Obtaining Consistent Parameter Estimators for Second-Order Modulus Models

Fredrik Ljungberg and Martin Enqvist

Tweet

Obtaining Consistent Parameter Estimators for Second-order Modulus Models

Fredrik Ljungberg and Martin Enqvist

Abstract—This work deals with the issue of obtaining consistent parameter estimators in nonlinear regression models where the regressors are second-order modulus functions, which is a structure that is often used in models of marine vessels. It is shown that the accuracy of an instrumental variable estimator can be improved by conducting experiments where the input signal has a static offset of sufficient amplitude and the instruments are forced to have zero mean. The proposed method is then evaluated in a simulation example.

Index Terms—Nonlinear system identification, Grey-box modeling, Maritime control.

I. INTRODUCTION

MATHEMATICAL modelling of marine vessels is a rather involved task. The couplings between different motion components in the six degrees of freedom are often complex and rarely negligible. The modelling is further complicated by the nonlinear hydrodynamic forces and moments affecting the vessel. System identification at sea can therefore prove to be challenging. Classical techniques for system identification applied to ship maneuvering include least squares [1], the extended Kalman filter [2], maximum likelihood [3] and model reference adaption [4]. During the past decades, these techniques have been refined in several ways.

In [5] identification of a high-speed trimaran ferry was done using a nonlinear prediction-error method with the unscented Kalman filter. In [6] an offline system identification algorithm was proposed that used a genetic algorithm to minimize a measure of the difference between the reference response and the response obtained with the identified parameters. Another recently suggested technique for parameter estimation is support vector machine regression (SVR). This was applied to ships in [7].

Sometimes the unknown model parameters are obtained in non-data-driven ways. In [8] a nonlinear model of a scale ship was obtained by first having some of the parameters being measured directly, during what is called towing tests. Very accurate parameter estimates can be obtained in this way but the experiments are often expensive and time consuming, especially when carried out in full scale. In [9] a nonlinear model of an underwater vehicle in the shape of a fish was developed. In that work a nonlinear prediction-error method was used to find values for the parameters connected to the frontal part of the robotic fish whereas the tail was modelled using beam theory. Recently there have also been advances in

development of methods using computational fluid dynamics for ship hydrodynamics. In [10] such a method was used to model maneuvers of both a model ship and its full-scale equivalent.

Two main approaches for dealing with nonlinearities in ship models exist in the literature. The first is using a truncated odd Taylor series expansion which was proposed by [11]. Only odd terms are considered because the model must behave in the same way for positive and negative relative velocities due to ship symmetry. The models usually include nonlinear terms of orders one and three.

The second alternative was first proposed in [12] and later in [13] and provides another nonlinear representation called second-order modulus models [14]. The second-order modulus models do, as the name suggests, include second-order terms. The constraint that the model must be based on an odd function is resolved by including absolute values. These models are not necessarily continuously differentiable, and strictly speaking they can therefore not represent the physical system. Experience has however shown that they can describe the water's damping effects quite accurately and they are therefore often used anyway [8].

The ship model

$$
\begin{aligned}
(m - X_{\dot{u}})\dot{u} &= X_{|u|u}|u|\,u + (1-t)T + (m + X_{vr})vr \\
&+ X_{\delta\delta}\delta^2 + X_{ext}, \\
(m - Y_{\dot{v}})\dot{v} + (mx_G - Y_{\dot{r}})\dot{r} &= -(m - Y_{ur})ur \\
&+ Y_{uv}uv + Y_{|v|v}|v|\,v + Y_{|v|r}|v|\,r + Y_\delta\delta + Y_{ext}, \\
(mx_G - N_{\dot{v}})\dot{v} + (I_z - N_{\dot{r}})\dot{r} &= -(mx_G - N_{ur})ur \\
&+ N_{uv}uv + N_{|v|v}|v|\,v + N_{|v|r}|v|\,r + N_\delta\delta + N_{ext},
\end{aligned}
$$

proposed in [15] is simpler than the earliest ones but retains most of the important properties regarding propulsion and steering. This model serves well as an example of a continuous-time second-order modulus model. Here u is the forward velocity (surge) and should not be confused with the propeller thrust, T, and the rudder angle, δ, which are the system's control signals. The other state variables are v, which is the side-ways velocity (sway) and the yaw rate, r. Furthermore, m is the ship's mass, x_G the distance to the center of gravity, t is called the thrust deduction number and X_{ext}, Y_{ext} and N_{ext} are external forces. The remaining parameters are in ship literature referred to as hydrodynamic derivatives, see for example the second chapter in [16] for further details about this. The nonlinear damping effects are similar for surface vessels and underwater vehicles. Second-order modulus models are therefore common in both cases.

A continuous-time second-order modulus model, like the one presented in [15], can be cast as a discrete-time model with the same type of terms, using a forward Euler scheme. The accuracy of the approximation depends on the length of the sampling time in comparison to the frequency of the signal variations. The remainder of the paper will deal with discrete-time models.

