Signed bounded confidence models for opinion dynamics

Claudio Altafini and Francesca Ceragioli

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Claudio Altafini [a], Francesca Ceragioli [b]

[a] *Division of Automatic Control, Dept. of Electrical Engineering, Linköping University, SE-58183, Linköping Sweden.*

[b] *Dip. di Scienze Mathematiche, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Italy.*

Abstract

The aim of this paper is to modify continuous-time bounded confidence opinion dynamics models so that "changes of opinion" (intended as changes of the sign of the initial states) are never induced during the evolution. Such sign invariance can be achieved by letting opinions of different sign localized near the origin interact negatively, or neglect each other, or even repel each other. In all cases, it is possible to obtain sign-preserving bounded confidence models with state-dependent connectivity and with a clustering behavior similar to that of a standard bounded confidence model.

Key words: Opinion dynamics; Bounded Confidence models; Signed Graphs; ODEs with discontinuous right-hand-side.

1 Introduction

A bounded confidence model is a model of consensus-like opinion dynamics in which the agents interact with each other only when their opinions are close enough. Such a class of models usually goes under the name of Hegselmann-Krause models [21] and has the peculiarity of expressing confidence as a function of the distance between the agents states. As a consequence, the graph that describes the interactions between the agents is itself state-dependent and varying in time. The emerging behavior of such a model is that the agents tend to form clusters, and a consensus value is achieved among the agents participating to a cluster. In the control literature, various aspects of such models have been studied: discrete-time [4,12,24], continuous-time [5,25,28], and stochastic [8] dynamics, convergence time [10,24], behavior of a continuum of agents [22], existence of interaction rules that allow to preserve the connectivity [30], presence of stubborn agents [14] etc. See [13,17,23] for an overview. In continuous time, if the confidence range is delimited by a sharp threshold, then the right hand side of the resulting ODEs is discontinuous. Existence and uniqueness analysis of the corresponding solutions have been carried out in [5,6]. In [6] approximations of the discontinuous dynamics are suggested.

In the social sciences literature, many models have been proposed to represent opinion dynamics and interpersonal influences in a social network of individuals [15,20,27,29]. A system-theoretical overview of some of these models, like for instance the French-DeGroot model (consensus-like behavior, without any distance-dependent bound, [16,11]) or its Friedkin-Johnsen generalization (mixture of consensus and stubborness, [19]) is given in [26], where many more pointers to relevant papers are provided. Alongside a vast theoretical research, the field of experimental social psychology has produced a number of empirical studies (mainly involving small social groups) meant to validate such social opinion change models. There is a wide consensus in this literature that the only experimental feature that can be consistently documented in this context is that opinions are constrained to the convex hull of the initial conditions, but that the sensitivity of an individual to influences is a subjective parameter, varying widely across a community of individuals [18]. Evidence of a threshold on the confidence range does not seem to be documented in this literature. In spite of the lack of empirical validation, from a dynamical point of view the behavior of a bounded confidence model is interesting as a mechanism for the formation of clusters of agents, according only to the initial conditions on the ODEs. It is in view of its rich dynamical behavior and of the nontrivial mathematics induced by state-dependence of the interaction graph that we have decided to adopt it in this paper.

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Email addresses: `claudio.altafini@liu.se` (Claudio Altafini), `d003432@polito.it` (Francesca Ceragioli).

For the bounded confidence model, there is a special situation in which confidence between the agents may be lost even if the opinions are in proximity, and it is when the signs of the opinions are different. It is intuitively clear that "changing opinion", intended as changing sign of an agent's opinion, is a fairly drastic process, a "mental barrier" not so likely to be trespassed in real scenarios. Currently available bounded confidence models only consider the value of the opinions relative to each other, and do not distinguish between the case of all opinions having the same sign or less, i.e., the opinions can freely cross zero while converging to a local consensus value. In other words, the bounded confidence models are translationally invariant.

The aim of this paper is to propose models of bounded confidence in which translational invariance is replaced by preservation of the signs of the original opinions. Several possible ways to implement this principle exist, and in fact in this paper we propose 3 different models. Their common basis is that opinions having the same sign attract each other, while opinions of different sign can lead to negative interactions, indifference or even repulsion. Consequently, the dynamics among opinions of different signs can be constructed according to different rules. The simplest possibility is to make use of the notion of bipartite consensus introduced in [1]. Under certain conditions on the graph of the signed interactions, the agents split into two groups converging to a consensus value which is equal in modulus but opposite in sign. The graphical condition that needs to be fulfilled, called structural balance [1], is naturally satisfied when initial conditions that have the same sign are associated to positive edges ("friends") and those having opposite signs to negative edges ("enemies"). The sign function used in the model to make this distinction implies that even when no bound on the confidence is present, the connectivity is state-dependent: the graph describing interactions among agents depends on the initial conditions. In spite of a discontinuous right-hand side, this model almost always has unique solutions. Only when one or more of the initial opinions are 0, then multiple Carathéodory solutions arise. When a bound is added on the confidence range, then the negative interactions among agents are only localized around the origin and do not affect the asymptotic behavior of opinions far from 0. Even with the negative interactions around the origin, almost all initial conditions are however *proper* (i.e., lead to a unique solution which can be prolonged to $+\infty$ without incurring in accumulation of nondifferentiability points). The overall behavior of the model is still to create clusters of agents achieving a common consensus value within each cluster while in addition preserving the sign of all initial conditions.

The behavior in terms of existence and uniqueness of the solutions, as well as in terms of the asymptotic clustering, is similar if in the model agents having opposite opinions simply ignore each other. Also in this case, in fact, a (Heaviside) sign function must be introduced in order to suppress the contribution of nearby opinions of different sign in the bounded confidence dynamics. The discontinuities of the sign function may give rise to multiple Carathéodory solutions. However, almost all initial conditions are still proper and lead to the formation of clusters of opinions.

Finally, when sign discordance is modeled as a repulsion term, the combination of sign preservation and bounded confidence can give rise to more complex behaviors in which solutions á la Carathéodory are not guaranteed to exist. In the third model we give, the repulsion dynamics may lead to discontinuities which are attractive, meaning that the opinion may stay on the discontinuity value while forming clusters. As in the previous models, the resulting solutions (now of Krasovskii type) have the property of preserving the sign of the original opinions, i.e., no agent has to change its mind during the time evolution of the system.

A preliminary version of this material was presented at the 2016 European Control Conference, see [7]. This conference paper deals only with the first of the three models discussed in the current manuscript. The other two variants are novel material presented here for the first time.

The rest of this paper is organized as follows. After recalling the necessary background material in Section 2, in Section 3 we introduce the three models of signed bounded confidence and describe their dynamical behavior in what is the main theorem of the paper. To illustrate their differences, in Section 4 the three models are studied in absence of any confidence bound. Finally Section 5 contains the proof of the main theorem and a series of examples.

2 Background material

2.1 Linear algebraic notions

A matrix $A \in \mathbb{R}^{n\times n}$ is said Hurwitz stable if all its eigenvalues $\lambda_i(A)$, $i = 1, \dots, n$, have $\mathrm{Re}[\lambda_i(A)] < 0$. It is said marginally stable if $\mathrm{Re}[\lambda_i(A)] \leqslant 0$, $i = 1, \dots, n$, and $\lambda_i(A)$ such that $\mathrm{Re}[\lambda_i(A)] = 0$ have an associated Jordan block of order one. A is said irreducible if there does not exist a permutation matrix Π such that $\Pi^T A \Pi$ is block triangular. The matrices A considered in this paper will always be symmetric: $A = A^T$. A matrix A is said *diagonally dominant* if

$$|A_{ii}| \geqslant \sum_{j\neq i} |A_{ij}|, \quad i = 1, \dots, n. \tag{1}$$

It is said *strictly diagonally dominant* when all inequalities of (1) are strict, and *weakly diagonally dominant* when at least one (but not all) of the inequalities (1) is strict. A is said *diagonally equipotent* [2] if

$$|A_{ii}| = \sum_{j\neq i} |A_{ij}|, \quad i = 1, \ldots, n.$$

2.2 Signed graphs

Given a matrix $A = A^T \in \mathbb{R}_+^{n\times n}$, consider the undirected graph $\Gamma(A)$ of A: $\Gamma(A) = \{\mathcal{V}, A\}$ where $\mathcal{V} = \{1, \ldots, n\}$ is the set of n nodes and A is its weighted adjacency matrix. Self weights are excluded from A: $A_{ii} = 0$. $\Gamma(A)$ is connected if there exists a path between each pair of nodes in $\mathcal{V}$. It is fully connected if $A_{ij} \neq 0 \; \forall i \neq j$. An adjacency matrix that can assume both positive and negative values is denoted A_s and its associated signed graph $\Gamma(A_s)$. An undirected signed graph $\Gamma(A_s)$ is said *structurally balanced* if all its cycles are positive (i.e., they have an even number of negative edges). $\Gamma(A_s)$ is structurally balanced if and only if there exists a vector $s = [s_1 \; \cdots \; s_n]$, $s_i = \pm 1$, such that the matrix $A = SA_sS$ is nonnegative definite, where $S = \mathrm{diag}(s)$ is the diagonal matrix having the entries of s on the diagonal.

2.3 Bipartite Consensus

Given a matrix A, $A_{ij} \geq 0$ for $i \neq j$, the *(standard) Laplacian* associated with A is the matrix L of elements

$$L_{ij} = \begin{cases} -A_{ij} & \text{if } i \neq j \\ \sum_{k\neq i} A_{ik} & \text{if } i = j \; . \end{cases}$$

The linear system

$$\dot{x} = -Lx. \tag{2}$$

describes a consensus problem. If A is irreducible, its solution corresponding to the initial condition $x(0)$ converges to $x^* = (1/n)\sum_j x_j(0)\mathbb{1}$, where $\mathbb{1}$ is the right eigenvector relative to $\lambda_1(L) = 0$, i.e. consensus is asymptotically reached. The *signed Laplacian* L_s of A_s is given by

$$L_{s,ij} = \begin{cases} -A_{s,ij} & \text{if } i \neq j \\ \sum_{k\neq i} |A_{s,ik}| & \text{if } i = j. \end{cases} \tag{3}$$

For nonnegative adjacency matrices the two definitions coincide. In any case, the two Laplacians are diagonally equipotent matrices. L is always singular, while L_s may or may not be [1]. $\Gamma(A_s)$ is structurally balanced if and only if L_s is a singular matrix, see [1,2]. If $\Gamma(A_s)$ is structurally balanced, then L_s is marginally stable and a bipartite consensus problem is given by the following linear system:

$$\dot{x} = -L_s x \tag{4}$$

whose solution is $x^* = (1/n)\sum_j |x_j(0)|S\mathbb{1}$, corresponding to a bipartite consensus value: $|x_i^*| = |x_j^*|$.

