with respect to the contact line, an uncoupled frictional contact takes place. (By considering a shift $\delta$ of the crack along the height of the plate, coupled frictional contact can also be enforced.) The plate of unit thickness, under plane stress condition, is loaded by a system of self-equilibrated tractions along the boundaries, whose normal and tangential (with respect to boundaries) resultants are indicated by $P$ and $Q$, respectively. The resultant $P$ corresponds to a uniform compression pressing the crack faces, while the resultant $Q$ is uniformly distributed along the four sides of the plate so as to produce an uniform shear within it. The following load path, expressed in terms of $P$ and $Q$, is considered:

$$P = P_{0} + \beta P_{0} \tan \gamma \cdot g(t), \hspace{2cm} Q = \beta P_{0} g(t),$$  \hspace{1cm} (32)

where $g(t) \in [0, 1]$ is a general oscillating time function and $\gamma$ ($\gamma \in [0, \pi/4]$) is the angle defining the direction of the oscillating resultant acting along the plate widths.

A finite element model with 8-node plane isoparametric elements (with $2 \times 2$ integration points) is considered. The finite element model consists of 368 elements (to reduce the size of the problem the plastic behaviour is limited to a region - constituted by 236 elements - surrounding the crack) and 1,188 nodes (of which 35 are contact nodes). Thus,
the size of the vector \( w \) is 70, of \( u \) 2,376 and of \( p \) 2,832. The adopted geometrical dimensions of the plate are such that \( b/a = 10 \) and \( h/a = 4 \).

In Fig. 2, some results for constant compression and oscillating shear (\( \gamma = 0 \)) are illustrated as the coefficient of friction \( \mu \) is made to vary. The shakedown limit \( \beta_S^{(E)} \) against \( \mu \), obtained from the optimization procedure in the case of elastic behaviour of the material (\( \sigma_0/\sigma_M \to \infty \), where \( \sigma_0 = \) yield stress and \( \sigma_M = (P_0/2b) \sqrt{(1 + \beta_S^{(E)} \tan \gamma)^2 + 3(\beta_S^{(E)})^2} \)), is shown, together with some results of incremental analysis for a value of the ratio \( \sigma_0/\sigma_M \) equal to 1.1. In particular, the results of incremental analysis, for each selected value of the coefficient of friction, are related to two different load levels corresponding to non-dissipative and dissipative (due to frictional slip) steady-state conditions. It can be seen that a (slight) increase of the shakedown limit due to plasticity in comparison to that for the elastic material occurs (note that the shakedown limit curve for elastic material indicates a load factor above which shakedown is impossible). Figure 3 shows the same type of results of Fig. 2, but with respect to the inclination angle \( \gamma \).

5. Conclusions

A fairly general problem of discrete systems involving friction and plasticity is presented. The formulation of the problem is discussed with reference to Coulomb friction and convex yield function, by pointing the elastic normal-tangential coupling at the contact surface and, in turn, the coupling between friction and plasticity. Some preliminary
results, in terms of optimization procedure and time-marching analysis, related to a finite element model of a plate with a frictional crack are presented for illustrative purposes.

References