Identification of nonlinear ocean vehicle models has been done before, *e.g.* in [5], [6], [7], [8] and [9]. In this paper the focus is instead on obtaining consistent parameter estimators for general second-order modulus models. The work serves as a continuation of [17], where the instrumental variable (IV) method was successfully applied for estimating the parameters of a linear ship model. It will be shown that the accuracy of an IV estimator can be improved by conducting experiments where the input signal has a static offset of sufficient amplitude and the instruments are forced to have zero mean.

II. MOTIVATING EXAMPLE

Obtaining a consistent parameter estimator of even a small scale single-input single-output second-order modulus system can prove to be a bit cumbersome. Consider the system

$$x(k+1) = n_0 x(k)|x(k)| + f_0 u(k) + w(k),$$
$$y(k) = x(k) + e(k),$$

where the two noise sources are mutually independent, stationary, uniformly distributed stochastic processes with zero mean, $-\eta_w < w(k) < \eta_w$, $-\eta_e < e(k) < \eta_e$. Assume that the input is Gaussian and white, also with zero mean and that the system is operating in open loop, *i.e.* that the input, $u(k)$, is not dependent of the measured state, $y(k)$, and consequently assumed to be independent of the noise signals, $w(k)$ and $e(k)$. Let $\theta_0 = \begin{bmatrix} n_0 & f_0 \end{bmatrix}^T$ denote the true system parameter vector. A simple way of modelling this system is to ignore the fact that only noisy measurements of $x(k)$ are available and to consider the one-step ahead predictor model

$$\hat{y}(k|\theta) = ny(k-1)|y(k-1)| + fu(k-1) \triangleq \varphi^T(k)\theta.$$

Here $\varphi(k) = \begin{bmatrix} y(k-1)|y(k-1)| & u(k-1) \end{bmatrix}^T$ denotes the regression vector and $\theta = \begin{bmatrix} n & f \end{bmatrix}^T$ is the unknown parameter vector. The least squares (LS) estimate can be formed as

$$\hat{\theta}_N^{LS} = \operatorname*{argmin}_{\theta} \sum_{k=1}^{N} [y(k) - \varphi^T(k)\theta]^2, \qquad (1)$$

where N is the number of data points. Thus, the asymptotic LS estimate for the system parameters can be obtained as

$$\lim_{N\to\infty} \hat{\theta}_N^{LS} = \lim_{N\to\infty} [\frac{1}{N}\sum_{k=1}^{N}\varphi(k)\varphi^T(k)]^{-1}[\frac{1}{N}\sum_{k=1}^{N}\varphi(k)y(k)]$$
$$= \bar{E}\{\varphi(k)\varphi^T(k)\}^{-1}\bar{E}\{\varphi(k)y(k)\}.$$

The notation $\bar{E}\{.\} = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}E\{.\}$ was adopted from [18]. Due to the causality of the system and the fact that

the input and noise sequences are zero mean, it is assumed to be the case that

$$\bar{E}\{u(k)x(l)\} = 0 \ \forall \ k \geq l,$$
$$\bar{E}\{e(k)x(l)\} = 0, \ \forall \ k, \ l,$$
$$\bar{E}\{w(k)x(l)\} = 0, \ \forall \ k \geq l.$$

Using this and the fact that the system is operating in open loop gives

$$\bar{E}\{\varphi(k)\varphi^T(k)\} = \begin{bmatrix} \bar{E}\{y(k-1)^4\} & 0 \\ 0 & \bar{E}\{u(k-1)^2\} \end{bmatrix},$$
$$\bar{E}\{\varphi(k)y(k)\} = \begin{bmatrix} \bar{E}\{y(k)y(k-1)|y(k-1)|\} \\ \bar{E}\{u(k-1)y(k)\} \end{bmatrix}.$$

It is the case that

$$\bar{E}\{y(k-1)^4\} = \bar{E}\{(x(k-1)^4 + 6x(k-1)^2 e(k-1)^2 + e(k-1)^4\},$$

because the expected value of any odd moment of a zero-symmetric distribution (if it exists) is zero. This holds for $x(k)$ and $e(k)$ alike. Also,

$$\bar{E}\{y(k)y(k-1)|y(k-1)|\} = \bar{E}\{(n_0 x(k-1)|x(k-1)| + f_0 u(k-1) + w(k-1) + e(k))(x(k-1) + e(k-1)) \cdot |x(k-1) + e(k-1)|\} = n_0\bar{E}\{x(k-1)|x(k-1)| \cdot (x(k-1) + e(k-1))|x(k-1) + e(k-1)|\},$$

where the properties that $u(k)$, $w(k)$ and $e(k)$ are mutually independent and white have been used. Lastly,

$$\bar{E}\{u(k-1)y(k)\} = \bar{E}\{u(k-1)(n_0 x(k-1)|x(k-1)| + f_0 u(k-1) + w(k-1) + e(k))\} = f_0\bar{E}\{u(k-1)^2\},$$

making use of the same properties. All in all this means that

$$\lim_{N\to\infty} \hat{\theta}_N^{LS} = \begin{bmatrix} \frac{n_0\bar{E}\{x|x|\cdot(x+e)|x+e|\}}{\bar{E}\{x^4 + 6x^2 e^2 + e^4\}} \\ f_0 \end{bmatrix},$$

where the time index was dropped for simplified notation. This means that in presence of measurement noise, $e \neq 0$, the LS estimator is inconsistent. It is however a known fact that the LS estimate is inconsistent under errors-in-variables (EIV) conditions, see for example [19]. An estimation technique that has proven to work well under EIV conditions before, is the IV method [20].