2.4 Solutions of ODEs

Given the system

$$\dot{x} = g(x), \quad x(0) = x_o \tag{5}$$

with $g : \mathbb{R}^n \to \mathbb{R}^n$, a *classical solution* of (5) on the interval $[0, t_1)$ is a map $\phi : [0, t_1) \to \mathbb{R}^n$ such that (i) ϕ is differentiable in $[0, t_1)$; (ii) $\phi(0) = x_o$; and (iii) $\dot{\phi}(t) = g(\phi(t))$ for all $t \in [0, t_1)$. When a function satisfies the equation (5) except for a set of measure zero, then we can use the notion of Carathéodory solution. More formally, a *Carathéodory solution* of (5) on the interval $[0, t_1)$ is a map $\phi : [0, t_1) \to \mathbb{R}^n$ such that (i) ϕ is absolutely continuous in $[0, t_1)$; (ii) $\phi(0) = x_o$; (iii) $\dot{\phi}(t) = g(\phi(t))$ for almost all $t \in [0, t_1)$. See [9,6] for more details.

Following [5], we say that $x_o \in \mathbb{R}^n$ is a *proper initial condition* if it satisfies the following conditions:

(a) there exists a unique Carathèodory solution $\phi : \mathbb{R}^+ \to \mathbb{R}^n$, $t \to \phi(t)$ satisfying $\phi(0) = x_o$,
(b) the subset of $\mathbb{R}^+$ on which ϕ is not differentiable is at most countable and has no accumulation point,

For the confidence models discussed in this paper a third condition can be added.

(c) if $\phi_i(t) = \phi_j(t)$ then $\phi_i(t') = \phi_j(t')$ for all $t' \geq t$.

Notice that condition (c) may not be required for existence of proper initial conditions in general. We list it here for convenience, as it is always needed for the confidence models considered in this manuscript. In the terminology of [5], proper initial conditions yield proper Carathéodory solutions of (5).

A Krasowskii solution of (5) on the interval $[0, t_1)$ is a map $\phi : [0, t_1) \to \mathbb{R}^n$ such that (i) ϕ is absolutely continuous; (ii) for almost every t, ϕ satisfies

$$\dot{\phi}(t) \in Kg(\phi(t)),$$

where $Kg(x) = \cap_{\delta>0}\overline{co}\{g(y) : y \text{ such that } \|x - y\| < \delta\}$ and $\overline{co}$ denotes the closed convex hull.

3 Signed bounded confidence

In this Section we first recall the properties of the standard bounded confidence model, as can found in [5,6]. Then we introduced three different variants of what we call *signed bounded confidence* model, i.e., a bounded confidence model which preserves the sign of the initial conditions. The dynamical properties of these three models are described in what is the main theorem of this paper.

3.1 Standard bounded confidence model

A bounded confidence model is given by the following consensus-like scheme in $\mathbb{R}^n$

$$\dot{x}_i(t) = \sum_{j \text{ s.t. } |x_j(t)-x_i(t)|<1} (x_j(t) - x_i(t)). \qquad (6)$$

The interpretation of (6) is that only nodes whose opinion is closed enough to that of node i contribute to the summation at each t, see Fig. 1, left panel.

If $\Gamma(A(x(t))) = \{\mathcal{V}, A(x(t))\}$ is the graph given by the pattern of active connections of (6) at time $t \geq 0$, then for the adjacency matrix $A(x(t))$ one has $A_{ij}(x(t)) = 1$ if and only if $|x_j(t) - x_i(t)| < 1$. $A(x(t))$ is in general time-varying and discontinuous in time.

The behavior of (6) is well-known. For example, we have that in spite of the discontinuous righthand side, (6) has a unique Carathéodory solution for almost all initial conditions. In fact, it is shown in [5] that except for at most a set of Lebesgue measure zero all initial conditions are proper initial conditions. This is listed as property **P1** in the following Theorem, that summarizes the behavior of the bounded confidence model of [5].

Theorem 1 *Consider the system* (6). *Its solutions have the following properties:*

P1: *Almost all $x_o \in \mathbb{R}^n$ are proper initial conditions.*

Furthermore, for any solution $x(t)$ issuing from a proper initial condition x_0:

P2: $x_i(\tau) \leq x_j(\tau) \Longrightarrow x_i(t) \leq x_j(t)$ *for all* $t \geq \tau$;
P3: *The average opinion*

$$c(t) = \frac{1}{n}\sum_i x_i(t) \qquad (7)$$

is constant for all $t \geq 0$;
P4: *The function* $W(x(t)) = \sum_i (x_i(t) - c)^2$ *is non-increasing;*
P5: *If there exists* $\tau \geq 0$ *such that* $\Gamma(A(x(\tau)))$ *is fully connected, then* $\Gamma(Ax((t)))$ *is fully connected* $\forall\, t \geq \tau$.
P6: *If there exists* $\tau \geq 0$ *such that* $\Gamma(A(x(\tau)))$ *is not connected, then* $\Gamma(A(x(t)))$ *is not connected for all* $t \geq \tau$.
P7: $\lim_{t\to+\infty} x(t) = x^*$ *where* x^* *is such that for all* $i, j \in \{1, \ldots, n\}$ *either* $x_i^* = x_j^*$ *or* $|x_i^* - x_j^*| \geq 1$.

For all properties, a proof is available in the literature (see e.g. [5,6,30]) or immediately deducible from it. Hence it is omitted here. The meaning of properties **P5**-**P7** is that opinions tend to cluster into "local" consensus values distant at least 1 from each other, see Fig. 1, left panel (in practice this distance typically is bigger, close to 2, see [5]). Clearly this entails a splitting of the graph $\Gamma(A(x(t)))$ into disjoint connected components,.

In a model like (6), the sign of the opinions does not matter but only their distance does, i.e., nearby opinions of different sign are treated as those of equal sign, and the opinions can freely cross zero while converging to a local consensus value, see Fig. 1, left panel.

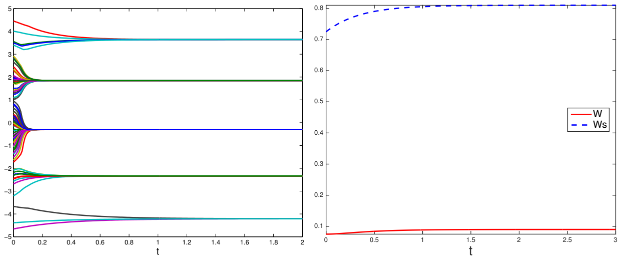

Fig. 1. Left panel: Standard bounded confidence model. Opinions cluster in local "consensus" values, regardless of the sign of the opinion. Right panel: Example 1: the functions $W(x(t))$ (red, solid) and $W_s(x(t))$ (blue, dashed) are both increasing.

3.2 Signed bounded confidence models and main result

The three models proposed in this Section combine sign invariance with bounded confidence. Sign invariance means that initial opinions that are strictly positive or negative have to remain so during the entire evolution. On the other hand, initially null opinions may become positive or negative. The three models correspond to three different ways of achieving sign preservation of the opinions. Their specific features will be described at length in Section 4.

Version 1: Bipartite consensus with bounded confidence. As in a bipartite consensus [1], if agent i and j have opinions of different signs, agent i's opinion is attracted by the opposite of agent j' opinion. If their opinions are too far, namely if their difference exceeds the confidence threshold, agent i and j do not influence each other. The model is

$$\dot{x}_i(t) = \sum_{j \text{ s.t. } |x_j(t)-x_i(t)|<1} \Big(\mathrm{sgn}\big(x_j(t)x_i(t)\big)x_j(t) - x_i(t)\Big) \qquad \text{(V1)}$$

where $\mathrm{sgn}(\cdot)$ is the sign function

$$\mathrm{sgn}(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ -1 & \text{if } z < 0. \end{cases}$$

Version 2: Same sign bounded confidence. The second model describes the case of agents with opinions of different signs ignoring each other. This interaction rule, combined with bounded confidence gives

$$\dot{x}_i(t) = \sum_{j \text{ s.t. } |x_j(t)-x_i(t)|<1} \text{ssgn}\big(x_j(t)x_i(t)\big)\Big(x_j(t) - x_i(t)\Big) \quad \text{(V2)}$$

where the "*same sign function*" ssgn(·) is the left-continuous Heaviside function

$$\text{ssgn}(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z \leq 0. \end{cases}$$

Version 3: Homogeneous repulsion with bounded confidence. In the third model, an agent's opinion is repelled by opinions of different sign, and again this interaction is combined with bounded confidence:

$$\dot{x}_i(t) = \sum_{j \text{ s.t. } |x_j(t)-x_i(t)|<1} \text{sgn}\big(x_j(t)x_i(t)\big)\Big(x_j(t) - x_i(t)\Big). \quad \text{(V3)}$$

The properties of the models (V1)-(V3) are summarized in the following theorem.