The IV estimate is

$$\hat{\theta}_N^{IV} = \operatorname{sol}\left\{\frac{1}{N}\sum_{k=1}^{N}\zeta(k)[y(k) - \varphi^T(k)\theta] = 0\right\}, \qquad (2)$$

where $\zeta(t)$ is called the instrument vector and the notation $\operatorname{sol}\{f(x) = 0\}$ is used for the solution to the equation $f(x) = 0$. It has been shown in [21] that having the instrument vector represent a noise-free version of the regression vector is a good choice, at least in the linear case. Assume for a moment that completely noise-free regressors are available for use as instruments

$$\zeta(k) = \begin{bmatrix} x(k-1)|x(k-1)| & u(k-1) \end{bmatrix}^T.$$

If $\bar{E}\{\zeta(k)\varphi^T(k)\}$ is nonsingular, the asymptotic IV estimate can be obtained as

$$\lim_{N\to\infty}\hat{\theta}_N^{IV} = \bar{E}\{\zeta(k)\varphi^T(k)\}^{-1}\bar{E}\{\zeta(k)y(k)\}$$

$$= \begin{bmatrix} \dfrac{\bar{E}\{y(k)x(k-1)|x(k-1)|\}}{\bar{E}\{x(k-1)|x(k-1)|\cdot y(k-1)|y(k-1)|\}} \\ \dfrac{\bar{E}\{u(k-1)y(k)\}}{\bar{E}\{u(k-1)^2\}} \end{bmatrix}.$$

For the new expressions it holds that

$$\bar{E}\{y(k)x(k-1)|x(k-1)|\} = \bar{E}\{(n_0 x(k-1)|x(k-1)| + f_0 u(k-1) + w(k-1) + e(k))x(k-1)|x(k-1)|\}$$
$$= n_0\bar{E}\{x(k-1)^4\},$$

and

$$\bar{E}\{x(k-1)|x(k-1)|\cdot y(k-1)|y(k-1)|\} = \bar{E}\{x(k-1) \cdot |x(k-1)|\cdot(x(k-1)+e(k-1))|x(k-1)+e(k-1)|\},$$

so that, again omitting the time index

$$\lim_{N\to\infty}\hat{\theta}_N^{IV} = \begin{bmatrix} \dfrac{n_0\bar{E}\{x^4\}}{\bar{E}\{x|x|\cdot(x+e)|x+e|\}} \\ f_0 \end{bmatrix},$$

which is inconsistent as well. From (2) it can be noted that for the IV estimate to be consistent it must be the case that

$$\bar{E}\{\zeta(k)\varphi^T(k)\} \text{ is nonsingular,}$$
$$\bar{E}\{\zeta(k)[y(k) - \varphi^T(k)\theta_0]\} = 0,$$

i.e. that the instruments are correlated with the regressors but uncorrelated with the optimal residual. For this system the second requirement means that

$$\bar{E}\{\zeta(k)[n_0 x(k-1)|x(k-1)| + w(k-1) + e(k) - n_0(x(k-1)+e(k-1))|x(k-1)+e(k-1)|]\} = 0.$$

If the instruments are uncorrelated with w and e this implies that

$$n_0\bar{E}\{\zeta(k)[x(k-1)|x(k-1)| - (x(k-1)+e(k-1))|x(k-1)+e(k-1)|]\} = 0,$$

which is not easily fulfilled, while keeping $\bar{E}\{\zeta(k)\varphi^T(k)\}$ nonsingular, for any choice of $\zeta(k)$. This means that despite the fact that the instruments are uncorrelated with the noise sequences, the IV estimator is inconsistent.

An inconsistent estimate can however be avoided by using an input with an offset. Remember that the amplitude of the measurement noise has an upper bound, $|e(k)| < \eta_e$. The input should be such that it excites the system to the extent that its state, $x(k)$ has an amplitude that is bigger than the worst-case amplitude of the measurement noise. Consider the case where the input $u(k) = \bar{u} + \tilde{u}(k)$ is applied, where $\tilde{u}(k)$ is uniformly distributed with zero mean, $-\eta_{\tilde{u}} < \tilde{u}(k) < \eta_{\tilde{u}}$. If the system is stable this will yield an output that, with the notation that $\bar{E}\{x(k)\} = \bar{x}$, can be written like

$$x(k) = \bar{x} + \tilde{x}(k), \quad \bar{E}\{\tilde{x}(k)\} = 0.$$

Further assume that $\bar{u} > 0$, $\bar{x} > 0$ and

$$x(k) = \bar{x} + \tilde{x}(k) > \eta_e > 0.$$

Under the stated circumstances can it be concluded that $x(k) + e(k) > 0$ and consequently that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \zeta(k)[y(k) - \varphi^T(k)\theta_0] = \bar{E}\{\zeta(k)[w(k-1) + e(k) - 2n_0 x(k-1)e(k-1) - e(k-1)^2]\}.$$

This will equal zero for all instruments that are independent of the noise sequences, w and e, while also fulfilling $\bar{E}\{\zeta(k)\} = 0$. For example the instrument vector

$$\zeta(k) = \begin{bmatrix} x(k-1)^2 - \bar{E}\{x(k-1)^2\} & u(k-1) - \bar{E}\{u(k-1)\} \end{bmatrix}^T$$
$$= \begin{bmatrix} \tilde{x}(k-1)^2 + 2\bar{x}\tilde{x}(k-1) - \bar{E}\{\tilde{x}(k-1)^2\} & \tilde{u}(k-1) \end{bmatrix}^T,$$

gives

$$\bar{E}\{\zeta(k)\varphi^T(k)\} = \begin{bmatrix} m_{\tilde{x}} & 0 \\ 0 & \bar{E}\{\tilde{u}^2\} \end{bmatrix},$$

$$\bar{E}\{\zeta(k)y(k)\} = \begin{bmatrix} n_0 m_{\tilde{x}} \\ f_0\bar{E}\{\tilde{u}^2\} \end{bmatrix},$$

where $m_{\tilde{x}} = \bar{E}\{\tilde{x}^4\} + 4\bar{x}\bar{E}\{\tilde{x}^3\} + 4\bar{x}^2\bar{E}\{\tilde{x}^2\} - \bar{E}\{\tilde{x}^2\}^2$. These computations are also based on the fact that $\tilde{x}(k)$ and $\tilde{u}(k)$ are independent. $\bar{E}\{\zeta(k)\varphi^T(k)\}$ is clearly invertible as long as the input is persistently exciting, so the asymptotic parameter estimator is

$$\lim_{N\to\infty}\hat{\theta}_N^{IV} = \bar{E}\{\zeta(k)\varphi^T(k)\}^{-1}\bar{E}\{\zeta(k)y(k)\} = \begin{bmatrix} n_0 \\ f_0 \end{bmatrix} = \theta_0,$$

i.e. the consistency has been shown.

III. THE GENERAL CASE

As mentioned in the introduction, the couplings between different motion components in a ship's six degrees of freedom, are often not negligible. The presented method of obtaining consistency generalizes to multiple-input and multiple-output (MIMO) systems, provided that a couple of assumptions are fulfilled. First a definition is needed.

Definition 3.1: A second-order modulus function is a function, $f : \mathbb{R}^{n+p} \to \mathbb{R}^m$ that can be written as

$$f(\mathbf{x}, \theta) = \Phi^T(\mathbf{x})\theta,$$

where each element of the $p \times m$ matrix $\Phi(\mathbf{x})$ is on one of the forms $x_i, |x_i|, x_i x_j, x_i|x_j|$ for $i, j \leq n$ or zero and $\theta \in \mathbb{R}^p$ is a vector of coefficients.

Consider a nonlinear discrete-time state-space system with n states, m inputs and n outputs

$$\mathbf{x}(k+1) = f(\begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{bmatrix}, \theta_0) + \mathbf{w}(k),$$

$$\mathbf{y}(k) = \mathbf{x}(k) + \mathbf{e}(k),$$

where all the states are measured directly (with noise) and the following assumptions are imposed.

A1. f is a second-order modulus function and its structure is known.

A2. $\mathbf{w}(k)$ and $\mathbf{e}(k)$ are mutually independent stationary white noise sequences with zero mean and well-defined moments of any order. The amplitude of the measurement noise is limited, $-\eta < \mathbf{e}(k) < \eta$.

A3. The system is globally identifiable according to the definition in [18].

Following Definition 3.1, this system can be expressed as

$$\mathbf{x}(k+1) = \Phi^T(\begin{bmatrix} \mathbf{x}(k) \\ \mathbf{u}(k) \end{bmatrix})\theta_0 + \mathbf{w}(k),$$

$$\mathbf{y}(k) = \mathbf{x}(k) + \mathbf{e}(k).$$

Since the structure of the true system is known by Assumption A1, it is reasonable to consider the one-step ahead predictor model

$$\hat{\mathbf{y}}(k|\theta) = \Phi^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta.$$

Furthermore some premises regarding the data collection are assumed to be imposed.

A4. The system is operating in open loop, *i.e.* the input, **u**, does not depend on the measured states, **y**, and is consequently assumed to be independent of the noise signals **w** and **e**.

A5. N_E experiments are performed, where in each N_D data points are collected. The input in each experiment is such that it excites the system to the extent that each of its states, $x_1(k)$, ... $x_n(k)$, continuously has an amplitude that is bigger than the corresponding worst-case amplitude of the measurement noise

$$\left| x_i(k) \right| > \eta_i > \left| e_i(k) \right|, \; k = 1, \; \dots \; N_D, \; i = 1, \; \dots \; n.$$

Assume that for each experiment, E, there is an $p \times n$ instrument matrix

$$\mathbf{Z}_E(k) = \begin{bmatrix} \zeta_{E,1}(k) & \dots & \zeta_{E,n}(k) \end{bmatrix},$$

that fulfills the following assumptions.