Theorem 2 *For the three signed bounded confidence models* (V1)-(V3)*, we have:*

- *The model* (V1) *satisfies the property* **P1***. For solutions issuing from proper initial conditions, the model* (V1) *satisfies* **P2***, but not* **P3***,* **P4***,* **P5***,* **P6***,* **P7** *and furthermore:*
  P8*: For all solutions $x(t)$ of* (V1) *there exists $\lim_{t\to+\infty} x(t) = x^*$ where x^* is such that for all $i, j \in \{1, \dots, n\}$ such that $\text{sgn}(x_i^* x_j^*) > 0$ either $x_i^* = x_j^*$ or $|x_i^* - x_j^*| \geq 1$ and for all $i, j \in \{1, \dots, n\}$ such that $\text{sgn}(x_i^* x_j^*) < 0$ it holds either $x_i^* = -x_j^*$ or $|x_i^* - x_j^*| \geq 1$.*
  P9*: If $x_i(0) \neq 0\ \forall\, i = 1, \dots, n$, then the average of the absolute values, c_s, is constant for all $t \geq 0$.*
  P10*: Let x_o be such that $(x_o)_i = 0$ for some i. There exists a Carathéodory solution $x(t)$ of* (V1) *such that $x(0) = x_o$ and $x_i(t) \equiv 0$.*
- *The model* (V2) *satisfies the property* **P1***. For solutions issuing from proper initial conditions it satisfies* **P2***,* **P3***,* **P4***,* **P5***,* **P6***,* **P9***,* **P10***. Furthermore, it does not satisfy* **P7***,* **P8** *but instead it holds:*
  P11*: For all solutions $x(t)$ of* (V2) *there exists $\lim_{t\to+\infty} x(t) = x^*$ where x^* is such that for all $i, j \in \{1, \dots, n\}$ such that $\text{sgn}(x_i^* x_j^*) > 0$ either $x_i^* = x_j^*$ or $|x_j^* - x_i^*| \geq 1$.*
- *The model* (V3) *does not satisfy* **P1** *but it satisfies the property*
  P12*: For any initial condition there exists a Krasovskii solution and it is complete.*
  With respect to Krasowskii solutions, the model (V3) *satisfies* **P3***,* **P10***, but not* **P2***,* **P4***,* **P5***,* **P6***,* **P7***,* **P8***,* **P9***. Furthermore, it holds:*
  P13*: For all solutions $x(t)$ of* (V3) *there exists $\lim_{t\to+\infty} x(t) = x^*$ where x^* is such that for all $i, j \in \{1, \dots, n\}$ such that $\text{sgn}(x_i^* x_j^*) > 0$ either $x_i^* = x_j^*$ or $|x_i^* - x_j^*| \geq 1$ and for all $i, j \in \{1, \dots, n\}$ such that $\text{sgn}(x_i^* x_j^*) < 0$ it holds $|x_i^* - x_j^*| \geq 1$.*

The proof of this Theorem will be given in Section 5. Before that, we need to describe the characteristics of the unbounded confidence version of three models (V1)-(V3). This is done in the next Section.

4 Signed unbounded confidence

In order to analyze the dynamical properties of the models (V1)-(V3), it is useful to disentangle the sign-preservation property from the effect of a bounded confidence. For this reason in this section we analyze the analogous of models (V1)-(V3) with an infinite confidence interval. We call them *signed unbounded confidence* models.

The results obtained in this section serve as preliminaries to the proof of Theorem 2.

4.1 Version 1: bipartite consensus (with unbounded confidence)

This model is inspired by the notion of bipartite consensus introduced in [1]. Nodes having opinions of different signs are connected by a negative edge, and those having the same sign by a positive edge. By construction, then, the resulting graph $\Gamma(A(x(t)))$ is structurally balanced. The signed Laplacian (3) is used for the dynamics. For constant graphs, it is known that this Laplacian leads to the formation of two opinion clusters of opposite signs and of equal modulus. For our state-dependent case, the unbounded confidence version of (V1) is the following:

$$\dot{x}_i(t) = \sum_{j\neq i} \Big(\text{sgn}\big(x_j(t)x_i(t)\big)x_j(t) - x_i(t)\Big). \quad (8)$$

The presence of the sign function means that the system (8) has a discontinuous right hand side when one or more of the x_i are equal to 0. It will sometimes be convenient to write the equations (8) as

$$\dot{x} = f(x) \quad (9)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the discontinuous vector field whose components are defined by

$$f_i(x) = \sum_{j \neq i} \Big(\operatorname{sgn}\big(x_j x_i\big) x_j - x_i \Big). \tag{10}$$

The following proposition shows that in spite of the discontinuities, the solutions of (9) corresponding to almost all initial conditions are proper.

Proposition 1 *The system* (8) *satisfies property* **P1**.

Proof. In order to prove that solutions corresponding to almost all initial conditions exist and that they can be continued on a whole half line, we prove that the discontinuity surfaces $x_i = 0$ are repellent with respect to at least one of the limit values of the vector field $f(\cdot)$. Consider the function $\sigma_i(x) = x_i$ and the surface $\sigma_i(x) = 0$. Let x be a point of this surface and let $f^-(x)$ be the limit value of $f(x)$ as x approaches the surface with $x_i < 0$. It holds $\nabla\sigma_i(x) \cdot f^-(x) = -\sum_{j \neq i} |x_j| < 0$ if at least one j is such that $x_j \neq 0$, i.e. the point does not coincide with the origin. Analogously if $f^+(x)$ is the limit value of $f(x)$ as x approaches the surface with $x_i > 0$ one has $\nabla\sigma_i(x) \cdot f^+(x) = \sum_{j \neq i} |x_j| > 0$ if at least one j is such that $x_j \neq 0$. This means that solutions issuing from points with $x_i \neq 0$ for all i cannot reach the discontinuity surfaces. ∎

A consequence of Proposition 1 is the following sign-preservation property.

Proposition 2 *Consider the system* (8)*. If for all* $i = 1, ..., n$ $x_i(0) \neq 0$*, then* $\operatorname{sgn}(x_i(t)) = \operatorname{sgn}\big(x_i(0)\big)$ *for all* $t \in [0, \infty)$.

Proof. Assuming without loss of generality that for the $x_i(t)$ sorted in absolute value it holds $0 < |x_1(t)| \leq |x_2(t)| \leq \ldots \leq |x_n(t)|$, if $x_1(t) > 0$, then

$$\begin{aligned} \dot{x}_1 &= \sum_j \big(\operatorname{sgn}\big(x_j(t)x_1(t)\big)x_j(t) - x_1(t)\big) \\ &= \sum_j \left(|x_j(t)| - |x_1(t)|\right) \geq 0, \end{aligned} \tag{11}$$

while if $x_1(t) < 0$

$$\begin{aligned} \dot{x}_1 &= \sum_j \left(-|x_j(t)| - x_1(t)\right) \\ &= \sum_j \left(-|x_j(t)| + |x_1(t)|\right) \leq 0. \end{aligned} \tag{12}$$

In both cases, $x_1(t)$ is repelled from the origin, (or at least does not approach it), meaning that $\operatorname{sgn}(x_i(0)) = \operatorname{sgn}(x_i(t))\ \forall\, t \geq 0$ and $\forall\, i = 1, \ldots, n$. ∎

Proposition 3 *Consider the system* (8)*. If* $x_i(0) \neq 0$ *for all* $i = 1, \ldots, n$*, then the system converges to bipartite consensus:* $\lim_{t \to +\infty} x_i(t) = (1/n)\operatorname{sgn}(x_i(0)) \sum_j |x_j(0)|$.

Proof. Sign invariance of $x(t)$ follows from Proposition 2. Hence, denoting $s_i = \operatorname{sgn}(x_i(0))$ and $S = \operatorname{diag}(s)$, if we apply the change of basis $y = Sx$, then $y(t) > 0\ \forall\ t \geq 0$. The resulting system $\dot{y}_i(t) = \sum_{j \neq i} (y_j(t) - y_i(t))$ is an ordinary consensus problem on a fully connected undirected graph. For it $\lim_{t \to +\infty} y_i(t) = \frac{\sum_j y_j(0)}{n}$, from which the result follows. ∎

Remark 1 From (11) and (12), if $x_i(0) \neq 0\ \forall\, i = 1, \ldots, n$, it is straightforward to show that the following conservation law holds for the system (8):

$$c_s = \frac{1}{n} \sum_j |x_j(0)| = \frac{1}{n} \sum_j |x_j(t)| \quad \forall\ t \geq 0. \tag{13}$$

Note that we can ignore the fact that the absolute value is a nondifferentiable function as $x_i(t)$ does not change sign. On the contrary, c of (7) is not a conservation law. In fact, denoting $\mathcal{I}_+(x(0)) = \{i \in \mathcal{V} \text{ s.t. } x_i(0) > 0\}$ and $\mathcal{I}_-(x(0)) = \{i \in \mathcal{V} \text{ s.t. } x_i(0) < 0\}$,

$$\begin{aligned} n\dot{c} &= \sum_i \dot{x}_i = \sum_i \sum_{j \neq i} \Big(\operatorname{sgn}\big(x_j x_i\big) x_j - x_i \Big) \\ &= \sum_{i,\, j \in \mathcal{I}_+(x(0))} (x_j - x_i) + \sum_{i \in \mathcal{I}_+(x(0))} \sum_{j \in \mathcal{I}_-(x(0))} (-x_j - x_i) \\ &\quad + \sum_{i \in \mathcal{I}_-(x(0))} \sum_{j \in \mathcal{I}_+(x(0))} (-x_j - x_i) + \sum_{i,\, j \in \mathcal{I}_-(x(0))} (x_j - x_i) \\ &= -2 \sum_{i \in \mathcal{I}_+(x(0))} \sum_{j \in \mathcal{I}_-(x(0))} (x_j + x_i). \end{aligned}$$

which is in general $\neq 0$ (unless $\sum_{i \in \mathcal{I}_+(x(0))} x_i = -\sum_{j \in \mathcal{I}_-(x(0))} x_j$). In the previous computation we have used the fact that if $i, j \in \mathcal{I}_+(x(0))$, then in the sum $\sum_{i,\, j \in \mathcal{I}_+(x(0))} (x_j - x_i)$ one has both terms $x_j - x_j$ and $x_i - x_j$, whose sum gives zero, and analogously for $i, j \in \mathcal{I}_-(x(0))$. As a consequence, it follows that $W(x(t))$ need not be non-increasing, see Example 1.

Example 1 Consider the system (8) with the $n = 3$ initial opinions $x(0) = [-0.15\ 0.2\ -0.1]^T$. The corresponding $W(x(t))$ is shown in red in Fig. 1, right panel. Also $W_s(x(t)) = \sum_i (x_i(t) - c_s)^2$ is increasing in this example (blue dashed curve in Fig. 1, right panel).

Remark 2 When $x_i(0) \neq 0\ \forall\, i = 1, \ldots, n$, the system (8) corresponds to a fully connected bipartite consensus problem, with bipartition given by $s(x(t))$, $s_i(x(t)) = \operatorname{sgn}(x_i(t))$. To see it, it is enough to observe that the signed adjacency matrix $A_s(x(t))$ of (8) is $A_s(x(t)) = S(x(t))AS(x(t))$, $S(x(t)) = \mathrm{diag}(s(x(t)))$, where the entries of A are

$$A_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

and those of $A_s(x(t))$ are

$$A_{s,ij}(x(t)) = \begin{cases} \mathrm{sgn}(x_i(t)x_j(t)) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \tag{14}$$

From Proposition 2, $s(x(t)) = s(x(0)) \ \forall t \geq 0 \implies A_s(x(t)) = A_s(x(0)) \ \forall t \geq 0$. Hence $\Gamma_s(A_s(x(t)))$ is a structurally balanced (and constant) graph $\forall t$. If L is the Laplacian of A, then the signed Laplacian of $A_s(x)$, $L_s(x) = S(x)LS(x)$, has entries

$$L_{s,ij}(x) = \begin{cases} -\mathrm{sgn}(x_i x_j) & \text{if } i \neq j \\ n-1 & \text{if } i = j. \end{cases} \tag{15}$$

It is straightforward to check that (15) and (3) coincide.

From Propositions 2-3 and Remark 2, the case of all non-zero initial conditions behaves exactly like a bipartite linear consensus problem on a structurally balanced graph [1]. The following special case has however no counterpart in linear bipartite consensus. It deals with non-proper initial conditions, in correspondence of which multiple Carathéodory solutions exist.

Proposition 4 *Consider the system* (8). *If $x_i(0) = 0$ for some $i = 1, \dots, n$, then for $t > 0$ the system has multiple Carathéodory solutions, corresponding to the limit values of the i-th component of the vector field f as $x_i \to 0$, namely*

$$\begin{cases} (f^+)_i(x) = \sum_{j \neq i} |x_j| & \text{(16a)} \\ (f^0)_i(x) = 0 & \text{(16b)} \\ (f^-)_i(x) = -\sum_{j \neq i} |x_j| \,. & \text{(16c)} \end{cases}$$

In particular, there exists a classical solution $x(t)$ corresponding to (16b)*, with all opinions collapsing to the origin:*

$$\lim_{t \to +\infty} x_i(t) = 0 \quad \forall\, i = 1, \dots, n.$$

Proof. Denote $\mathcal{I}_0(x(0)) = \{i \in \mathcal{V} \text{ s. t. } x_i(0) = 0\}$, and let n_0 be its cardinality. For a given $i \in \mathcal{I}_0(x(0))$, the vector fields $f^-(\cdot)$ and $f^+(\cdot)$ were introduced already in the proof of Proposition 1 and shown to yield solutions that are repelled away from the discontinuity surface $x_i = 0$. Besides these, there are solutions which follow $f^0(\cdot) = f(\cdot)\big|_{x_i=0}$ remaining on the discontinuity surface. For these:

$$\dot{x}_i(0) = \sum_j \Big(\underbrace{\mathrm{sgn}\big(x_j(0)x_i(0)\big)}_{=0} x_j(0) - \underbrace{x_i(0)}_{=0} \Big) = 0.$$

$f^0(\cdot)$ leads in particular to a solution which is everywhere continuous and differentiable, hence a classical solution. Consider this solution $x(t)$ corresponding to $f^0(\cdot)$, i.e. such that $\dot{x}_i(t) = 0 \ \forall\, t \geq 0$. If $x_j(0) \neq 0$, then $\forall\, t \geq 0$, it holds

$$\begin{aligned} \dot{x}_j = & \sum_{k \notin \mathcal{I}_0(x(0))} \Big(\mathrm{sgn}\big(x_j x_k\big) x_k - x_j \Big) \\ & + \sum_{k \in \mathcal{I}_0(x(0))} \Big(\underbrace{\mathrm{sgn}\big(x_j x_k\big) x_k}_{=0} - x_j \Big) \\ = & \sum_{k \notin \mathcal{I}_0(x(0))} \big(\mathrm{sgn}\big(x_j x_k\big) x_k - x_j \big) - \sum_{k \in \mathcal{I}_0(x(0))} x_j \\ = & \sum_{k \notin \mathcal{I}_0(x(0))} \big(\mathrm{sgn}\big(x_j x_k\big) x_k - x_j \big) - n_0 x_j \end{aligned} \tag{17}$$

i.e., apart from the consensus-like terms for $k \notin \mathcal{I}_0(x(0))$, for $k \in \mathcal{I}_0(x(0))$ the terms $-x_j$ appear, which vanish only at $x_j = 0$. Consider the function $V(x(t)) = \sum_j |x_j(t)|$. Note that $V(x(t)) \geq 0 \ \forall t$, with $V(x(t)) = 0$ if and only if $x(t) = 0$. From Proposition 2, for any j such that $x_j(0) \neq 0$ the values of $x_j(t)$ do not change sign, hence $V(x(t)) = \sum_{j \notin \mathcal{I}_0(x(0))} |x_j(t)|$. We can then differentiate: $\frac{d}{dt} V(x(t)) = \sum_{j \notin \mathcal{I}_0(x(0))} \frac{d}{dt} |x_j(t)|$, where

$$\frac{d}{dt}|x_j(t)| = \begin{cases} \dot{x}_j(t) & \text{if } j \in \mathcal{I}_+(x(0)) \\ -\dot{x}_j(t) & \text{if } j \in \mathcal{I}_-(x(0)) \,. \end{cases}$$

From the proof of Proposition 2, if $x_j \neq 0$, rewriting (17) as in (11) and (12),

$$\dot{x}_j = \begin{cases} \sum_{k \notin \mathcal{I}_0(x(0))} \big(|x_k| - |x_j| \big) - n_0 |x_j| & \text{if } j \in \mathcal{I}_+(x(0)) \\ -\Big(\sum_{k \notin \mathcal{I}_0(x(0))} \big(|x_k| - |x_j| \big) - n_0 |x_j| \Big) & \text{if } j \in \mathcal{I}_-(x(0)). \end{cases}$$

Hence

$$\begin{aligned} \frac{d}{dt} V(x(t)) = & \sum_{j \notin \mathcal{I}_0(x(0))} \frac{d}{dt} |x_j(t)| = \sum_{j \in \mathcal{I}_+(x(0))} \dot{x}_j(t) - \sum_{j \in \mathcal{I}_-(x(0))} \dot{x}_j(t) \\ = & \sum_{j \in \mathcal{I}_+(x(0))} \left(\sum_{k \notin \mathcal{I}_0(x(0))} (|x_k(t)| - |x_j(t)|) - n_0 |x_j| \right) \\ & + \sum_{j \in \mathcal{I}_-(x(0))} \left(\sum_{k \notin \mathcal{I}_0(x(0))} (|x_k(t)| - |x_j(t)|) - n_0 |x_j| \right) = \\ = & \sum_{j \notin \mathcal{I}_0(x(0))} \left(\sum_{k \notin \mathcal{I}_0(x(0))} (|x_k(t)| - |x_j(t)|) - n_0 |x_j| \right) \\ = & -n_0 \sum_{j \notin \mathcal{I}_0(x(0))} |x_i(t)| = -n_0 V(x(t)). \end{aligned}$$

Therefore $\frac{d}{dt} V(x(t)) = -n_0 V(x(t)) < 0$ and $V(x(t)) \to 0$ as $t \to +\infty$, i.e. $x_j(t) \to 0$ as $t \to +\infty$ for all j. ∎

Remark 3 Note that the case of multiple intersection of hyperplanes of the form $x_i = 0$ can be treated analogously.

Remark 4 In (16), the Carathéodory solutions which follow (16a) and (16c) are not classical as their derivatives are not defined at the time they leave $x_i = 0$.

Example 2 Consider (8) with $n = 2$ and initial condition $x(0) = [0\ 1]^T$. The (classical) solution issuing from $x(0)$ and following $f^0(\cdot)$ asymptotically tends to $[0\ 0]^T$. Two other solutions issuing from $x(0)$, tend to the points $[-1/2\ 1/2]^T$ and $[1/2\ 1/2]^T$.

Remark 5 When $\mathcal{I}_0(x(0))$ is nonempty, the graph $\Gamma_s(A(x(t)))$ is state-dependent and changes with the solution considered, i.e., different graph evolutions may originate from the same initial condition. For the classical solution following $f^0(\cdot)$, the subgraph of $\Gamma_s(A(x(t)))$ of nodes $\mathcal{V} \setminus \mathcal{I}_0(x(0))$ is still fully connected and structurally balanced, although the entire Γ_s is no longer fully connected.

Remark 6 When $\mathcal{I}_0(x(0)) = \emptyset$, the observation that $x_i(t)$ does not change sign $\forall\, t \geq 0$ (Proposition 2) implies that (8) is equivalent to

$$\dot{x} = -L_s(x)x \tag{18}$$

where $L_s(x)$ is still given by (15), as it is straightforward to verify. In this case $L_s(x)$ is a constant along the solutions of the system. When instead $\mathcal{I}_0(x(0)) \neq \emptyset$ and the classical solution corresponding to $f^0(\cdot)$ is chosen, (18) still holds, but $L_s(x)$ is no longer diagonally equipotent. It is instead strictly diagonally dominant (some of the functions $\text{sgn}(x_i x_j)$ are equal to 0).

4.2 Version 2: same sign consensus (with unbounded confidence)

In order to describe interaction among opinions of different signs, an alternative model to (8) is characterized by "indifference", i.e. nodes having opinions of different signs are disconnected in the confidence model. In this case, positive opinions will cluster together into their average consensus value, and so will the negative opinions, but the two consensus values will normally be different in modulus. From (V2), the model we consider in the unbounded confidence case is the following:

$$\dot{x}_i(t) = \sum_{j \neq i} \text{ssgn}\big(x_j(t)x_i(t)\big)\Big(x_j(t) - x_i(t)\Big). \tag{19}$$

Since in (19) consensus-like terms exist if and only if x_i and x_j have the same (nonzero) sign, the following proposition is obvious.

Proposition 5 *For system* (19) **P1** *holds.*

The proof is completely analogous to the proof of Proposition 1. Also in this case, there are multiple Carathéodory solutions for the initial conditions issuing from the hyperplanes $\sigma_i(x) = 0$, $i \in \mathcal{I}_0(x(0))$.