A6. $\mathbf{Z}_E(k)$ and $\mathbf{w}(k)$ are mutually independent.

A7. $\mathbf{Z}_E(k)$ and $\mathbf{e}(k)$ are mutually independent.

A8. $\bar{E}\{\mathbf{Z}_E(k)\} = 0$ and the moments of any higher order are well-defined.

The IV estimate is the least-squares solution to the system of pN_E equations

$$\begin{cases} \frac{1}{N_D} \sum_{k=1}^{N_D} \mathbf{Z}_1(k)[\mathbf{y}(k) - \Phi^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta] = 0, \\ \qquad\qquad\vdots \\ \frac{1}{N_D} \sum_{k=(N_E-1)N_D+1}^{N_E N_D} \mathbf{Z}_{N_E}(k)[\mathbf{y}(k) - \Phi^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta] = 0. \end{cases}$$

Lastly assume that when $N_D \to \infty$, the data from all the experiments combined is informative such that the parameters can be determined uniquely.

A.9 $\bar{E}\{\begin{bmatrix} \mathbf{Z}_1(k) \\ \vdots \\ \mathbf{Z}_{N_E}(k) \end{bmatrix} \Phi^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\}$ is full rank.

Lemma 3.1: For a system that fulfills Assumptions A1-A3, an experiment design that fulfills A4-A5 and an instrument matrix that fulfills A6-A9, the IV method defined above is a consistent estimator of θ.

Proof: Under Assumptions A1-A9, the consistency of the IV method can be investigated by analyzing the unbiasedness of the asymptotic IV estimator. Due to Assumption A9, a sufficient condition for consistency is that

$$\bar{E}\{\mathbf{Z}_E(k)[\mathbf{y}(k) - \Phi^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta_0]\} = 0, \; E = 1, \; \dots \; N_E. \tag{3}$$

Denoting the columns of the regression matrix as

$$\Phi(.) = \begin{bmatrix} \varphi_1(.) & \dots & \varphi_n(.) \end{bmatrix},$$

it can be seen that (3) is fulfilled if

$$\bar{E}\{\zeta_{E,i}(k)[y_i(k) - \varphi_i^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta_0]\} = 0,$$

for $i = 1, \; \dots \; n$ and $E = 1, \; \dots \; N_E$. Here

$$y_i(k) - \varphi_i^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta_0 = \varphi_i^T(\begin{bmatrix} \mathbf{x}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta_0$$
$$+ w_i(k-1) + e_i(k) - \varphi_i^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})\theta_0.$$

Since $\bar{E}\{\zeta_{E,i}(k)w_i(k-1)\} = \bar{E}\{\zeta_i(k)e_i(k)\} = 0$, by Assumptions A2, A6 and A7, it remains to show that

$$\bar{E}\{\zeta_{E,i}(k)[\varphi_i^T(\begin{bmatrix} \mathbf{x}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix}) - \varphi_i^T(\begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{u}(k-1) \end{bmatrix})]\} = 0, \quad (4)$$

holds for all $i = 1, \dots, n$. and $E = 1, \dots, N_E$. This residual vector will consist of a combination of different kinds of elements. Elements on the form $u_j, |u_j|, u_j u_l$ or $u_j|u_l|$ are trivially zero since the input is assumed to be perfectly known. Elements on the form $|x_j|$ give

$$\bar{E}\{\zeta_{E,i}(k)[\;|x_j(k-1)| - |x_j(k-1) + e_j(k-1)|]\}$$
$$= \bar{E}\{\zeta_{E,i}(k)[x_j(k-1) - (x_j(k-1) + e_j(k-1))]\} = 0,$$

if $x_j > \eta_j$. This follows by A2, A5 and A7. For the case when $x_j < -\eta_j$ only the sign of the expression changes. Cross-elements on the form $x_j|u_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[x_j(k-1)|u_l(k-1)| - (x_j(k-1) + e_j(k-1))$$
$$\cdot |u_l(k-1)|]\} = -\bar{E}\{\zeta_{E,i}(k)e_j(k-1)|u_l(k-1)|\} = 0,$$

which follows from A2, A4 and A7. Cross-elements on the form $u_j|x_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[u_j(k-1)|x_l(k-1)| - u_j(k-1)|x_l(k-1)$$
$$+ e_l(k-1)|]\} = \bar{E}\{\zeta_{E,i}(k)[u_j(k-1)(x_l(k-1) - (x_l(k-1)$$
$$+ e_l(k-1)))]\} = -\bar{E}\{\zeta_{E,i}(k)u_j(k-1)e_l(k-1)\} = 0,$$

if $x_l > \eta_l$. This follows by A2, A4, A5 and A7. For the case when $x_l < -\eta_l$ only the sign of the expression changes. Finally elements on the form $x_j|x_l|$ give

$$\bar{E}\{\zeta_{E,i}(k)[x_j(k-1)|x_l(k-1)| - (x_j(k-1) + e_j(k-1))$$
$$\cdot |x_l(k-1) + e_l(k-1)|]\} = \bar{E}\{\zeta_{E,i}(k)[x_j(k-1)x_l(k-1)$$
$$- (x_j(k-1) + e_j(k-1))(x_l(k-1) + e_l(k-1))]\}$$
$$= -\bar{E}\{\zeta_{E,i}(k)[x_j(k-1)e_l(k-1) + e_j(k-1)x_l(k-1)$$
$$+ e_j(k-1)e_l(k-1)]\} = 0,$$

if $x_l > \eta_l$. This follows by A2, A4, A5, A7 and A8. For the case when $x_l < -\eta_l$ only the sign of the expression changes.