Proposition 6 *Consider the system* (19)*. For all* $i \notin \mathcal{I}_0(x(0))$*,* $\text{sgn}(x_i(t)) = \text{sgn}\big(x_i(0)\big)$ *for all* $t \geq 0$*.*

Proof. It is enough to note that for each i the surface $\sigma_i(x) = 0$ is repelling with respect to the limit values of the vector field defined by the right-hand side of (19) as x_i tends to 0 from right and left. ∎

Proposition 7 *Consider the system* (19)*. For any initial condition such that* $\mathcal{I}_0(x(0)) = \emptyset$

$$\lim_{t\to+\infty} x_i(t) = \begin{cases} \frac{1}{n_+}\sum_{j\in\mathcal{I}_+x(0))} x_j(0) & \text{if } i \in \mathcal{I}_+(x(0)) \\ \frac{1}{n_-}\sum_{j\in\mathcal{I}_-(x(0))} x_j(0) & \text{if } i \in \mathcal{I}_-(x(0)). \end{cases} \tag{20}$$

where n_+ *and* n_- *denote the cardinalities of* $\mathcal{I}_+(x(0))$ *and* $\mathcal{I}_-(x(0))$*.*

Proof. By construction, the graph $\Gamma(A(x(t)))$ of (19) is split into the two disjoint connected components $\Gamma(A_+(x(t)))$ and $\Gamma(A_-(x(t)))$, where $A_{+,ij}(x(t)) = 1$ if $i, j \in \mathcal{I}_+(x(t))$ and $A_{-,ij}(x(t)) = 1$ if $i, j \in \mathcal{I}_-(x(t))$. Both subgraphs are constant for all t and one can set up on each of them a standard consensus problem, yielding the value in (20). ∎

Remark 7 When $\mathcal{I}_0(x(0)) \neq \emptyset$, different Carathéodory solutions corresponding to initial conditions in $\sigma_i(x) = 0$, $i \in \mathcal{I}_0(x(0))$ converge to different equilibria. Note that the set of equilibria of (19) is $\{x \in \mathbb{R}^n : x_i = x_j\ \forall i, j \in \mathcal{I}_+(x) \text{ and } x_i = x_j\ \forall i, j \in \mathcal{I}_-(x)\}$.

Example 3 Consider system (19) in dimension 3, with initial condition at the point $x(0) = [0\ 1\ -1]^T$. $x(0)$ is an equilibrium, but besides the constant solution, there are other solutions: among these, there is one which asymptotically goes to $[1/2\ 1/2\ -1]^T$ and another one which goes to $[-1/2\ 1\ -1/2]^T$.

Remark 8 The quantity (13) is a conservation law also for (19). In the case $\mathcal{I}_0(x(0)) = \emptyset$ it is enough to observe that

$$c_s(t) = \sum_{j\in\mathcal{I}_+(x(t))} x_j(t) - \sum_{j\in\mathcal{I}_-(x(t))} x_j(t)$$

where both quantities on the right hand side are conservation laws for, respectively, $\Gamma(A_+(x(t)))$ and

$\Gamma(A_-(x(t)))$. Since consensus on $\Gamma(A_+(x(t)))$ is achieved independently of what happens on $\Gamma(A_-(x(t)))$ and viceversa, also the average value $c(t)$ of (7) is a conservation law for (19). Consider now a solution $x(t)$ of (19) such that $\mathcal{I}_0(x(0)) \neq \emptyset$ and let $i \in \mathcal{I}_0(x(0))$. Depending on the limit value of the vector field followed by $x(t)$ one can have $x_i(t) > 0$, $x_i(t) < 0$, or $x_i(t) = 0$ for $t > 0$. In the first case the ith-component joins the connected component of the graph $\Gamma(A_+(x(t)))$, in the second case it joins $\Gamma(A_-(x(t)))$, and in the third case it remains constant. In any of these cases consensus on $\Gamma(A_+(x(t)))$ is achieved independently of what happens on $\Gamma(A_-(x(t)))$ and viceversa, and both $c(t)$ and $c_s(t)$ remain constant.

4.3 Version 3: homogeneous repulsion (with unbounded confidence)

The homogeneous form of the dynamics used in (19) can also be endowed with a repulsive action for opinions of different sign, for instance considering the following model (unbounded confidence equivalent of (V3)):

$$\dot{x}_i(t) = \sum_{j \neq i} \operatorname{sgn}\big(x_j(t)x_i(t)\big)\Big(x_j(t) - x_i(t)\Big). \tag{21}$$

Nodes having opinions that differ in sign are connected by a negative edge and exercise a repulsive "force" on each other.

It can be useful in the following to denote by $\ell(x)$ the vector field defined by the righthand side of (21), i.e. $\ell_i(x) = \sum_{j\neq i} \operatorname{sgn}\big(x_j(t)x_i(t)\big)\Big(x_j(t) - x_i(t)\Big)$

Proposition 8 *The system* (21) *satisfies* **P1**.

Proof. The set of discontinuities of $\ell(x)$ is the union of the hyperplanes $\sigma_i(x) = 0$, which are repellent for $\ell(x)$ in the sense that $\lim_{x_i\to 0+} \ell_i(x) = \sum_{j\in\mathcal{I}_+(x)} x_j - \sum_{j\in\mathcal{I}_-(x)} x_j \geq 0$ and $\lim_{x_i\to 0-} \ell_i(x) = -\sum_{j\in\mathcal{I}_+(x)} x_j + \sum_{j\in\mathcal{I}_-(x)} x_j \leq 0$. ■

We remark that Carathéodory solutions corresponding to initial conditions with some null initial components may have such components null, positive or negative, as in Example 4.

Example 4 Let $n = 2$ and consider $x(0) = [1\ 0]^T$. There are multiple Carathéodory solutions issuing from this point: among these, $x(t) = [1\ 0]^T$ is a classical solution; a Carathéodory solution moves on the line $x_2 = -x_1+1$ and asymptotically tends to the point $[1/2\ 1/2]^T$, and another one moves on the same line and $x_1(t) \to +\infty$, $x_2(t) \to -\infty$.

We will show that this model is diverging in general, but we first prove that opinions are sign invariant.

Proposition 9 *Consider the system* (21)*. For all* $i = 1, ..., n$ *such that* $x_i(0) \neq 0$*,* $\operatorname{sgn}(x_i(t)) = \operatorname{sgn}\big(x_i(0)\big)$ *for all* $t \in [0, \infty)$*.*

Proof. We have already observed in the proof of Proposition 8 that hyperplanes $\sigma_i(x) = 0$ are repellent for $\ell(x)$. ■

When an initial condition has both negative and positive components we have the following.

Proposition 10 *Let* $x(t)$ *be a solution of* (21)*. If* $\mathcal{I}_+(x(0)) \neq \emptyset$ *and* $\mathcal{I}_-(x(0)) \neq \emptyset$*, then* $\lim_{t\to+\infty} x_i(t) = \operatorname{sgn}(x_i(0))\infty$ *for all* $i \in \mathcal{I}_+(x(0)) \cup \mathcal{I}_-(x(0))$*.*

Proof. Divergence follows from the fact that whenever $x_i(t)x_j(t) < 0$, the repulsive interaction between i and j never vanishes, not even at large distances. In Proposition 9 we have proved that $x_i(0) > 0$ implies $x_i(t) > 0$ for all $t \geq 0$. Let $m(t) \in \{1, ..., N\}$ be such that $x_{m(t)}(t) = \min\{x_i(t) : x_i(t) > 0\}$. $x_{m(t)}(t)$ is differentiable for almost all $t \geq 0$ and

$$\dot{x}_{m(t)}(t) = \sum_{j\in\mathcal{I}_+(x(t))} (x_j(t) - x_{m(t)}(t)) - \sum_{j\in\mathcal{I}_-(x(t))} (x_j(t) - x_{m(t)}(t)) \geq x_{m(t)}(t).$$

This implies that $x_{m(t)} \to +\infty$ as $t \to +\infty$ and then $x_i(t) \to +\infty$ for all $i \in \mathcal{I}_+(x(0))$. Analogously it can be proved that $x_i(t) \to -\infty$ for all $i \in \mathcal{I}_-(x(0))$.

■

Remark 9 When $\mathcal{I}_0(x(0)) = \emptyset$, the adjacency matrix corresponding to (21) is still (the constant) $A_s(x(0))$ of (14), but the "Laplacian" corresponding to (21) is

$$L^3_{s,ij} = \begin{cases} n_+ - n_- & \text{if } i = j \in \mathcal{I}_+(x(0)) \\ n_- - n_+ & \text{if } i = j \in \mathcal{I}_-(x(0)) \\ -A_{s,ij}(x(0)) & \text{if } i \neq j \end{cases}$$

which is no longer diagonally dominant (hence the instability). When $\mathcal{I}_0(x(0)) \neq \emptyset$, then $\Gamma(A_s(x(t)))$ is no longer constant in time, but varies according to the specific solution followed. In particular, the classical solution in which $x_i(0) = 0 \Longrightarrow x_i(t) = 0 \,\forall\, t$ has

$$L^{3'}_{s,ij} = \begin{cases} n_+ - n_- & \text{if } i = j \in \mathcal{I}_+(x(0)) \\ n_- - n_+ & \text{if } i = j \in \mathcal{I}_-(x(0)) \\ 0 & \text{if } i = j \in \mathcal{I}_0(x(0)) \\ -A_{s,ij}(x(0)) & \text{if } i \neq j. \end{cases}$$

Remark 10 The quantity (13) is no longer a conservation law for (21), as it is straightforward to show. Instead the average (7) is conserved, as for almost every t we have

$$\dot{c}(t) = \sum_i \dot{x}_i(t) = \sum_i \sum_{j \neq i} \mathrm{sgn}(x_i(t)x_j(t))\big(x_j(t) - x_i(t)\big) = 0.$$

A final remark is that owing to the divergence of the opinions, this model does not make sense *per se*, but only in presence of a confidence bound that restricts the repulsive action to an interval around the origin, as in (V3).

5 Proof of Theorem 2 and counterexamples

Proof of Theorem 2. Let us rewrite (V1) as $\dot{x} = h(x)$, where $h(x)$ is the vector field of components

$$h_i(x) = \sum_{j \text{ s.t. } |x_j(t)-x_i(t)|<1} \Big(\mathrm{sgn}\big(x_j(t)x_i(t)\big)x_j(t) - x_i(t)\Big),$$

and (V3) as $\dot{x} = l(x)$., where $l(x)$ is the vector field of components

$$l_i(x) = \sum_{j \text{ s.t. } |x_j(t)-x_i(t)|<1} \mathrm{sgn}\big(x_j(t)x_i(t)\big)\Big(x_j(t) - x_i(t)\Big).$$

To avoid trivial cases, assume that $\mathcal{I}_+(x(0)) \neq \emptyset$, and $\mathcal{I}_-(x(0)) \neq \emptyset$.

Let us now show that the model (V1) obeys the properties listed in Theorem 2.

V1 - P1. First of all, let us observe that for the model (V1) the opinions are sign invariant for solutions issuing from proper initial conditions. In fact, from Propositions 1 and 4 the discontinuity surfaces $\sigma_i(x) = 0$ do not correspond to proper initial conditions. These surfaces are (locally) repelling even when the confidence is bounded. Furthermore, we can remark that for any initial condition outside $\sigma_i(x) = 0$ and $|x_i - x_j| = 1$ for all i, j, there exists a unique local solution. For such initial conditions Proposition 2 still holds once the summations are reduced to the opinions fulfilling the condition $|x_j - x_i| < 1$, hence an $x_i \neq 0$ can never cross the origin.

We then have to prove that existence and uniqueness of any such solution is not lost in case it reaches the discontinuity surface at a point $\overline{x}$ such that $\overline{x}_i - \overline{x}_j = 1$ (the case $\overline{x}_i - \overline{x}_j = -1$ is analogous). The case $\overline{x}_i, \overline{x}_j > 0$ is the same treated in [3]. We then have to examine the cases $\overline{x}_i = 1, \overline{x}_j = 0$ and $\overline{x}_j < 0 < \overline{x}_i$.

Considering the first one, as the surfaces $\sigma_j(x) = 0$ are repelling (Proposition 1), the set of points reaching them has measure zero.

We then consider the case $\overline{x}_j < 0 < \overline{x}_i$. Let $\sigma_{ij}(x) = x_i - x_j$, $\Sigma_{ij} = \{x \in \mathbb{R}^n : x_i - x_j = 1\}$, $\Sigma_{ij}^- = \{x \in \mathbb{R}^n : x_i - x_j < 1\}$ and $\Sigma_{ij}^+ = \{x \in \mathbb{R}^n : x_i - x_j \geq 1\}$. Assume that the solution is approaching the surface Σ_{ij} from Σ_{ij}^-. In this case, at $x \in \Sigma_{ij}^-$ it must be

$$\begin{aligned}\nabla\sigma_{ij}(x) \cdot h(x) &= h_i(x) - h_j(x) \\ &= \sum_{|x_r - x_i|<1} (|x_r| - x_i) - \sum_{|x_k - x_j|<1} (-|x_k| - x_j) > 0.\end{aligned}$$

Since $x \in \Sigma_{ij}^-$, the edge (i, j) is present in $\Gamma_s(A(x(t)))$. Emphasizing it in the previous expression:

$$\begin{aligned}\nabla\sigma_{ij}(x) \cdot h(x) = &\sum_{\substack{|x_r - x_i|<1 \\ r \neq j}} (|x_r| - x_i) + |x_j| - x_i \\ &+ \sum_{\substack{|x_k - x_j|<1 \\ k \neq i}} (-|x_k| - x_j) - (-|x_i| - x_j) > 0.\end{aligned}$$

Consider the situation of at least one of the nodes i and j having two or more edges. When x approaches $\overline{x} \in \Sigma_{ij}$, then $x_i > 0$ and $x_j < 0$, hence $|x_j| - x_i + |x_i| + x_j = 0$, which means that $\nabla\sigma_{ij}(\overline{x}) \cdot h(\overline{x}) > 0$, i.e., in the case $\mathrm{sgn}(x_i x_j) = -1$ the discontinuity surface is always crossed when it is approached from Σ_{ij}^-. An analogous argument holds when Σ_{ij} is approached from Σ_{ij}^+. Hence as long as transitions are "simple" (i.e., in $\overline{x}$ only one of the Σ_{ij} is crossed) and the crossing does not result in both nodes i and j becoming completely disconnected, the solution exists and it is unique.

When instead both nodes i and j do not have any other connection in $\Gamma_s(A(x(t)))$ than the edge (i, j), then since at the transition the edge disappears it becomes $h_i(\overline{x}) = 0$ and $h_j(\overline{x}) = 0$, i.e., $\nabla\sigma_{ij}(\overline{x}) \cdot h(\overline{x}) = 0$. This means that the solution stays on the surface Σ_{ij} thereafter. Also in this case, however, the solution exists and it is unique. Notice that it is enough that one of the two nodes i and j has at least another edge to guarantee that $\nabla\sigma_{ij}(\overline{x}) \cdot h(\overline{x}) > 0$ at the transition. Combining all these considerations, we obtain that the subset of $\mathbb{R}^+$ in which the solution of (V1) is not differentiable is at most countable and cannot have accumulation points, i.e., condition (b) in the definition of proper initial conditions holds. Condition (c) of the same definition follows from uniqueness of solutions. In fact if $x_i(t) = x_j(t)$ then $\dot{x}_i(t) = \dot{x}_j(t)$.

V1 - P2. Follows directly from condition (c) of the definition of proper initial conditions.

V1 - not P3. Follows from Remark 1.

V1 - not P4. A counterexample is in Example 1.

V1 - not P5. A counterexample is in Example 5 below.

V1 - not P6. A counterexample is in Example 6 below.

V1 - P8. As we are interested only in solutions issuing from a proper initial condition, it is enough to consider the case $\mathcal{I}_0(x(0)) = \emptyset$. The proof is similar for what is possible to that of Theorem 2 of [5]. At t let us assume the components of $x(t)$ obey the following: $0 < |x_1(t)| \leqslant \ldots \leqslant |x_n(t)|$ (notice that the order of absolute values can change over time). From (11) and (12) one has:

$$\frac{d}{dt}|x_i| = \sum_{\substack{j \text{ s.t.} \\ |x_i - x_j| < 1}} \big(|x_j| - |x_i|\big) \tag{22}$$

When the expression (22) is computed for x_n then $\frac{d}{dt}|x_n| \leqslant 0$, hence all $|x_i(t)|$ are bounded for all $t \geqslant 0$. Let us observe that when $\mathcal{I}_0(x(0)) = \emptyset$, because of symmetry, the following partial sums vanish for any k:

$$\sum_{i=1}^{k} \sum_{\substack{j \leqslant k \text{ s.t.} \\ |x_i - x_j| < 1}} \big(|x_j| - |x_i|\big) = 0. \tag{23}$$

Hence, almost always

$$\sum_{i=1}^{k} \frac{d}{dt}|x_i| = \sum_{i=1}^{k} \sum_{\substack{j > k \text{ s.t.} \\ |x_i - x_j| < 1}} \big(|x_j| - |x_i|\big) \geqslant 0$$

because $j > k \geqslant i$ implies $|x_j| \geqslant |x_k| \geqslant |x_i|$. From the boundedness of $|x_i|$ (and of $\sum_{i=1}^{k} |x_i|$), it follows that the summations must converge monotonically for any k, and hence so must the $|x_i|$ and the x_i almost always. To show that in x^* either $x_i^* = x_j^*$ or $|x_i^* - x_j^*| \geqslant 1$, the same contradictory argument of [5] can now be used.

V1 - P9. From (22) and (23), if $\mathcal{I}_0(x(0)) = \emptyset$ then $\sum_{i=1}^{n} \frac{d}{dt}|x_i| = 0$, hence $c_s(t) = \text{const}\ \forall\, t \geqslant 0$.

V1 - P10. Let $x_o = x(0)$ be such that $x_i(0) = 0$. The limit values of the i-th component of the vector field h as $x_i \to 0$ are:

$$\dot{x}_i = \begin{cases} (h^+)_i(x) = \sum_{\substack{j \neq i \text{ s.t.} \\ |x_i - x_j| < 1}} |x_j| & \text{(24a)} \\ (h^0)_i(x) = 0 & \text{(24b)} \\ (h^-)_i(x) = -\sum_{\substack{j \neq i \text{ s.t.} \\ |x_i - x_j| < 1}} |x_j| \,. & \text{(24c)} \end{cases}$$

By the same argument used in the proof of Proposition 4 one deduce existence of a Caratéodory solution whose i-th component follows $(h^0)_i(x)$, and, by the same argument used in **P1**, can be extended to a Carathéodory solution on the entire half-line.

V2 - P1. Let us consider initial conditions such that $x_i(0) \neq 0$ for all $i = 1, ..., n$. The equations read

$$i \in \mathcal{I}_+(x(0)) : \ \dot{x}_i = \sum_{j \in \mathcal{I}_+(x(0)) \text{ s.t. } |x_j(t) - x_i(t)| < 1} x_j - x_i$$

$$i \in \mathcal{I}_-(x(0)) : \ \dot{x}_i = \sum_{j \in \mathcal{I}_-(x(0)) \text{ s.t. } |x_j(t) - x_i(t)| < 1} x_j - x_i.$$

As Theorem 1 can be applied to the two blocks of components, then almost all $x(0) \in \mathbb{R}^n$ are proper initial conditions.

In other words, the model (V2) (with initial conditions such that $x_i(0) \neq 0$ for all $i = 1, ..., n$) corresponds to considering two disjoint "parallel" bounded confidence problems, one on $\Gamma(A_+(x(t)))$ and the other on $\Gamma(A_-(x(t)))$, where $A_{+,ij}(x(t)) = 1$ if $i, j \in \mathcal{I}_+(x(t)) \cap \{|x_i(t) - x_j(t)| < 1\}$, $A_{-,ij}(x(t)) = 1$ if $i, j \in \mathcal{I}_-(x(t)) \cap \{|x_i(t) - x_j(t)| < 1\}$ and $A_{ij}(x(t)) = 0$ in the other cases. Note that initial conditions with some null components are, in general, non proper initial conditions.

V2 - P2. Property **P2** holds for solutions corresponding to proper initial conditions thanks to Theorem 1 and to the fact that for such solutions the states components are split as noticed in the proof of V2-**P1**. Monotonicity of the components then follows from **P2** of Theorem 1.

V2 - P3. Let $c_+(t) = \frac{1}{n_+} \sum_{i \in \mathcal{I}_+(x(t))} x_i(t)$, $c_-(t) = \frac{1}{n_-} \sum_{i \in \mathcal{I}_-(x(t))} x_i(t)$. The quantities $c_+(t)$ and $c_-(t)$ are conserved quantities thanks to **P3** of Theorem 1 applied to the two subsystems whose graphs are $\Gamma(A_+(x(t)))$ and $\Gamma(A_-(x(t)))$, as noticed in the proof of V2-**P2**. Then also $c(t) = \frac{1}{n}(n_+ c_+(t) + n_- c_-(t))$ is conserved.