First and second-order elements without the modulus operator can be seen to equal zero following to the same type of reasoning. Hence, all elements in (4) will be zero, regardless of i, j, l and E. Conclusively (3) is fulfilled so the estimator for θ is consistent. This concludes the proof. ∎

Remark 1: It is important that the input is informative such that the parameters can be determined uniquely. If for example both the regressors $x_j|x_l|$ and $x_l|x_j|$ are present in φ_i, both experiments where x_j and x_l are of the same sign and experiments where they are of opposite sign are needed.

Remark 2: If a whole experiment does not fulfill the requirement that all the states continuously has an amplitude that is bigger than the corresponding worst-case amplitude of the measurement noise, it is possible to form new shorter datasets, only including the parts that do. Only two sequential data points are needed from each set in order to contribute to the estimate. This is due to the nature of the system following Definition 3.1, where the state at sample time k only depends on the state and the input at sample time $k-1$.

IV. SIMULATION STUDY

As shown, the proposed method also works in the MIMO case. However, in order to make succinct comparisons between the proposed IV approach and the LS method, two simulation studies have been carried out using the small-scale system

$$r(k+1) = a_0 r(k) + n_0 r(k)\big|r(k)\big| + f_0 \tau(k) + w(k),$$
$$y(k) = r(k) + e(k),$$

which is describing the yaw dynamics of an underwater vehicle [16]. Here r is the yaw rate, τ is an input force that can be generated either with thrusters or using a rudder and w is some disturbance force, for example caused by underwater currents. It is assumed that the input is perfectly known and that the yaw rate is measured with an additive disturbance. Similar to earlier, the one-step ahead predictor model

$$\hat{y}(k|\theta) = a y(k-1) + n y(k-1)\big|y(k-1)\big| + f\tau(k-1),$$

is considered.

In the motivating example it was unrealistically assumed that exact noise-free versions of the regressors were available for use as instruments. A common way of obtaining instruments in practice is by simulation of a nominal model. In this work it was assumed that a nominal model

$$\hat{r}'(k) = a'\hat{r}'(k-1) + n'\hat{r}'(k-1)\big|\hat{r}'(k-1)\big| + f'\tau(k-1),$$

with crude parameter values was available for this.

Two sets of simulations were carried out, corresponding to two different experiment designs. The baseline input, $\tilde{\tau}(k)$, was a set of pulses with amplitudes between -0.3 and 0.3. The pulses were of varying width and excited the system well. In the first set of simulations this input was used right away $\tau(k) = \tilde{\tau}(k)$, *i.e.* the input had zero mean. In the second set it was combined with a static positive offset

$$\tau(k) = \bar{\tau} + \tilde{\tau}(k), \ \bar{\tau} = 0.4,$$

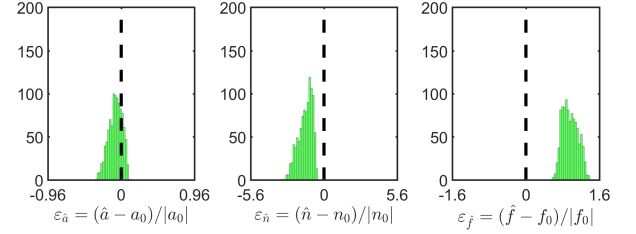| a_0	n_0	f_0	a'	n'	f'
[0.85, 0.95]	[-0.15, -0.05]	[0.75, 1.25]	0.8	-0.2	1.5

Fig. 1. Normalized LS estimation errors for the set of Monte Carlo runs without excitation offset ($\bar{\tau} = 0$). The mean errors plus/minus one standard deviation are $\varepsilon_{\hat{a}} = -0.0766 \pm 0.0883$, $\varepsilon_{\hat{n}} = -1.4237 \pm 0.5312$, $\varepsilon_{\hat{f}} = 0.9603 \pm 0.1670$.
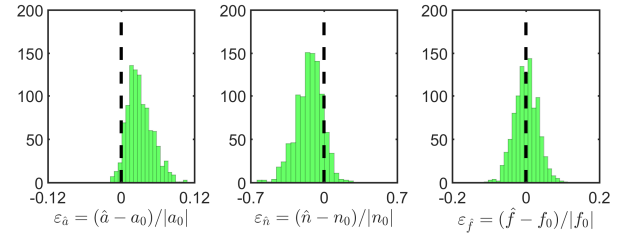