V2 - P4. For each solution, we compute the derivative with respect to time of the function $W(t)$:

$$\begin{aligned} \frac{d}{dt} W(x(t)) &= 2 \sum_i (x_i(t) - c(t))(\dot{x}_i(t) - \dot{c}(t)) = \\ &= 2 \sum_i x_i(t) \sum_{j \text{ s.t. } |x_j(t) - x_i(t)| < 1} \text{ssgn}\big(x_j(t) x_i(t)\big) \Big(x_j(t) - x_i(t)\Big) \\ &= 2 \sum_{i \in \mathcal{I}_+(x(t))} x_i(t) \sum_{j \in \mathcal{I}_+(x(t)) \text{ s.t. } |x_j(t) - x_i(t)| < 1} \big(x_j(t) - x_i(t)\big) \\ &\quad + 2 \sum_{i \in \mathcal{I}_-(x(t))} x_i(t) \sum_{j \in \mathcal{I}_-(x(t)) \text{ s.t. } |x_j(t) - x_i(t)| < 1} \big(x_j(t) - x_i(t)\big). \end{aligned}$$

The two terms in the sum are negative thanks to property **P4** of Theorem 1 as they correspond to the derivatives of the functions $W_+(t) = \sum_{i \in \mathcal{I}_+(x(t))} (x_i(t) - c_+(t))^2$ and $W_-(t) = \sum_{i \in \mathcal{I}_-(x(t))} (x_i(t) - c_-(t))^2$ corresponding the two bounded confidence systems associated to $\Gamma(A_+(x(t)))$ and $\Gamma(A_-(x(t)))$.

V2 - P5, P6. For initial conditions with $x_i(0) \neq 0$ for all $i = 1, ..., n$, **P5** is trivially satisfied as $\Gamma(A(x(t)))$ can be fully connected only if $x_i(0)$ have all the same sign. P6 follows from Theorem 1 applied to $\Gamma(A_+(x(t)))$ and to $\Gamma(A_-(x(t)))$.

V2 - P9. Consider initial conditions such that $x_i(0) \neq 0$ for all $i = 1, ..., n$, and let $c_+(t) = \sum_{i\in\mathcal{I}_+(x(t))} x_i(t)$ and $c_-(t) = \sum_{i\in\mathcal{I}_-(x(t))} x_i(t)$. These are conserved quantities for, respectively, $\Gamma(A_+(x(t)))$ and $\Gamma(A_-(x(t)))$, and so must be $c_s(t) = \frac{1}{n}[c_+(t) - c_-(t)]$ for the model (V2).

V2 - P10. This property is trivially satisfied due to the form of the right-hand side of (V2).

V2 - P11, not P7. Since $\Gamma(A_+(x(t)))$ and $\Gamma(A_-(x(t)))$ are disjoint (and so are the nodes in the origin), the statement follows readily from Theorem 1 when $\mathrm{sgn}(x_i x_j) = +1$, while there is no requirement on $|x_i^* - x_j^*|$ when $\mathrm{sgn}(x_i x_j) \neq +1$. Hence **P11** holds but **P7** can be violated.

V3 - not P1. A counterexample is in Example 8.

V3 - P12. The righthand side of system (V3) is measurable and locally bounded, hence Krasovskii solutions exist for any initial condition. Next we prove that they are bounded: from this fact it follows that they can be continued up to $+\infty$. Boundedness of solutions is a consequence of "bounded confidence". Let $x(t)$ be any Krasovskii solution of (V3) and let $M \in \{1, .., n\}$ be any index such that $x_M(t) = \max\{x_i(t) : i = 1, ..., n\}$. Assume $x_M(t) > 0$ in order to avoid trivial cases. If moreover $x_M(t) \geq 1$, the nodes in $\mathcal{I}_-(x(t))$ do not affect the dynamics of M and it follows from the system equations that $\dot{x}_M(t) \leq 0$. If $x_M(t) \leq 1$, $x_M(t)$ may increase, but as soon as it reaches 1, we get again to the previous case. This means that $x_M(t) \leq \max\{x_M(0), 1\}$, and $x(t)$ is bounded. An identical argument holds for the lower bound.

V3 - not P2. Since condition (c) of the definition of proper initial conditions does not hold, opinions that are identical at a certain τ need not stay so. For instance if $x_i(\tau) = x_j(\tau) \;\not\Longrightarrow\; x_i(t) = x_j(t)\ \forall\ t > \tau$, it means that at least one of the two possible formulations of **P2** ($x_i(\tau) \leq x_j(\tau) \Longrightarrow x_i(t) \leq x_j(t)\ \forall\ t > \tau$, and $x_i(\tau) \geq x_j(\tau) \Longrightarrow x_i(t) \geq x_j(t)\ \forall\ t > \tau$) is necessarily violated. See Example 9.

V3 - P3. The quantity $c(t)$ is preserved thanks to the symmetry of the matrix $L_s^3(x)$, where $L_s^3(x)$ is the state dependent Laplacian matrix associated to (V3).

V3 - not P4. A counterexample is in Example 11.

V3 - not P5. A counterexample is in Example 11.

V3 - not P6. A counterexample is in Example 12.

V3 - not P9. A counterexample is in Example 11.

V3 - P10. This property is satisfied as $(l(x))_i|_{x_i=0} = 0$.

V3 - P13. Let $l^q(x)$, $q \in Q(x) \subset \mathbb{N}$ be the limit values of the vector field $l(x)$ at x, where $Q(x) \subset \mathbb{N}$ is an enumeration of the regions delimited by the discontinuity hyperplanes in a neighborhood of the point x. To each region there correspond a set of edges in the communication graph, so that for any $q \in Q(x)$

$$
\begin{aligned}
(l^q(x))_i &= \sum_{j\in\mathcal{I}^q(x)} \mathrm{sgn}(x_i x_j)(x_j - x_i) \\
&= \sum_{j\in\mathcal{I}_+^q(x)} (x_j - x_i) - \sum_{j\in\mathcal{I}_-^q(x)} (x_j - x_i).
\end{aligned}
$$

Let $x(t)$ be any Krasovskii solution of (V3). For any component i one has $\dot{x}_i \in (Kl(x))_i$ where $(Kl(x))_i = \sum_{q\in Q(x)} \alpha_q l^q(x)$ and α_q depend on t and are such that $\sum_{q\in Q(x(t))} \alpha_q(t) = 1$. Let $m^+(t)$ be any index such that $x_{m^+(t)}(t) = \min\{x_i(t), i \in \mathcal{I}_+(x(t))\}$. In the following we will omit explicit dependence of $m^+(t)$ on t. One has

$$
\begin{aligned}
\dot{x}_{m^+} \in (Kl(x))_{m^+} &= \sum_{q\in Q(x)} \alpha_q(t) l^q_{m^+}(x) \\
&= \sum_{q\in Q(x)} \alpha_q(t) \sum_{j\in\mathcal{I}:|x_j - x_{m^+}|<1} \mathrm{sgn}(x_j x_{m^+})(x_j - x_{m^+}) \\
&= \sum_{q\in Q(x)} \alpha_q(t) \Big(\sum_{j\in\mathcal{I}_+^q(x):|x_j - x_{m^+}|<1} (x_j - x_{m^+}) \\
&\quad - \sum_{j\in\mathcal{I}_-^q(x):|x_j - x_{m^+}|<1} (x_j - x_{m^+}) \Big).
\end{aligned}
$$

This shows that $\dot{x}_{m^+}(t) > x_{m^+}(t) \geq x_{m^+}(0) > 0$ as far as there are negative nodes that communicate with i and implies that the node m^+ disconnects from negative nodes in finite time. Afterwards the dynamics is the usual bounded confidence dynamics so that positive components converge either to the same value or to values whose distances are grater than 1. Analogous considerations can be repeated for negative components.

■

Remark 11 Although it obeys **P9**, the model (V1) does not obey the equivalent of **P4**, i.e., $\frac{dW_s}{dt}$ need not be negative almost always. See Example 1 (blue dashed curve in Fig. 1, right panel).

Example 5 (V1: not P5) Consider the model (V1) with the $n = 3$ initial opinions $x(0) = [0.1\ -0.85\ -0.89]^T$ and evolution shown in Fig 2, left panel. At $t = 0$,

$\Gamma(A(x(t)))$ is fully connected, however x_1 (blue) becomes disconnected from x_3 already at $t = 0.01$, and also from x_2 at $t = 0.35$, i.e., **P5** does not hold.

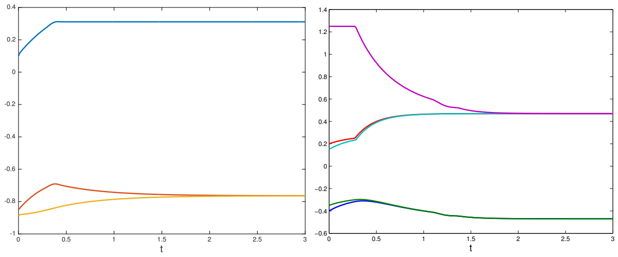

Fig. 2. Left panel: Example 5. For the model (V1) the graph $\Gamma(A(x(t)))$ is fully connected at $t = 0$ but x_1 becomes disconnected at later times. Right panel: Example 6. For the model (V1) property **P6** does not hold. In fact, one of the $n = 5$ agents (x_5, purple) is disconnected from the remaining 4 at $t = 0$, but all become connected at later t.

Example 6 (V1: not P6) Consider the $n = 5$ model (V1) with $x(0) = [-0.4\ -0.35\ 0.2\ 0.15\ 1.25]^T$. For it, the property **P6** does not hold, i.e., a disconnected $\Gamma(A(x(t)))$ can become connected because of the repulsion, see Fig. 2, right panel.

Example 7 In Fig. 3, various possible outcomes of the clustering obtained with the model (V1) are shown. If the $n = 100$ agents have all nonzero initial conditions, then they will cluster to local consensus values while keeping the same sign of their initial conditions (first panel). None of them will approach 0. If one of the initial conditions is equal to 0, then the nearest positive and negative groups of agents can converge to the origin (second panel). However, it can also happen that only a group on one side converges to 0 (third panel) or that no group at all does (fourth panel). Fig 3 shows that for the model (V1) there is a sufficiently neat separation of time scales between the consensus-like convergence within a group and the convergence of a group to the origin due to diagonal dominance, when it happens.