Fig. 2. Normalized IV estimation errors for the set of Monte Carlo runs without excitation offset ($\bar{\tau} = 0$). The mean errors plus/minus one standard deviation are $\varepsilon_{\hat{a}} = 0.0307 \pm 0.0207$, $\varepsilon_{\hat{n}} = -0.1416 \pm 0.1289$, $\varepsilon_{\hat{f}} = 0.0002 \pm 0.0321$.

which means that the pulse amplitudes were instead varying between 0.1 and 0.7. The two noise sources were in both cases sampled from zero mean Gaussian distributions $e(k) \sim \mathcal{N}(0, 0.1)$, $w(k) \sim \mathcal{N}(0, 0.01)$. This means that the distribution of the measurement noise did not have finite support, a choice made in order to test the robustness of the method. Each of the simulation sets used $N = 10^4$ data points for each parameter estimation step, and was repeated 1000 times, using new system parameters and new noise sequences, in a Monte Carlo manner. Both the simulation studies were carried out under the premises stated in Table I. The true system parameters were varied uniformly within the provided intervals whereas the nominal model was fixed.

For the first set of simulations, where $\bar{\tau} = 0$, histograms showing the parameter errors of LS and IV estimates are provided in Figure 1 and 2, respectively. The IV estimates were obtained using the instrument vector

$$\zeta(k) = \left[\hat{r}'(k-1) \ \ \big|\hat{r}'(k-1)\big|\hat{r}'(k-1) \ \ \tau(k-1)\}\right]^T. \quad (5)$$

Neither the LS nor the IV estimator seems to be consistent in this setup. The bias is however more substantial for the LS estimator.

For the second set of simulations, where $\bar{\tau} = 0.4$, the LS method gave the estimation errors in Figure 3. For the IV method with (5) as instruments, the estimation errors are provided in Figure 4. By comparing these results with those
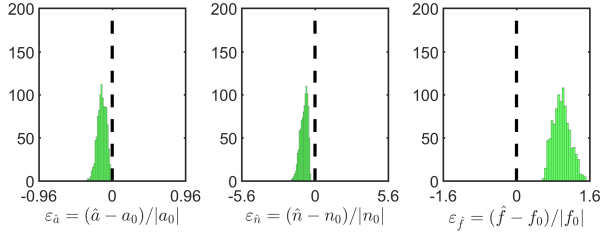
Fig. 3. Normalized LS estimation errors for the set of Monte Carlo runs with excitation offset ($\bar{\tau} = 0.4$). The mean errors plus/minus one standard deviation are $\varepsilon_{\hat{a}} = -0.1369 \pm 0.0574$, $\varepsilon_{\hat{n}} = -0.8528 \pm 0.2860$, $\varepsilon_{\hat{f}} = 0.9853 \pm 0.1879$.
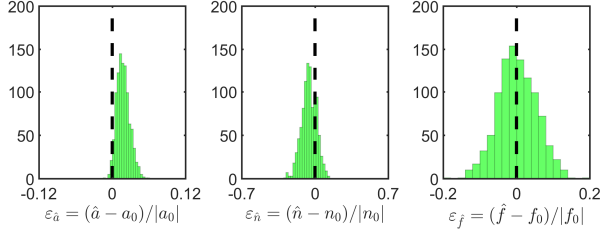

Fig. 4. Normalized IV estimation errors for the set of Monte Carlo runs with excitation offset ($\bar{\tau} = 0.4$). The mean errors plus/minus one standard deviation are $\varepsilon_{\hat{a}} = 0.0180 \pm 0.0107$, $\varepsilon_{\hat{n}} = -0.0560 \pm 0.0707$, $\varepsilon_{\hat{f}} = 0.0020 \pm 0.0532$.

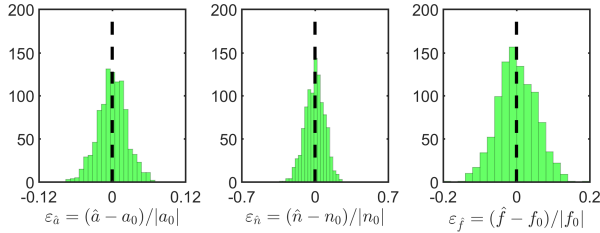Fig. 5. Normalized zero-mean-instrument IV estimation errors for the set of Monte Carlo runs with excitation offset ($\bar{\tau} = 0.4$). The mean errors plus/minus one standard deviation are $\varepsilon_{\hat{a}} = 0.0002 \pm 0.0236$, $\varepsilon_{\hat{n}} = -0.0026 \pm 0.0927$, $\varepsilon_{\hat{f}} = 0.0011 \pm 0.0531$.

obtained earlier, it can be noted that just applying the static offset reduces the bias for the IV estimator. In Figure 5 the IV estimation errors, where the instrument vector

$$\zeta(k) = \begin{bmatrix} \hat{r}'(k-1) - \frac{1}{N}\sum_{k=1}^{N}\hat{r}'(k-1) \\ \left|\hat{r}'(k-1)\right|\hat{r}'(k-1) - \frac{1}{N}\sum_{k=1}^{N}\left|\hat{r}'(k-1)\right|\hat{r}'(k-1) \\ \tau(k-1) - \bar{\tau} \end{bmatrix}$$

was used, are presented. This choice gives instruments with zero mean. It can be noted that removing the mean from the instruments did eliminate the bias. At the same time it increased the variance of the estimates.