Example 8 (V3: not P1) Consider the model (V3). A state $x(0) = [0.05\ -0.1\ -0.2\ -0.3\ -0.88]^T$ moves towards the discontinuity surface $\sigma_{15}(x) = x_1 - x_5 = 1$, which it reaches at around $t_1 = 0.065$, see Fig. 4. The limit values of the vector field at the point $x(t_1)$ arriving from $\sigma_{15}(x) < 1$ or from $\sigma_{15}(x) > 1$ are $v^-(t_1) = [2.32\ -1.25\ -1\ -0.75\ 0.68]^T$ and $v^+(t_1) = [1.32\ -1.25\ -1\ -0.75\ 1.68]^T$. One has $\nabla\sigma_{15}(x(t_1)) \cdot v^-(t_1) = 1.64 > 0$ and $\nabla\sigma_{15}(x(t_1)) \cdot v^+(t_1) = -0.36 < 0$. This means that the two vectors "point towards" the discontinuity surface and that there is no Carathèodory solution issuing from $x(t_1)$. In Fig. 4, this is shown as v_1 and v_5 that chatter. The solution remains on this surface until $t_2 = 0.086$, where both $\nabla\sigma_{15}(x(t_2)) \cdot v^-(t_2) > 0$ and $\nabla\sigma_{15}(x(t_2)) \cdot v^+(t_2) > 0$ and the system can exit the surface $\sigma_{15}(x) = 1$. By slightly modifying the initial condition $x(0)$ we find a segment of points in the same situation. Following backwards v^- and v^+ we find a positive measure set of initial conditions which are not proper.

Example 9 (V3: not P2) Consider system (V3) with the initial condition $x_o = [0\ 0\ 1]^T$. Since $Kl(x_o) = \overline{co}\{[0\ 1\ -1]^T, [1\ 1\ -2]^T, [1\ 0\ -1]^T, [0\ 0\ 0]^T\}$ we have a Krasovskii solution such that $x_1(t) = 0$ for all $t \geq 0$ whereas $x_2(t) > 0$ when $t > 0$. Note that such solution tends to the equilibrium $[0\ 1/2\ 1/2]^T$. Hence if we formulate **P2** as $x_1(0) \geq x_2(0) \Longrightarrow x_1(t) \geq x_2(t)$, the property is violated (obviously no violation occurs if we write **P2** with reversed signs: $x_1(0) \leq x_2(0) \Longrightarrow x_1(t) \leq x_2(t)$).

Example 10 In the example shown in Fig. 5, the model is (V3) and the initial condition is $x(0) = [-0.073\ 0.76\ -0.1\ -0.17\ 0.006]^T$. The opinions x_1 and x_3 collapse into each other and stay identical thereafter. x_3 and then x_1 reach a distance 1 from x_2, and both pairs remain on the discontinuity surfaces $\sigma_{23}(x) = 1$ and $\sigma_{12}(x) = 1$ for a while before exiting them. A further sliding on multiple discontinuity surfaces happens later ($\sigma_{15}(x) = 1$ and $\sigma_{35}(x) = 1$). Notice that, in spite of the non-proper solution and of the sliding on multiple discontinuity surfaces, monotonicity of the opinions is preserved in this case.

Remark 12 For the model (V3), in Example 10 we have a case of opinions collapsing into each other in finite time, due to the repulsive action. Example 9, instead, shows that identical opinions can split when passing through a discontinuity. Hence it is in principle not guaranteed that a strict version of the monotonicity property **P2** may hold for the model V3.

Example 11 (V3: not P4, not P5 not P9) In dimension 2 consider the solution $\varphi(t)$ starting from the initial condition $[-0.2\ 0.2]^T$ which reaches the equilibrium point $[-1/2\ 1/2]^T$ in finite time T. The quantity $W(x(t))$ is not decreasing, in fact $W(0) = 0.08 < W(T) = 1/2$ (not **P4**). $\Gamma(A(\varphi(t)))$ is fully connected until $t < T$ but $\Gamma(A(\varphi(T)))$ is not connected (not **P5**). This example also shows that $c_s(t)$ is not conserved along trajectories: $c_s(0) = 0.2$ and $c_s(T) = 1/2$ (not **P9**).

Example 12 (V3: not P6) In dimension 3 consider the solution $\varphi(t)$ corresponding to the initial condition $[-1/4\ 1/4\ 5/4]^T$. $\Gamma(A(\varphi(0)))$ is not connected as the nodes 3 does not communicate with the nodes 1 and 2. Thanks to the repulsive action of node 1 on node 2, the nodes 2 and 3 do communicate for $t > 0$, so that $\Gamma(A(\varphi(t)))$ is connected in a interval $(0, T)$. At time T the solution reaches the equilibrium $(1 - x^*, x^*, x^*)$, where $x^* = 3/4$ can be computed taking into account the fact that the average of initial conditions is preserved. We remark that $\Gamma(A(\varphi(T)))$ is not connected.

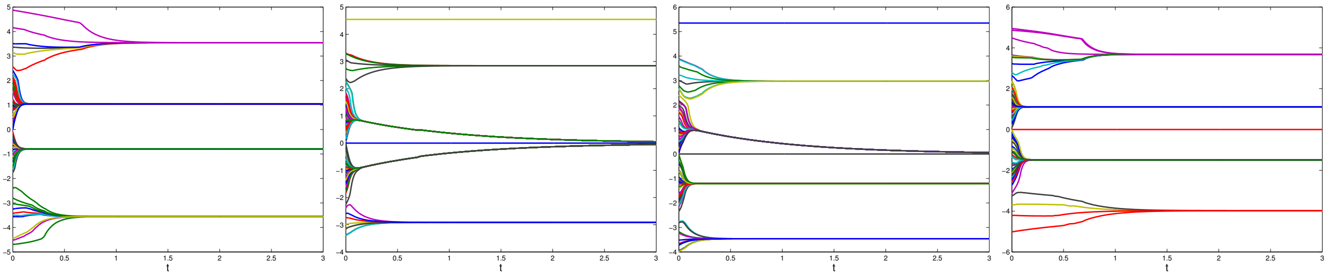

Fig. 3. Example 7 (model (V1)).

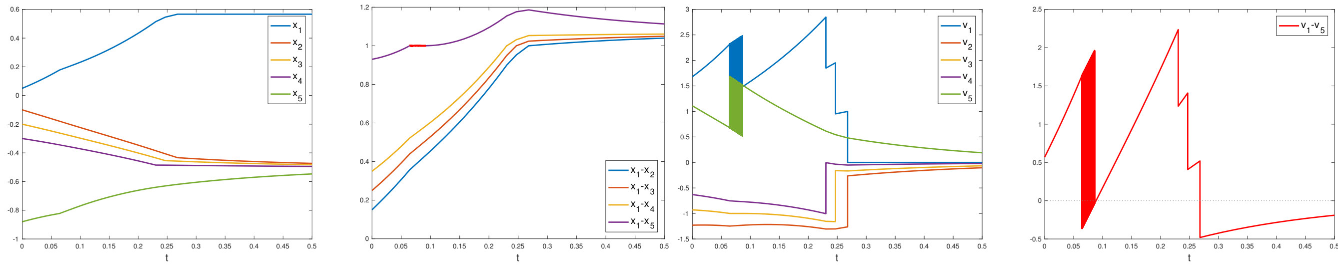

Fig. 4. Example 8 (model V3). Top left: the opinions. Top right: distances among the opinions (in red the interval in which the system lies on the discontinuity surface $\sigma_{15}(x) = 1$). Bottom left: velocities of the opinions. Bottom right: velocity difference showing chattering.

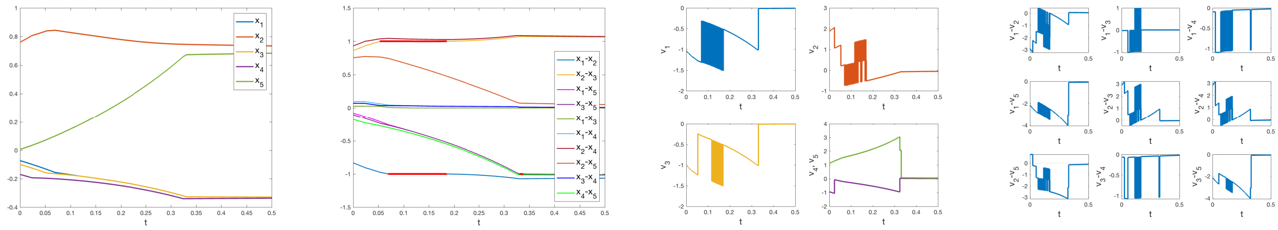

Fig. 5. Example 10 (model V3). Top left: the opinions. Top right: distances among the opinions, The intervals in which the system lies on the discontinuity surfaces are shown in thick red. These surfaces are $\sigma_{12}(x) = 1$ and $\sigma_{23}(x) = 1$ in the first part, and then $\sigma_{15}(x) = 1$ and $\sigma_{35}(x) = 1$. Bottom left: velocities of the opinions. Bottom right: velocity differences showing chattering.

6 Conclusion

In an effort to expand the scope and the applicability of existing bounded confidence models for opinion dynamics, the signed bounded confidence models proposed in this paper combine the clustering behavior of a standard bounded confidence model with sign invariance of the agents opinions. The three variants we propose correspond to three different ways to describe sign invariance. All introduce a state-dependence in the interaction graph, dependence with adds up to that introduced by bounded confidence.

Among the various phenomena we have observed for our discontinuous ODEs, it is worth mentioning the convergent behavior of one of the solutions of our model (V1) (the one inspired by bipartite consensus), in the case of initial conditions that vanish for one or more agents. Given that the signed graphs are by construction structurally balanced, this is an intrinsically nonlinear phenomenon, due to the presence of discontinuities induced by the sign functions, and with no counterpart in linear bipartite consensus. Another interesting feature appears in the model (V3): due to the repulsion, opinions can collapse into each other in finite time, rather than asymptotically as observed in standard bounded confidence models.

In spite of the added complexity, we believe that models featuring sign preservation of the opinions are more suitable than existing ones to describe phenomena like social cleavage and polarization, appearing frequently in opinion dynamics. As future work, we plan to extend the idea also to other classes of models like the French-DeGroot and the Friedkin-Johnsen models.

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