V. CONCLUSIONS

A method for obtaining consistent parameter estimators for second-order modulus models has been proposed. This is primarily of interest for maritime applications, where these type of models are common. The tradeoff between variance and bias is unavoidable in system identification and in this work the primary focus has been to eliminate the latter. A potential future work is a more thorough investigation of the

usefulness of the proposed method for smaller datasets. It would also be of interest to estimate a more complete ship model using real data.

REFERENCES

[1] W.-W. Zhou and M. Blanke, "Identification of a class of non-linear state space models using RPE techniques," *IEEE Transactions on Automatic Control*, vol. 34, no. 3, pp. 312–316, 1989.

[2] H. K. Yoon and K. P. Rhee, "Identification of hydrodynamic coefficients in ship maneuvering equations of motion by estimation-before-modeling technique," *Ocean Engineering*, vol. 30, no. 18, pp. 2379–2404, 2003.

[3] K. J. Åström and C. G. Källström, "Identification of ship steering dynamics," *Automatica*, vol. 12, no. 1, pp. 9–22, 1976.

[4] J. Van Amerongen, "Adaptive steering of ships—a model reference approach," *Automatica*, vol. 20, no. 1, pp. 3–14, 1984.

[5] E. R. Herrero and F. J. V. Gonzalez, "Two-step identification of non-linear manoeuvring models of marine vessels," *Ocean Engineering*, vol. 53, pp. 72–82, 2012.

[6] S. Sutulo and C. G. Soares, "An algorithm for offline identification of ship manoeuvring mathematical models from free-running tests," *Ocean Engineering*, vol. 79, pp. 10–25, 2014.

[7] W. Luo, C. G. Soares, and Z. Zou, "Parameter identification of ship maneuvering model based on support vector machines and particle swarm optimization," *Journal of Offshore Mechanics and Arctic Engineering*, vol. 138, no. 3, p. 031101, 2016.

[8] R. Skjetne, Ø. N. Smogeli, and T. I. Fossen, "A nonlinear ship manoeuvering model: Identification and adaptive control with experiments for a model ship," *Modeling, Identification and Control*, vol. 25, no. 1, pp. 3–27, 2004.

[9] V. Kopman, J. Laut, F. Acquaviva, A. Rizzo, and M. Porfiri, "Dynamic modeling of a robotic fish propelled by a compliant tail," *IEEE Journal of Oceanic Engineering*, vol. 40, no. 1, pp. 209–221, 2015.

[10] P. M. Carrica, F. Ismail, M. Hyman, S. Bhushan, and F. Stern, "Turn and zigzag maneuvers of a surface combatant using a URANS approach with dynamic overset grids," *Journal of Marine Science and technology*, vol. 18, no. 2, pp. 166–181, 2013.

[11] M. A. Abkowitz, "Lectures on ship hydrodynamics – steering and manoeuvrability," Hydro- and Aerodynamics Laboratory, Lyngby, Denmark, Tech. Rep., 1964.

[12] K. Fedyaevsky and G. Sobolev, *Control and stability in ship design*. State Union Ship-building House, 1964.

[13] N. H. Norrbin, "Theory and observation on the use of a mathematical model for ship maneuvering in deep and confined waters," in *Proceedings of the 8th symposium on naval hydrodynamics*, Pasadena, California, 1970, pp. 807—904.

[14] D. Clarke, "The foundations of steering and maneuvering," in *Proceedings of the IFAC Conference on Manoeuvering and Control Marine Crafts*, Girona, Spain, 2003, pp. 10–25.

[15] M. Blanke, "Ship propulsion losses related to automatic steering and prime mover control," Ph.D. dissertation, Technical University of Denmark, Lyngby, 1981.

[16] T. I. Fossen, *Guidance and control of ocean vehicles*. John Wiley & Sons Inc, 1994.

[17] J. Linder, M. Enqvist, T. I. Fossen, T. A. Johansen, and F. Gustafsson, "Modeling for IMU-based online estimation of a ship's mass and center of mass," in *Proceedings of the 10th IFAC Conference on Manoeuvring and Control of Marine Craft (MCMC)*, Copenhagen, Denmark, 2015, pp. 198–203.

[18] L. Ljung, *System identification: theory for the user, 2nd edition*. Upper Saddle River, N. J.: Prentice Hall PTR, 1999.

[19] T. Söderström, "Errors-in-variables methods in system identification," *Automatica*, vol. 43, no. 6, pp. 939–958, 2007.

[20] S. Thil, M. Gilson, and H. Garnier, "On instrumental variable-based methods for errors-in-variables model identification," in *Proceedings of the 17th IFAC World Congress*, Seoul, Korea, 2008, pp. 426–431.

[21] M. Gilson, H. Garnier, P. C. Young, and P. M. Van den Hof, "Optimal instrumental variable method for closed-loop identification," *IET control theory & applications*, vol. 5, no. 10, pp. 1147–1154, 2011.